Factoring Zero-dimensional Ideals of Linear Partial Differential Operators

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Abstract

This paper presents an algorithm for factoring a zero-dimensional left ideal in the ring $\bar{\mathbf{Q}}(x,y)[\partial_x,\partial_y]$, i.e. factoring a linear homogeneous partial differential system whose coefficients belong to $\bar{\mathbf{Q}}(x,y)$, and whose solution space is finite-dimensional over $\bar{\mathbf{Q}}$. The algorithm computes all the zero-dimensional left ideals containing the given ideal. It generalizes the Beke-Schlesinger algorithm for factoring linear ordinary differential operators, and uses an algorithm for finding hyperexponental solutions of such ideals.

1 Introduction

For various reasons *linear* differential equations have been of particular importance in the history of mathematics. First of all, the problems connected with them are much easier than those for nonlinear equations. Second, many nonlinear problems may be linearized in some way such that the results of the former theory may be applied to them. This is especially true for Lie's symmetry analysis of differential equations which reduces the problem of solving nonlinear ordinary differential equations (ode's) with a sufficiently large number of symmetries to the study of certain systems of linear partial differential equations (pde's). The problem of finding conservation laws for nonlinear pde's also leads to systems of linear pde's.

It has been possible to generalize many concepts from commutative algebra suitably such that they may be applied to linear ode's, e.g. the greatest common divisor and the least common multiple, the concept of reducibility and factorization which finally led to the theory of Picard and Vessiot and differential Galois theory. This is true to a much less extent for systems of linear pde's. In order to obtain manageable problems we have to specialize them further. The constraint that the general solution depends on a finite number of constants, i.e. that it may be represented as a linear combination with constant coefficients of a finite number of special solutions which form a fundamental system, turns out to be appropriate. It allows us to generalize many concepts from the theory of linear ode's in an almost straightforward manner to these pde's. Furthermore, they have important applications in symmetry analysis.

It is the purpose of this paper to describe a generalization of the Beke-Schlesinger factorization algorithm [1, 13] to systems of linear homogeneous pde's in one dependent and two independent variables with a finite-dimensional solution space. The base field consists of the rational functions in the independent variables with algebraic number coefficients. In principle, most problems related to such systems of pde's reduce, as shown by S. Lie, to corresponding problems for linear ode's. However, such "reduction" may be nontrivial and usually leads to solving or factoring linear ode's with parameters. This makes many known algorithms fail. The algorithm presented in this paper avoids such complications.

Throughout this paper the following notation will be used: $\bar{\mathbf{Q}}$ stands for the algebraic closure of the field of rational numbers, $\mathbf{K}$ for the differential field $\bar{\mathbf{Q}}(x,y)$ with usual derivation operators ∂_x and ∂_y, and $\mathbf{K}[\partial_x,\partial_y]$ for the ring of linear partial differential operators generated by ∂_x and ∂_y over $\mathbf{K}$. By an ideal in $\mathbf{K}[\partial_x,\partial_y]$ we mean a left ideal in $\mathbf{K}[\partial_x,\partial_y]$. A system of linear homogeneous pde's in ∂_x and ∂_y over $\mathbf{K}$ can be naturally identified with an ideal in $\mathbf{K}[\partial_x,\partial_y]$ (see, e.g. [9, §1.1]). A solution of a system is understood as an element of a universal differential field extension $\mathbf{E}$ of $\mathbf{K}$, which is annihilated by the operators in the corresponding ideal of $\mathbf{K}[\partial_x,\partial_y]$. A system has a finite-dimensional solution space over $\bar{\mathbf{Q}}$ if and only if the corresponding ideal is zero-dimensional. This point of view enables us to make statements concisely.

In the language of differential operators just described, our factorization algorithm computes all ideals (in $\mathbf{K}[\partial_x,\partial_y]$) containing a given zero-dimensional ideal. Besides applications to solving differential equations and symmetry analysis, the algorithm is also applicable to holonomic systems, for example, finding Weyl closures (w.r.t. $\mathbf{K}[\partial_x,\partial_y]$) that

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properly contain a given holonomic D-ideal in the Weyl algebra $\bar{\mathbf{Q}}\langle x, y, \partial_x, \partial_y\rangle$ (see [9, §1.4]). Factoring a zero-dimensional ideal in $\mathbf{K}[\partial_x, \partial_y]$ is more involved than factoring a linear ode. For instance, the notions of factors and quotients need to be carefully examined; the leading derivatives of a factor are not as obvious as in the ode case; the notion of Wronskians will be extended; and the normal form of a factor has to be considered. Most of the results stated in this paper hold for several variables. We confine ourselves to the case of two variables, because a generalization of the algorithm in [7] for several variables is still on the way.

The paper is based on several known results. The theory of linear differential ideals [6] supplies useful conclusions about dimension and linear dependence over a constant field. The notion and computation of differential Gröbner bases in $\mathbf{K}[\partial_x, \partial_y]$ (see [5, 3, 9]) make sure that the system to be factored and the factors to be sought are of required dimensions. The algorithm in [7] enables us to compute first-order factors. The idea of associated equations [1, 13, 12, 2] inspires us to reduce our factorization problem to that of finding first-order factors. The factorization problem in this paper has been considered by Tsarev [14], in which chains of factors of such systems are investigated from the point of view of the Jordan-Hölder theorem. But the results about algorithms therein are fairly sketchy. We shall clarify a few points in that paper.

The paper is organized as follows. Section 2 specifies the notation and states our factorization problem. Section 3 discusses quotient systems. Section 4 presents some simple and useful facts about factorization. Sections 5 and 6 generalize the notions of Wronskians and associated systems, respectively. Section 7 describes ideas about factorization. Section 8 presents a factorization algorithm. Concluding remarks are given in Section 9.

2 Preliminaries

We denote by Θ the commutative monoid generated by ∂_x and ∂_y. An element of Θ is called a derivative. For a subset L of $\mathbf{K}[\partial_x, \partial_y]$, $\mathrm{sol}(L)$ stands for the solution space of L, which is contained in $\mathbf{E}$ and is a vector space over $\bar{\mathbf{Q}}$. The (left) ideal generated by L is denoted by (L). An ideal I of $\mathbf{K}[\partial_x, \partial_y]$ is zero-dimensional if the left $\mathbf{K}$-linear space $\mathbf{K}[\partial_x, \partial_y]/I$ is finite-dimensional. It follows from [6, Chap. IV, §5] that $\dim_{\mathbf{K}}(\mathbf{K}[\partial_x, \partial_y]/I) = \dim_{\bar{\mathbf{Q}}} \mathrm{sol}(I)$ if the ideal I is zero-dimensional.

For convenience, we fix a term order

$$1 < \partial_x < \partial_y < \partial_x^2 < \partial_x\partial_y < \partial_y^2 < \cdots$$

on $\mathbf{K}[\partial_x, \partial_y]$ throughout the paper. The reader may easily find that all conclusions in this paper are valid for any term order. A differential Gröbner basis in $\mathbf{K}[\partial_x, \partial_y]$ will be simply called a Gröbner basis. By "given an ideal", we mean to be given its reduced Gröbner basis. By "computing an ideal", we mean to compute its reduced Gröbner basis. In this paper, Gröbner bases may be replaced by coherent autoreduced sets in the theory of linear differential ideals [8, 6], or by Janet bases in symmetry analysis [4, 11, 10], both of which appeared earlier in the literature than Gröbner bases. We choose Gröbner bases because they are well-known and the computer algebra implementations to compute them are widely available. For brevity, all Gröbner bases are assumed to be reduced. It seems improper to factor an arbitrary system, for such a system may possibly be inconsistent.

