On Stability and Convergence of Multi-Commodity Networks and Services

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***Abstract*—The rise of distributed services and user-driven networking concepts in recent years poses the critical question of stability. Can a system operating under non-cooperation and self-interest converge to a stable state? and how fast? The answers to these questions readily lend themselves to game theory analysis, and to the study of congestion games in particular. In the past, much work have been done on establishing the existence of pure Nash equilibria in congestion games, and has shown that finding a pure Nash equilibrium is PLS-complete [1] and hence convergence to a pure Nash equilibrium is very difficult (exponential time in worst case). Furthermore, much of the convergence analysis have been carried out on simple single-commodity game models. In this paper, we attempt to construct a more realistic multi-commodity congestion game model suited for distributed service and user-driven networking scenarios. We introduce the desirability of equilibrium concept that is helpful in determining whether a system state meets the quality requirements of the users and services. Desirability is an alternative concept to price of anarchy. In fact we show the desirability ratio is a special case of price of anarchy. We then define the α-threshold congestion game whose minimum potential state corresponds to a desirable equilibrium (if the system permits one) and we bound its convergence to polynomial time through game transformation. Finally, we present a mechanism for partial simultaneous moves. To the best of our knowledge, there has been no prior establishment of the desirability concept and no bound given on the convergence of asymmetric multi-commodity congestion games with exponential cost function.**

***Index Terms*—Game theory, stability, convergence**

I. INTRODUCTION

In recent years, there has been an emergence of distributed services (e.g. distributed applications and service oriented architecture), user-driven networking concepts (e.g. overlays, peer-to-peer networks, etc.), and self-stabilizing system designs[1]. In general, they operate under an environment of non-cooperation: a finite distributed population of selfish users sharing a common collection of resources and playing for the maximization of their own utility. This raises the question of stability: are there global stable states of operation in such systems? and if so, how quickly can the system converge to such a stable state from an arbitrary initial state? The question readily lends itself to game theory analysis, in particular to the study of congestion games [2]. In the past, much analysis have been conducted to establish the existence of pure Nash equilibria in congestion games and to determine their complexity (e.g. [1][2][3][4]). In general, there is no guarantee that a pure Nash equilibrium exists in all congestion games [2], and when it does, the number of steps it takes for the system to converge is exponential in worst case [1]. Furthermore, the models used in studying convergence are often over simplified. For instance, the popular K-P model [5] is a single commodity model that assumes all players have a common source and destination, choose a single resource from a shared resource collection, and the resources are independent (e.g. parallel links).

In this paper, we consider the application scenarios in distributed services and user-driven networking by constructing a more realistic congestion game model. In particular, we address the issue of multi-commodity, players with splittable and varying load, and the condition for resource independency. We show our game model is an exact potential game and hence there exists at least one pure Nash equilibrium. Furthermore, we introduce the desirability of equilibrium concept based on a threshold metric that evaluates whether a system state could accommodate the requirements of the services and applications. The intuition is that sometimes it is more important to ensure the players' service requirements are satisfied and to achieve system-wide load balancing than trying to bound the distance to optimality. This is an alternative concept to the traditional price of anarchy [6] and provides an alternative target of convergence. In fact we show that the desirability ratio is a special case of price of anarchy. Since the equilibrium obtained from the minimization of a potential function is not necessarily a desirable equilibrium, we define a α-threshold congestion game in which the minimization of the potential function corresponds to a desirable equilibrium if the system permits one. Finally, we show that convergence to such an equilibrium could be achieved in $O(L^{\lambda}Cn)$ time based on a game transformation that is isomorphic in desirability. We further show that assuming the size of each player's strategy set is a small subset of the resource collection, the player population could be partitioned into groups where all

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players in a group could move simultaneously. We show that problem of finding the minimum number of groups needed to partition the entire population is equivalent to the classic graph coloring problem. The main contribution of our work are: the construction of a more realistic multi-commodity congestion game model; the concept of desirability as an alternative evaluation of the system states; the definition of a α-threshold congestion game that converges to a desirable equilibrium; a convergence bound of at most $O(L^{\lambda}Cn)$ steps; and a mechanism for partial simultaneous moves. To the best of our knowledge, there has been no prior establishment of the desirability concept and no bound given on the convergence of asymmetric multi-commodity congestion games with exponential cost function.

The rest of the paper is organized as follows. Section II gives background and related work in congestion games and their convergence. Section III defines the game model we are interested in and its applications. Section IV establishes the desirability of equilibrium concept and details the construction of the α-threshold congestion game. Section V bounds the convergence of α-threshold congestion game to polynomial time with a game transformation that is isomorphic in desirability. Section VI concludes the paper with summary and open problems.

II. Background and related work

Congestion games were first introduced by Rosenthal [7] and later formalized by Monderer and Shapley [3]. It's a class of games in which the cost of a resource is a non-decreasing function depending on the number of players sharing the resource. A game is called a potential game when there exists a potential function such that the increase in utility of a player (or drop in cost) causes a decrease in potential. All potential games has at least one pure Nash equilibrium. Monderer and Shapley showed that potential games are isomorphic to congestion games. However, Milchtaich [2] showed that this is only the case for unweighed congestion games. A weighed congestion game in which players may have different impacts on the cost function (i.e. varied load among players) may not possess any pure Nash equilibrium. Fabrikant, Papadimitriou and Talwar [1] have shown that the complexity of finding a pure Nash equilibrium in asymmetric congestion games is PLS-complete, and thus under the best-reply dynamic, there exist convergence paths of exponential length.

