Graph Isomorphism Completeness for Perfect Graphs and Subclasses of Perfect Graphs

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Abstract

A problem is said to be GI-complete if it is provably as hard as graph isomorphism; that is, there is a polynomial-time Turing reduction from the graph isomorphism problem. It is known that the GI problem is GI-complete for some special graph classes including regular graphs, bipartite graphs, chordal graphs and split graphs. In this paper, we prove that deciding isomorphism of double split graphs, the class of graphs exhibiting a 2-join, and the class of graphs exhibiting a balanced skew partition are GI-complete. Further, we show that the GI problem for the larger class including these graph classes–that is, the class of perfect graphs–is also GI-complete.

1 Introduction

The *Graph Isomorphism* (GI) problem consists of deciding whether two given graphs are isomorphic and thus, consists of determining whether there exists a bijective mapping from the vertices of one graph to the vertices of the second graph such that the edge adjacencies are respected. The GI problem is a well-known open problem that was first listed as an important open problem in Karp's paper over three decades ago [Kar72]. GI is of great interest since it is one of the few problems contained in NP that is neither known to be computable in polynomial time nor to be NP-complete. Presently, there is no known polynomial-time algorithm for graph isomorphism and further, there is strong evidence that the problem is not NP-complete. Mathon [Mat79] demonstrates that the problem of counting the number of isomorphisms between two labeled graphs is Turing reducible to GI; this gives indication that GI is unlikely to be NP-complete since for almost all NP-complete problems their counting versions are of much higher complexity than themselves. The inability to find a polynomial-time algorithm for the GI problem demonstrates evidence that it is unlikely that the problem is in P.

A problem is said to be *GI-complete* if it is provably as hard as graph isomorphism; that is, there is a Turing reduction to the graph isomorphism problem. Graph isomorphism remains GI-complete even when restricted to a number of "hard" special classes, including regular graphs, bipartite graphs, chordal graphs, comparability graphs, split graphs, and k-trees with unbounded k. Recently, Uehara et al. [UTN05] showed that deciding the isomorphism of strongly chordal graphs is GI-Complete. Specific subclasses of bipartite graphs have been shown to be GI-complete–namely, isomorphism is GI-complete for chordal bipartite graphs [NUT93], which shows a distinction between the class of convex graphs and chordal bipartite graphs.

On the other hand, there exists specific cases of the GI problem that have efficient, polynomial-time algorithms; such cases require restrictions upon the class of graphs considered. Examples of such restricted classes include, but are not limited to: planar graphs, interval graphs, convex graph [Che99] and permutation graphs.

Given the current divide between graph classes that exhibit a polynomial-time algorithm for GI and those where GI is provable to be GI-complete, significant attention has been given towards investigating classes for which the relative complexity of the GI problem is not known. Many classes have been proposed and widely investigated; we refer the reader to Brandstädt, Le, and Spinrad [BLS99] for a comprehensive survey of this topic. We prove that deciding isomorphism for double split graphs, the class of graphs admitting a balanced skew partition, and the class of graphs admitting a 2-join is GI-complete. We consider the implications of these results to the related graph classes–namely, QP and SQP.

2 Preliminaries

In order to be self-contained we describe all definitions and results from graph theory and computational complexity required for the comprehension of our results. We restrict attention to finite simple graphs and use standard graph-theoretic notation $G = (V, E)$, where V is the vertex set and $E \subseteq V \times V$. We denote the complement of a given graph G as $\overline{G}$. The complement of a graph G can trivially be computed in polynomial time. A graph $G = (V, E)$ is *bipartite* if V can be divided into two sets A and B such that every edge joins exactly one vertex in A to another vertex in B. A *clique* of a graph G is a subgraph H of G such that every pair of vertices vertices in H are adjacent. We denote $\omega(G)$ as the size of the largest clique in the graph G. A *stable set* is a set A of n vertices such that $G(A) = N_n$, where $G(A)$ is the induced subgraph of A and N_n is the null graph with n vertices.

A *path* has distinct nodes $x_i, \ldots, x_k$ and edges $x_i x_{i+1}$ for $1 \leq i < k$. If $P = x_1, \ldots, x_k$ and $k \geq 3$, then the graph $C = P \cup \{x_k x_1\}$ is called a *cycle*. The number of edges in a path or cycle is its *length*. We denote a path of length k by P_k. An edge which joins two nodes of a path or cycle but is not itself an edge of the path or cycle, is called a *chord*. We refer to a *hole* as an induced subgraph that is a chordless cycle of length at least four and an *antihole* as the complement of a hole. An *even pair* is any pair of non-adjacent vertices such that every chordless path between them has even length.