For a Gröbner basis L, we denote by $\mathrm{lder}(L)$ the set of leading derivatives of L, and by $\mathrm{pder}(L)$ the set of parametric derivatives of L. Recall that a leading derivative of L is the highest derivative in an element of L, and a parametric derivative of L is a derivative not divisible by any element of $\mathrm{lder}(L)$. By [6, Chap. IV, §5] or [9, §1.4], $\dim_{\bar{\mathbf{Q}}} \mathrm{sol}(L) = |\mathrm{pder}(L)|$ if (L) is zero-dimensional. Note that $|\mathrm{pder}(L)|$ is also called holonomic rank of L. Both $|\mathrm{lder}(L)|$ and $|\mathrm{pder}(L)|$ can be easily obtained from L.

This paper provides an algorithm for solving

Problem F Given a Gröbner basis L with $\dim_{\bar{\mathbf{Q}}} \mathrm{sol}(L) = d$ and an integer n with $1 \le n < d$, compute all Gröbner bases F in $\mathbf{K}[\partial_x, \partial_y]$ such that

1. $\dim_{\bar{\mathbf{Q}}} \mathrm{sol}(F) = n$, or, equivalently, $|\mathrm{pder}(F)| = n$,
2. $\mathrm{sol}(F) \subset \mathrm{sol}(L)$, or, equivalently, $(L) \subset (F)$.

We call F an nth-order *factor* of L over $\mathbf{K}$.

Remark 1 If two Gröbner bases F_1 and F_2 have the same solution space, F_1 and F_2 are equal as sets [6, page 151].

Remark 2 As pointed out in the proof of Theorem 2 in [14], the solution spaces of systems in $\mathbf{K}[\partial_x, \partial_y]$ contained in $\mathrm{sol}(L)$ form a modular lattice $\mathbf{V}$ in which the partial ordering is inclusion. For $V_1, V_2 \in \mathbf{V}$, both $(V_1 \cap V_2)$ and $(V_1 + V_2)$ are also in $\mathbf{V}$. Indeed, they correspond to the sum (gcd) and intersection (lcm) of respective defining ideals. Hence, for two maximal chains of proper factors $F_1, \ldots, F_k$ and $G_1, \ldots, G_m$ with

$$\{0\} \subset \mathrm{sol}(F_1) \subset \cdots \subset \mathrm{sol}(F_k) \subset \mathrm{sol}(L)$$

and

$$\{0\} \subset \mathrm{sol}(G_1) \subset \cdots \subset \mathrm{sol}(G_m) \subset \mathrm{sol}(L),$$

we have $m = k$ and the quotient spaces $(\mathrm{sol}(F_i)/\mathrm{sol}(F_{i-1}))$ and $(\mathrm{sol}(G_j)/\mathrm{sol}(G_{j-1}))$ have the same dimension pairwise after a rearrangement of indices.

Example 3 Let L be the Gröbner basis:

$$\begin{aligned}\{L_1 &= \partial_x^2 + \tfrac{y^2}{4x}\partial_y + \tfrac{2-xy}{4x}\partial_x - \tfrac{y}{4x},\\ L_2 &= \partial_x\partial_y + \tfrac{y}{4}\partial_y - \tfrac{2+xy}{4y}\partial_x - \tfrac{1}{4},\\ L_3 &= \partial_y^2 + \tfrac{x}{4}\partial_y + \tfrac{2x-x^2y}{4y^2}\partial_x - \tfrac{x}{4y}\}.\end{aligned} \quad (1)$$

We find $\mathrm{lder}(L) = \{\partial_x^2, \partial_x\partial_y, \partial_y^2\}$ and $\mathrm{pder}(L) = \{\partial_x, \partial_y, 1\}$. Hence, $\mathrm{sol}(L)$ is of dimension 3 over $\bar{\mathbf{Q}}$. The Gröbner basis

$$F = \left\{\partial_x^2 + \frac{1}{2x}\partial_x - \frac{y}{4x},\ \partial_y - \frac{x}{y}\partial_x\right\}$$

is a second-order factor of L, since L_1, L_2 and L_3 can be reduced to zero by F. Notice that $(\mathrm{lder}(L) \cap \mathrm{lder}(F))$ is nonempty, which never happens in the ode case. One may observe that (1) also has a first-order factor $\left\{\partial_y - \frac{1}{y},\ \partial_x\right\}$.

In the rest of this paper, $L \subset \mathbf{K}[\partial_x, \partial_y]$ is assumed to be a Gröbner basis with $\dim_{\bar{\mathbf{Q}}} \mathrm{sol}(L) = d > 0$.

To save space, we use exterior algebra notation to denote determinants. For $\theta_1, \ldots, \theta_n \in \Theta$, the exterior product

$$\lambda = \theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n$$

is understood as a multi-linear mapping from $\mathbf{E}^n$ to $\mathbf{E}$:

$$\lambda(\vec{z}) = \begin{vmatrix} \theta_1 z_1 & \theta_1 z_2 & \cdots & \theta_1 z_n \\ \theta_2 z_1 & \theta_2 z_2 & \cdots & \theta_2 z_n \\ \vdots & \vdots & \cdots & \vdots \\ \theta_n z_1 & \theta_n z_2 & \cdots & \theta_n z_n \end{vmatrix}$$

for $\vec{z} = (z_1, \ldots, z_n) \in \mathbf{E}^n$. Besides being multi-linear and anti-symmetric, λ also has the property that, for all $a \in \mathbf{K}$,

$$\partial_x (a\lambda) = (\partial_x a)\lambda + a \sum_{j=1}^{m} (\theta_1 \wedge \cdots \wedge \partial_x \theta_j \wedge \cdots \wedge \theta_m). \quad (2)$$

In this way we may regard any $\mathbf{K}$-linear combination of n-fold exterior products formed by elements of Θ as a multi-linear function from $\mathbf{E}^n$ to $\mathbf{E}$. Clearly, a derivation operator can be applied to such a combination.

For a subset S of Θ, we denote by $\Lambda^n(S)$ the $\mathbf{K}$-linear space generated by all the n-fold exterior products formed by the elements of S. In particular, $\Lambda^n(\Theta)$ is a $\mathbf{K}$-linear space closed under ∂_x and ∂_y. For every $\lambda \in \Lambda^n(\Theta)$, there exists $\lambda_L \in \Lambda^n(\text{pder}(L))$ such that,

$$\lambda(z_1, \ldots, z_n) = \lambda_L(z_1, \ldots, z_n) \quad (3)$$

for $z_1, \ldots, z_n \in \text{sol}(L)$. The exterior expression λ_L can be computed by replacing each derivative appearing in λ by its normal form w.r.t. L. Equation (3) is crucial in this paper.

3 Quotient Systems

A factor F of L helps us to find a subspace of $\text{sol}(L)$. How can we use the factor to describe all the solutions of L? An answer is to use the quotient system of L w.r.t. F, which is briefly described in [14] without proofs. We shall spell out the details.

