Closed form solutions of linear odes having doubly periodic coefficients

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Abstract

We consider the problem of finding closed form solutions of linear differential equations having doubly periodic coefficients. We give a decision procedure for determining if such equations have doubly periodic solutions and study algorithms for solving such a decision procedure. In addition, in the case of a second order equation we show how to find the general solution to such an ode.

1 Introduction

In this paper we look at linear differential equations of the form

$$a_n(x)y^{(n)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0 \tag{1}$$

where the coefficients $a_i(x)$ are *doubly-periodic*. Doubly periodic functions are complex-valued functions having two independent periods, that is, two distinct constants T and T' such that

$$f(x+T) = f(x) \text{ and } f(x+T') = f(x) \text{ for all } x.$$

Classical examples of doubly-periodic functions include the Weierstrass $\wp$ and $\wp'$ functions, the Jacobi sn, cn and dn functions and ratios of Theta functions [2]. A doubly-periodic function that is analytic everywhere (except at poles) is called an *elliptic function*. Elliptic functions, under this definition, must not have any essential singularities. We remark that the term doubly-periodic is often used when elliptic is really meant, it being assumed that the functions under discussion are also analytic.

Linear differential equations with doubly periodic coefficients appear historically in many instances (cf. [8]). For example, solving Laplace's equation in three dimensions in ellipsoidal coordinates using the separation of variables method gives the classical Lamé equation

$$y''(x) - (n(n+1)\wp(x) + B)y(x) = 0 \tag{2}$$

where n is a positive integer and B is arbitrary. Additional interesting examples can be found in many texts, for example Kamke [13] and Forsyth [8].

There are two classical representations of doubly periodic functions, the Weierstrass form and the Jacobi form. In either case, we can convert (1) to a linear ode having coefficients from $K(x, \sqrt{\omega(x)})$ where $\omega(x)$ is a polynomial of degree 3 or 4 and K is a field of constants. Using standard modern methods from computer algebra we obtain a decision procedure for determining when (1) has doubly periodic solutions.

Picard's theorem [8] implies that when solutions of (1) are single-valued then in general the ode has a basis of solutions which are *doubly-periodic of the second kind.* These functions can be expressed in terms of the Weierstrass Zeta and Sigma functions and the Jacobi Zeta function [2]. In this paper we show how we can find such a general solution in the case of second order linear odes. We do this by finding doubly-periodic solutions of the symmetric power of the original ode. This is similar to finding a basis of exponential solutions for a particular ode having algebraic coefficients. Finally, we study the problem of ensuring that our procedures can be converted into practical algorithms.

The remainder of the paper is organized as follows. In the next section we show how to obtain a decision procedure for solutions of (1). Section 3 looks at the case of general solutions for our ode, and provides details of Picard's theorem which describes such solutions. Section 4 gives a method for solving the order two problem while Section 5 discusses a number of different strategies for solving the linear odes that arise in this study. The paper ends with a conclusion along with topics for future research.

2 Finding Doubly Periodic Solutions of Arbitrary Order Equations

In this section we consider the problem of finding doubly periodic solutions for linear odes having doubly periodic coefficients. We show that this problem can be converted into one of solving a 2×2 system of linear odes of rational functions. The existence of decision procedures for finding rational solutions of such systems (c.f. Singer [16, 17]) then results in a decision procedure for doubly periodic solutions of (1).

There are two classical representations of doubly periodic functions, the Weierstrass form and the Jacobi form. Let $\wp(x) = \wp(x; g_2, g_3)$ denote the Weierstrass $\wp$ function where g_2 and g_3 are constants which are determined by the periods. Then $\wp'(x) = \sqrt{4\wp(x)^3 - g_2\wp(x) - g_3}$ and every doubly-periodic function can be represented as a rational function of $\wp$ and $\wp'$ (cf. [3]). In the case of Jacobi forms there are many representations for doubly-periodic functions. For example, let $sn(x) = sn(x, k)$ where k is a constant determined by the periods. Then $sn'(x) = cn(x) \cdot dn(x) = \sqrt{(1 - sn^2(x)) \cdot (1 - k^2 sn^(x))}$ and every doubly-periodic function can be represented as a rational function of sn and sn'. A similar statement holds for all the other 11 forms of the Jacobi elliptic functions.

In order to cover the principal representations for such problems we do the following.

Lemma 2.1 *Let L be a linear differential operator having coefficients from the domain $\mathbb{K}(f, f')$ with $\mathbb{K}$ a field of constants and where f satisfies $(f')^2 = \omega(f)$ for some polynomial $\omega(z) \in \mathbb{K}[z]$. Then one has a decision procedure for finding solutions of $L(y) = 0$ in the domain $\mathbb{K}(f, f')$.*

Proof: Let

$$L = a_n D^n + \cdots a_1 D + a_0$$

with the coefficients $a_i \in \mathbb{K}(f, f')$ and where $D = \frac{d}{dx}$. Elements in $\mathbb{K}(f, f')$ can be written in the form $r(f) + s(f) \cdot f'$ with $r(f), s(f) \in \mathbb{K}(f)$. Assume that the coefficients of L can be written as

$$a_i = r_i(f) + s_i(f) \cdot f'$$

for each i and that we are looking for a solution y of the form $y = b(f) + c(f) \cdot f'$. Then for each i we have

$$\frac{d^i y}{dx^i} = b_i(f) + c_i(f) f'$$

with $b_0(f) = b(f), c_0(f) = c(f)$ and

$$b_i(f) = \frac{dc_{i-1}}{df} \cdot \omega + \frac{c_{i-1}}{2} \cdot \frac{d\omega}{df} \text{ and } c_i = \frac{db_{i-1}}{df}$$

for $i > 0$. From $(f')^2 = w(f)$ we have that $2 \cdot f \cdot f' = w' \cdot f'$ and hence

$$f'' = \frac{w'}{2} \in \mathbb{K}(f, f').$$

Similarly, f''', f'''' are given in terms of elements in $\mathbb{K}(f, f')$. Substituting $y = b(f) + c(f) f'$ in our ode and simplifying results in a 2×2 system of linear odes for $b(f)$ and $c(f)$

$$\begin{bmatrix} L_{1,1} & L_{1,2} \\ L_{2,1} & L_{2,2} \end{bmatrix} \cdot \begin{bmatrix} b(x) \\ c(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

where $L_{i,j}$ is a linear differential operator in $D_f = \frac{d}{df}$ of order at most the order of L. The result then follows from the fact that there exist decision procedures for solving systems of equations given by (3) (c.f. [17, Prop. 3.2]). □

Remark 2.2 *Lemma 2.1 can be used for cases where one has a linear ode with coefficients which are periodic functions rational in* sin *and* cos *when one is looking for periodic functions which are also rational in* sin *and* cos*. Thus for Mathieu's equation*

$$y''(x) + (p - k^2 \cdot v^2 \cdot \cos^2(x)) \cdot y(x) = 0 \tag{4}$$

one can determine that there are no solutions rational in sin *and* cos*. Unfortunately, this is not the same as saying that there are no periodic solutions for Mathieu's equation since not all periodic functions are rational in* sin *and* cos*.*

Let us return to the case where we are looking for doubly-periodic solutions of linear odes having doubly-periodic coefficients. Assume first that our coefficients are in Weierstrass form and that we are looking for solutions in the same form. Setting $f(x) = \wp(x; g_2, g_3)$[1], the Weierstrass $\wp$ function, allows us to make use of the above formalism using $\omega(z) = 4z^3 - g_2 z - g_3$. Hence the problem of finding a doubly-periodic solution to a linear ode with doubly-periodic coefficients reduces to finding rational solutions of a 2×2 system of linear odes. A similar statement can be made for doubly-periodic functions represented in Jacobi form.