A *colouring* of a graph $G = (V, E)$ is a mapping $c : V \to \mathcal{C}$ such that $c(x) \neq c(y)$ for all $xy \in E$. If $|\mathcal{C}| = k$ then c is a *k-colouring*. The smallest k such that G admits a k-colouring is the *chromatic number* $\chi(G)$. The gap between the lower bound $\omega(G)$ and the chromatic number $\chi(G)$ can be arbitrary large [Car01]. Berge [Ber61] called a graph G *perfect* if and only if the $\omega(H)$ coincides with $\chi(H)$ for all proper induced subgraphs H of G. All other graphs are *imperfect*.

We say that G is a *double split graph* if $V(G)$ can be partitioned into four sets $\{a_1, \ldots a_m\}$, $\{b_1, \ldots b_m\}$, $\{c_1, \ldots c_n\}$, $\{d_1, \ldots d_n\}$ for some $n, m \geq 2$, such that:

- a_i is adjacent to b_i for $1 \leq i \leq m$ and c_j is nonadjacent to d_j for $1 \leq j \leq n$;
- there are no edges between $\{a_i, b_i\}$ and $\{a_{i'}, b_{i'}\}$ for $1 \leq i < i' \leq n$ and all four edges between $\{c_j, d_j\}$ and $\{c_{j'}, d_{j'}\}$ for $1 \leq j < j' \leq n$;
- there are exactly two edges between $\{a_i, b_i\}$ and $\{c_j, d_j\}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and these two edges have no common end.

Further, we aim to show GI for the class of graphs admitting a *2-join* is GI-complete. We let A, B be a partition of $V(G)$, A_1, A_2 be non-empty disjoint subsets of A and B_1, B_2 be non-empty disjoint subsets of B. If for $i = 1, 2$, every vertex of A_i is adjacent to every vertex of B_i and there are no other edges between A and B, then we say that G admits a 2-join. This concept was introduced by Cornuéjols and Cunningham [CC85] in 1985. Both double split graphs and graphs exhibiting a 2-join are perfect graphs [C+06]. We will show that GI for these graph classes is GI-complete and that GI for perfect graphs is also GI-complete.

2.1 Graph Isomorphism

We define two graphs G_1 and G_2 to be *isomorphic* if there is a bijection $\varphi : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(\varphi(u), \varphi(v)) \in E_2$. We write $G_1 \cong G_2$ and call φ an isomorphism. Hence, the GI problem is to determine if $G_1 \cong G_2$ for given input graphs G_1 and G_2. If G_1 and G_2 are isomorphic, it follows from the definition that there exists a bijection $\varphi : V_1 \to V_2$ which preserves edge adjacencies. Let $e = \{v_1, v_2\}$ and $e' = \varphi^{-1}(e) = \{\varphi^{-1}(v_1), \varphi^{-1}(v_2)\}$. It follows that $e \in \overline{G}_1$ if and only if $e \notin G_1$, and further because φ is an isomorphism for G_1 and G_2 we have that $e \notin G_1$ if and only if $e' \notin G_2$. Finally, $e' \notin G_2$ if and only if $e' \in \overline{G}_2$ proves that $e \in \overline{G}_1$ if and only if $e' \in \overline{G}_2$. If G_1 and G_2 are not isomorphic, a similar result can be shown. Thus, we have the following fact.

Observation 1 *Given two graphs G_1 and G_2 and their respective complements $\overline{G}_1$ and $\overline{G}_2$, $G_1 \cong G_2$ if and only if $\overline{G}_1 \cong \overline{G}_2$.*

An early result by Booth and Lueker [BL79] will be central to a simple proof we give for showing that the class of perfect graphs are GI-complete. They prove that graph isomorphism is polynomially-reducible to chordal graph isomorphism [BL79] by defining a mapping M from an arbitrary graph to a chordal graph. Trivially, this mapping can be carried out in polynomial time and perserves the isomorphism property, demonstrating that GI for the class of chordal graphs is GI-complete.