On the topic of convergence to Nash equilibria in congestion games, the single-commodity K-P model [5] is often used. Milchtaich [2] has shown polynomial time convergence exists for players with varied payoff functions. Goldberg [8] bounded the convergence in such games to polynomial time, and Even-Dar and Mansour [9] considered the case in which all players can move simultaneously according to a Nash rerouting policy and have found a polynomial time convergence bound. In multi-commodity congestion games with simultaneous moves, whether convergenece to a pure Nash equilibrium could be bounded is still an open question and examples could be found in which convergence doesn't occur. Because of its complexity, the study on convergence in general congestion games has been mainly focused on finding convergence bound to approximate solutions. Christodoulou, Mirrokni and Sidiropoulos [10] bounded the solution after one round of best-response walk by all players to Θ(n)-approximate in general case. Chien and Sinclair [11] showed that when the increase in cost of adding a player is bounded ("bounded jump" condition), convergence to ϵ-Nash occurs in polynomial time. In the game model we establish in this paper, the cost jump by adding a player is unbounded, and hence it is difficult to provide a bounded approximation because of the possibility of a player with low payoff gain is unwilling to move and thus "locks out" another player's chance of obtaining a high payoff gain. Goemans, Mirrokni and Vetta [12] studied convergence of Nash dynamics to "sink equilibrium", which is not an approximate of a pure Nash equilibrium. In fact, a sink equilibrium could be formed by a group of cyclic states in some cases.

In bounding the optimality of an equilibrium, price of anarchy (PoA) is frequently used [6]. It defines the ratio between the social cost of an equilibrium and the social optimal. Some research (e.g. [13][14]) have been conducted on bounding the PoA ratio in congestion games and in some cases tight bounds are found. In this paper, we introduce the desirability of equilibrium as an alternative concept. It evaluates whether a system state could accommodate the requirements of the services and applications rather than attempts to bound its social cost distance to the optimal. Under this concept, we are able to obtain polynomial time convergence to desirable equilibria in multi-commodity congestion games.

Some research work have looked into Bayesian congestion games [15][16] for games with incomplete information. Facchini et. al. [15] have shown Bayesian congestion games with uniform player loads exhibit a potential function and hence has pure Bayesian Nash equilibria under the assumption of common prior. Gairing, Monien and Tiemann [16] gave a polynomial time algorithm to compute a pure Bayesian Nash equilibrium for the model of identical links and independent type distribution. In this paper, we do not model the scenarios as Bayesian games for a number of reasons. First, the complexity of Bayesian congestion games are typically higher than congestion games due to the need of each player to know the other players' types and their strategies, and the distribution among player types must be a consistent and common knowledge among players. Second, the required knowledge of each player with regard to other players' strategies, type distribution, and the common prior is difficult to obtain for a player in distributed environment. Third, few work on convergence in asymmetric multi-commodity Bayesian congestion games have been conducted thus far and it's complexity class is an open question.

III. Model Application and Definition

In this section, we define a particular congestion game suitable for distributed services and user-driven networking applications. We will show that such a congestion game has an exact potential and hence there exist pure Nash equilibria. Consider the following scenarios:

Scenario 1: an overlay network consists of a finite number of users (i.e. players). Each user has a set of overlay network paths she may switch to depending on the end-to-end network quality over each path. The end-to-end network quality (e.g. delay) can be represented as the sum of the quality over each overlay link, which is an increasing function depending on the number of user on the link. Each user wants to avoid congestive links in the network and is interested in maximizing her network quality (e.g. minimizing end-to-end delay).

Scenario 2: a service environment hosts a finite number of web service composites (i.e. players). Each web service composite has a set of candidate compositions, where each composition uses a set of service components. The response time of each service component is an increasing function depending on the number of users, and the total response time of a web service composition is the sum of response time of its service components. In choosing one of its candidate compositions, each web service composite wants to ensure each of the selected service component meets some minimum service quality level, while minimizing the total response time.

With the presence of a central controller, the solutions to the above scenarios could be obtained readily. However, when each of the players are allowed to make their own decisions under the assumption of non-cooperation and without knowledge of the global state, there is no guarantee that the overall system have any stable states, and if so, whether convergence to a stable state is possible, and how long it would take. In providing answers to these questions through game theory, we require the description of a game model. Consider the following game model specification:

Let $\Gamma_D = \langle N, \{Y_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a game in strategic form. N is the finite set of players $\{1, \ldots, n\}$, Y_i is the finite set of strategies available to player i and $u_i : Y \rightarrow \Re_+$ where $Y = Y_1 \times Y_2 \ldots \times Y_n$ is the payoff function of player i. Given a finite set of resources $T = \{t_1, \ldots, t_m\}$, define $Y_i \subset 2^T$. Let $A_i \in Y_i$ be a strategy of player i, $A \in Y$ be a strategy profile, c_j be the cost function of resource t_j, and l_j be the normalized serving capacity of t_j, then

$$u_i(A) = \textstyle\sum_{j \in A_i} c_j(A)$$
$$c_j(A) = \frac{x_j(A)}{l_j}$$
$$x_j(A) = \#\{i \in N : t_j \in A_i\}$$

