Simplification of Definite Sums of Rational Functions

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Abstract

We propose an algorithm for simplification of definite sums of rational functions which, for a given input rational function $F(n,k)$, constructs two rational functions $G(n)$ and $T(n,k)$ such that

$$\sum_{k=0}^{n} F(n,k) = G(n) + \sum_{k=0}^{n} T(n,k),$$

where the degree of the denominator w.r.t. k of $T(n,k)$ is "small".

1 Preliminaries

Let K be a field of characteristic 0, $F(n,k) \in K(n,k)$. Set

$$F(n,k) = \frac{a(n,k)}{b(n,k)},\ a(n,k),\, b(n,k) \in K[n,k]. \tag{1}$$

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$F(n,k)$ is defined to be *proper* if $\deg_k a(n,k) < \deg_k b(n,k)$, and in *reduced form* if $\gcd(a(n,k), b(n,k)) = 1$. Applying to $F(n,k)$ an algorithm to solve the indefinite sums of rational functions [1, 2, 6, 7] w.r.t. k results in

$$F(n,k) = \Delta_k S(n,k) + T(n,k), \tag{2}$$

where $S(n,k), T(n,k) \in K(n,k)$, $T(n,k)$ is proper, in reduced form, and the denominator of $T(n,k)$ has the minimal degree w.r.t. k. The operator Δ_k denotes the forward difference operator which acts on k, i.e., $\Delta_k f(k) = f(k+1) - f(k)$ for any function $f(k)$. An *indefinite additive decomposition* of $F(n,k)$ consists of the pair (S,T) such that (2) holds. S is called the *summable part* of the decomposition, and T the *non-summable part.* As a special case, if $T(n,k)$ in (2) vanishes, then $F(n,k)$ is said to be *indefinite summable*, i.e.,

$$\sum_k F(n,k) = S(n,k),\ S(n,k) \in K(n,k). \tag{3}$$

Note that $S(n,k)$ is defined up to a term $p(k) \in K(n)$. Thus, $\sum_k F(n,k)$ is a set.

For any fixed $n \in \mathbb{N}$, if $F(n,k)$ does not have any pole for all $0 \le k_0 \le n$ (throughout the paper, we assume that $F(n,k)$ has this property), then it follows from (2) that

$$\sum_{k=0}^{n} F(n,k) = G(n) + \sum_{k=0}^{n} T(n,k), \tag{4}$$

where $G(n) = S(n, n+1) - S(n,0)$. A *definite additive decomposition* of $F(n,k)$ consists of a pair $(G(n), T(n,k))$ where $G(n) \in K(n), T(n,k) \in K(n,k)$ are such that G, T satisfy (4), and the denominator of $T(n,k)$ has the minimal degree w.r.t. k. As a special case, if $T(n,k)$ in (4) vanishes, then $F(n,k)$ is said to be *definite summable*, i.e.,

$$\sum_{k=0}^{n} F(n,k) = G(n),\ G(n) \in K(n). \tag{5}$$

Note that if $F(n,k) \in K(n,k)$ is indefinite summable, then F is definite summable. However, the reverse is not necessarily true. As an example, consider the rational function

$$F(n,k) = \frac{1}{nk + 3n + 1} - \frac{1}{n^2 - nk + 1}.$$

An indefinite additive decomposition of F is

$$(S,T) = \left(0, \frac{-2\,nk + n(n-3)}{(nk+3\,n+1)(n^2-nk+1)}\right),$$

i.e., F is not indefinite summable. However, applying our proposed algorithm (see Example 2) to $F(n,k)$ results in the pair

$$(G,T) = \left(-\frac{n\,(2\,n^6+16\,n^5+45\,n^4+60\,n^3+45\,n^2+18\,n+3)}{(2\,n+1)(n^2+n+1)(n^2+2\,n+1)(n^2+3\,n+1)}, 0\right),$$

i.e., $F(n,k)$ is definite summable.

It is worth mentioning that Zeilberger's algorithm [8], a very useful tool for proving combinatorial identities which involve definite sums of hypergeometric terms (including rational functions as a special case), is not applicable to $F(n,k)$. In other words, it does not terminate for the input $F(n,k)$ [3]. It is not applicable to a large class of rational functions which are shown to be definite summable either [3, Sect. 7].