Theorem 1 *([BL79]) The GI problem for arbitrary graphs is polynomially reducible to chordal graphs.*

Using this result we prove two important results–one that proves that the GI-completeness of a subclass of a graph class implies the GI-completeness of the encompasing class and another that demonstrates that the GI problem restricted to any arbitrary graph class is in GI. The following lemma has important ramfications for many restricted graph classes since the inclusion of chordal graphs is not severe; thus, implying that GI for many restricted graph classes is GI-complete.

Lemma 1 *Given the graph classes α and β such that $\beta \subseteq \alpha$, if GI for β is GI-complete then GI for α is GI-complete.*

Proof. Given that β is GI-complete then β is GI-hard, implying that there there exists a polynomial-time Turing reduction from the GI problem of arbitrary graphs to the GI problem of graphs in the class β. This same reduction can then be applied to graphs in α, thus, showing that the GI problem for class α is in GI. Next, we demonstrate that isomorphism for graphs in α is at least as hard as isomorphism for graphs in β. Suppose otherwise, that there exits a polynomial-time algorithm for solving the GI problem when attention is restricted to graphs in α, then it follows that this algorithm can be applied to any graph in β (since if $G \in \beta$ then $G \in \alpha$). Hence, the same algorithm that is a polynomial-time algorithm for the class α exists for the class β, contradicting the fact that the class β is GI-hard. □

The following lemma demonstrates that in order to demonstrate that any graph class is GI-complete we need only give a polynomial-time reduction from any known GI-comlpete graph class to the considered class that preserves graph isomorphism.

Lemma 2 *The GI problem for any restricted graph class is in the class GI.*

Proof. We define a mapping M from an arbitrary graph to a chordal graph, corresponding to the mapping of Booth and Leuker [BL79]. It is apparent that this mapping can be carried out in polynomial time and preserves the isomorphism property. Since any graph in any given graph class can be considered to be an arbitrary graph, the mapping M can be applied. Hence, this demonstrates that there exists a polynomial-time reduction from any given graph class to the class of chordal graphs that preserves the isomorphism property and therefore is in the class GI. □

3 Main Results

As previously mentioned, the set of known GI-complete graph classes includes bipartite graphs and line graphs [BC79]. Further, when attention is restricted to the class of comparability graphs and chordal graphs isomorphism remains GI-complete [UTN05]. The polynomial-time reduction of isomorphism for arbitrary graphs to isomorphism for chordal graphs then demonstrates that isomorphism for the class of perfect graphs is GI-complete since that reduction can be applied to perfect non-chordal graphs. Further, we restrict interest to two specific subclasses of perfect graphs and show that GI for these classes is GI-complete.

3.1 Perfect Graphs and GI-Completeness

We have the following lemma that demonstrates that the class of perfect graphs is GI-complete. Using Lemma 1 we can explicitly show that the class of perfect graphs is GI-complete.

Lemma 3 *The GI problem for the class of perfect graphs is GI-complete.*

Proof. Clearly, chordal graphs are a subclass of perfect graphs. It follows then from the fact that GI for chordal graphs is GI-complete [BL79] and Lemma 1, that perfect graphs are GI-complete. □

3.2 Reduction for Double Split Graphs

A *split* graph is a graph whose vertex set can be partitioned into a non-empty stable set and a non-empty clique. It is known that the GI problem is GI-complete if attention is restricted to split graphs [UTN05]. We form a double split graph $\mathcal{G}$ by taking a *split graph* $G = (Q \cup S, E)$ where Q is a clique, S is a stable set, and E contains edges between Q and S, replacing every node $x_i \in Q$ by two non-adjacent nodes x'_i, x''_i and every node $y_j \in S$ by two adjacent nodes y'_j, y''_j, and for every edge $\{x_i, y_j\} \in E$ we have $\{x''_i, y''_j\} \in \mathcal{E}$, $\{x'_i, y'_j\} \in \mathcal{E}$, and for every edge $\{x_i, y_j\} \notin E$ we have $\{x''_i, y'_j\} \in \mathcal{E}$, $\{x'_i, y''_j\} \in \mathcal{E}$. We let A be the set of all y'_j, B the set of all y''_j, C the set of all x'_i and D the set of all x''_i. Finally, for every $x'_i, x'_j \in C$, and $x''_i, x''_j \in D$ with $i \neq j$, we have all four edges between $\{x'_i, x''_i\}$ and $\{x'_j, x''_j\}$. See figure 1 for an example of this reduction.