Γ_D is a multi-commodity game that models the *Scenarios* described. Unlike single-commodity models, a multi-commodity game does not limit the players from choosing more than one resource in a single strategy. We observe that the cost function c_j of resource t_j is a strictly increasing function solely depending on the number of players using t_j. Two assumptions must stand for this cost function to hold in practice: 1) each user of a resource t_j adds equal amount of load to t_j (i.e. unweighed game); 2) the resources are independent (i.e. load condition on one resource does not affect load condition on another resource). We now address these two assumptions.

Let N be the finite set of players each with varied load $d_1, \ldots, d_n$, we define a load constant ϑ and a mapping function $SPLIT : N \rightarrow N'$ such that

$$SPLIT(N) = \begin{cases} i \in N', d'_i = \vartheta & : d_i \leq \vartheta \\ \{i_1, \ldots, i_k\} \subset N', k = \lceil \frac{d_i}{\vartheta} \rceil & : d_i > \vartheta \\ d'_{i_1}, d'_{i_2}, \ldots, d'_{i_k} = \vartheta & \end{cases}$$

for all $i \in N$

In effect, for players with load less than ϑ, we tax the load to be ϑ and for players with load greater than ϑ, the load is split into multiple loads of value ϑ, each of which is played independently. the tail of the split i_k is also taxed to be load ϑ. the new game $\Gamma'_D = \langle N', \{Y_i\}_{i \in N}, \{u_i,\}_{i \in N} \rangle$ could then be played as an unweighed game where the serving capacity of each resource is normalized with respect to ϑ. We observe that any state of Γ'_D has a corresponding state in Γ_D with lower cost, and the maximum amount of tax a player might pay is bounded by ϑ. In *Scenario 1*, network flows are typically splittable. In *Scenario 2*, we restrict ourself to look at cases where each service request needs the same amount of processing at a service component. Or for the case of variable workload, the load is splittable. We say Γ'_D is game Γ_D in its taxed form.

We now examine the conditions for resource independency. Consider a pipeline model of two consecutive resources A and B respectively (e.g. a network path). When the aggregate load in A is too high, it is expected that resource B will not be subjected to the same amount of load. Take the network path example, excess load at A would be dropped and thus B is subjected to a lower load than A. w.l.o.g., we say that the assumption of resource independency in a system only hold if the system is in a state with the following condition:

$$\forall t_j \in T, c_j(A) = \frac{x_j(A)}{\alpha l_j} \leq 1, \ \textit{where } 0 < \alpha \leq 1 \qquad (1)$$

The parameter α is used to control the load threshold of resources. For sake of analysis, let the threshold be $\alpha_T \approx 1$. In Section IV, we will show how α is used to define a threshold level for desirability. With respect to the modified cost function of eq.1, we establish the following theorem:

THEOREM 3.1: *A congestion game in its taxed form* $\Gamma_D = \langle N, \{Y_i\}_{i \in N}, \{u_i,\}_{i \in N} \rangle$ *has an exact potential.*

Define $\phi = \sum_{j=1}^{T} \sum_{k=0}^{x_j(A)} \frac{k}{\alpha l_j}$. Given player i changes strategy from A_i to A_i', the change in potential is:

$$\begin{aligned}
\phi_i - \phi_i' &= \sum_{j \in (A_i \cup A_i')} \sum_{k=0}^{x_j(A)} \frac{k}{\alpha l_j} - \sum_{j \in (A_i' \cup A_i)} \sum_{k=0}^{x_j(A')} \frac{k}{\alpha l_j} \\
&= \sum_{j \in (A_i - A_i')} \sum_{k=0}^{x_j(A')} \frac{k}{\alpha l_j} + \sum_{j \in (A_i - A_i')} \frac{x_j(A)}{\alpha l_j} \\
&\quad + \sum_{j \in (A_i' - A_i)} \sum_{k=0}^{x_j(A)} \frac{k}{\alpha l_j} - \sum_{j \in (A_i' - A_i)} \sum_{k=0}^{x_j(A)} \frac{k}{\alpha l_j} \\
&\quad - \sum_{j \in (A_i' - A_i)} \frac{x_j(A')}{\alpha l_j} - \sum_{j \in (A_i - A_i')} \sum_{k=0}^{x_j(A')} \frac{k}{\alpha l_j} \\
&= \sum_{j \in (A_i - A_i')} \frac{x_j(A)}{\alpha l_j} - \sum_{j \in (A_i' - A_i)} \frac{x_j(A')}{\alpha l_j} \\
&= u_i(A) - u_i(A')
\end{aligned}$$

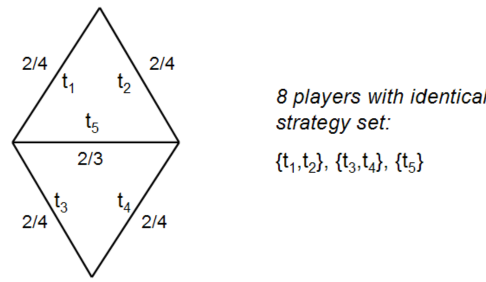
□

Thus, ϕ is an exact potential function of Γ_D. Since a solution always exist when minimizing the value of ϕ over Y, there must exist a pure Nash equilibrium in Γ_D.