In order to compute $\sum_{k=0}^{n} F(n,k)$ for $F(n,k) \in K(n,k)$, a standard method is to compute an indefinite additive decomposition $(S(n,k), T(n,k))$ of $F(n,k)$, and then to express the part $\sum_{k=0}^{n} T(n,k)$ in (4) in terms of the Digamma and Polygamma functions. For the example mentioned above,

$$\begin{aligned}\textstyle\sum_{k=0}^{n} F(n,k) &= \tfrac{1}{n}\Psi(n+1+(3n+1)/n) + \tfrac{1}{n}\Psi(n+1-(n^2+1)/n) - \\ &\quad \tfrac{1}{n}\Psi((3n+1)/n) - \tfrac{1}{n}\Psi(-(n^2+1)/n)\end{aligned}$$

while $F(n,k)$ is indeed definite summable. As another example (see Example 4), consider the rational function

$$F(n,k) = \frac{1}{k+2} - \frac{1}{k+1} + \frac{1}{2\,k+5} - \frac{1}{2\,n-2\,k+1} + \frac{1}{nk+1}.$$

An indefinite additive decomposition of F is

$$(S,T) = \left(\frac{1}{k+1}, \frac{(4\,n+4)k^2 - (2\,n^2-12)k - (12\,n+1)}{(nk+1)(2\,k+5)(2\,k-2\,n-1)}\right).$$

Note that the degree of the denominator of $T(n,k)$ w.r.t. k is 3. Applying our proposed algorithm results in the pair

$$(G,T) = \left(-\frac{1}{3}\,\frac{28\,n^3+144\,n^2+233\,n+117}{(n+2)\,(2\,n+3)\,(2\,n+5)}, \frac{1}{nk+1}\right)$$

where the degree of the denominator of $T(n,k)$ w.r.t. k is 1.

For a given $F(n,k) \in K(n,k)$, denote by $denom(F(n,k)) \in K[n,k]$ the denominator of F. Let $(S(n,k), T(n,k))$ be an indefinite decomposition of $F(n,k)$. We present in this paper an algorithm to construct a pair $(G(n), T'(n,k))$ such that

$$\sum_{k=0}^{n} F(n,k) = G(n) + \sum_{k=0}^{n} T'(n,k), \text{ and}$$

either $T'(n,k)$ vanishes or $\deg_k denom(T'(n,k))$ is "small" in the sense that

$$\deg_k denom(T'(n,k)) \leq \deg_k denom(T(n,k)).$$

The pair $(G(n), T'(n,k))$ is called a *simplification* of $\sum_{k=0}^{n} F(n,k)$.

2 Indefinite Sums of Rational Functions

Let $F(n,k) \in K(n,k)$. Denote by $F(n;k)$ an element from $K(n)(k)$, and when suitable, as an element from $K(n)[k]$. We also consider elements of the rings $\overline{K(n)}[k]$, and denote these polynomials as $\varphi(n;k)$, $\mu(n;k)$, et cetera.

Let the pair (S,T) be an indefinite additive decomposition of $F(n,k)$. Set the non-summable part

$$T(n,k) = \frac{f(n,k)}{g(n,k)},\ f(n,k),\ g(n,k) \in K[n,k]. \tag{6}$$

If $T(n,k) \neq 0$, then $g(n,k)$ has the following property [1]:

P1. If $\varphi_1(n;k), \varphi_2(n;k)$ are factors of $g(n;k)$ irreducible over $\overline{K(n)}$ then $\varphi_1(n;k+h) \neq \varphi_2(n;k)$ for all $h \in \mathbb{Z} \setminus \{0\}$.

The following corollary follows directly.

Corollary 1 *$g(n, n-k)$ has property* **P1**.

If $T(n,k) = 0$, let $F(n,k)$ be written in the reduced form (1). Then the denominator $b(n;k)$ of $F(n,k)$ has the following property [1]:

P2. If $\mu_1(n;k)$ is a factor of $b(n;k)$ irreducible over $\overline{K(n)}$ then there exist a factor $\mu_2(n;k)$ irreducible over $\overline{K(n)}$ of $b(n;k)$ and a non-zero integer h such that $\mu_1(n;k+h) = \mu_2(n;k)$.