Lemma 4 *$\mathcal{G}$ is a double split graph.*

Proof. First we note that $|A| = |B|$ and $|C| = |D|$, and let $m = |A|, n = |C|$. Our sets can be rewritten as $A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_m\}, C = \{c_1, \ldots, c_n\}, D = \{d_1, \ldots, d_n\}$. Note that $|Q|, |S| \geq 1$ implies that $n, m \geq 2$. Thus, from our construction we see that $\mathcal{G}$ is a double split graph.

Lemma 5 *Given split graphs G_1 and G_2, $G_1 \cong G_2$ if and only if $\mathcal{G}_1 \cong \mathcal{G}_2$.*

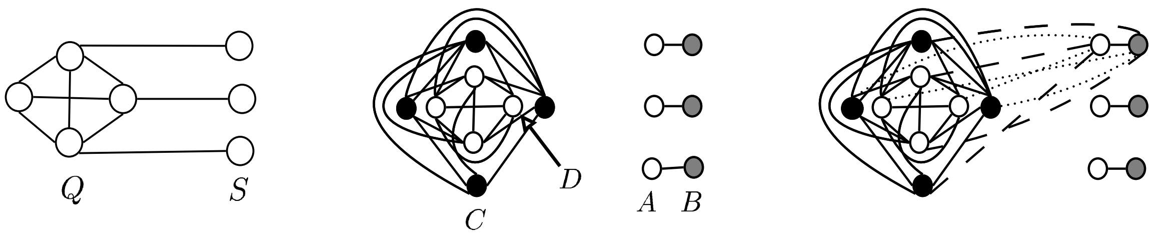

Figure 1: Reduction from split graph G to double split graph $\mathcal{G}$

Proof. It is clear that if $G_1 \cong G_2$ then $\mathcal{G}_1 \cong \mathcal{G}_2$. It remains to show the other direction. By construction, if $\{x_i'', y_j''\} \in \mathcal{E}$ then three things follow: $\{x_i', y_j'\} \in \mathcal{E}$, no other edges exist between $\{x_i', x_i''\}$ and $\{y_j', y_j''\}$, and $\{x_i, y_j\} \in E$. Further, if $\{x_i'', y_j'\} \in \mathcal{E}$ then $\{x_i', y_j''\} \in \mathcal{E}$, no other edges exist between $\{x_i', x_i''\}$ and $\{y_j', y_j''\}$, and $\{x_i, y_j\} \notin E$. It follows that every split graph reduces to an unique double split graph, and that the split graph can be uniquely recovered from our constructed double split graph. Therefore, if $\mathcal{G}_1 \cong \mathcal{G}_2$ then $G_1 \cong G_2$. □

The first of the two main theorems in the paper is the following.

Theorem 2 *The GI problem for double split graphs is GI-complete.*

Proof. Graph isomorphism for the class of double split graphs is trivially in the class GI. It is apparent that the reduction from split graphs to double split graphs can be done in polynomial time and therefore, if follows from lemmas 4 and 5 that this claim for double split graphs holds. □

3.3 Reduction for 2-join Graphs

The reduction from any given non-trivial bipartite graph to a 2-join graph works as follows: given the bipartite graph $G = (A_1 \cup A_2, E)$ with non-empty vertex partitions, the vertex set $\mathcal{V}$ of the reduced graph $\mathcal{G} = (\mathcal{V} = A_1 \cup A_2 \cup B_1 \cup B_2, \mathcal{E})$ contains $A_1 \cup A_2$ and $a_1' \in B_1$ for every $a_1 \in A_1$ and $a_2' \in B_2$ for every $a_2 \in A_2$. The edge set of the reduced graph contains an edge $\{a_1, b_1'\}$ for all $a_1 \in A_1$ and $b_1' \in B_1$ and similarly for each $a_2 \in A_2$ and $b_2' \in B_2$. Finally, $\{a_1, a_2\} \in \mathcal{E}$ if $\{a_1, a_2\} \in E$. See figure 2 for an example of this reduction.