Write $L = \{L_1, \ldots, L_p\}$ and $F = \{F_1, \ldots, F_q\}$. Since F is a factor of L,

$$L_i = \sum_{j=1}^{q} Q_{ij} F_j \quad \text{for some } Q_{ij} \in \mathbf{K}[\partial_x, \partial_y]. \quad (4)$$

Since F is a Gröbner basis, all the S-pairs of F

$$(\delta_a F_a - \delta_b F_b) = \sum_{j=1}^{q} P_{abj} F_j, \text{ for some } P_{abj} \in \mathbf{K}[\partial_x, \partial_y] \quad (5)$$

where δ_a and δ_b are the derivatives to form the S-pair of F_a and F_b. Let $u_0, u_1, \ldots, u_q$ be differential indeterminates over $\mathbf{K}$. For $A \in \mathbf{K}[\partial_x, \partial_y]$, the action of A on u_i is denoted by $A(u_i)$ which is an element of $\mathbf{K}\{u_0, u_1, \ldots, u_q\}_1$ (see [6]). The *quotient system* of L w.r.t. F is defined to be $Q = \{Q_i = 0, T_{ab} = 0 | 1 \le i \le p, 1 \le a < b \le q\}$ where

$$Q_i = \sum_{j=1}^{q} Q_{ij}(u_j), \; T_{ab} = (\delta_a(u_a) - \delta_b(u_b)) - \sum_{j=1}^{q} P_{abj}(u_j).$$

Proposition 1 *Let $G(u_0, u_1, \ldots, u_q)$ denote the system*

$$\{F_1(u_0) = u_1, \ldots, F_q(u_0) = u_q\}.$$

Then

- *if $(v_1, \ldots, v_q) \in \text{sol}(Q)$, then there exists $v_0 \in \mathbf{E}$ such that $(v_0, v_1, \ldots, v_q) \in \text{sol}(G)$, so that $v_0 \in \text{sol}(L)$;*
- *if $v_0 \in \text{sol}(L)$, then $(F_1(v_0), \ldots, F_q(v_0)) \in \text{sol}(Q)$;*
- *$\text{sol}(Q)$ is of dimension $(d-n)$ over $\bar{\mathbf{Q}}$.*

Proof To prove the first assertion, we regard

$$G(u_0) = G(u_0, v_1, \ldots, v_q)$$

as a differential system in u_0. Its integrability conditions $T_{ab}(v_1, \ldots, v_q)$ $(1 \le a < b \le q)$ vanish, since T_{ab} is in Q. In other words, $\{F_1(u_0) - v_1, \ldots, F_q(u_0) - v_q\}$ is a coherent autoreduced set. Hence, $G(u_0)$ has a solution v_0. It follows from (4) that $v_0 \in \text{sol}(L)$. The second assertion is direct from (4) and (5).

To prove the last assertion, we let h be the dimension of $\text{sol}(Q)$ over $\bar{\mathbf{Q}}$, and $z_1, \ldots, z_n, w_1, \ldots, w_{n-d}$ a fundamental system of L, in which $z_1, \ldots, z_n \in \text{sol}(F)$. Then the vectors $\vec{v}_i = (F_1(w_i), \ldots, F_q(w_i))$, where $1 \le i \le (d-n)$, are nontrivial solutions of Q by the second assertion. If these vectors are $\bar{\mathbf{Q}}$-linearly dependent, then a nontrivial $\bar{\mathbf{Q}}$-linear combination of the w_i is a solution of all the F_i, a contradiction to the selection of the w_i. Thus, $h \ge (d-n)$. For nonzero $\vec{v} \in \text{sol}(Q)$, let v_0 satisfy $G(u_0, \vec{v})$. Since $v_0 \in \text{sol}(L)$, it can be expressed as a nontrivial $\bar{\mathbf{Q}}$-linear combination of the z_j and w_i. Applying each F_k in F to the linear combination yields that $\vec{v}$ is a $\bar{\mathbf{Q}}$-linear combination of the $\vec{v}_i$. Hence $h \le (d-n)$. □

Example 4 Consider the Gröbner basis

$$H = \left\{ \partial_x^2 - \frac{yx-1}{x}\partial_x - \frac{y}{x}, \; \partial_y - \frac{x}{y}\partial_x \right\}.$$

It has a first-order factor $\{F_1 = \partial_y - x, \; F_2 = \partial_x - y\}$. The quotient system Q of H w.r.t. F is

$$\left\{ \partial_x u_2 = \frac{-1}{x} u_2, \; u_1 = \frac{x}{y} u_2, \; \partial_x(u_1) - \partial_y(u_2) = y u_1 - x u_2 \right\}.$$

We find that $\left(u_1 = \frac{1}{y}, \; u_2 = \frac{1}{x}\right)$ is a nonzero solution of Q. Let v_0 be a solution of $G(u_0) = \left\{F_1(u_0) = \frac{1}{y}, F_2(u_0) = \frac{1}{x}\right\}$. It follows from Proposition 1 that H has a fundamental system of solutions $\{\exp(xy), \; v_0\}$. If both x and y are positive real variables, then v_0 can be chosen as

$$-\exp(xy)\text{Ei}(1, xy) = -\exp(xy) \int_0^\infty \frac{\exp(-xyt)}{t} \text{d}t.$$

4 Useful facts about factorization

We show that finding first-order factors of L is equivalent to finding its hyperexponential solutions.

A nonzero element h of $\mathbf{E}$ is said to be *hyperexponential* over $\mathbf{K}$ if both $(\partial_x h)/h$ and $(\partial_y h)/h$ belong to $\mathbf{K}$. Two hyperexponential elements are said to be *equivalent* if their ratio belongs to $\mathbf{K}$. Hyperexponential elements play a key role in algorithms for factoring differential operators. An easy calculation shows

Proposition 2 *The set of all hyperexponential elements over $\mathbf{K}$ is closed under multiplication and division.*

Proposition 3 *The Gröbner basis L has a first-order factor if and only if L has a hyperexponential solution.*

Proof If h is a hyperexponential solution of L, then

$$F = \left\{\partial_x - \frac{\partial_x h}{h},\, \partial_y - \frac{\partial_y h}{h}\right\}$$

is a first-order factor of L over $\mathbf{K}$. Conversely, let a first-order factor of L be $F = \{\partial_x - u, \partial_y - v\} \subset \mathbf{K}[\partial_x, \partial_y]$. For a nonzero $h \in \mathrm{sol}(F)$, both $(\partial_x h)/h = u$ and $(\partial_y h)/h = v$ imply that h is hyperexponential. □

The algorithm presented in [7] computes all first-order factors of a given system L such as the first-order factors appearing in examples 3 and 4.

For a dth order linear ode w.r.t. ∂_x, its nth order right factors have leading derivative ∂_x^n. What is $\mathrm{lder}(F)$ if F is a factor of L ? The following lemma provides an answer.

Lemma 4 *If F is specified as above, then*

$$\mathrm{lder}(F) \subset (\mathrm{lder}(L) \cup \mathrm{pder}(L)),\ \mathrm{pder}(F) \subset \mathrm{pder}(L).$$

and, for every $\theta \in \mathrm{lder}(L)$, there exists $\delta \in \mathrm{lder}(F)$ such that θ is divisible by δ.