Lemma 1 *For each monic irreducible factor* $\varphi_0(n;k) = k - \alpha_0(n)$ *over* $\overline{K(n)}$ *of* $g(n;k)$*, there exists at most one monic irreducible factor* $\varphi_1(n;k) = k - \alpha_1(n)$ *over* $\overline{K(n)}$ *of* $g(n;k)$ *such that*

$$\varphi_0(n, k+h) = -\varphi_1(n, n-k),\ \ h \in \mathbb{Z}.$$

Proof : Suppose there exist $h_1, h_2 \in \mathbb{Z}$, and an irreducible factor $\varphi_2(n;k) = k - \alpha_2(n)$ of $g(n;k)$ such that

$$\varphi_0(n, k+h_1) = -\,\varphi_1(n, n-k),\ \text{ and } \varphi_0(n, k+h_2) = -\,\varphi_2(n, n-k). \qquad (7)$$

It follows from (7) that

$$\varphi_2(n, n-k) = \varphi_1(n, n-k+(h_1-h_2)).$$

Since $T(n,k)$ is the non-summable part of $F(n,k)$, it follows from Corollary 1 that $g(n, n-k)$ has property **P1**. Consequently, $h_1 = h_2$, and hence, $\alpha_1(n) = \alpha_2(n)$. ∎

3 A General Algorithm

3.1 Polynomial Splitting

For a given polynomial $P(n,k) \in K[n,k]$, consider the problem of splitting $P(n,k)$ into

$$P(n,k) = U(n,k)\,V(n,k),\ \ U(n,k),\ V(n,k) \in K[n,k], \qquad (8)$$

where for each irreducible factor $u_i(n,k)$ from $K[n,k]$ of $U(n,k)$, there exist an $h \in \mathbb{Z}$ and an irreducible factor $v_j(n,k)$ of $V(n,k)$ such that

$$u_i(n, k+h) = c\,v_j(n, n-k),\ \ c \in K. \qquad (9)$$

Definition 1 *Let* $a, b \in K[n][k] \setminus \{0\}$*. Define the function spread as follows:*

$$spr_k(a(k), b(k)) = \{h \mid h \in \mathbb{Z},\ \deg\gcd(a(k+h), b(k)) > 0\}.$$

The function *spread* of a and b can be computed as the set of integer roots of the polynomial $R(h) = \mathrm{Res}_k(a(k+h), b(k))$. Another algorithm based on factorization of polynomials is given in [5].

Corollary 2 *For a given* $P(n,k) \in K[n,k]$, *set* $Q(n,k) = P(n,n-k)$. *Let* $\mathcal{S} = spr_k(P(n,k), Q(n,k))$. *If* $\mathcal{S} = \{\}$, *then* $(U(n,k),\ V(n,k)) = (1,\ P(n,k))$. *Otherwise, let* $h \in \mathbb{Z}$ *be an element of* $\mathcal{S}$. *Set*

$$f(n,k) = \gcd\left(P(n,k+h), Q(n,k)\right).$$

Then the polynomial $f(n,k-h) \in K[n,k]$, *which is a factor of* $P(n,k)$, *can be split into*

$$f_1(n,k)\, f_2(n,k),\ \ f_1,\ f_2 \in K[n,k] \tag{10}$$

such that $f_1(n,k+h) = c\, f_2(n,n-k)$, $c \in K$.

Corollary 2 provides an algorithm to split $P(n,k) \in K[n,k]$ into the desired form (8). Note that to obtain a split in (10), one only needs to factor f in $K[n,k]$, i.e., a complete factorization into irreducibles over $\overline{K(n)}$ is not required.

Example 1 Let

$$\begin{aligned} P(n,k) \ &= \ (n+k-4)\,(2\,n-k-1)\,(n^2k^2 - 2\,n^2k + n^2 - k + 3) \\ &\quad\ (n^4 - 2\,n^3k + n^2k^2 - n + k + 2)\,(nk+1). \end{aligned}$$