IV. Desirability of Equilibrium

A. Definition of desirability

A congestion game with an exact potential function not only is guaranteed to have a pure Nash equilibrium, but also has the finite improvement property (FIP). Hence from an arbitrary state, following the best-reply path, the game is guaranteed to converge to the pure Nash equilibrium over time, albeit exponential in worst case. Figure 1 shows a running example of the congestion game model we have defined in Section III. In this example, there are 8 players with an identical strategy set, 6 of the players are in equilibrium with respect to each other as depicted in the graph. The table shows the payoffs of the two remaining players under different strategy profiles. The bold numbers are computed potential of the system under the specific profile. A '*' symbol besides a payoff indicates it is the player's best-response to the other player's particular strategy. We see that indeed the state with the lowest potential is an equilibrium between the two players. And in fact, this strategy profile is a pure Nash equilibrium for this 8-player game.



Player 2

Player 1	$\{t_1,t_2\}$	$\{t_3,t_4\}$	$\{t_5\}$
$\{t_1,t_2\}$	(2,2) **7.5**	(1.5,1.5) **7**	(1.5,1*) **6.5**
$\{t_3,t_4\}$	(1.5,1.5) **7**	(2,2) **7.5**	(1.5,1*) **6.5**
$\{t_5\}$	(1*,1.5) **6.5**	(1*,1.5) **6.5**	(1.3*,1.3*) **6.3**

Fig. 1. A Running Example of Γ_D

Under this equilibrium state, we observe resource t_5 is in an overloaded condition, yet no player is willing to play another strategy. We argue such a state is a bad configuration for the overall system especially in regard to the scenarios the game is modeled for (e.g. excessive delay and response time, high packet dropping rate and dropped service requests, etc.). Hence, we define the following:

DEFINITION 4.1: *Let* $\Gamma = \langle N, \{Y_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ *be a finite player game in strategic form over a finite collection of resource T=*$\{t_1, \ldots, t_m\}$*, where* l_j *is the capacity of* t_j*. Let* A *be a strategy profile and* A_i *be the strategy of player* i *in* A*.* A *is a* ***desirable equilibrium*** *iff. the following conditions are true:*

$$u_i(A_i, A_{-i}) \leq u_i(A_i', A_{-i}), \forall i \in N (\forall A_i' \in Y_i, A_i' \neq A_i)$$
$$\textit{and}$$
$$\frac{\#\{i \in N : t_j \in A_i\}}{\alpha l_j} \leq 1, \forall t_j \in T, 0 < \alpha \leq 1$$

This is a more strict form of the pure Nash equilibrium definition and therefore all desirable equilibria are pure Nash equilibria. As a weaker argument, we define desirable state as:

DEFINITION 4.2: *Let* $\Gamma = \langle N, \{Y_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ *be a finite player game in strategic form over a collection of resources T=*$\{t_1, \ldots, t_m\}$*, where* l_j *is the capacity of* t_j*. Let* A *be a* *strategy profile and A_i be the strategy of player i in A. A is a **desirable state** iff. the following condition is true:*

$$\frac{\#\{i \in N : t_j \in A_i\}}{\alpha l_j} \leq 1, \forall t_j, 0 < \alpha \leq 1 \quad (2)$$

We call Eq. 2 the *desirability condition*. It is apparent that a desirable equilibrium is also a desirable state. We note that Eq. 2 is in the same form as the resource independency condition established in Eq. 1, hence the link independency assumption holds for a system in desirable state. In general, the parameter α is set to be a threshold level of the resource capacity such that the minimum quality of service required by the player is guaranteed when the number of users on the resource does not exceed this threshold. This is of particular interest to players that require each of their resources to have certain service quality. For example, players in *Scenario 1* may care about the degree of congestion at each link while players in *Scenario 2* may care about the minimum number of processor cycles they may obtain at all the service components they are using. In a congestion game, not all Nash equilibria satisfy the desirability condition, because the player's objective is to minimize her total cost rather than to meet the desirability condition. This is depicted in the example of Figure 1 where the equilibrium is not a desirable state.

In essence, our definition of desirability lends itself to study the problem of distributed load balancing under which the maximum load to capacity ratio on any resource in the system is bounded. This is in effect a special case under price of anarchy where the social cost being considered is the maximum load to capacity ratio of the system. Thus, we can define the *desirability ratio* as:

$$\frac{MAX(\frac{x_j(A)}{l_j})_{j \in T}}{SOC_{min}} \quad (3)$$

Where $x_j(A)$ is the number of users using resource t_j in a state A and SOC_{min} is the optimal $MINMAX$ (i.e. lowest maximum load to capacity ratio of a resource) achievable in the system given the players and their strategy sets.