Proof If $\delta \in \mathrm{lder}(F)$ and $\delta \notin (\mathrm{lder}(L) \cup \mathrm{pder}(L))$, then δ is divisible by some θ in $\mathrm{lder}(L)$. As each member of L can be reduced to zero by F, θ is divisible by an element of $\mathrm{lder}(F)$, so is δ, contradicting to the fact that F is reduced. The last two assertions follow from the same argument. □

This lemma tells us that there are only finitely many choices for $\mathrm{lder}(F)$. As shown in Example 3, the intersection of $\mathrm{lder}(L)$ and $\mathrm{lder}(F)$ may be nonempty. This point is neglected in [14].

5 Wronskian representations

A key idea in the Beke-Schlesinger algorithm is to look for right factors whose coefficients are Wronskian-like determinants. To use this idea, we extend the notion of Wronskians. Let F be a Gröbner basis in $\mathbf{K}[\partial_x, \partial_y]$ with n-dimensional solution space. Let $\mathrm{lder}(F) = \{\theta_1, \ldots, \theta_k\}$ and $\mathrm{pder}(F) = \{\xi_1, \ldots, \xi_n\}$, where ξ_i is lower than ξ_j for $1 \le i < j \le n$. Assume that $F = \{F_1, \ldots, F_k\}$ in which each F_i is monic with leading derivative θ_i. We call the element $\omega_F = (\xi_1 \wedge \cdots \wedge \xi_n)$ the *Wronskian operator* of F. It follows from (3) (replacing L by F) and $\Lambda^n(\mathrm{pder}(F)) = \{r\omega_F \mid r \in \mathbf{K}\}$ that, for every $\lambda \in \Lambda^n(\Theta)$, there exists $r_\lambda \in \mathbf{K}$ such that

$$\lambda(z_1, \ldots, z_n) = r_\lambda \omega_F(z_1, \ldots, z_n) \tag{6}$$

for $z_1, \ldots, z_n \in \mathrm{sol}(F)$.

Lemma 5 *For all $z_1, \ldots, z_n \in \mathrm{sol}(F)$, $z_1, \ldots z_n$ are $\bar{\mathbf{Q}}$-linearly independent if and only if $\omega_F(\vec{z}) \neq 0$. Moreover, let $z_1, \ldots, z_n$ form a fundamental system of solutions of F and denote $(z_1, \ldots z_n)$ by $\vec{z}$. Then*

$$(\omega_F \wedge \theta_i)(\vec{z}, \cdot) = \omega_F(\vec{z}) F_i, \quad i = 1, \ldots, k,$$

where $(\omega_F \wedge \theta_i)(\vec{z}, \cdot)$ means the $(n+1)\times(n+1)$ determinant

$$\begin{vmatrix} \xi_1 z_1 & \xi_1 z_2 & \cdots & \xi_1 z_n & \xi_1 \\ \xi_2 z_1 & \xi_2 z_2 & \cdots & \xi_2 z_n & \xi_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_n z_1 & \xi_n z_2 & \cdots & \xi_n z_n & \xi_n \\ \theta_i z_1 & \theta_i z_2 & \cdots & \theta_i z_n & \theta_i \end{vmatrix}$$

in which derivatives are placed at the right-hand side in a product.

Proof If $z_1, \ldots, z_n$ are linearly independent over $\bar{\mathbf{Q}}$, Theorem 1 in [6, page 86] implies that there exists λ in $\Lambda^n(\Theta)$ such that $\lambda(z_1, \ldots, z_n) \neq 0$. The first assertion then follows from (6). The converse is true by the same theorem.

The expression $(\omega_F \wedge \theta_i)(\vec{z}, \cdot)$ can be reduced to zero by F, because $(\omega_F \wedge \theta_i)(\vec{z}, z_j) = 0$ for $j = 1, \ldots, n$. On the other hand, expanding $(\omega_F \wedge \theta_i)(\vec{z}, \cdot)$ according to its last column yields, for all i with $1 \le i \le k$,

$$(\omega_F \wedge \theta_i)(\vec{z}, \cdot) = \omega_F(\vec{z})\theta_i + \sum_{j=1}^{d} \underbrace{(-1)^{d+j+1}(\eta_j \wedge \theta_i)(\vec{z})}_{w_{ij}} \xi_j, \tag{7}$$

where $\eta_j = \xi_1 \wedge \cdots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \cdots \wedge \xi_n$. It follows that $((\omega_F \wedge \theta_i)(\vec{z}, \cdot) - \omega_F(\vec{z})F_i)$ equals zero. □

We call $\{(\omega_F \wedge \theta_1)(\vec{z}, \cdot), \ldots, (\omega_F \wedge \theta_k)(\vec{z}, \cdot)\}$ a *Wronskian representation* of F. Any two Wronskian representations of F can only differ by a multiplicative constant in $\bar{\mathbf{Q}}$, because any two sets of fundamental solutions of F can be transformed from one to the other by a matrix over $\bar{\mathbf{Q}}$.

Corollary 6 *The Wronskian representation of the Gröbner basis F is a Gröbner basis over $\mathbf{K} < z_1, \ldots, z_n >$.*

Example 5 The Wronskian representation of a Gröbner basis F with $\mathrm{lder}(F) = \{\partial_x^2, \partial_y\}$ is

$$\{\,(1 \wedge \partial_x)(\vec{z})\partial_x^2 - (1 \wedge \partial_x^2)(\vec{z})\partial_x + (\partial_x \wedge \partial_x^2)(\vec{z}),$$

$$(1 \wedge \partial_x)(\vec{z})\partial_y - (1 \wedge \partial_y)(\vec{z})\partial_x + (\partial_x \wedge \partial_y)(\vec{z})\,\}$$

where $\vec{z} = (z_1, z_2)$ and z_1, z_2 form a fundamental system of solutions of F. The Wronskian representation of F with leading derivatives ∂_y^2 and ∂_x is

$$\{\,(1 \wedge \partial_y)(\vec{z})\partial_y^2 - (1 \wedge \partial_y^2)(\vec{z})\partial_y + (\partial_y \wedge \partial_y^2)(\vec{z}),$$

$$(1 \wedge \partial_y)(\vec{z})\partial_x + (\partial_y \wedge \partial_x)(\vec{z})\,\},$$

since $(1 \wedge \partial_x)(\vec{z})$ has to be zero, for, otherwise, the representation would not be reduced, a contradiction to Corollary 6.

The next proposition reveals that the coefficients w_{ij} are quite simple although they may be outside $\mathbf{K}$.

Proposition 7 *Let F be a Gröbner basis in $\mathbf{K}[\partial_x, \partial_y]$, $z_1, \ldots, z_n$ form a fundamental system of solutions of F, and denote $(z_1, \ldots, z_n)$ by $\vec{z}$. Then $\omega_F(\vec{z})$ is hyperexponential over $\mathbf{K}$. Furthermore, for all $\lambda \in \Lambda^n(\Theta)$, $\lambda(\vec{z})$ is either zero or hyperexponential over $\mathbf{K}$, so is every w_{ij} given in (7).*

Proof Lemma 5 implies that $\omega_F(\vec{z})$ is nonzero. It follows from (6) that the ratio of $\partial_x \omega_F(\vec{z})$ and $\omega_F(\vec{z})$ belongs to $\mathbf{K}$. The same conclusion holds for the ratio of $\partial_y \omega_F(\vec{z})$ and $\omega_F(\vec{z})$. Hence, $\omega_F(\vec{z})$ is hyperexponential over $\mathbf{K}$. Proposition 2 and (6) then imply that any nonzero $\lambda(\vec{z})$ is hyperexponential. □

We show how to compute $\omega_F(\vec{z})$ and the coefficients w_{ij} in the next section.