Set $Q(n,k) = P(n,n-k)$. Then $\mathcal{S} = spr(P,Q) = \{1,3\}$.
For $h = 3$,

$$f(n,k) = \gcd(P(n,k+h), Q(n,k)) = f_1'\, f_2'$$

where $f_1' = n + k - 1, f_2' = 2\,n - k - 4$. Since $f_1'(n,k-h) = f_2'(n,n-k)$, $f(n,k-h)$ can be split into $f_1\, f_2$ where $f_1 = f_1'(n,k-h)$, $f_2 = f_2'(n,k-h)$. Similarly, for $h = 1$,

$$g(n,k) = \gcd(P(n,k+h), Q(n,k)) = g_1'\, g_2',$$

where $g_1'(n,k) = n^2k^2 - k + 2$, and

$$g_2'(n,k) = n^4 - 2\,n^3k - 2\,n^3 + n^2k^2 + 2\,n^2k + n^2 - n + k + 3.$$

Since $g_1'(n,k-h) = g_2'(n,n-k)$, $g(n,k-h)$ can be split into $g_1\, g_2$ where $g_1 = g_1'(n,k-h)$, $g_2 = g_2'(n,k-h)$. Consequently, the polynomial $P(n,k)$ is split into $U(n,k)V(n,k)$ where $U = f_1\, g_1$, $V = (nk+1)\, f_2\, g_2$.

Note that if $P(n,k)$ has property **P1**, then it follows from Lemma 1 that for each irreducible factor $u(n,k)$ in $K[n,k]$ of $P(n,k)$, there exists at most one irreducible factor $v(n,k)$ in $K[n,k]$ of $P(n,k)$ such that

$$u(n,k+h) = c\, v(n,n-k),\ \ h \in \mathbb{Z},\ \ c \in K.$$

3.2 Algorithm Description

Proposition 1 *For any* $F(n,k) \in K(n,k)$,

$$\sum_{k=0}^{n} (F(n,k) - F(n,n-k)) = 0.$$

Proof :

$$\begin{aligned}\textstyle\sum_{k=0}^{n} (F(n,k) - F(n,n-k)) &= (F(n,0) + \cdots + F(n,n)) - \\ &\quad (F(n,n) + \cdots + F(n,0)) = 0.\end{aligned}$$

■

Corollary 3

$$\sum_{k=0}^{n} F(n,k) = \frac{1}{2} \sum_{k=0}^{n} (F(n,k) + F(n,n-k)).$$

For a given $P(n,k) \in K[n,k]$, let *split* be the algorithm to split P into the form (8) as described in subsection 3.1. Define an interface for the algorithm, called *indefdecomp*, which solves the indefinite additive decomposition problem as follow (see [2, 6] for instance for detailed descriptions of the algorithm).

`algorithm` *indefdecomp*;
`input:` $F(n,k) \in K(n,k)$;
`output:` an indefinite additive decomposition $(S(n,k),\, T(n,k))$ of F;

Consider the following description of an algorithm, called *defdecomp*.

`algorithm` *defdecomp*;
`input:` $F(n,k) \in K(n,k)$;
`output:` the pair $(G(n),\, T'(n,k))$ such that
(i) $\sum_{k=0}^{n} F(n,k) = G(n) + \sum_{k=0}^{n} T'(n,k)$,
(ii) $\deg_k denom(T'(n,k))$ is small;

$(S(n,k), T(n,k)) := indefdecomp(F(n,k))$;
$G_1(n) := S(n,n+1) - S(n,0)$;
if $T(n,k) = 0$ **then**

    return $(G_1(n), 0)$;
fi;
$F_1(n,k) = \frac{1}{2}\,(T(n,k) + T(n,n-k))$;
$(S_1(n,k), T_1(n,k)) := \mathit{indefdecomp}(F_1(n,k))$;
if $S_1(n,k) = 0$ **then**
    return $(G_1(n), T(n,k))$;
fi;
$G_2(n) := S_1(n, n+1) - S_1(n, 0)$;
if $T_1(n,k) = 0$ **then**
    return $(G_1(n) + G_2(n), 0)$;
fi;
Set $T_1(n,k) = U(n,k)/V(n,k)$, $U,\, V \in K[n][k]$, $\gcd(U,V) = 1$;
$(q_1(n,k),\, q_2(n,k)) := \mathit{split}(V(n,k))$;
represent $T_1(n,k)$ in the form
    $T_1(n,k) = \frac{w_1(n,k)}{q_1(n,k)} + \frac{w_2(n,k)}{q_2(n,k)}$, $w_1,\, w_2 \in K(n)[k]$;
$F_2(n,k) := T_1(n,k) - \frac{w_2(n,k)}{q_2(n,k)} + \frac{w_2(n,n-k)}{q_2(n,n-k)}$;
$(S'(n,k), T'(n,k)) := \mathit{indefdecomp}(F_2(n,k))$;
$G_3(n) := S'(n,n+1) - S'(n,0)$;
return $(G_1(n) + G_2(n) + G_3(n),\; T'(n,k))$;