[1] As in classical texts, we drop the g_2 and g_3 arguments for $\wp$ in cases where this is obvious

Example 2.3 *Consider Lamé's equation (2) given in Weierstrass form. In this case, the linear system of equations is already separated into the two linear odes:*

$$\omega(\wp)\cdot\frac{d^2b(\wp)}{d\wp^2}+(6\wp^2-\frac{g_2}{2})\frac{db(\wp)}{d\wp}-((n(n+1)\wp+B)b(\wp)=0$$

$$\omega(\wp)\cdot\frac{d^2c(\wp)}{d\wp^2}+\frac{(36\wp^2-3\cdot g_2)}{2}\frac{dc(\wp)}{d\wp}+((12-n(n+1))\wp-B)c(\wp)=0.$$

In general these do not have rational solutions and hence the Lamé equation does not have doubly-periodic solutions. However, if $n=3$ and $B=0$ then Lamé's equation is just $y''(x)=12\wp(x)y(x)$. It is easy to see that this has the doubly-periodic solution $y(x)=\wp'(x)$. Similarly, if $n=2$, $B=g_2=0$ then $y''(x)=6\wp(x)y(x)$ has the doubly-periodic solution $y(x)=\wp(x)$. □

3 Finding a General Solution of Arbitrary Order Equations

A second class of functions which are *almost* doubly-periodic also plays an important role in the study of linear odes of the form (1). A function $F(x)$ is said to be *doubly-periodic of the second kind* if there exist two periods T, T', and two constants s, s' such that

$$F(x+T)=s\cdot F(x),\ \ F(x+T')=s'\cdot F(x)$$

for all x in the complex plane. These functions are not, in general, truly periodic, as the value of $F(x)$ changes by a constant factor each time x changes by the period.

A classical result in the theory of linear odes with doubly-periodic coefficients is the following observation given by Picard (see Ince [12]). Picard's Theorem tells us that in the case of uniform solutions we can find a general solution to our ode in terms of a basis of doubly-periodic functions of the second kind. As such we will look for solutions of (1) which are doubly periodic of second kind. For our purposes it is necessary to see why Picard's theorem holds, particularly as some of the details will help guide the development of our new algorithm. Hence we will also sketch a proof here.

Theorem 3.1 (Picard's Theorem) *When the coefficients of a homogeneous linear differential equation are doubly-periodic functions of the independent variable, the equation possesses a fundamental set of solutions which, if uniform (that is, single-valued), are in general doubly-periodic functions of the second kind.*

Proof: Let the given n-th order linear homogeneous ode be given by (1) where the coefficients $a_i(x)$ are all doubly-periodic functions of the first kind, with periods T, T'. Note that if $f(x)$ is a solution of the linear ode (1), then $f(x+T)$ and $f(x+T')$ also satisfy the equation.

Let $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ be a basis (or fundamental system) of solutions of (1). Then $f_1(x+T)$, $f_2(x+T)$, ..., $f_n(x+T)$, and $f_1(x+T')$, $f_2(x+T')$, ..., $f_n(x+T')$ are also solutions of (1). Hence each can be expressed as a linear combination of the basis functions:

$$f_j(x+T)=a_{1j}f_1(x)+a_{2j}f_2(x)+\cdots+a_{nj}f_n(x) \quad (5)$$

and

$$f_j(x+T') = b_{1j}f_1(x) + b_{2j}f_2(x) + \cdots + b_{nj}f_n(x) \tag{6}$$

where all the a_{ij}, b_{ij} are constants, independent of x.

These equations may be rewritten in matrix form as

$$\tilde{f}(x+T) = \tilde{f}(x)A \text{ and } \tilde{f}(x+T') = \tilde{f}(x)B \tag{7}$$

where

$$\tilde{f}(x) = [f_1(x), \cdots, f_n(x)],\ A = [a_{ij}]_{n\times n},\ B = [b_{ij}]_{n\times n}. \tag{8}$$

Now, not only is $\{f_1(x), \ldots, f_n(x)\}$ a basis for (1), but so are both $\{f_1(x+T), \ldots, f_n(x+T)\}$ and $\{f_1(x+T'), \ldots, f_n(x+T')\}$ (cf. [7]). From this it follows that both A and B are non-singular.

We now wish to show the existence of a non-trivial solution to (1) that is doubly-periodic of the second kind, that is, a solution $F(x)$, not identically zero, such that $F(x+T) = sF(x)$, $F(x+T') = s'F(x)$, for some non-zero constants s, $s' \in C$. Let us first find a solution $F(x)$ which satisfies the first condition, $F(x+T) = sF(x)$, and consider the second period T' later.

Since $F(x)$ is a non-trivial solution of (1), it can be expressed as a linear combination of the basis $\{f_1(x), f_2(x), \ldots, f_n(x)\}$:

$$F(x) = \lambda_1 f_1(x) + \lambda_2 f_2(x) + \cdots + \lambda_n f_n(x), \tag{9}$$

where λ_1, λ_2, ..., λ_n are constants, not all zero. Then

$$\begin{aligned} F(x+T) = &\ \lambda_1 f_1(x+T) + \lambda_2 f_2(x+T) + \cdots + \lambda_n f_n(x+T) \\ = &\ \lambda_1 \left(\textstyle\sum_{j=1}^n a_{j1} f_j(x)\right) + \lambda_2 \left(\textstyle\sum_{j=1}^n a_{j2} f_j(x)\right) + \cdots + \lambda_n \left(\textstyle\sum_{j=1}^n a_{jn} f_j(x)\right) \\ = &\ \left(\textstyle\sum_{i=1}^n a_{1i}\lambda_i\right) f_1(x) + \left(\textstyle\sum_{i=1}^n a_{2i}\lambda_i\right) f_2(x) + \cdots + \left(\textstyle\sum_{i=1}^n a_{ni}\lambda_i\right) f_n(x). \end{aligned} \tag{10}$$

We want

$$\begin{aligned} F(x+T) &= sF(x) \\ &= s\lambda_1 f_1(x) + s\lambda_2 f_2(x) + \cdots + s\lambda_n f_n(x). \end{aligned} \tag{11}$$

Equating the two expressions for $F(x+T)$ gives

$$\left(\sum_{i=1}^n a_{1i}\lambda_i - s\lambda_1\right) f_1(x) + \left(\sum_{i=1}^n a_{2i}\lambda_i - s\lambda_2\right) f_2(x) + \cdots + \left(\sum_{i=1}^n a_{ni}\lambda_i - s\lambda_n\right) f_n(x) = 0. \tag{12}$$

Since $\{f_1(x), f_2(x), \ldots, f_n(x)\}$ is a linearly independent set, each coefficient must be zero, i.e., it is necessary and sufficient that $s, \lambda_1, \ldots, \lambda_n$ satisfy

$$\begin{aligned} (a_{11}-s)\lambda_1 + a_{12}\lambda_2 + \cdots + a_{1n}\lambda_n &= 0 \\ a_{21}\lambda_1 + (a_{22}-s)\lambda_2 + \cdots + a_{2n}\lambda_n &= 0 \\ &\vdots \\ a_{n1}\lambda_1 + a_{n2}\lambda_2 + \cdots + (a_{nn}-s)\lambda_n &= 0, \end{aligned} \tag{13}$$

where not all of $\lambda_1, \ldots, \lambda_n$ are zero. We see that s is an eigenvalue, and $\lambda = (\boldsymbol{\lambda_1}, \ldots, \boldsymbol{\lambda_n})^{\mathbf{T}}$ a corresponding (non-zero) eigenvector, of the matrix A. Since A is nonsingular we also have that $s \neq 0$.

Similarly, working with the second period T', we can show the existence of a solution $G(x)$, not identically zero, such that $G(x+T') = s'G(x)$, where $s' \neq 0$. We let

$$G(x) = \lambda_1^{'} f_1(x) + \lambda_2^{'} f_2(x) + \cdots + \lambda_n^{'} f_n(x), \tag{14}$$

where not all of $\lambda_1^{'},\ldots,\lambda_n^{'}$ are zero. We then find the necessary and sufficient condition is that s' be an eigenvalue, and $\lambda^{'} = (\lambda_1^{'},\ldots,\lambda_n^{'})^{\mathbf{T}}$ a corresponding (non-zero) eigenvector, of the matrix B. Again, since B is nonsingular $s' \neq 0$.

However, we require a solution $F(x) = \lambda_1 f_1(x) + \cdots + \lambda_n f_n(x)$ to satisfy both $F(x+T) = sF(x)$ and also $F(x+T') = s'F(x)$. This could only occur if there exists some vector $\lambda = (\lambda_1,\ldots,\lambda_n)^{\mathbf{T}}$ which is an eigenvector of both A and B (although the corresponding eigenvalues s, s' could be different, of course).