6 Associated Systems

In this section we generalize the notion of associated equations for factoring linear ode's. As in the previous sections, let n be an integer with $1 \le n < d$. We want to regard every element of $\Lambda^n(\Theta)$ as a function on $\mathrm{sol}(L)^n$. Two elements of $\Lambda^n(\Theta)$ are said to be *equivalent* if they are identical (as functions) on $\mathrm{sol}(L)^n$. For an element λ of $\Lambda^n(\Theta)$, its equivalence class is denoted by $\bar{\lambda}$. It is easy to verify that the equivalence relation is compatible with linear operations and differentiations on $\Lambda^n(\Theta)$. The $\mathbf{K}$-linear space consisting of the equivalence classes is called the *nth Beke space* relative to L, and denoted by B_n when L is clear from the context. From (3) it follows that each equivalence class contains an element of $\Lambda^n(\mathrm{pder}(L))$. Consequently, B_n can be $\mathbf{K}$-linearly generated by the elements in form

$$\overline{(\theta_1 \wedge \theta_2 \wedge \cdots \wedge \theta_n)} \tag{8}$$

where the θ_i belong to $\mathrm{pder}(L)$ and θ_i is lower than θ_j for all i, j with $1 \le i < j \le n$. The elements in (8) are called *canonical generators* of B_n. They are not necessarily $\mathbf{K}$-linearly independent. The following lemma is evident.

Lemma 8 *The nth Beke space B_n is of dimension less than or equal to $\binom{d}{n}$ and closed under differentiation.*

Example 6 If $\mathrm{lder}(L) = \{\partial_x^2, \partial_x\partial_y, \partial_y^2\}$, then $\mathrm{pder}(L) = \{1, \partial_x, \partial_y\}$. The second Beke space B_2 relative to L is generated by $\overline{(1 \wedge \partial_x)}$, $\overline{(1 \wedge \partial_y)}$, and $\overline{(\partial_x \wedge \partial_y)}$.

Set $m = \binom{d}{n}$. For an element $\bar{\lambda}$ of B_n, the ideal consisting of all annihilators of $\bar{\lambda}$ in $\mathbf{K}[\partial_x, \partial_y]$ is denoted by $\mathrm{ann}(\bar{\lambda})$. It follows from Lemma 8 that the solution space of $\mathrm{ann}(\bar{\lambda})$ is no greater than m. A finite subset of $\mathrm{ann}(\bar{\lambda})$ with finite-dimensional solution space is called a *system associated with* $\bar{\lambda}$. The following method computes an associated system by linear algebra and differential reduction. Lemma 8 implies that both $\bar{\lambda}, \partial_x\bar{\lambda}, \ldots, \partial_x^m\bar{\lambda}$ and $\bar{\lambda}, \partial_y\bar{\lambda}, \ldots, \partial_y^m\bar{\lambda}$ are $\mathbf{K}$-linearly dependent. Suppose that p and q are smallest nonnegative integers such that $\partial_x^p\bar{\lambda} + \sum_{i=0}^{p-1} f_i\partial_x^i\bar{\lambda} = 0$ and $\partial_y^q\bar{\lambda} + \sum_{j=0}^{q-1} g_j\partial_y^j\bar{\lambda} = 0$. where $f_{p-1}, \ldots, f_0, g_{q-1}, \ldots, g_0 \in \mathbf{K}$. We find the system

$$\left\{\partial_x^p + \sum_{i=0}^{p-1} f_i\partial_x^i,\ \partial_y^q + \sum_{j=0}^{q-1} g_j\partial_y^j\right\} \tag{9}$$

annihilating $\bar{\lambda}$. The solution space of (9) is of finite dimension, because parametric derivatives of a Gröbner basis for the ideal generated by (9) are contained in

$$D_\lambda = \{\partial_x^i\partial_y^j \mid 0 \le i < p,\ 0 \le j < q\}.$$

Hence, (9) is a system associated with $\bar{\lambda}$. Considering all possible $\mathbf{K}$-linear combinations of $(m+1)$ elements of D_λ, we may obtain an associated system with m-dimensional solution space (see the proof of Lemma 1 in [14]).

For factorization, we need systems associated with the canonical generators. The following method for constructing these systems in form (9) is an easy generalization of the method described in [2]. Let $\vec{b}_n$ be the m-dimensional vector consisting of the canonical generators of B_n. Differentiating this vector and using (3), we obtain two $(m \times m)$ matrices M_n and N_n over $\mathbf{K}$ such that

$$\partial_x\vec{b}_n = \vec{b}_n M_n \quad \text{and} \quad \partial_y\vec{b}_n = \vec{b}_n N_n. \tag{10}$$

Set $M_{n,0} = N_{n,0}$ to be the $(m \times m)$ unit matrix, and set $M_{n,1} = M_n$ and $N_{n,1} = N_n$. Define two sequences of matrices:

$$M_{n,i} = M_2 M_{n,i-1} + \partial_x M_{n,i-1},\ \ N_{n,i} = N_2 N_{n,i-1} + \partial_y N_{n,i-1}, \tag{11}$$

where $i \ge 2$, respectively. An easy induction leads to

$$\partial_x^i\vec{b}_n = \vec{b}_n M_{n,i} \quad \text{and} \quad \partial_y^i\vec{b}_n = \vec{b}_n M_{n,i} \quad (i \ge 0).$$

For $j = 1, \ldots, m$, let $\vec{g}_{n,j}$ and $\vec{h}_{n,j}$ be two $(m+1)$-dimensional vectors whose kth vector is the $(k-1)$th derivatives of the jth component of $\vec{b}_n$ w.r.t. x and y, respectively. Then we have

$$\vec{g}_{n,j} = \vec{b}_n U_{n,j} \quad \text{and} \quad \vec{h}_{n,j} = \vec{b}_n V_{n,j} \tag{12}$$

where the kth column of $U_{n,j}$ $(V_{n,j})$ is the jth column of $M_{n,k-1}$ $(N_{n,k-1})$, for $k = 1, \ldots, m+1$. Since both $U_{n,j}$ and $V_{n,j}$ have more columns than rows, their columns are $\mathbf{K}$-linearly dependent. From these linear relations we can extract a system associated with the jth component of $\vec{b}_n$. Briefly, the systems associated with the canonical generators of B_n can be constructed by solving $2m$ $\mathbf{K}$-linear homogeneous algebraic systems whose coefficient matrices are $U_{n,j}$ and $V_{n,j}$ for $j = 1, \ldots, m$.