B. α-threshold congestion game

We now focus on developing a form of congestion game with desirable equilibria (if the system permits one). Our intuition is to define the resource cost as a function of its users such that when the threshold capacity is exceeded, the resource cost becomes a dominating term in the total cost of the players' strategies. Formally, we define α-threshold congestion game Γ_α as:

Let $\Gamma_\alpha = \langle N, \{Y_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ be a game in strategic form. N is the finite set of players $\{1, \ldots, n\}$, Y_i is the finite set of strategies available to player i and $u_i : Y \rightarrow \Re_+$ where $Y = Y_1 \times Y_2 \ldots \times Y_n$ is the payoff function of player i. Given a finite set of resources $T = \{t_1, \ldots, t_m\}$, define $Y_i \subset 2^T$. Let $A_i \in Y_i$ be a strategy of player i, $A \in Y$ be a strategy profile, c_j be the cost function of resource t_j, l_j be the normalized serving capacity of t_j, and α be the threshold parameter, then

$$u_i(A) = \sum_{j \in A_i} c_j(A)$$
$$c_j(A) = (\frac{x_j(A)}{\alpha l_j})^\lambda, \lambda > 1, 0 < \alpha \leq 1$$
$$x_j(A) = \#\{i \in N : t_j \in A_i\}$$

Γ_α is an asymmetric congestion game of finite unweighed players. The cost function has the property that when λ is high, a resource with load level exceeding threshold has its cost exponentially increased, while the cost of a resource below threshold is exponentially reduced (Figure 2). Thus, any resource exceeding the threshold is a dominating term in the cost of a player's strategy that includes it. Furthermore, the value of α could be resource specific.

In practice, we can achieve similar effect using the following cost function:

$$\begin{cases} (\frac{x_j(A)}{\alpha l_j})^\lambda, & x_j(A) \leq l_j \\ (\frac{x_j(A)}{\alpha l_j})^\lambda + \xi, & x_j(A) > l_j \end{cases}$$

ξ is some integer constant (e.g. $\xi \approx 100$). Hence λ can be kept low (e.g. $\lambda \approx 3$). For sake of analysis, we use the function $(\frac{x_j(A)}{\alpha l_j})^\lambda$ in the rest of the paper, we note that the results obtained in this paper still follow when the above equation is used.

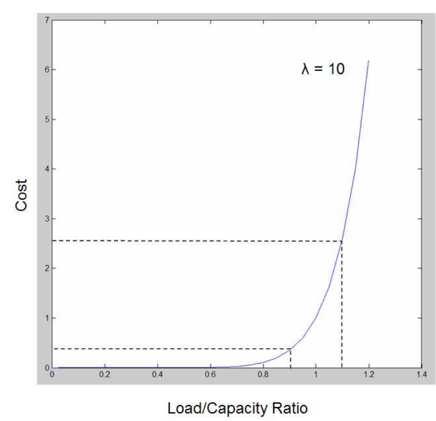


Fig. 2. The Cost Function

LEMMA 4.1: *The α-threshold congestion game Γ_α has an exact potential.*

Define $\phi = \sum_{j=1}^{T} \sum_{k=0}^{x_j(A)} (\frac{k}{\alpha l_j})^\lambda$ as the potential function. The proof follows similar to that of Theorem 3.1.

□

THEOREM 4.2: *A α-threshold congestion game Γ_α has at least one desirable equilibrium if there exist desirable states in the system.*

Proof by contradiction. Assume this is not the case. Let $\lambda \to \infty$. By Lemma 4.1, there exists an equilibrium state A^* induced by the minimization of the potential function, such that $\phi(A^*) \le \phi(A)$, $\forall A \in Y$, $A \neq A^*$. Let A' be a desirable state. Since A' contains no resource with load over the threshold, $\phi(A') \le 1$. Since A^* is not a desirable equilibrium and thus contains at least one resource with load exceeding the threshold, $\phi(A^*) \to \infty$. Therefore, $\phi(A^*) > \phi(A')$, $A^* \neq A'$. We arrive at a contradiction.

□

Figure 3 shows the potential values of the 8-player game depicted in Figure 1 in α-threshold congestion game form. Compared with Γ_D, the α-threshold congestion game Γ_α has its lowest potential corresponding to a state of balanced load in the system and avoids overloading any resources in the system.

$\lambda = 3$

Player 1 \ Player 2	$\{t_1,t_2\}$	$\{t_3,t_4\}$	$\{t_5\}$
$\{t_1,t_2\}$	3.74	*2.58*	2.74
$\{t_3,t_4\}$	*2.58*	3.74	2.74
$\{t_5\}$	2.74	2.74	4.27

Fig. 3. The potentials of 8-player example with $\lambda = 3$

V. Convergence to Desirable Equilibrium

In this section, we address the topic of convergence to desirable equilibria in α-threshold congestion games. Two issues are addressed in particular. First, we attempt to bound the convergence time of the game to polynomial time. We first transform the game Γ_α into a binary factor form Γ_α^T and show there is a polynomial time convergence bound of $O(L^\lambda Cn)$. Furthermore, we establish the isomorphic in desirability property between Γ_α and Γ_α^T and hence show stability obtained through Γ_α^T is a desirable state in Γ_α. Second, we find a mechanism to enable partial simultaneous moves among players. Under the assumption that each player has a strategy set that is a small subset of the resource collection, we can construct a neighborhood graph of the players and show players in different neighborhood can in fact move simultaneously. Effectively, this finding reduces the problem to the classic coloring problem, where all players sharing the same color could move simultaneously and the number of steps needed for all player to move once is equal to the number of colors required to color the entire population.