Now, if matrices A and B were completely independent of each other, this could not be guaranteed to occur. But in fact, because we have assumed that our solutions are single-valued and because the doubly-periodicity implies that all doubly-periodic solutions satisfy

$$F(x+T+T') = F(x+T) = F(x) \text{ and } F(x+T+T') = F(x+T') = F(x)$$

we find that A and B commute (for details see [7]). Classical results from linear algebra then implies that there exists at least one common eigenvector for both A and B. Therefore for any eigenvalue s of A, there exists an eigenvector of A corresponding to s, $\lambda = (\lambda_1,\ldots,\lambda_n)^{\mathbf{T}}$, that is also an eigenvector of B corresponding to some eigenvalue s' of B. Hence a solution of (1) will exist which is doubly-periodic of the second kind, as claimed by Picard's Theorem. □

We remark that in the general case we could expect all n eigenvalues of A to be distinct, and the same to be true for B. Now, since eigenvectors corresponding to different eigenvalues will be linearly independent, there exist n linearly independent eigenvectors, which are also eigenvectors of B. Then the n solutions corresponding to these eigenvectors will also be linearly independent. Hence, in this general case, there will exist a basis of solutions of (1), each of which is a doubly-periodic function of the second kind.

When the eigenvalues of A are not distinct, we can no longer guarantee that there exist n linearly independent eigenvectors, common to both A and B, and hence there is no guarantee that there exists a basis of n solutions, all of which are doubly-periodic of the second kind. But there will always be at least one non-zero eigenvalue s of A, and at least one non-zero eigenvalue s' of B, which have a corresponding eigenvector λ in common. Hence, if the general solution is uniform, there will always exist at least one solution which is doubly-periodic of the second kind.

From the proof of Picard's Theorem, we may deduce the following:

Lemma 3.2 *If an ode satisfies the conditions of Picard's Theorem, then for the two periods T and T', the products of the eigenvalues, taking multiplicities into account, are* $\exp\left(-\int_x^{x+T}\frac{a_{n-1}(u)}{a_n(u)}\,du\right)$ *and* $\exp\left(-\int_x^{x+T'}\frac{a_{n-1}(u)}{a_n(u)}\,du\right)$*, respectively. In particular, if $a_{n-1}(u) = 0$, then the two products are both 1.*

Proof: Consider the period T, and the corresponding multipliers s_1,s_2,…,s_n. As shown earlier, the s_i are precisely the eigenvalues of A so $\det(A) = s_1 s_2 \cdots s_n$. Now, consider (7) for the period T:

$$\tilde{f}(x+T) = \tilde{f}(x)A \tag{15}$$

Differentiating $n-1$ times w.r.t. x gives $n-1$ additional equations of the form

$$\tilde{f}^{'}(x+T) = \tilde{f}^{'}(x)A,\ \ldots,\ \tilde{f}^{(n-1)}(x+T) = \tilde{f}^{(n-1)}(x)A. \tag{16}$$

Combining these into a single matrix equation, we have

$$[\ \tilde{f}(x+T)\ \mid\ \cdots\ \mid\ \tilde{f}^{(n-1)}(x+T)\ ] = [\ \tilde{f}(x)\ \mid\ \cdots\ \mid\ \tilde{f}^{(n-1)}(x)\ ]A. \tag{17}$$

Taking determinants and noting that $\det\left[\tilde{f}(x)|\cdots|\tilde{f}^{(n-1)}(x)\right]$ is just $W(\tilde{f}(x))$, the Wronskian of the basis $\tilde{f}(x)$, and similarly the left hand side is $W(\tilde{f}(x+T))$ we have that

$$\det(A) = \frac{W(\tilde{f}(x+T))}{W(\tilde{f}(x))}. \tag{18}$$

By Abel's identity, the ratio of the two Wronskians is $\exp\left(-\int_x^{x+T}\frac{a_{n-1}(u)}{a_n(u)}\,du\right)$. Hence we have,

$$s_1 s_2\cdots s_n = \det(A) = \frac{W(\tilde{f}(x+T))}{W(\tilde{f}(x))} = \exp\left(-\int_x^{x+T}\frac{a_{n-1}(u)}{a_n(u)}\,du\right), \tag{19}$$

as desired. Similarly, for the other period T' we have

$$s_1^{'} s_2^{'}\cdots s_n^{'} = \det(B) = \frac{W(\tilde{f}(x+T'))}{W(\tilde{f}(x))} = \exp\left(-\int_x^{x+T'}\frac{a_{n-1}(u)}{a_n(u)}\,du\right). \tag{20}$$

Finally, if $a_{n-1}(x)=0$, so that the term of order $n-1$ is not present in the ode (1), then the equations reduce to

$$s_1 s_2\cdots s_n = 1 \text{ and } s_1^{'} s_2^{'}\cdots s_n^{'} = 1. \tag{21}$$

□

We can sometimes make make use of Lemma 3.2 for helping to find solutions of (1).

Lemma 3.3 *Assume that equation (1) has $a_{n-1}(x) = 0$ and that $\{y_1(x),\ldots,y_n(x)\}$ is a basis of solutions with each $y_i(x)$ doubly-periodic of the second kind. Then $z(x) = y_1(x)\cdots y_n(x)$ is doubly-periodic.*

Proof: Let T and T' be the periods for the coefficients of the linear ode and suppose that $y_i(x+T) = s_i\cdot y_i(x)$, $y_i(x+T') = s_i^{'}\cdot y_i(x)$, for $i=1,\ldots,n$. Then $s_1\cdots s_n = 1$ so that

$$\begin{aligned} z(x+T) &= y_1(x+T)\cdots y_n(x+T)\\ &= (s_1 y_1(x))\cdots(s_n y_n(x))\\ &= (s_1\cdots s_n)\,y_1(x)\cdots y_n(x)\\ &= z(x). \end{aligned} \tag{22}$$

Similarly, $z(x+T') = z(x)$ for the second period T', hence $z(x)$ is doubly periodic. □

Remark 3.4 *One can use the transform $u(x) = exp(-\int\frac{a_{n-1}(x)}{na_n(x)}dx)\cdot y(x)$ to convert an ode in $u(x)$ into an ode in $y(x)$ having the second highest term 0. In the case of second order equations this converts*

$$a_2(x)\cdot u''(x) + a_1(x)\cdot u'(x) + a_0(x)\cdot u(x) = 0$$

into

$$y''(x) + a(x)\cdot y(x) = 0 \tag{23}$$

with $a(x) = \frac{a_0(x)}{a_2(x)} - \left(\frac{a_1(x)}{2a_2(x)}\right)' - \left(\frac{a_1(x)}{2a_2(x)}\right)^2$.

If $\{y_1(x), \ldots, y_n(x)\}$ is a basis for a given linear ode, then the n-th symmetric product gives a linear ode having a basis the products $y_1(x)^{m_1} \cdots y_n(x)^{m_n}$ with $m_i \geq 0$ and $m_1 + \cdots + m_n = n$. From Picard's theorem and Lemma 3.3 we can therefore find a doubly periodic solution to the symmetric power ode. If one could recover the individual solutions $y_i(x)$ to (1) from the doubly periodic solution of the n-th symmetric product ode then one would have a method for finding the general solution for an arbitrary order linear ode having doubly-periodic coefficients. Unfortunately, the recovery process is only known for the case of second order equations.

4 Finding a General Solution of Second Order Equations

From Remark 3.4 we can assume that we want to find a general solution of a second order linear ode of the form (23). We show in Lemma 4.1 that solutions of such an equation can be determined by looking for solutions of

$$z'''(x) + 4 \cdot a(x) \cdot z'(x) + 2 \cdot a'(x) \cdot z(x) = 0. \tag{24}$$

the second symmetric power of equation (23). It is well known that a basis for solutions of equation (24) is given by $\{y_1(x)^2, y_1(x) \cdot y_2(x), y_2(x)^2\}$ where $\{y_1(x), y_2(x)\}$ is a basis for equation (23). We make use of the converse.