Example 7 Let L be given in (1) and $n = 2$. Write

$$L = \{\partial_x^2 + l_1,\ \partial_x\partial_y + l_2,\ \partial_y^2 + l_3\}$$

where the l_i's are $\mathbf{K}$-linear combinations of 1, ∂_x, and ∂_y. Three canonical generators of B_2 are

$$b_1 = \overline{(1 \wedge \partial_x)},\ b_2 = \overline{(1 \wedge \partial_y)},\ b_3 = \overline{(\partial_x \wedge \partial_y)}.$$

By differentiation we get

$$\partial_x b_1 = \overline{1 \wedge \partial_x^2} = -\overline{1 \wedge l_1}.$$

$$\partial_x b_2 = \overline{\partial_x \wedge \partial_y} + \overline{1 \wedge \partial_y^2} = \overline{\partial_x \wedge \partial_y} - \overline{1 \wedge l_3}.$$

$$\partial_x b_3 = \overline{\partial_x^2 \wedge \partial_y} + \overline{\partial_x \wedge \partial_x\partial_y} = -\overline{l_1 \wedge \partial_y} - \overline{\partial_x \wedge l_2}.$$

Substituting concrete expressions of the l_i yields

$$M_2 = \begin{pmatrix} \frac{xy-2+xy}{4x} & \frac{2+xy}{4y} & \frac{-1}{4} \\ \frac{-y^2}{4x} & \frac{-y}{4} & \frac{y}{4x} \\ 0 & 1 & \frac{-1}{2x} \end{pmatrix}$$

as described in (10). In the same vein, we have

$$N_2 = \begin{pmatrix} \frac{2+xy}{4y} & \frac{x(xy-2)}{4y^2} & \frac{-x}{4y} \\ \frac{-y}{4} & \frac{-x}{4} & \frac{1}{4} \\ -1 & 0 & \frac{1}{2y} \end{pmatrix}$$

Using (11) to find M_{22}, M_{23}, N_{22} and N_{23}, and constructing U_{2j} and V_{2j} for $j = 1, 2, 3$, we find that the systems associated with b_1, b_2 and b_3 are, respectively,

$$\Big\{\partial_x^3 + \tfrac{3}{x}\partial_x^2 - \tfrac{-3+xy}{4x^2}\partial_x - \tfrac{y}{8x^2},$$

$$\partial_y^3 - \tfrac{2xy-6}{y(xy-6)}\partial_y^2 - \tfrac{42-23\,xy+x^2y^2}{4y^2(xy-6)}\partial_y + \tfrac{72-30\,xy+x^2y^2}{8y^3(xy-6)}\Big\},$$

$$\Big\{\partial_x^3 + \tfrac{2}{x(2+xy)}\partial_x^2 - \tfrac{xy+1}{4x^2}\partial_x + \tfrac{y^2}{8x(2+xy)},$$

$$\partial_y^3 + \tfrac{xy-6}{y(-2+xy)}\partial_y^2 - \tfrac{-12+x^3y^3-12\,xy-x^2y^2}{4y^2\left(4-4\,xy+x^2y^2\right)}\partial_y$$

$$-\tfrac{x^2(-10+xy)}{8y\left(4-4\,xy+x^2y^2\right)}\Big\},$$

and

$$\left\{\partial_x^3 + \frac{15+xy}{6x}\partial_x^2 - \frac{y^2}{24x},\, \partial_y^2 - \frac{1}{2y}\partial_y - \frac{xy-2}{4y^2}\right\}.$$

7 Idea about factorization

As before, we let F be a Gröbner basis in $\mathbf{K}[\partial_x, \partial_y]$ such that $\mathrm{sol}(F) \subset \mathrm{sol}(L)$ and $\dim_{\bar{\mathbf{Q}}} \mathrm{sol}(F) = n$, where $1 \le n < d$. Let $\vec{z}$ be the vector $(z_1, \ldots, z_n)$ where $z_1, z_2, \ldots, z_n$ form a fundamental system of solutions of F.

7.1 Compute candidates for the canonical generators of B_n

Let $b_1, \ldots, b_m$ be the canonical generators of B_n with respective associated systems $A_1, \ldots, A_m$. Lemma 4 implies that $\bar{\omega}_F$ is a canonical generator of B_n. Hence, the function value $\omega_F(\vec{z})$ is a hyperexponential solution of its associated systems by the first assertion of Proposition 7. For every b_i, the function value $b_i(\vec{z})$ is a solution of A_i, which is either 0 or hyperexponential by the second assertion of Proposition 7. Applying the algorithm in [7], we can find all possible candidates for $b_1(\vec{z}), \ldots, b_m(\vec{z})$.

Example 8 Consider the system L given in (1). According to the associated systems presented in Example 7, we find the candidates of $b_1(\vec{z})$, $b_2(\vec{z})$, and $b_3(\vec{z})$ are

$$\left\{\frac{c_1 y}{\sqrt{xy}} \,|\, c_1 \in \bar{\mathbf{Q}}\right\},\ \left\{\frac{c_2 x}{\sqrt{xy}} \,|\, c_2 \in \bar{\mathbf{Q}}\right\},\ \text{and } \{0\}, \qquad (13)$$

respectively.

7.2 Compute candidates for the Wronskian representation of a factor

Assume that

$$\mathrm{lder}(F) = \{\theta_1, \ldots, \theta_k\} \quad \text{and} \quad \mathrm{pder}(F) = \{\xi_1, \ldots, \xi_n\}.$$

The Wronskian coefficients of F

$$w_{ij} = (-1)^{d+j+1}(\eta_j \wedge \theta_i)(\vec{z})$$

given in (7) are $\mathbf{K}$-linear combinations of the $b_i(\vec{z})$'s, because $\overline{\eta_j \wedge \theta_i} \in B_n$ and $z_1, \ldots, z_n \in \mathrm{sol}(L)$. Hence, we can obtain all candidates for the w_{ij}'s as long as the candidates for the $b_i(\vec{z})$'s are known.

Example 9 Again, let L be given in (1). The Wronskian representation of F given in Example 5 is

$$\{(1 \wedge \partial_x)(\vec{z})\partial_x^2 - (1 \wedge \partial_x^2)(\vec{z})\partial_x + (\partial_x \wedge \partial_x^2)(\vec{z}),$$

$$(1 \wedge \partial_x)(\vec{z})\partial_y - (1 \wedge \partial_y)(\vec{z})\partial_x + (\partial_x \wedge \partial_y)(\vec{z})\}.$$

Let b_1, b_2, b_3 be the same as in Example 8. In B_2 we have

$$\overline{1 \wedge \partial_x^2} = -\frac{2-xy}{4x}b_1 - \frac{y^2}{4x}b_2,\ \overline{\partial_x \wedge \partial_x^2} = -\frac{y}{4x}b_1 - \frac{y^2}{4x}b_3.$$

As the components z_1 and z_2 of $\vec{z}$ belong to $\mathrm{sol}(L)$, the above equalities and Example 8 imply that the Wronskian representation of F must be in form

$$\Big\{\tfrac{c_1 y}{\sqrt{xy}}\partial_x^2 + \left(\tfrac{2-xy}{4x}\tfrac{c_1 y}{\sqrt{xy}} + \tfrac{y^2}{4x}\tfrac{c_2 x}{\sqrt{xy}}\right)\partial_x - \tfrac{y}{4x}\tfrac{c_1 y}{\sqrt{xy}},$$
$$\tfrac{c_1 y}{\sqrt{xy}}\partial_y - \tfrac{c_2 x}{\sqrt{xy}}\partial_x\Big\} \qquad (14)$$

where $c_1, c_2 \in \bar{\mathbf{Q}}$ with $c_1 \ne 0$.