A. Game transformation and convergence

We now define the transformed game Γ_α^T as a congestion game in strategic form identical to Γ_α except for a transformed resource collection and a transformed strategy set. Let $L = MAX\{\alpha l_j\}_{j\in T}$, construct the binary factor set $B = \{2^0, 2^1, \ldots, 2^{\lfloor \log_2 L \rfloor}\}$. For each resource t_j in T, associate a resource set $t_j^T \subset B$ in Γ_α^T, such that $\sum_{k\in t_j^T} l_k = \alpha l_j$. Hence, the set of resources in Γ_α^T is a binary factoring of the resources in Γ_α. For each strategy A_i of player $i \in N$, associate to Y_i^T the set of strategies $\prod_{t_j \in A_i} t_j^T$. Thus, each strategy A_i of a player i in Γ_α is expanded into a set of strategies over the binary factoring of the resources in A_i.

THEOREM 5.1: *convergence in the transformed game Γ_α^T is bounded by $O(L^\lambda Cn)$.*

Γ_α^T is a α-threshold congestion game ($\alpha = 1$) and hence it has a pure Nash equilibrium. When a player makes a move to a new state, it must be that the new state has a better payoff (i.e. lower total cost) than the old state. Since Γ_α^T has an exact potential (Theorem 4.1), by definition of the exact potential property, there must be a equal drop in potential. We claim that in Γ_α^T, the smallest drop in potential when a player makes a move is $2^{-\lambda\lfloor \log_2 L \rfloor}$. Suppose player i makes a move, let A be the state before the move and A' be the state after the move, by Theorem 3.1, the drop in potential is:

$$\begin{aligned} \Delta\phi &= u_i(A) - u_i(A') \\ &= \sum_{j\in(A_i - A_i')} (\frac{x_j(A)}{l_j})^\lambda - \sum_{k\in(A_i' - A_i)} (\frac{x_k(A)+1}{l_k})^\lambda \end{aligned}$$

Since all l_j are factors of 2, there must exist a sequence of constants $a_1, a_2, \ldots, a_m$ such that for all l_j, $a_j l_j = 2^{\lfloor \log_2 L \rfloor}$. It then follows,

$$\begin{aligned} &\Delta\phi \\ &= \sum_{j\in(A_i - A_i')} (\frac{a_j x_j(A)}{a_j l_j})^\lambda - \sum_{k\in(A_i' - A_i)} (\frac{a_k(x_k(A)+1)}{a_k l_k})^\lambda \\ &= \sum_{j\in(A_i - A_i')} \frac{(a_j x_j(A))^\lambda}{2^{\lambda\lfloor \log_2 L \rfloor}} - \sum_{k\in(A_i' - A_i)} \frac{(a_k(x_k(A)+1))^\lambda}{2^{\lambda\lfloor \log_2 L \rfloor}} \end{aligned}$$

Since all the terms in the above equation has a common denominator, the result of the arithmetic is guaranteed to be some multiples of $2^{-\lambda\lfloor \log_2 L \rfloor}$. Therefore, the smallest potential drop from a move in Γ_α^T is bounded by $2^{-\lambda\lfloor \log_2 L \rfloor}$. Let ϕ_{max} and ϕ_{min} be the upper and lower bounds on the potential values respectively, and C be the upper bound on the cost of any player, then the maximum number of steps to convergence is bounded by:

$$\frac{\phi_{max} - \phi_{min}}{2^{-\lambda\lfloor \log_2 L \rfloor}} \le \frac{nC}{2^{-\lambda\lfloor \log_2 L \rfloor}} = O(L^\lambda Cn)$$

□

THEOREM 5.4: *A desirable equilibrium exists in Γ_α iff. a desirable equilibrium exists in Γ_α^T.*

Given a desirable equilibrium A^* in Γ_α, it must be the case that for all $t_j \in T$, $x_j(A^*) \leq \alpha l_j$. By the method of transformation from t_j to the factored set t_j^T, we know the total capacity does not change. i.e. $\sum_{k \in t_j^T} l_k = \alpha l_j$. Therefore, $x_j(A^*)$ could easily be distributed among the factored resources in t_j^T such that none of the resources exceed threshold. In fact, there's an optimal distribution under the best-reply dynamic. Therefore the transformed state A^{*T} is a desirable state in Γ_α^T. By Theorem 4.2, it is then clear Γ_α^T must have a desirable equilibrium.

Given a desirable equilibrium A^{*T} in Γ_α^T, it must be the case that for all $t_k \in T$, $x_k(A^{*T}) \leq l_k$. With similar reasoning as above, one can assign $\sum_{k \in t_j^T} x_k(A^{*T})$ to the corresponding resource t_j in Γ_α and it is guaranteed that the number of users of resource t_j will not exceed threshold. Hence the transformed state A^* is a desirable state in Γ_α. By Theorem 4.2, Γ_α then must have a desirable equilibrium.