Lemma 4.1 *Let $\{y_1(x), y_2(x)\}$ be a basis of solutions for (23) and let $z(x) = y_1(x) \cdot y_2(x)$ a solution of (24). Then*

$$y_1(x) = \sqrt{z(x)} \cdot \exp\left(-\frac{C}{2}\int \frac{dx}{z(x)}\right) \text{ and } y_2(x) = \sqrt{z(x)} \cdot \exp\left(\frac{C}{2}\int \frac{dx}{z(x)}\right). \tag{25}$$

where C is a nonzero constant given by

$$C^2 = z'(x)^2 - 2 \cdot z(x) \cdot z''(x) - 4 \cdot a(x) \cdot z(x)^2. \tag{26}$$

Proof: Let $z(x) = y_1(x) \cdot y_2(x)$ and C be the Wronskian of $y_1(x), y_2(x)$. Note that C is a constant by Abel's identity [12] and is nonzero since $y_1(x)$ and $y_2(x)$ are independent. From $z'(x) = y_1'(x) \cdot y_2(x) + y_1(x) \cdot y_2'(x)$ we have that

$$\frac{z'(x)}{z(x)} = \frac{y_1'(x)}{y_1(x)} + \frac{y_2'(x)}{y_2(x)} \tag{27}$$

while $C = y_2'(x) \cdot y_1(x) - y_2(x) \cdot y_1'(x)$ implies that

$$\frac{C}{z(x)} = \frac{y_1'(x)}{y_1(x)} - \frac{y_2'(x)}{y_2(x)}. \tag{28}$$

Taking equations (27) and (28) together gives

$$\frac{y_1'(x)}{y_1(x)} = \frac{z'(x) - C}{2 \cdot z(x)} \text{ and } \frac{y_2'(x)}{y_2(x)} = \frac{z'(x) + C}{2 \cdot z(x)}$$

which gives (25). In addition, differentiating $\frac{y_1'(x)}{y_1(x)}$ gives

$$\frac{y_1''(x)}{y_1(x)} - \left(\frac{y_1'(x)}{y_1(x)}\right)^2 = \frac{2 \cdot z(x) \cdot z''(x) - 2 \cdot z'(x)^2 + 2 \cdot C \cdot z'(x)}{4 \cdot z(x)^2}$$

so that

$$\frac{y_1''(x)}{y_1(x)} = \frac{2 \cdot z(x) \cdot z''(x) - z'(x)^2 + C^2}{4 \cdot z(x)^2}.$$

Since $\frac{y_1''(x)}{y_1(x)} = -a(x)$ we get identity (26). □

Remark 4.2 *Lemma 4.1 appears to have been known in the 1800's by Hermite [11], Brioschi [4], and Halphen [10] at least in the case of Lamé's equation. Brioschi also used the result to analyze Mathieu's equation (where $a(x) = p - k^2 v^2 cos^2(x)$). Indeed, in this case equation (26) is known as Brioschi's Identity.*

In the case of second order linear odes having doubly periodic coefficients, Lemma 4.1 provides a simple method for finding a general solution for the ode since we are always able to find a doubly-periodic solution for the linear ode defined by the symmetric power. Classically these solutions are given in terms of the Weierstrass Zeta and Sigma functions which are defined in terms of $\wp$ via

$$\zeta'(x) = -\wp(x) \text{ and } \frac{\sigma'(x)}{\sigma(x)} = \zeta(x). \tag{29}$$

These Weierstrass functions are not periodic but quasi-periodic (cf. [2]).

Example 4.3 *Consider Lamé's equation (2), with n any positive integer, B, g_2, and g_3 arbitrary constants. We search for solutions that are doubly-periodic of the second kind. The corresponding symmetric product is given by*

$$z'''(x) - 4\left(n(n+1)\wp(x) + B\right) z'(x) - 2n(n+1)\wp'(x)z(x) = 0. \tag{30}$$

and this must have a doubly-periodic solution (obtain via the methods of Section 2) if our equation has a solution doubly-periodic of the second kind.

We can illustrate with some examples for n small. For example, if $n = 1$, then a basis for solutions $z(x)$ of (30) which are rational in $\wp$ and $\wp'$ is $\{\wp - B\}$. Then $C^2 = z'(x)^2 - 2z(x)z''(x) - 4a(x)z(x)^2 = 4B^3 - g_2 B - g_3$, a constant. If $C \neq 0$, two independent solutions are

$$\begin{aligned} \tilde{y_1}(x) &= \exp\left(\int \frac{z_1'(x) - C}{2z_1(x)} dx\right) = \exp\left(\frac{1}{2}\int \frac{\wp'(x) - \sqrt{4B^3 - g_2 B - g_3}}{\wp(x) - B} dx\right) \\ \tilde{y_2}(x) &= \exp\left(\int \frac{z_1'(x) + C}{2z_1(x)} dx\right) = \exp\left(\frac{1}{2}\int \frac{\wp'(x) + \sqrt{4B^3 - g_2 B - g_3}}{\wp(x) - B} dx\right). \end{aligned} \tag{31}$$

We can return the final solution in a form found in standard texts as follows. We have

$$\frac{1}{2}\int \frac{\wp'(x) - \sqrt{4B^3 - g_2 B - g_3}}{\wp(x) - B} dx = \frac{1}{2}\int \frac{\wp'(x)}{\wp(x) - B} dx - \frac{1}{2}\int \frac{\sqrt{4B^3 - g_2 B - g_3}}{\wp(x) - B} dx. \tag{32}$$

In this case, the first integral on the right is $\frac{1}{2}\ln(\wp(x) - B)$, while if a is a constant such that $\wp(a) = B$, then (cf. [2]):

$$\int \frac{dx}{\wp(x) - \wp(a)} = -\frac{1}{\wp'(a)} \left(\ln \sigma(x+a) - \ln \sigma(x-a) - 2x\zeta(a)\right). \tag{33}$$

Since $\wp(a) = B$, $\wp'(a) = \sqrt{4\wp^3(a) - g_2\wp(a) - g_3} = \sqrt{4B^3 - g_2 B - g_3}$. Thus the second integral is

$$\begin{aligned} \frac{1}{2}\int \frac{\sqrt{4B^3 - g_2 B - g_3}}{\wp(x) - B} dx &= \frac{1}{2}\int \frac{\wp'(a)}{\wp(x) - \wp(a)} dx \\ &= \frac{\wp'(a)}{2}\left(-\frac{1}{\wp'(a)}\left(\ln \sigma(x+a) - \ln \sigma(x-a) - 2x\zeta(a)\right)\right) \\ &= -\frac{1}{2}\left(\ln \sigma(x+a) - \ln \sigma(x-a) - 2x\zeta(a)\right) \end{aligned} \tag{34}$$

and so (32) becomes

$$\frac{1}{2}\left(\ln(\wp(x)-\wp(a))+(\ln\sigma(x+a)-\ln\sigma(x-a)-2x\zeta(a))\right)=\frac{1}{2}\ln\left(\frac{(\wp(x)-\wp(a))\sigma(x+a)}{\sigma(x-a)}\right)-x\zeta(a), \tag{35}$$

where B has been replaced by $\wp(a)$. Using the identity [2]

$$\wp(x)-\wp(a)=-\frac{\sigma(x+a)\sigma(x-a)}{\sigma^2(x)\sigma^2(a)}. \tag{36}$$

we can substitute into (35) and simplify the integral (32) to be

$$\frac{1}{2}\ln\left(-\frac{\sigma^2(x+a)}{\sigma^2(x)\sigma^2(a)}\right)-x\zeta(a)=\frac{1}{2}\ln(-1)+\ln\left(\frac{\sigma(x+a)}{\sigma(x)\sigma(a)}\right)-x\zeta(a). \tag{37}$$

Similarly,

$$\frac{1}{2}\int\frac{\wp'(x)+\sqrt{4B^3-g_2B-g_3}}{\wp(x)-B}dx=\frac{1}{2}\ln(-1)+\ln\left(\frac{\sigma(x-a)}{\sigma(x)\sigma(a)}\right)+x\zeta(a). \tag{38}$$

Hence, from (31), we have the solutions

$$\begin{aligned} y_1(x) &= \exp\left(\tfrac{1}{2}\ln(-1)+\ln\left(\tfrac{\sigma(x+a)}{\sigma(x)\sigma(a)}\right)-x\zeta(a)\right)=i\tfrac{\sigma(x+a)}{\sigma(x)\sigma(a)}e^{-x\zeta(a)}\\ y_2(x) &= i\tfrac{\sigma(x-a)}{\sigma(x)\sigma(a)}e^{x\zeta(a)}, \end{aligned} \tag{39}$$

where a is such that $\wp(a)=B$ and $i^2=-1$.