3.3 Algorithm Correctness

Proposition 2 *For a given $F(n,k) \in K(n,k)$, algorithm $\mathit{defdecomp}$ returns a pair $(G(n), T'(n,k))$ such that*

$$\sum_{k=0}^{n} F(n,k) = G(n) + \sum_{k=0}^{n} T'(n,k). \qquad (11)$$

Proof : By $\mathit{defdecomp}$,

$$\begin{aligned}
\textstyle\sum_{k=0}^{n} F(n,k) &= \textstyle\sum_{k=0}^{n} (\Delta_k\, S(n,k) + T(n,k)) \\
&= S(n,n+1) - S(n,0) + \textstyle\sum_{k=0}^{n} \frac{1}{2}(T(n,k) + T(n,n-k)) \\
&= G_1(n) + \textstyle\sum_{k=0}^{n} (\Delta_k\, S_1(n,k) + T_1(n,k)) \\
&= G_1(n) + S_1(n,n+1) - S_1(n,0) + \textstyle\sum_{k=0}^{n} F_2(n,k) + \\
&\quad \textstyle\sum_{k=0}^{n} \left(\frac{w_2(n,k)}{q_2(n,k)} - \frac{w_2(n,n-k)}{q_2(n,n-k)} \right) \\
&= G_1(n) + G_2(n) + \textstyle\sum_{k=0}^{n} (\Delta_k\, S'(n,k) + T'(n,k)) + 0 \\
&= G(n) + \textstyle\sum_{k=0}^{n} T'(n,k)
\end{aligned}$$

where $G(n) = G_1(n) + G_2(n) + G_3(n),\ G_3(n) = S'(n, n+1) - S'(n, 0)$. ■

Proposition 3 *If there exists an $h \in \mathbb{Z}$ such that*

$$k + h - \alpha_i(n) = k + \alpha_j(n) - n,\ \alpha_i,\ \alpha_j \in \overline{K(n)}, \tag{12}$$

then there exists an $h_1 \in \mathbb{Z}$ such that

$$k + h_1 - \alpha_j(n) = k + \alpha_i(n) - n.$$

Proof : The claim is proven by setting $k = n - k$ in (12) and $h_1 = h$. ■

Let $T(n, k)$ of the form (6) be a non-summable part of $F(n, k)$. Consider $F_1(n, k) = T(n, k) + T(n, n-k)$. Let $(S_1(n, k), T_1(n, k))$ be an indefinite additive decomposition of $F_1(n, k)$. Set

$$T_1(n, k) = \frac{f_1(n, k)}{g_1(n, k)},\ f_1,\ g_1 \in K[n, k].$$

Observe that all monic irreducible factors of $g(n; k)$ and $g_1(n; k)$ over $\overline{K(n)}$ are of the form $k - \alpha_i(n)$ and $k + \alpha_j(n) - n,\ \alpha_i, \alpha_j \in \overline{K(n)}$, respectively.

The following theorem verifies the correctness of algorithm *defdecomp*.

Theorem 1 *For a given $F(n, k) \in K(n, k)$, algorithm defdecomp returns a pair $(G(n), T'(n, k))$ such that (11) holds and $\deg_k denom(T'(n, k)) \leq \deg_k denom(T(n, k))$.*

Proof : The returned pair $(G(n), T'(n, k))$ satisfies (11) follows from Proposition 2. Consider the case where $S_1(n, k)$ does not vanish. Since $g(n; k)$ has property **P1**, and $g_1(n; k)$ also has property **P1** (by Corollary 1), it follows from [2] that there exists a non-empty set of monic irreducible factors $\varphi_i(n; k) = k - \alpha_i(n)$ over $\overline{K(n)}$ of $g(n; k)$ such that for each simple fraction of the form

$$\frac{\beta_{i_s}(n)}{\varphi_i(n, k)^s},\ \beta_{i_s} \in \overline{K(n)},\ s \in \mathbb{N} \setminus \{0\} \tag{13}$$