By computing hyperexponential solutions of systems associated with the canonical generators and performing linear operations in B_n, we can get all possible candidates for the Wronskian representation of F. There are only finitely many such candidates, each of which may involve a finite number of unspecified constants in $\bar{\mathbf{Q}}$. This is due to the structure of hyperexponential solutions of L (see [7, Theorem 3.2]).

7.3 Determine coefficients

It remains to determine which candidate leads to a factor of L. A monic associate of a candidate is the set consisting of monic associates of all elements in the candidate. According to (6), the ratio of $\omega_F(\vec{z})$ and any coefficient appearing in some equation of a candidate belongs to $\mathbf{K}$. Therefore, we exclude those candidates whose monic associates do not belong to $\mathbf{K}[\partial_x, \partial_y]$. Let M be a monic associate of a candidate contained in $\mathbf{K}[\partial_x, \partial_y]$. The system M is a factor of L with $\dim_{\bar{\mathbf{Q}}} \mathrm{sol}(M) = n$ if and only if

- M is a Gröbner basis with $\mathrm{lder}(M) = \{\theta_1, \ldots, \theta_k\}$.
- For each element of L, its normal form w.r.t. M is zero.

Example 10 The monic associate of (14) is

$$\left\{\partial_x^2 + \left(\tfrac{2-xy}{4x} + \tfrac{cy}{4}\right)\partial_x - \tfrac{y}{4x},\ \partial_y - \tfrac{cx}{y}\partial_x\right\} \qquad (15)$$

where $c = c_2/c_1$. This system has leading derivatives ∂_x^2 and ∂_y. A simple calculation shows that (15) is a Gröbner basis if and only if $c = 1$. Setting $c = 1$ and computing the normal form of every element of L given in (1) w.r.t. (15), we confirm that L has a factor F as given in Example 3.

Example 11 Let us determine factors of L given in (1) with leading derivatives ∂_y^2 and ∂_x. Assume that G is such a factor with a set of fundamental solutions z_1 and z_2, and denote (z_1, z_2) by $\vec{z}$. In this case the Wronskian operator of G becomes $\omega_G = (1 \wedge \partial_y)$. The candidates for $\omega_G(\vec{z})$ are $\left\{\frac{c_2 x}{\sqrt{xy}} \,|\, c_2 \in \bar{\mathbf{Q}},\, c_2 \ne 0\right\}$ as given in (13). As shown in Example 5, the Wronskian representation of G is

$$\{(1 \wedge \partial_y)(\vec{z})\partial_y^2 - (1 \wedge \partial_y^2)(\vec{z})\partial_y + (\partial_y \wedge \partial_y^2)(\vec{z}),$$

$$(1 \wedge \partial_y)(\vec{z})\partial_x + (\partial_y \wedge \partial_x)(\vec{z})\}.$$

The candidates for the Wronskian representation are

$$\left\{\frac{c_2 x}{\sqrt{xy}}\partial_y^2 + \left(\frac{x^2 c_2}{4\sqrt{xy}}\right)\partial_y - \frac{x}{4y}\frac{c_2 x}{\sqrt{xy}},\ \frac{c_2 x}{\sqrt{xy}}\partial_x\right\}$$

with monic associate $M = \left\{ \partial_y^2 + \left(\frac{x}{4}\right)\partial_y - \frac{x}{4y},\, \partial_x \right\}$. But M is not a Gröbner basis. So F has no factor with leading derivatives ∂_y^2 and ∂_x.

The system (1) has only one second-order factor by Examples 10 and 11.

8 Factorization algorithm

For simplicity, we describe an algorithm for finding nth-order factors F of L under the assumption that $\text{lder}(F)$ is given. It is easy to adjust the algorithm to compute all factors of L by Lemma 4.

Algorithm F Given a Gröbner basis L with d-dimensional solution space, and $\Delta \subset (\text{lder}(L) \cup \text{pder}(L))$ whose elements are mutually reduced, compute all factors F of L with $\text{lder}(F) = \Delta$.

1. [compute $\text{pder}(F)$] Find $\Delta^- \subset \Theta$ consisting of all derivatives not divisible by any elements of Δ. If $|\Delta^-| \geq d$, exit [no such factors exist]. Set n to be $|\Delta^-|$.

2. [candidates for the Wronskian] Construct the system A_1 associated with ω_F, and compute hyperexponential solutions of A_1. If no hyperexponential solution is found, exit [no such factors exist]. Organize the solutions as equivalence classes:

$$\begin{aligned} \{h_{11} &= p_{11} \exp\left(\textstyle\int (f_{11}\mathrm{d}x + g_{11}\mathrm{d}y)\right), \cdots, \\ h_{1k} &= p_{1k} \exp\left(\textstyle\int (f_{1k}\mathrm{d}x + g_{1k}\mathrm{d}y)\right)\} \end{aligned} \tag{16}$$

where the f_{1i} and g_{1i} are in $\mathbf{K}$, and the p_{1i} are polynomials whose coefficients are elements of $\bar{\mathbf{Q}}$ and unspecified constants (see [7, Theorem 3.2]).

3. [candidates for other canonical generators] Construct the systems $A_2, \ldots, A_m$ associated with other canonical generators, and compute their hyperexponential solutions equivalent to some h_{1i} $(1 \leq i \leq k)$. Set h_{ji} to be the hyperexponential solution of A_j equivalent to h_{1i} if such a solution exists. Otherwise, set h_{ji} to be zero. Let

$$H = \{(h_{11}, h_{21}, \ldots, h_{m1}),\, \ldots,\, (h_{1k}, h_{2k}, \ldots, h_{mk})\}$$

where the h_{1i} are in (16), and the h_{ji} with $j > 1$ are either zero or hyperexponential elements equivalent to h_{1i}.

4. [candidates for factors] Construct the Wronskian representation defined by Δ. Construct the matrix transforming the canonical generators to the Wronskian coefficients. Use this matrix and the elements of H to get all rational monic associates $\{F_1, \ldots, F_k\}$ of the candidates for factors.

5. [true factors] Check if each F_i satisfies the two conditions given in Section 7.3, and solve algebraic equations in unspecified constants when necessary. Return the true factors.

A few words need to be said about Algorithm **F**. The first step is clear. The second step is a direct application of the algorithm in [7] whose outputs are distinct equivalence classes of hyperexponential solutions (after a formal integration). If no hyperexponential solution is found, then factors with leading derivatives Δ do not exist by Proposition 7. In the third step, (6) implies that we need only hyperexponential solutions equivalent to some h_{1i}. Since these solutions belong to one equivalence class, all of them can be expressed as $q_i h_{1i}$, where q_i is a polynomial in x and y whose coefficients are elements of $\bar{\mathbf{Q}}$ and unspecified constants. Thus, H contains at most k elements. Finding these solutions amounts to computing rational solutions of some zero-dimensional ideals, which is easier than computing all hyperexponential solutions of other associated systems. This technique is introduced in [2] for the ode case, and is extended to the pde case in [14]. The last two steps have been explained in Sections 7.2 and 7.3, respectively.

The techniques to use invertible matrices to avoid finding hyperexponential solutions in [2] can also be generalized to our case, in which the matrices U_{nj} and V_{nj} in (12) play the same role as A_S in [2]. The results in Remark 2 also help to detect factors, as will be shown in the next example.