For higher values of n, solutions for $z(x)$ and C, and hence for $y_1(x)$ and $y_2(x)$, are still obtained. However, it becomes more difficult to evaluate the integrals for increasing values of n so we just give $z(x)$ and C^2, for $n=2$ to 5:

$$\begin{aligned}
n=2: z(x) =\ & -\tfrac{1}{4}g_2+\tfrac{1}{9}B^2-\tfrac{1}{3}B\wp+\wp^2\\
C^2 =\ & 7\tfrac{1}{324}(6\wp+B)(9g_2-4B^2+12B\wp-36\wp^2)^2\\[1em]
n=3: z(x) =\ & -\tfrac{1}{4}g_3+\tfrac{1}{15}Bg_2-\tfrac{1}{225}B^3-\tfrac{1}{4}g_2\wp+\tfrac{2}{75}B^2\wp-\tfrac{1}{5}B\wp^2+\wp^3\\
C^2 =\ & \tfrac{1}{202500}(12\wp+B)(-225g_3+60Bg_2-4B^3-225g_2\wp+24\wp B^2-180B\wp^2+900\wp^3)^2\\[1em]
n=4: z(x) =\ & \tfrac{11}{252}Bg_3+\tfrac{9}{400}g_2^2-\tfrac{113}{22050}g_2B^2+\tfrac{1}{11025}B^4-\tfrac{2}{9}g_3\wp+\tfrac{53}{1260}Bg_2\wp-\tfrac{2}{2205}B^3\wp\\
& -\tfrac{3}{10}g_2\wp^2+\tfrac{3}{245}B^2\wp^2-\tfrac{1}{7}B\wp^3+\wp^4\\
C^2 =\ & \tfrac{1}{7779240000}(20\wp+B)(7700Bg_3+3969g_2^2-904g_2B^2+16B^4-39200g_3\wp+7420g_2B\wp\\
& -160B^3\wp-52920g_2\wp^2+2160B^2\wp^2-25200B\wp^3+176400\wp^4)^2\\[1em]
n=5: z(x) =\ & \tfrac{27}{560}g_2g_3-\tfrac{29B^2}{11340}g_3-\tfrac{44B}{11025}g_2^2+\tfrac{53B^3}{297675}g_2-\tfrac{B^5}{893025}+\tfrac{B}{36}\wp g_3+\tfrac{25}{784}g_2^2\wp-\tfrac{191}{79380}g_2B^2\wp\\
& +\tfrac{1}{59535}B^4\wp-\tfrac{1}{4}g_3\wp^2+\tfrac{47}{1260}Bg_2\wp^2-\tfrac{1}{2835}B^3\wp^2-\tfrac{5}{14}g_2\wp^3+\tfrac{4}{567}B^2\wp^3-\tfrac{1}{9}B\wp^4+\wp^5\\
C^2 =\ & \tfrac{1}{51039593640000}(30\wp+B)(-688905g_2g_3+36540g_3B^2+57024g_2^2B-2544g_2B^3+16B^5\\
& -396900Bg_3\wp-455625g_2^2\wp+34380g_2B^2\wp-240\wp B^4+3572100g_3\wp^2-532980Bg_2\wp^2\\
& +5040B^3\wp^2+5103000g_2\wp^3-100800B^2\wp^3+1587600B\wp^4-14288400\wp^5)^2
\end{aligned} \tag{40}$$

Using the computer algebra system Maple, the solutions $y_1(x)$, $y_2(x)$ generated from these solutions for $z(x)$ and C have all been verified to be solutions of the corresponding instance of Lamé's equation. □

5 Algorithms for Finding Doubly-Periodic Solutions

In this section we consider the computational problem of finding an answer for the decision procedure given in Section 2. We will discuss two methods for solving this problem, one using methods from Ore algebra, a second based on converting the problem to one of finding algebraic solutions to a linear ode having algebraic coefficients. In the latter case, our procedure is a direct extension of methods for finding rational solutions to one which finds algebraic solutions.

5.1 Ore Methods

The first method builds on the discussion in Section 2 where the decision procedure reduced our problem to one of solving a 2×2 linear system of differential operators $M \cdot v = 0$ where $M = [L_{i,j}]$ with each $L_{i,j} \in \mathbb{K}(x)[D]$ a differential operator in $D = \frac{d}{dx}$ and v a 2×1 vector of elements from $K(x)$. The method is described in Singer [16, 17].

If the matrix M was made up of polynomials and we were searching for polynomial solutions to $M \cdot v = 0$ then the classical methods for solving such a problem is to convert the matrix M to its Hermite (i.e. triangular) or Smith (i.e. diagonal) normal forms. Such a conversion can be done when the coefficient domain is a Euclidean domain. It can also be done if the entries of the matrix come from an Ore domain where there are one-sided Euclidean operations. Thus, the linear system of differential operators can be solved by using row and column operations in the Ore domain $\mathbb{K}(x)[D]$ in order to diagonalize the 2×2 coefficient matrix L. Thus there are invertible matrices $U, V \in \mathbb{K}(x)[D]$ such that $U^{-1}, V^{-1} \in \mathbb{K}(x)[D]$ and where $U \cdot L \cdot V = \hat{L}$ is a diagonal matrix. In this case a solution to $L \cdot v = 0$ is given by $v = V^{-1}\hat{v}$ where $\hat{v}$ solves $\hat{L}\hat{v} = 0$. Since $\hat{L}$ is a diagonal matrix we have reduced our problem to two problems of finding a rational solution of linear odes have rational solutions.

This method is particularly convenient when the 2×2 system is already in diagonal or anti-diagonal form as in Example 2.3. In this case it reduces the problem to 2 calls to a decision procedure for linear odes having rational function coefficients, both of the same order. However, in other cases one has to finding rational solutions of equations of higher order. In addition, the elimination process increases the size of the coefficients of the differential operators and hence increases the difficulty in finding rational solutions of the associate linear odes. For these reasons, the usefulness of this method is considerably reduced since the cost of finding rational solutions depends significantly on the order of the linear ode along with the size of its coefficients.

Example 5.1 *Let*

$$\begin{aligned}(-2\wp^3(x) - \tfrac{g_2}{2}\wp(x) - g_3 - 2\wp'(x))y''(x) \quad &+(12\wp^2(x) - g_2 + 6\wp^2(x)\wp'(x) + \tfrac{g_2}{2}\wp'(x))y'(x) \\ &-(12\wp^4(x) - 6g_2\wp^2(x) - 12g_3\wp(x) - \tfrac{g_2^2}{4})y(x) = 0\end{aligned}$$

In this case the 2×2 linear system of differential operators does not separate into two linear odes. Using elimination via Ore algebras one can decide if doubly-periodic solutions exist. In this case we have a basis of doubly-periodic solutions given by

$$y_1(x) = \wp(x) \text{ and } y_2(x) = \wp'(x) - 2.$$

The elimination reduces the problem to finding rational solutions of two linear odes, each of order 4 and both having coefficients approximately twice the size of the original operators (both in terms of *degrees of the coefficients and also in terms of the sizes of the coefficients of the rational functions). Details of the computation can be found in [7].* □

5.2 An Algebraic Rational Solver

Throughout this subsection we will assume that our problem is given in an algebraic form. That is, we are given a linear ode having coefficients in $\mathbb{K}(x, \sqrt{w(x)})$ and wish to find a solution to the equation which lies in the same coefficient domain (but with perhaps the field of constants being enlarged). Using the algebraic form of the equation allows us to make use of many well known constructions that parallel those used for finding rational solutions of linear odes having rational functions as coefficients.

Let $L \in \mathbb{K}(x, \sqrt{w(x)})[D]$ with $w(x)$ a polynomial, square-free, and suppose that we are looking for a solution to $Ly = 0$ of the form $y = A(x) + B(x)\sqrt{w(x)}$ with $A(x), B(x) \in \mathbb{K}(x)$. By clearing denominators we can assume that $L \in \mathbb{K}[x, \sqrt{w(x)}][D]$. Furthermore, if $a_n(x) + b_n(x) \cdot \sqrt{w(x)}$ is the leading coefficient of L then if $b_n(x) \neq 0$ we can multiply the equation by the conjugate $a_n(x) - b_n(x) \cdot \sqrt{w(x)}$ to ensure that the leading coefficient lies in $\mathbb{K}[x]$. We will find a solution to our homogenous equation in $\hat{\mathbb{K}}(x, \sqrt{w(x)})$ (if it exists), with $\hat{\mathbb{K}}$ an algebraic extension of $\mathbb{K}$, by following the main steps used in the rational function case. The main difference is that we will work over an algebraic curve rather than a line. In this case this translates into a singular point α (i.e. a root of the leading coefficient) mapping to possibly two points of the algebraic curve.