$\square$

PROPOSITION 5.5: *For every desirable state in Γ_α there is a corresponding desirable state in Γ_α^T and vice versa.*

This is a stronger claim that follows from the proof of Theorem 5.4. Given a desirable state in Γ_α, by applying the transformation technique, we obtain a desirable state in Γ_α^T, and vice versa.

$\square$

To formalize our findings, we define the following property concerning Γ_α and Γ_α^T:

DEFINITION 5.6: *given a congestion game Γ and a congestion game Γ'. We say Γ and Γ' are* ***isomorphic in desirability*** *if the following two conditions are true:*

- *There exist desirable equilibria in Γ and Γ'.*
- *There exist transformation functions ζ and ζ', such that for every desirable state A in Γ, $\zeta(A)$ is a desirable state in Γ'; and for every desirable state A' in Γ', $\zeta'(A')$ is a desirable state in Γ.*

It is clear that Γ_α and Γ_α^T are isomorphic in desirability. This implies that an equilibrium state reached in Γ_α^T is a desirable state in Γ_α. In other words, a finite set of players participating in a game Γ, could in fact obtain convergence to stability by playing an isomorphic (hopefully simpler) game Γ', and still arrive at a desirable state in Γ. This is an interesting finding under the concept of desirability, because our target of convergence here is to ensure the system can stabilize in a desirable state rather than converge to a state near optimality which may be quite expensive. Many practical scenarios can benefit from this approach under the desirability concept, such as load balancing, congestion avoidance, quality assured services, etc.

B. Simultaneous move among players

Throughout our analysis, we have considered a system where only one player moves at a time. This property must hold for the system to guarantee convergence under the finite improvement property (FIP). Although there are special cases in single-commodity congestion games [9] that allow for simultaneous player moves and still converge in polynomial time, bounding convergence in asymmetric multi-commodity congestion game in which players move simultaneously is still an open problem. Consider the game presented in Figure 1, if the last two players are allowed to move simultaneously under the best-reply dynamic, they may both choose the strategy set $\{t_1, t_2\}$ in round 1 and then both deterministically switch to the strategy set $\{t_3, t_4\}$ in round 2, and then back to strategy set $\{t_1, t_2\}$ in round 3, and so forth. Hence even with the existence of pure Nash equilibria, the system is not guaranteed to converge to a equilibrium. Mixed strategies may be applied to resolve this problem where a player chooses each of her strategies with some fixed probability. However, mixed strategy games introduce additional complexity in analysis, as the finite improvement property no longer holds in simultaneous moves and it is assumed that the mixed strategy set of each player is common knowledge to the other players in the system, or can be learned over time using some mechanisms such as fictitious play.

Thus far, research in pure congestion games have used various method of facilitating the sequential movement mechanism, such as random selection, round-robin, or highest improvement first. We now consider an approach to enable partial simultaneous moves. Assuming each player's strategy set is a small subset of the common resource collection, let T_i be the set of resource used by player i's strategy set Y_i (i.e. $T_i = \{t_j : t_j \in A_i\}$). Then, we define the *neighborhood* of player i as

$$NB_i = \{k : T_k \cap T_i \neq \varnothing\}_{k \in N}$$

We claim that if a player k is not in the neighborhood of player i or vice versa, then players i and k may move simultaneously in the system and the resulting potential change is as if they have moved in sequence, w.l.o.g. say i moves then k moves.

THEOREM 5.7: *In a α-threshold congestion game, if player i is not a neighbor of player k, then the potential change of players i and k moving simultaneously is equal to the total potential change of player i and player k moving in sequence.*

By Theorem 4.1, it is suffice to study the change in player payoffs. Let A be the state before players i and k move, A'

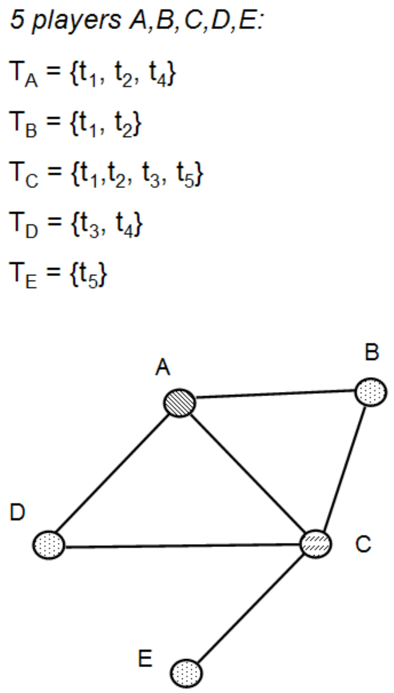


Fig. 4. An Example of Neighborhood Graph and Its Coloring

be the state after player i moves but before k moves, and A'' be the state after both i and k moves, we can construct the total change in potential in state A'' as:

$$
\begin{aligned}
&(u_i(A) - u_i(A')) + (u_k(A') - u_k(A'')) \\
&= \left(\sum_{t_j \in A_i} (\frac{x_j(A)}{\alpha l_j})^\lambda - \sum_{t_j \in A_i'} (\frac{x_j(A')}{\alpha l_j})^\lambda \right) \\
&+ \left(\sum_{t_j \in A_k'} (\frac{x_j(A')}{\alpha l_j})^\lambda - \sum_{t_j \in A_k''} (\frac{x_j(A'')}{\alpha l_j})^\lambda \right)
\end{aligned}
$$