of $F_1(n, k)$, there exist an $h \in \mathbb{Z}$ and exactly one monic irreducible factor $\varphi_j(n; k) = k + \alpha_j(n) - n$ of $g_1(n; k)$ such that $\varphi_i(n, k + h) = \varphi_j(n, k)$,

$$\frac{\beta_{j_s}(n)}{\varphi_j(n, k)^s},\ \beta_{j_s} \in \overline{K(n)}$$

is a simple fraction of $F_1(n,k)$, and

$$\left(\frac{\beta_{i_s}(n)}{\varphi_i(n,k)^s} + \frac{\beta_{j_s}(n)}{\varphi_j(n,k)^s}\right) \in \overline{K(n)}.$$

Considering the sets of simple fractions of $T(n,k)$ and $T(n,n-k)$, it follows from Proposition 3 that if we extract a simple fraction $p(n,k)$ of the form (13) from $T(n,k)$, we also extract the corresponding simple fraction $p(n,n-k)$ from $T(n,n-k)$. As a consequence, for each simple fraction

$$p_1 = \frac{\beta_{i_s}(n)}{(k-\alpha_i(n))^s},\ \beta_{i_s},\ \alpha_i \in \overline{K(n)},\ s \in \mathbb{N}\setminus\{0\}$$

of $T_1(n,k)$, there exists exactly one simple fraction also of $T_1(n,k)$ of the form $p_2(n,k) = p_1(n,n-k)$. By *defdecomp*, the fraction p_1 belongs to the term $w_1(n,k)/q_1(n,k)$ of T_1, while the fraction p_2 belongs to the term $w_2(n,k)/q_2(n,k)$. Also, for each pair (p_1,p_2),

$$p_1(n,k) + p_2(n,k) - p_2(n,k) + p_2(n,n-k) = 2\,p_1(n,k).$$

(We just applied the step

$$F_2(n,k) := T_1(n,k) - \frac{w_2(n,k)}{q_2(n,k)} + \frac{w_2(n,n-k)}{q_2(n,n-k)}$$

in *defdecomp* locally.) Since $p_1(n,k)$ is a simple fraction of T_1 which is a non-summable part of F_1, p_1 is a non-summable part of p_1.

We have shown that when $S_1(n,k) \neq 0$,

$$\deg_k denom(T'(n,k)) < \deg_k denom(T(n,k)).$$

By following the same argument, we have

$$\deg_k denom(T'(n,k)) = \deg_k denom(T(n,k))$$

for the case where $S_1(n,k) = 0$. In this case, one can return the result obtained from an indefinite additive decomposition of the input $F(n,k)$ right away. ∎

4 Examples

Example 2 Consider the following rational function as mentioned in section 1.

$$F(n,k) = \frac{1}{nk+3n+1} - \frac{1}{n^2-nk+1}.$$

An indefinite additive decomposition $(S(n,k), T(n,k))$ of $F(n,k)$ is

$$(S,T) = \left(0, \frac{-2nk+n(n-3)}{(nk+3n+1)(n^2-nk+1)}\right).$$

Set $F_1(n,k) = \frac{1}{2}(T(n,k)+T(n,n-k))$. An indefinite additive decomposition $(S_1(n,k), T_1(n,k))$ of $F_1(n,k)$ has $T_1 = 0$, and $S_1 = \frac{1}{2}\frac{u(n,k)}{v(n,k)}$ where

$$\begin{aligned} u(n,k) &= n(-2k+n+1)\left((6k+2+3k^2)n^6 + (-6k^3+28k+16)n^5 + \right. \\ &\quad (-6k^3+45-19k^2+3k^4+40k)n^4 + (60-18k^2+24k)n^3 + \\ &\quad \left.(45-6k^2+6k)n^2+18n+3\right), \end{aligned}$$

and

$$\begin{aligned} v(n,k) &= (nk+2n+1)(n^2-nk+n+1)(nk+n+1) \\ &\quad (n^2-nk+2n+1)(nk+1)(n^2-nk+3n+1). \end{aligned}$$

Since $T_1(n,k) = 0$, $F(n,k)$ is definite summable, and

$$\sum_{k=0}^{n} F(n,k) = -\frac{n(2n^6+16n^5+45n^4+60n^3+45n^2+18n+3)}{(2n+1)(n^2+n+1)(n^2+2n+1)(n^2+3n+1)}.$$

Note that the "sum" command in Maple 7 returns the answer in terms of the Ψ function as shown in Section 1. The "Sum" command in Mathematica 4 returns an answer (of rather large size) which involves the cot function.