Let $a(x)$ be an irreducible factor of the leading coefficient of L and assume that $\alpha \in \mathbb{K}(\alpha)$ satisfies $a(\alpha) = 0$ where $\mathbb{K}(\alpha) = \mathbb{K}[z]/(a(z))$. Then there are three possibilities for singular points associated with α. If $z^2 - w(\alpha)$ factors in $\mathbb{K}(\alpha)[z]$ into $(z-\beta)(z+\beta)$ with $\beta \in \mathbb{K}(\alpha)$ nonzero, then both (α, β) and $(\alpha, -\beta)$ are finite singular points of L on the algebraic curve defined by $\sqrt{w(x)}$. If on the other hand $z^2 - w(\alpha)$ is irreducible in $\mathbb{K}(\alpha)[z]$ let $\beta \in \mathbb{K}(\alpha, \beta)$ satisfy $\beta^2 = w(\alpha)$ where $\mathbb{K}(\alpha, \beta) = \mathbb{K}(\alpha)[z]/(z^2 - w(\alpha))$. Then (α, β) is a finite singular point of L on the algebraic curve. Finally, if $w(\alpha) = 0$, then $(\alpha, 0)$ is a singular point of L.

For a given finite singular point $P = (\alpha, \beta)$, let T be a local parameterization about the point on the algebraic curve. Then any expression of the form $y = A(x) + B(x)\sqrt{w(x)}$ can be expanded in the form

$$A(x) + B(x)\sqrt{w(x)} = u_m T^m + u_{m+1} T^{m+1} + \cdots, \tag{41}$$

with $u_m \neq 0$. Let

$$E_{L,P,T}(m) = \ \text{tcoeff}_T\left(\frac{L(T^m)}{T^m}\right). \tag{42}$$

be the *indicial* equation for L at α with local parameterization T. Then it is well known that when y solves $Ly = 0$ then m is an integer root of $E_{L,P,T}(m) = 0$. Define $v(P)$ to be the smallest negative integer root of $E_{L,P,T}(m) = 0$. We can extend this definition to the irreducible factor by setting

$$v(a) = \begin{cases} v(P), & \text{P} = (\alpha, 0) \\ v(P), & \text{P} = (\alpha, \beta) \text{ with } z^2 - w(\alpha) \text{ irreducible in } \mathbb{K}(\alpha)[z] \\ \min(v(P_1), v(P_2)), & \text{P}_1 = (\alpha, \beta),\ \text{P}_2 = (\alpha, -\beta) \text{ with } z^2 - w(\alpha) = (z-\beta)(z+\beta), \beta \in \mathbb{K}(\alpha) \end{cases} \tag{43}$$

As done in the standard rational case, our algorithm for the algebraic case will first find denominators of the algebraic expression, in this case two polynomials in an extension field of $\mathbb{K}$. The numerators of both $A(x)$ and $B(x)$ are then determined by bounding their degrees by analyzing the singularity at infinity. Once upper bounds for the degrees have been found, we find the undetermined coefficients for the numerator polynomials by plugging in our expression into the ode and solving the resulting linear system of equations.

Theorem 5.2 *Let $a_1(x), \ldots, a_k(x)$ be the irreducible factors of the leading coefficient of L and suppose that $y = A(x) + B(x)\sqrt{w(x)}$ solves $Ly = 0$ with $A(x), B(x)$ rational functions over an extension of $\mathbb{K}$. Then there are integers $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ such that*

$$\operatorname{denom}(A(x)) \mid a_1(x)^{p_1} \cdots a_k(x)^{p_k} \text{ and } \operatorname{denom}(B(x)) \mid a_1(x)^{q_1} \cdots a_k(x)^{q_k}. \tag{44}$$

In particular,

(a) if $w(x) \neq 0 \bmod a_i(x)$ then $p_i = q_i = \max(0, -v(a_i))$;

(b) if $w(x) = 0 \bmod a_i(x)$ then $p_i = \max(0, -\lfloor \frac{v(a_i)}{2} \rfloor)$ and $q_i = \max(0, -\lfloor \frac{1+v(a_i)}{2} \rfloor)$.

Proof: Let $a_i(x)$ be an irreducible factor of the leading coefficient of L and assume that $\alpha \in \mathbb{K}(\alpha)$ satisfies $a_i(\alpha) = 0$ where $\mathbb{K}(\alpha) = \mathbb{K}[z]/(a(z))$. Assume first that $z^2 - w(\alpha)$ is irreducible in $\mathbb{K}(\alpha)$ and let $\beta \in \mathbb{K}(\alpha, \beta)$, nonzero, satisfy $\beta^2 = w(\alpha)$ where $\mathbb{K}(\alpha, \beta) = \mathbb{K}(\alpha)[z]/(z^2 - w(\alpha))$. Since $\beta = w(\alpha) \neq 0$, one can expand $\sqrt{w(x)}$ as a power series in the local parameter $T = (x - \alpha)$. Therefore any solution $y = A(x) + B(x)\sqrt{w(x)}$ can be expanded in the form (41) with $m \geq v(a_i)$. In particular, if an algebraic solution exists for L then one must be able to write the solution in the form (41) for the smallest negative integer root m of the indicial equation, implying that

$$A(x) = (x - \alpha)^m \hat{A}(x) \text{ and } B(x) = (x - \alpha)^m \hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x).$$

Since $a_i(x)$ is irreducible, we also then have that

$$A(x) = a_i(x)^m \hat{A}(x) \text{ and } B(x) = a_i(x)^m \hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x). \tag{45}$$

A similar statement is true in the case that $z^2 - \alpha$ factors over $\mathbb{K}(\alpha)$. Since (45) is true for all irreducible factors part (a) holds.

When $\beta = w(\alpha) == 0$ then $\sqrt{w(x)}$ no longer has a Laurent series in x about α. In this case one can use T with $T^2 = (x - \alpha)$ as a local parameter so that

$$\sqrt{w(x)} = c_1 T + c_2 T^2 + \cdots \text{ with } c_i \in \mathbb{K}(\alpha).$$

We can repeat the previous arguments to determine that

$$A(x) = T^m \hat{A}(x) \text{ and } B(x) = T^{m-1} \hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x).$$

In terms of $(x - \alpha)$ we then have that for $v(a_i) = 2s$ even

$$A(x) = (x - \alpha)^s \hat{A}(x) \text{ and } B(x) = (x - \alpha)^s \hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x)$$

so

$$A(x) = a_i(x)^s \hat{A}(x) \text{ and } B(x) = a_i(x)^s \hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x). \tag{46}$$

If $v(a_i) = 2s + 1$ is odd then

$$A(x) = (x - \alpha)^{s+1} \hat{A}(x) \text{ and } B(x) = (x - \alpha)^{s-1} \hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x)$$

so

$$A(x) = a_i(x)^{s+1}\hat{A}(x) \text{ and } B(x) = a_i(x)^{s-1}\hat{B}(x) \text{ with } \hat{A}(x), \hat{B}(x) \in \mathbb{K}(\alpha, \beta)(x). \tag{47}$$

Equations (46) and (46) then give part (b). □

It remains to determine the numerators of an algebraic solution of $Ly = 0$. To this end, let L_∞ be the linear ode in T obtained by transforming L via $x = 1/T$ (if $w(x)$ has even degree) or $x = 1/T^2$ (if $w(x)$ has odd degree). Then 0 is a finite singular point of L_∞ on the algebraic curve defined by $\sqrt{w^*(T)}$ where $w^*(T)$ is $w(T)$ but with the coefficients in reversed order. Set $v_\infty = v(T)$ where $v(T)$ is defined by (43).