It is apparent that $A_k = A_k'$ and $A_i' = A_i''$. Since $T_i \cap T_k = \varnothing$, then $A_i^{\diamondsuit} - A_k^{\diamondsuit} = A_i^{\diamondsuit}$ and $A_k^{\diamondsuit} - A_i^{\diamondsuit} = A_k^{\diamondsuit}$ for all $A^{\diamondsuit} \in Y$. And a move made by i would not have any impact on the load level of the resources used by k, vice versa. The following equalities hold,

$$
\begin{aligned}
\textstyle\sum_{t_j \in A_i'} (\frac{x_j(A')}{\alpha l_j})^\lambda = \sum_{t_j \in A_i''} (\frac{x_j(A'')}{\alpha l_j})^\lambda \\
\textstyle\sum_{t_j \in A_k'} (\frac{x_j(A')}{\alpha l_j})^\lambda = \sum_{t_j \in A_k} (\frac{x_j(A)}{\alpha l_j})^\lambda
\end{aligned}
$$

Thus we have,

$$
\begin{aligned}
&(u_i(A) - u_i(A')) + (u_k(A') - u_k(A'')) \\
&= \left(\sum_{t_j \in A_i} (\frac{x_j(A)}{\alpha l_j})^\lambda - \sum_{t_j \in A_i''} (\frac{x_j(A'')}{\alpha l_j})^\lambda \right) \\
&+ \left(\sum_{t_j \in A_k} (\frac{x_j(A)}{\alpha l_j})^\lambda - \sum_{t_j \in A_k''} (\frac{x_j(A'')}{\alpha l_j})^\lambda \right) \\
&= \sum_{t_j \in (A_i - A_k)} (\frac{x_j(A)}{\alpha l_j})^\lambda + \sum_{t_j \in (A_k - A_i)} (\frac{x_j(A)}{\alpha l_j})^\lambda \\
&+ \sum_{t_j \in (A_i \cap A_k)} (\frac{x_j(A)}{\alpha l_j})^\lambda - \sum_{t_j \in (A_i'' - A_k'')} (\frac{x_j(A'')}{\alpha l_j})^\lambda \\
&- \sum_{t_j \in (A_k'' - A_i'')} (\frac{x_j(A'')}{\alpha l_j})^\lambda - \sum_{t_j \in (A_k'' \cap A_i'')} (\frac{x_j(A'')}{\alpha l_j})^\lambda
\end{aligned}
$$

We observe this is indeed the potential change for players i and k moving simultaneously.

□

We construct a neighborhood graph $G = \langle N, E \rangle$. G is undirected and an edge exists between two players iff. they are in the neighborhood of each other. The nodes (i.e. players) in G can be partitioned into a number of sets, where in each set, there does not exist an edge connecting any two nodes in the set. It follows from Theorem 5.7 that all nodes in a set could move simultaneously. Thus the minimum number of rounds it takes for all players to have a chance to move is equal to the minimum number of color needed to color the graph. Therefore, this is a reduction to the classic graph coloring problem which is NP-complete. Any distributed heuristic in graph coloring could be applied here. Figure 4 shows an example of the neighborhood graph and the coloring of its nodes.

VI. Conclusion

In this paper, we have established a α-threshold multi-commodity congestion game for distributed services and user-driven networks under the desirability of equilibrium concept. In general, convergence to pure Nash equilibria in asymmetric multi-commodity congestion games are hard, as shown by its PLS-completeness. Existing approximation approaches attempt to converge the game to an approximate state of the equilibria. We observe that in many practical cases, especially in the scenarios we are considering, it is sufficient and often important to assure the system is in a stable state and the requirements of the services and users are met. Hence, we establish desirability as a threshold evaluation of the system state. We have shown in this paper that the existence of a pure Nash equilibrium as ensured by a potential function does not guarantee the desirability of said equilibrium. This motivated our construction of the α-threshold multi-commodity congestion game which has its minimum potential state corresponds to a desirable equilibrium (if the system permits one). Through a transformed game that is isomorphic in desirability, we have provided a polynomial time convergence bound of $O(L^{\lambda}Cn)$. Interestingly, we find games that are isomorphic in desirability could be used "in place" of each other such that a equilibrium reach in one game is a desirable state in another. Hence, players participating in a difficult game, may in effect play an isomorphic but simpler game, whose stable state outcome remains applicable in the difficult game they participate in. Finally, we establish a mechanism to allow simultaneous player moves under the weak assumption that the strategy set of each player is a small subset of the resource collection. Because of the similarities between the game scenarios we explore and the application of self-stabilizing systems, we think the results we obtain in this context could also be useful in designing self-stabilizing systems.

Quite a number of open problems extend from this work. For example, the game model we have used in this paper replies on a taxation mechanism for splitting players with varied load into multiple independent unweighed players. It is interesting to consider the case of weighed atomic players which is more difficult. Furthermore, as we having established in the paper, the desirability ratio is in fact a special case of price of anarchy. Hence a correlation could be made to determine the bound on desirability between the stable state achieved in a game and the system minimal.

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