Example 3 Consider the rational function

$$F(n,k) = \frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k-n-2} + \frac{1}{k-n-3}.$$

Applying $indefdecomp$ to $F(n,k)$ results in

$$(S(n,k), T(n,k)) = \left(\frac{n-2k+2}{(k+1)(n-k+3)},\ 2\,\frac{n-2k+2}{-k^2+2k+kn+n+3}\right).$$

Set $F_1(n,k) = \frac{1}{2}(T(n,k)+T(n,n-k))$, and applying *indefdecomp* to $F_1(n,k)$ results in the pair $(S_1(n,k), T_1(n,k))$ where $T_1 = 0$, and

$$S_1 = -\frac{8+11\,n-14\,k-6\,k^2-2\,kn-6\,k^2n+2\,kn^2+3\,n^2+4\,k^3}{(kn+2\,n-k^2+4)(-k^2+2\,k+kn+n+3)}.$$

Since $T_1(n,k)=0$, $F(n,k)$ is definite summable, and

$$\sum_{k=0}^{n} F(n,k) = S(n,n+1)+S_1(n,n+1)-S(n,0)-S_1(n,0) = \frac{1}{2}\,\frac{3\,n^2+11\,n+8}{(n+2)(n+3)}.$$

Maple 7 is unable to compute this definite sum, while Mathematica 4 returns a very long answer which involves the Polygamma function, the RootSum structure (sum of the roots of a given expression), Euler's constant, et cetera.

Example 4 Consider the rational function

$$F(n,k) = \frac{1}{k+2} - \frac{1}{k+1} + \frac{1}{2\,k+5} - \frac{1}{2\,n-2\,k+1} + \frac{1}{nk+1}.$$

Following the same steps as those in Example 1 results in

$$G_1(n)+G_2(n) = -\frac{1}{3}\,\frac{28\,n^3+144\,n^2+233\,n+117}{(n+2)\,(2\,n+3)\,(2\,n+5)}, \text{ and}$$

$$T_1(n,k) = \frac{1}{2}\,\frac{n^2+2}{n^3k-n^2k^2+n^2+1}.$$

Applying *split* to the denominator of $T(n,k)$ results in

$$q_1(n,k) = nk+1,\ q_2(n,k) = n^2-nk+1.$$

Therefore,

$$\frac{w_1(n,k)}{q_1(n,k)} = \frac{1}{2}\,\frac{1}{nk+1},\ \frac{w_2(n,k)}{q_2(n,k)} = \frac{1}{2}\,\frac{1}{n^2-nk+1}, \text{ and}$$

$$F_2(n,k) = \frac{w_1(n,k)}{q_1(n,k)} + \frac{w_2(n,n-k)}{q_2(n,n-k)} = \frac{1}{nk+1}.$$

It is easy to see that the application of *indefdecomp* to $F_2(n,k)$ results in the pair $(S'(n,k), T'(n,k))$ where

$$S'(n,k) = 0,\ T'(n,k) = \frac{1}{nk+1}.$$

Therefore, a simplification of $\sum_{k=0}^{n} F(n,k)$ is

$$(G(n), T'(n,k)) = \left(-\frac{1}{3} \frac{28\,n^3 + 144\,n^2 + 233\,n + 117}{(n+2)\,(2\,n+3)\,(2\,n+5)},\ \frac{1}{nk+1} \right).$$

Equivalently, the value of $\sum_{k=0}^{n} F(n,k)$ is

$$-\frac{1}{3} \frac{28\,n^3 + 144\,n^2 + 233\,n + 117}{(n+2)\,(2\,n+3)\,(2\,n+5)} + \frac{1}{n} \left(\Psi(n + 1/n + 1) - \Psi(1/n)\right).$$

Note that Maple 7 returns

$$\frac{1}{n+2} - \frac{7}{3} + \frac{1}{2}\Psi(n+7/2) - \frac{1}{2}\Psi(-n-1/2) + \frac{1}{n}\Psi(n+1/n+1) - \frac{1}{n}\Psi(1/n).$$

Mathematica 4 returns a very long, complicated result.

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