Theorem 5.3 *Let $y = A(x) + B(x)\sqrt{w(x)}$ be a solution of $Ly = 0$. Then the numerators of $A(x)$ and $B(x)$ have the following degree bounds.*

(a) If $w(x)$ has odd degree then

$$\begin{aligned} \deg \operatorname{numer}(A(x)) &\leq \deg \operatorname{denom}(A(x)) + \lfloor \frac{-v_\infty}{2} \rfloor \\ \deg \operatorname{numer}(B(x)) &\leq \deg \operatorname{denom}(B(x)) - \lfloor \frac{v_\infty + \deg(w(x))}{2} \rfloor \end{aligned}$$

(b) If $w(x)$ has even degree then

$$\begin{aligned} \deg \operatorname{numer}(A(x)) &\leq \deg \operatorname{denom}(A(x)) - v_\infty \\ \deg \operatorname{numer}(B(x)) &\leq \deg \operatorname{denom}(B(x)) - v_\infty - \frac{\deg(w(x))}{2}. \end{aligned}$$

Proof: Since L has a polynomial as a leading coefficient, ∞ is a singular point. If $w(x)$ has even degree then we can use $T = 1/x$ as a local parameter. Finding the smallest negative integer root m of the indicial equation for L transformed by T gives

$$A(x) + B(x)\sqrt{w(x)} = c_m T^m + c_{m+1} T^{m+1} + \cdots.$$

Since $x = 1/T$ we have degree bounds for the numerators given by

$$\mathit{deg\,numer}(A(x)) - \mathit{deg\,denom}(A(x)) \leq -m \tag{48}$$

and

$$\mathit{deg\,numer}(B(x)) - \mathit{deg\,denom}(B(x)) + \frac{1}{2}\mathit{deg}\ \ w(x) \leq -m. \tag{49}$$

If now $w(x)$ has odd degree then we need to use $T = 1/x^2$ as a local parameter. Finding the smallest negative integer root m of the indicial equation for L transformed by T gives

$$A(x) + B(x)\sqrt{w(x)} = c_m T^m + c_{m+1} T^{m+1} + \cdots.$$

Since $x^2 = 1/T$ we have degree bounds for the numerators given by

$$\mathit{deg}(\mathit{numer}(A(x))) - \mathit{deg}(\ \mathit{denom}(A(x))) \leq -\frac{m}{2} \tag{50}$$

and

$$deg(\ numer(B(x))) - \ deg(\ denom(B(x))) + \frac{1}{2} deg\, w(x) \leq -\frac{m+1}{2}. \tag{51}$$

The theorem follows from (48), (49), (50) and (51). □

Theorem 5.2 implies that if there exists a solution $y = A(x) + B(x)\sqrt{w(x)}$ to $Ly = 0$ then one can write

$$A(x) = \frac{n_A(x)}{a_1(x)^{p_1} \cdots a_k(x)^{p_k}} \text{ and } B(x) = \frac{n_B(x)}{a_1(x)^{q_1} \cdots a_k(x)^{q_k}} \tag{52}$$

with $n_A(x), n_B(x)$ polynomials. Theorem 5.3 gives degree bounds for $n_A(x)$ and $n_B(x)$. One can therefore determine if there is a solution (and if so then find all of these solutions) by using a method of undetermined coefficients. That is, if d_A and d_B are degree bounds for $n_A(x)$ and $n_B(x)$, respectively, then one can set

$$n_A(x) = \sum_{i=0}^{d_A} a_i x^i \text{ and } n_B(x) = \sum_{i=0}^{d_B} b_i x^i.$$

Plugging in $y = A(x) + B(x)\sqrt{w(x)}$ into $Ly = 0$ then results in a system of linear equations for the $d_A + d_B + 2$ unknowns a_i and b_i. Solving such a system either finds all possible solutions or else determines that no solution exists.

Example 5.4 *Consider the following linear ode given by*

$$(4x^3-2x-3)^2(x^4-4x^3+2x+3)^3y'''(x) + 3(6x^2-1)(4x^3-2x-3)(x^4-4x^3+2x+3)^3y''(x)$$
$$-(x^4-4\,x^3+2\,x+3)(4\,x^3-2\,x-3)[(x^{10}-16\,x^9+108\,x^8-146\,x^7-122\,x^6-36\,x^5+583\,x^4+126\,x^3-135\,x^2-330\,x-9)$$
$$-2\,(x^8-10\,x^7+12\,x^6+13\,x^5+10\,x^4-54\,x^3-5\,x^2-21\,x-27)\sqrt{w(x)}]y'(x)$$
$$-((4\,x^3-2\,x-3)(x^{13}-10\,x^{12}+24\,x^{11}-63\,x^{10}-31\,x^9+252\,x^8-528\,x^7+932\,x^6-117\,x^5+269\,x^4-240\,x^3$$
$$+675\,x^2+249\,x-315)-(-6\,x^{14}+76\,x^{13}-375\,x^{12}+296\,x^{11}+700\,x^{10}-383\,x^9-610\,x^8-1376\,x^7+1227\,x^6+4610\,x^5$$
$$+795\,x^4-2386\,x^3-2949\,x^2+9\,x+54)\sqrt{w(x)})y(x) = 0 \tag{53}$$

where $w(x) = 4x^3 - 2x - 3$. Equation (53) is already in the form $Ly = 0$ with $L \in \mathbb{Q}[x, \sqrt{w(x)}]$ and with the leading coefficient in $\mathbb{Q}[x]$.

There are only two irreducible factors of the leading coefficient of the associated linear operator. In the case of the irreducible factor $4x^3 - 2x - 3$, the finite singular point is given by $(\alpha, 0)$ where $4\alpha^3 - 2\alpha - 3 = 0$ in the domain $\mathbb{Q}(\alpha)$. The indicial equation in this case is a constant times $(N-2)(N^2 - 4N - 6)$ and hence $v(4x^3 - 2x - 3) = 2$. Therefore this factor appears to the power 0 and hence is not part of the denominators of any algebraic solution.

In the case of the second irreducible factor $x^4 - 4x^3 + 2x + 3$ there are two finite singular points, (α, α^2) and $(\alpha, -\alpha^2)$ where $\alpha^3 - 2\alpha - 3 = 0$ in $\mathbb{Q}(\alpha)$. In the first case, the indicial equation is

$$c_1 \cdot N(N-1)(N-2)$$

while for the second integer root the indicial equation is given by

$$c_2 \cdot (N+1)(N-1)(N-3)$$

with c_1 and c_2 constants. Therefore $v(x^4 - 4x^3 + 2x + 3) = -1$.

By Theorem 5.2 an algebraic solution would be of the form

$$y = \frac{a(x)}{x^4 - 4x^3 + 2x + 3} + \frac{b(x)}{x^4 - 4x^3 + 2x + 3}\sqrt{x^3 - 2x - 3} \tag{54}$$

with $a(x)$ and $b(x)$ polynomials. It remains to find $a(x)$ and $b(x)$.

Since $w(x)$ has odd degree we make use of the local parameterization $T^2 = 1/x$. Transforming the linear ode one determines that the indicial equation is $16N - 32 = 0$ and so $v_\infty = v(T) = 2$. Hence, by Theorem 5.3

$$\deg\left(a(x)\right) \leq 3 \text{ and } \deg\left(b(x)\right) \leq 1.$$

Letting $a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and $b = b_0 + b_1 x$, substituting (54) into (53) and then setting the resulting rational function to 0 gives the solution $a_3 = -b_1$, arbitrary, and $a_0 = a_1 = a_2 = b_0 = 0$. Thus our linear ode has a single algebraic solution given by

$$y = \frac{x^3}{x^4 - 4x^3 + 2x + 3} - \frac{x}{x^4 - 4x^3 + 2x + 3}\sqrt{x^3 - 2x - 3}. \tag{55}$$

□

Example 5.5 *If we wish to find the general solution for*

$$(\wp^2(x) + \wp'(x))y''(x) - (6\wp^2(x) + \wp''(x) + \wp(x)\wp'(x))y'(x) - (\wp(x)\wp''(x) + \wp(x)^2\wp'(x) - \wp'(x)^2)y(x) = 0 \tag{56}$$

an equation that appears in Forsyth [8]. Then we can follow the steps of the previous section. This would require that we convert the second order to invariant form by removing the middle term and then look for doubly periodic solutions to its second symmetric power. Converting this third order equation to algebraic form results in equation (53), at last when $g_2 = 2$ and $g_3 = 3$ (the same results hold for arbitrary g_2 and g_3).