Example 12 Apply Algorithm **F** to find all factors of L:

$$\begin{aligned} \Big\{ &\partial_x^3 + \tfrac{x^3-3x}{x^2-1}\partial_x^2,\ \partial_y\partial_x + \tfrac{3yx^2}{x^2-1}\partial_x^2 + 3yx\partial_x + x\partial_y, \\ &\partial_y^2 + \tfrac{9y^2x+3x+y}{x^2-1}\partial_x^2 + 3y\partial_y + (3+9y^2)\partial_x \Big\}. \end{aligned} \tag{17}$$

All the c with subscripts are hereafter unspecified constants.

We start to compute first-order factors. By Proposition 3 the algorithm in [7] yields all the first-order factors F_1:

$$\left\{ \partial_x - \frac{2c_2}{2c_1 + 2c_2x - 3y^2c_2},\ \partial_y + \frac{6c_2y}{2c_1 + 2c_2x - 3y^2c_2} \right\}.$$

To find second-order factors, we need canonical generators of the second Beke space, which are:

$$\begin{array}{lll} b_1 = \overline{1 \wedge \partial_x}, & b_2 = \overline{1 \wedge \partial_y} & b_3 = \overline{1 \wedge \partial_x^2} \\ b_4 = \overline{\partial_x \wedge \partial_y} & b_5 = \overline{\partial_x \wedge \partial_x^2} & b_6 = \overline{\partial_y \wedge \partial_x^2}. \end{array}$$

We do not display their associated systems, for they are quite large. Note that associated systems only depend on L.

First, compute factors with leading derivatives $\{\partial_x^2, \partial_y\}$. Step 1 yields $\Delta^- = \{1, \partial_x\}$. Hence, the Wronskian operator is b_1. Step 2 finds candidates

$$\left\{ c_{11},\ (c_{11} + c_{12}x + c_{13}x^2)\exp\left(-\frac{x^2}{2}\right) \right\}$$

for b_1. Step 3 finds candidates $(c_{11}, c_{21}, 0, 0, 0, 0)$ and

$$\left((c_{11} + c_{12}x + c_{13}x^2)u, (c_{21} + c_{22}x + c_{23}y + c_{24}xy)u, 0, 0, 0, 0\right),$$

where $u = \exp\left(-\frac{x^2}{2}\right)$, for canonical generators. Step 4 gets the Wronskian representation

$$\left\{ b_1\partial_x^2 - b_3\partial_x + b_5,\ b_1\partial_y - b_2\partial_x + b_4 \right\}$$

and, thus, yields two candidates

$$\{\partial_x^2,\ \partial_y + c_{21}\partial_x\},\ \left\{ \partial_x^2,\ \partial_y + \frac{c_{21} + c_{22}x + c_{23}y + c_{24}xy}{c_{11} + c_{12}x + c_{13}x^2}\partial_x \right\}$$

for factors. In the last step we obtain a factor

$$F_{21}:\ \{\partial_x^2,\ \partial_y + 3y\partial_x\}$$

from the second candidate.

Next, we compute factors with leading derivatives ∂_y^2 and ∂_x. The Wronskian operator then becomes b_2. Step 2 yields six candidates

$$\Big\{c_{21},\ c_{21}\exp(-x^2),\ (c_{21}+c_{22}x+c_{23}y+c_{24}xy)\exp\left(\tfrac{-1}{2}x^2\right),\ c_{21}\exp\left(\tfrac{-3}{2}y^2-x^2\right),\ (c_{21}+c_{22}x)\exp\left(\tfrac{-3}{2}y^2+\tfrac{-1}{2}x^2\right),\ c_{21}\exp\left(\tfrac{-3}{2}y^2\right)\Big\}$$

for b_2. Note that b_1 has to be zero in this case, for otherwise, we would not have factors with desired leading derivatives. Step 3 gives six candidates for b_2. Deleting repetitions, we find that three of them are left. They are

$$(0,\ c_{21},\ 0,\ 0,\ 0,\ 0),\ \big(0,\ c_{21}\,u,\ 0,\ c_{41}xu,\ 0, c_{61}(1-x^2)u\big)$$

where $u=\exp\left(-\frac{3}{2}y^2-x^2\right)$, and

$$\big(0,\ (c_{21}+c_{22}x)v,\ 0,\ (c_{41}+c_{42}x)v,\ 0,\ c_{61}(1-x^2)v\big)$$

where $v=\exp\left(-\frac{3}{2}y^2-\frac{1}{2}x^2\right)$. In Step 4 we find the Wronskian representation

$$\left\{b_2\partial_y^2-\overline{(1\wedge\partial_y^2)}\partial_y+\overline{(\partial_y\wedge\partial_y^2)},\ b_2\partial_x-b_4\right\},$$

and transformations $\overline{(1\wedge\partial_y^2)}=rb_3-3yb_2-(9y^2-3)b_1$ and $\overline{(\partial_y\wedge\partial_y^2)}=rb_6+(9y^2+3)b_4$, where $r=\frac{9y^2x+3x+y}{1-x^2}$. They lead to two candidates for factors. Step 5 yields a factor

$$F_{22}:\ \{\partial_y^2+3y\partial_y+y,\ \partial_x+x\}.$$

We have found all second-order factors of L.

Instead of algorithm **F** we use the results in Remark 2 to find all third-order factors. A straightforward calculation shows that every factor F_{11} obtained from a specialization of unspecified constants in F_1 satisfies $\mathrm{sol}(F_{11})\subset\mathrm{sol}(F_{21})$, and that $\mathrm{sol}(F_{21})\cap\mathrm{sol}(F_{22})=\{0\}$. Therefore, F_{22} is irreducible. It follows that

$$\{0\}\subset\mathrm{sol}(F_{11})\subset\mathrm{sol}(F_{21})\subset\mathrm{sol}(L)$$

is a maximal chain, for, otherwise, there would be a chain starting with $\{0\}\subset\mathrm{sol}(F_{22})$ and having length greater than four. Hence, any third-order factor F_3 must satisfy $\mathrm{sol}(F_{22})\subset\mathrm{sol}(F_3)$. The dimension formula from linear algebra shows

$$\dim_{\bar{\mathbf{Q}}}(\mathrm{sol}(F_3)+\mathrm{sol}(F_{21}))+\dim_{\bar{\mathbf{Q}}}(\mathrm{sol}(F_3)\cap\mathrm{sol}(F_{21}))=5.$$

We conclude that $(\mathrm{sol}(F_3)\cap\mathrm{sol}(F_{21}))$ is of dimension one over $\bar{\mathbf{Q}}$, because $(\mathrm{sol}(F_3)+\mathrm{sol}(F_{21}))$ is contained in $\mathrm{sol}(L)$ whose dimension is four over $\bar{\mathbf{Q}}$. It follows that a third-order factor has a first-order factor, which is an instance of F_1. Hence, the ideal generated by a third-order factor is the intersection of the respective ideals generated by F_{22} and an instance of F_1. All third-order factors can then be computed by standard techniques from Gröbner basis arsenal.

9 Concluding remarks

The results of this article are a first step toward generalizing the theory of linear ode's to pde's. The limitation to two independent variables is justified by its applications to the symmetry analysis on nonlinear ode's. A more complete theory will deal with any number of dependent and independent variables. In this way it will be possible to generalize Loewy's decomposition of linear ode's to systems of linear pde's with a finite-dimensional solution space.

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