Lemma 6 *$\mathcal{G}$ admits a 2-join.*

Proof. Define $A = A_1 \cup A_2, B = B_1 \cup B_2$. For $i = 1, 2$ every vertex of A_i is adjacent to every vertex of B_i and there are no other edges between A and B. Thus, $\mathcal{G}$ admits a 2-join. □

Lemma 7 *Given non-trivial bipartite graphs G_1 and G_2, $G_1 \cong G_2$ if and only if $\mathcal{G}_1 \cong \mathcal{G}_2$.*

Proof. If $G_1 \cong G_2$ then the result follows. Let $G_1 = (A_{11} \cup A_{12}, E_1), G_2 = (A_{21} \cup A_{22}, E_2)$ and assume that $\mathcal{G}_1 \cong \mathcal{G}_2$ with isomorphism ψ. If $E_1 = E_2 = \emptyset$ then $G_1 \cong G_2$. Assume the edge set is non-empty and note that $\mathcal{G}_1 = (A_{11} \cup A_{12} \cup B_{11} \cup B_{12}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (A_{21} \cup A_{22} \cup B_{21} \cup B_{22}, \mathcal{E}_2)$. For any $b_1 \in B_{11}$ we know that $\psi(b_1) \in B_{21} \cup B_{22}$. Suppose otherwise, and assume that $a_1 = \psi(b_1) \in A_{21}$. b_1 was only adjacent to vertices in A_{11} and so ψ must map each vertex in A_{11} to either B_{21} or the set of vertices adjacent to a_1. Note that because $|B_{21}| > 0$ at least one vertex of A_{11} must be mapped to B_{21}. If $|B_{21}| > |A_{11}|$ this contradicts the ismorphic mapping because the degree of a_1 is not the same as the degree of b_1. If $|B_{21}| < |A_{11}|$ then the vertices from A_{11} mapped to $|B_{21}|$ have too small of a degree. Hence, $|B_{21}| = |A_{11}|$. This means that every vertex in A_{11} was mapped to a single vertex in B_{21}. From this, we see that every vertex in B_{11} was mapped to a single vertex in A_{21}. This contradicts our definition of a 2-join because vertices in B_{11} cannot be adjacent to vertices from A_{12} or B_{12}, which is what ψ must do in order to map the remaining vertices from $\mathcal{G}_1$ to $\mathcal{G}_2$. A similar argument holds if $a_2 = \psi(b_1) \in A_{22}$, and both arguments can be reapplied for any $b_2 \in B_{12}$. Thus, ψ must map vertices from $A_{11} \cup A_{12}$ to $A_{21} \cup A_{22}$ preserving edge adjacencies. Therefore, $G_1 \cong G_2$. □

Theorem 3 *The GI problem for the class of graphs admitting a 2-join is GI-complete.*

Proof. By Lemma 2 GI for the class of graphs admitting a 2-join is in the class GI. It is apparent that the reduction from bipartite graphs to graphs admitting a 2-join can be done in polynomial time and therefore, if follows from Lemma 6 and Lemma 7 that this claim for graphs admitting a 2-join holds. □

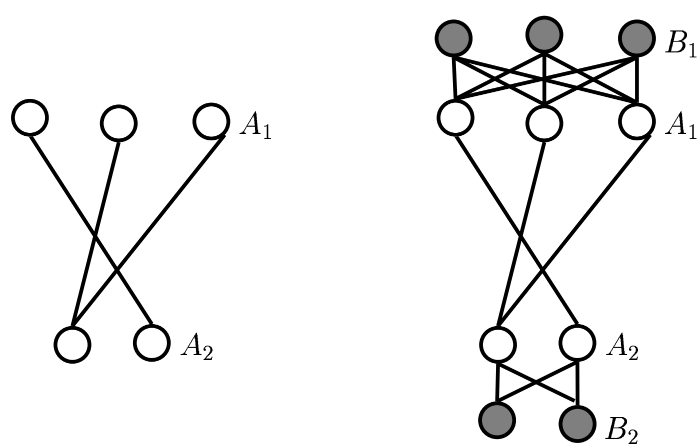

Figure 2: Reduction from bipartite graph G to 2-join graph $\mathcal{G}$

Lemma 8 *The GI problem for the class of graphs whose complement exhibits a 2-join is GI-complete.*

Proof. Again, by Lemma 2 that the GI problem for the class of graphs exhibiting a 2-join is in the class GI. If we let G_1 and G_2 be two graphs admitting a 2-join then their respective complements $\overline{G}_1$ and $\overline{G}_2$ can be computed in polynomial time. This construction together with Observation 1 then shows that the GI problem for the class of graphs whose complement exhibits a 2-join is GI-complete. □