Example 5.4 gives

$$y = \frac{\wp^3(x)}{\wp^4(x) - 4\wp^3(x) + 2\wp(x) + 3} - \frac{\wp(x)}{\wp^4(x) - 4\wp^3(x) + 2\wp(x) + 3}\wp'(x)$$

as the doubly periodic solution to the corresponding third order problem. This gives a solution basis to (56) as

$$\begin{aligned} y_1(x) &= \exp\left(\int^x \frac{\wp'(t)dt}{\wp(t)}\right), \\ y_2(x) &= \exp\left(\int^x \frac{\wp(t)(\wp^2(t) - \sqrt{4\wp^3(t) - 2\wp(t) - 3})}{4\wp^3(t) - g_2\wp(t) - g_3 - (4\wp^3(t) - g_2\wp(t) - g_3)^{1/2}\wp^2(t)}\sqrt{4\wp^3(t) - g_2\wp(t) - g_3}\,dt\right) \\ &= \exp\left(-\int^x \wp(t)dt\right) \end{aligned}$$

which simplifies to

$$y_1(x) = \wp(x) \text{ and } y_2(x) = \exp^{\zeta(x)}. \tag{57}$$

□

Remark 5.6 *While our procedure follows the same essential steps as the method for finding rational solutions of linear odes having rational coefficients, there are still technical difficulties that arise. In particular, one has the problem of finding the indicial equation without an apriori knowledge of how* *many terms of a Laurent expansion are needed for a given local parameterization. In our case, for a given number of terms h we compute the terms and set*

$$\sqrt{w(x)} = \sum_{i=0}^{h-1} w_i T^i + c \cdot T^h$$

where c is an indeterminiant. If there is a c present in the trailing coefficient computation of the indicial equation, we then know to increase the number of terms and repeat our computation.

6 Finding Doubly-Periodic Solutions of Second Kind

In this section we show how to find doubly-periodic solutions of the second kind for (1). As done in the last section it is convenient to first convert our input linear ode into one having algebraic coefficients.

The method that we describe in this section is based on finding first order right factors of the associated differential operator. Such a factorization is available for the case of linear differential operators having rational function coefficients [19]. Our approach is similar to Trager's algorithm for factoring polynomials over an algebraic number field. Trager's algorithm takes a polynomial $p(x)$ having coefficients in the algebraic number field, forms the polynomial $Norm(p(x))$ in order to remove the algebraic numbers, factors the resulting polynomial using standard methods and then obtains the individual factors using gcd operations. In some cases the algorithm needs to translate the parameter x by some random amount to ensure that $Norm(p(x))$ is square-free.

In our case, we replace the norm operation by an LCLM operation, the factorization by one which finds all right factors (called an LCLM factorization) and finally the gcd operation by a right gcd operation. The operation which is equivalent to ensuring a square-free Norm occurs when the LCLM operation produces an operator which has order less the sum of the two orders. In this case, if the coefficient domain is $\mathbb{K}(x, \sqrt{w(x)})$ we translate the variable via $x \to x + c\sqrt{w(x)}$ for some random c.

More precisely, let $\bar{L} \in K(x, \sqrt{w(x)})[D]$ be formed by replacing all occurrences of $\sqrt{w(x)}$ by $-\sqrt{w(x)}$ and let

$$\hat{L} = LCLM(L, \bar{L}). \tag{58}$$

Then $\hat{L} \in K(x)[D]$ rather than in $K(x, \sqrt{w(x)})[D]$ since all coefficients of $\hat{L}$ are invariant under the substitution $\sqrt{w(x)} \to -\sqrt{w(x)}$. All right factors of $\hat{L}$ can be found using LCLM factorization. Thus we have right factors $\hat{R}_1, \ldots, \hat{R}_k \in K(x)[D]$ such that in addition

$$\hat{L} = LCLM(\hat{R}_1, \ldots, \hat{R}_k).$$

Right factors of $\hat{L}$ are then obtained using

$$R_i = GCRD(L, \hat{R}_i) \in K(x, \sqrt{w(x)})[D].$$

Example 6.1 *Let*

$$L = D^2 + \frac{a_1}{b_1} D + \frac{a_0}{b_1} \tag{59}$$

where

$$
\begin{aligned}
a_1 = & \quad -(4\sqrt{w(x)}x^3 + 4x^5 + \sqrt{w(x)}g_2x - 3x^3g_2 \\
& \quad +2\sqrt{w(x)}g_3 - 4x^2g_3 - 2\sqrt{w(x)}x^4)x) \\
b_1 = & \quad 2w(x)(-4x^3 + g_2x + g_3 + x^4) \\
a_0 = & \quad (4\sqrt{w(x)}x^3 + 4x^5 + \sqrt{w(x)}g_2x - 3x^3g_2 \\
& \quad +2\sqrt{w(x)}g_3 - 4x^2g_3 - 2\sqrt{w(x)}x^4)
\end{aligned}
\tag{60}
$$

with $w(x) = 4x^3 - g_2x - g_3$.

Replacing all occurences of $\sqrt{w(x)}$ *by* $-w\sqrt{(x)}$ *to get a new differential operator* $\bar{L}$ *and taking the LCLM of the two operators gives*

$$
\begin{aligned}
\hat{L} = & \quad LCLM(L, \bar{L}) \\
= & \quad D^3 + \frac{8x^6-48x^5+10x^4g_2-40x^3g_2+16x^3g_3-96x^2g_3+xg_2^2+4g_2g_3}{2\left(8x^7-16x^6-2g_2x^5-2x^4g_3-4x^3g_3+x^2g_2^2+3xg_3g_2+2g_3^2\right)}D^2 \\
& -\frac{2x^2(x^4+3g_3+g_2x)}{8x^7-16x^6-2g_2x^5-2x^4g_3-4x^3g_3+x^2g_2^2+3xg_3g_2+2g_3^2}D \\
& +\frac{2x(x^4+3g_3+g_2x)}{8x^7-16x^6-2g_2x^5-2x^4g_3-4x^3g_3+x^2g_2^2+3\wp g_3g_2+2g_3^2}
\end{aligned}
$$

An LCLM factorization of $\hat{L}$ *is then given by*

$$
\hat{L} = \{R_1, R_2\} = \left\{D - \frac{1}{x}, \quad D^2 + \frac{4x^3 + g_2x + 2g_3}{2x\,(4x^3 - g_2x - g_3)}D - \frac{x^2}{4x^3 - g_2x - g_3}\right\}.
$$

Then two right factors of L *are*

$$
\begin{aligned}
F_1 & = GCRD(L, R_1) = D - \tfrac{1}{x}, \\
F_2 & = GCRD(L, R_2) = D - \frac{x(x^2-\sqrt{4x^3-g_2x-g_3})}{4x^3-g_2x-g_3-\sqrt{4x^3-g_2x-g_3}x^2}
\end{aligned}
\tag{61}
$$

Thus a basis of solutions to $Ly = 0$ *is* $\{y_1, y_2\}$, *where*

$$
\begin{aligned}
y_1(x) & = \exp\left(\int \tfrac{dx}{x}\right), \\
y_2(x) & = \exp\left(\int \frac{x(x^2-\sqrt{4x^3-g_2x-g_3})}{4x^3-g_2x-g_3-\sqrt{4x^3-g_2x-g_3}x^2}dx\right).
\end{aligned}
\tag{62}
$$

□

Remark 6.2 *Converting the linear ode (56) from Example 5.5 into algebraic form gives the algebraic linear ode of Example 6.1. Conversion of the results of the above example back into Weierstrass form and simplfying gives (57).*

7 Conclusions

For linear odes having doubly periodic coefficients, it is possible to determine if such an ode has doubly periodic solutions. We have shown how this can be used to find the general solution of such linear odes for order two in terms of functions which are doubly periodic of the second kind.

It is of interest to find a similar procedure for finding general solutions of higher order. Unfortunately, the method used in this paper (finding solutions of a symmetric power and then recovering a basis to the original ode) does not extend to higher order. In addition, the order of the symmetric power increases quickly and even finding doubly periodic solutions of these equations becomes a significant problem with our methods. We expect that it may be possible to do such an extension for order three using the methods given in Singer and Ulmer [18].

In some cases it is possible that a linear ode of the form (1) does not have solutions which are doubly periodic of the same period as the original ode, but does have solutions which are periodic of a multiple of the periods. For example, when our methods are applied to the linear ode

$$y'''(x) - 3\wp(x, g_2, g_3)\, y'(x) - \frac{3}{2}\wp'(x, g_2, g_3)y(x) = 0,$$

then we find that there are no doubly-periodic solutions. However there is a basis of solutions of this equation given by

$$\{\frac{1}{\tilde{\wp}'}, \frac{\tilde{\wp}}{\tilde{\wp}'}, \frac{\tilde{\wp}^2}{\tilde{\wp}'}\}$$

where $\tilde{\wp} = \wp(\frac{x}{2}; \frac{g_2}{2}, \frac{g_3}{2})$ and $\tilde{\wp}' = \wp'(\frac{x}{2}; \frac{g_2}{2}, \frac{g_3}{2})$, doubly-periodic but with twice the periods. Such a method is presented in [7]. However this method requires further investigation, particularly when a decision procedure is desired. We plan to investigate this in the future.

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