3.4 Implications to Other Graph Classes

On the basis of the concept of even pairs in a graph, we now define two classes of perfect graphs: *quasi-parity* (QP) and *strict quasi-parity* SQP. The class of SQP graphs is defined as the class of graphs where every induced subgraph is either a clique or contains an even pair, and is first defined by Meyniel in 1987 [Mey87]. Meyniel proves that no minimally imperfect graph has an even pair [Mey87] and as a consequence, shows that every SQP graph is perfect. A graph G is called QP if, for every induced subgraph H of G on at least two vertices, either H or its complement has an even pair. Meyniel further proves that QP graphs are perfect [Mey87]. The class of chordal bipartite graphs is contained in the class of strongly chordal graphs[UTN05] and further, strongly chordal graphs are a subclass of SQP graphs [LMR01]. Hence, we have the following inclusion:

$$\text{chordal bipartite} \subseteq \text{strongly chordal} \subseteq \text{SQP} \subseteq \text{QP}.$$

Uehara et al. [UTN05] show that the GI problem for both chordal bipartite graphs and strongly chordal graphs are GI-complete. As an implication of these results, Lemma 2 and inclusion of graph classes described above, we have that GI for SQP graphs is GI-complete. Further, we conclude that GI for QP graphs is GI-complete; this result follows from the above inclusion and an additional application of Lemma 2. In addition, Lemma 2 and inclusion properties of specific graph classes can be used to demonstrate that other graph classes are GI-complete. For example, a graph is *perfectly contractile* if for all induced subgraphs H there is a sequence $H = H_0, \ldots, H_k$ such that H_k is a clique and H_{i+1} is obtained from H_i by contracting an even pair. Therefore, in a similar manner to above it can be demonstrated that perfectly contractile graphs are GI-complete since strongly chordal graphs are a subclass of perfectly contractile graphs [Ber90].

Babel et al. [BPT96] demonstrate that GI for directed path graphs is GI-complete. Uehara et al. [UTN05] explicitly give a reduction from chordal bipartite graphs to a strongly chordal graphs in order to demonstrate that strongly chordal graphs are GI-complete. However, given that directed path graphs are a subclass of strongly chordal graphs and that GI for directed path graphs is GI-complete, Lemma 1 trivially implies that GI for strongly chordal graphs is GI-complete.

4 Concluding Remarks

There are many graph classes for which we do not know whether the GI problem is GI-complete or polynomial-time solvable. In order to obtain further insight into the complexity of the GI problem, the complexity of GI for other restricted graph classes would be advantegous. For example, clique separable graphs, which is a subclass of perfectly contractile graphs as shown by Bertschi [Ber90], is a subclass of perfect graphs where the complexity of the GI problem is unknown. Further, there does not exist a known graph class that is a subclass of clique separable graphs and for which GI is GI-complete, implying that Lemma 1 does not lead to any results concerning this graph class. However, showing GI for clique separable graphs is GI-complete will imply the results in Section 3.3. The implications of Lemma 1 are vast–that is, many graph classes can trivially be shown to be GI-complete.

Recently, Chudnovsky *et al.* [C+06] demonstrate that if G is perfect then either G belongs to one of the five basic classes defined, or one of $G, \overline{G}$ admits a 2-join, or G admits a homogeneous pair, or G admits a balanced skew partition. Chudnovsky later shows in her doctoral thesis that the class of graphs admitting a homogeneous pair can be eliminated [Chu03]. Given a split graph G we know that the vertex set can be partitioned into a stable set A and a clique B. Therefore, every split graph trivially exhibits a balanced skew partition; $G(A)$ is obviously not connected, $\overline{G(B)}$ is not connected, there does not exist any induced path of length greater than or equal to two with ends in B (since B is a clique), and there does not exist any induced path of length greater than or equal to two with ends in A in $\overline{G}$ (A is a stable set and B is a clique). Hence, the class of splits graphs is contained in the class of graphs exhibiting a balanced skew partition and it follows from Lemma 1 that the class of graphs exhibiting a balanced skew partition is GI-complete. It is worth remarking that in proving the GI-completness of GI for double split graphs, the class of graphs exhibiting a 2-join, and the class of graphs exhibiting a balanced skew partition, we demonstrate that each of the graph classes Chudnovsky *et al.* considered to prove the Strong Perfect Graph Theorem are GI-complete.

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