A DIRECT ALGORITHM TO CONSTRUCT THE MINIMAL TELESCOPERS FOR RATIONAL FUNCTIONS (Q-DIFFERENCE CASE)

H. Q. LE, S. A. ABRAMOV, AND K. O. GEDDES

ABSTRACT. In this paper we present a direct algorithm to construct the minimal telescopers for rational functions for the q-difference case.

1. Preliminaries

The well-known Zeilberger's algorithm [4, 11] is a useful tool for finding closed forms of definite hypergeometric sums, and for certifying the truth for large classes of identities that occur in combinatorics and in the theory of special functions. It is shown in [3, 5, 10] that Zeilberger's algorithm can be carried over to the q-difference case; and similar to the case of its difference counterpart, the q-analogue version of Zeilberger's algorithm (we name it hereafter as $q\mathcal{Z}$) also has a wide range of applications [9].

Let q be an indeterminate parameter. Denote by Q_n, Q_k the q-shift operators w.r.t. q^n and q^k, respectively, defined by $Q_n\, T(q^n, q^k) = T(q^{n+1}, q^k)$, $Q_k\, T(q^n, q^k) = T(q^n, q^{k+1})$. For a given q-hypergeometric term $T(q^n, q^k)$ of q^n and q^k, i.e., the consecutive-term ratios $T(q^{n+1}, q^k)/T(q^n, q^k)$ and $T(q^n, q^{k+1})/T(q^n, q^k)$ are elements from $\mathbb{C}(q)(q^n, q^k)$, $q\mathcal{Z}$ *tries* to construct for $T(q^n, q^k)$ a qZ-pair (L, G) which consists of a linear q-difference operator L with coefficients which are polynomials of q^n over $\overline{\mathbb{C}(q)}$

$$L = a_\rho(q^n)Q_n^\rho + \cdots + a_1(q^n)Q_n^1 + a_0(q^n)Q_n^0,\ a_i(q^n) \in \overline{\mathbb{C}(q)}[q^n], \tag{1}$$

and a q-hypergeometric term $G(q^n, q^k)$ such that

$$LT(q^n, q^k) = (Q_k - 1)\, G(q^n, q^k). \tag{2}$$

It can be shown that if there exist qZ-pairs for $T(q^n, q^k)$, then $q\mathcal{Z}$ terminates with one of the qZ-pairs and the telescoper L in the returned qZ-pair is of minimal possible order. Note that L is unique up to a factor $P(q^n) \in \overline{\mathbb{C}(q)}[q^n]$, and we name it *the minimal telescoper.* We also name the qZ-pair (L, G) where L is the minimal telescoper the *minimal qZ-pair.*

$q\mathcal{Z}$ uses an *item-by-item examination* of the order ρ of L. It starts with the value of 0 for ρ and increases ρ until it is successful in finding the minimal qZ-pair (L, G). For a given q-hypergeometric term $T(q^n, q^k)$, the question whether there exists a qZ-pair for $T(q^n, q^k)$ is not conclusively answered although a sufficient condition is known via the *fundamental theorem* (see [4, 8, 10]). If $T(q^n, q^k)$ is a *proper* q-hypergeometric term [8, 10], then $q\mathcal{Z}$ is applicable to T, and an upper bound for ρ can be computed [8]. This upper bound, however, is much too large in practice. This is the main reason for applying item-by-item examination strategy. As a consequence, when applying $q\mathcal{Z}$ to a q-hypergeometric term, we waste resources either trying to compute a qZ-pair in the case when no such qZ-pair exists or trying to compute without success a telescoper with $\operatorname{ord} L < \rho$ in the case when the qZ-pairs exist and the order of the minimal telescoper is ρ.

Let $T(q^n, q^k)$ be a rational function of q^n and q^k. The problem of determining a necessary and sufficient condition for the termination of $q\mathcal{Z}$ on $T(q^n, q^k)$ is solved and presented in [7]. First,

Supported in part by Natural Sciences and Engineering Research Council of Canada Grant No. CRD215442-98, by the French-Russian Lyapunov Institute under grant 98-03 and by Natural Sciences and Engineering Research Council of Canada Grant No. RGPIN8967-01.

it provides a decision procedure as to whether or not one should apply $q\mathcal{Z}$ to the input $T(q^n, q^k)$. Secondly, in the case when the termination of $q\mathcal{Z}$ is guaranteed, it helps to speed up the construction of the minimal qZ-pair by reducing the size of the problem to be solved. Note that an item-by-item examination technique is still used in this case.

In this paper we present a *direct algorithm* to construct the minimal qZ-pair (L, G) for a given rational function $T(q^n, q^k)$. By direct algorithm, we mean an algorithm which computes the minimal qZ-pair directly, without using an item-by-item examination. The algorithm is based on a special form of representation of $T(q^n, q^k)$, on direct construction of the minimal telescoper for each member of this representation. The minimal qZ-pair for $T(q^n, q^k)$ can then be obtained using Least Common Left Multiple (LCLM) computation.

Note that the same problem for the difference case is solved and presented in [6]. In fact, even though the details are different, we essentially use the same approach as the one that we used for the difference case in solving the problem for the q-difference case.

2. The Decomposition Problem and the Existence of a qZ-pair

In this section, we recall some known results relating to the decomposition problem [2] and the existence of a qZ-pair [7] for rational functions. They are needed in subsequent sections.

2.1. **The Decomposition Problem.** Let $T(q^k)$ be an element from $\mathbb{C}(q)(q^k)$. An algorithm to solve the decomposition problem constructs a rational function $S(q^k)$ and a proper rational function $F(q^k)$ such that

$$T(q^k) = (Q_k - 1)S(q^k) + F(q^k),\ S(q^k),\ F(q^k) \in \mathbb{C}(q)(q^k), \tag{3}$$

where the denominator $g(q^k)$ of $F(q^k)$ has the lowest possible degree in q^k.

If $F(q^k) = 0$, then $T(q^k)$ is said to be *rational summable.*

Consider the partial fraction decomposition of $T(q^k)$ w.r.t. the complete factorization of its denominator over $\overline{\mathbb{C}(q)}$:

$$T(q^k) = \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{\beta_{ij}}{(q^k - \alpha_i)^j},\ \alpha_i, \beta_{ij} \in \overline{\mathbb{C}(q)}.$$

Define $\alpha_i \sim \alpha_j$ iff $\alpha_i = q^l \alpha_j$ where l is an integer. It is easy to check that $\sim$ is an equivalence relation on the set $\{\alpha_1, \ldots, \alpha_m\}$. If α_i and α_j are in the same equivalence class, then the two monic irreducibles $q^k - \alpha_i$ and $q^k - \alpha_j$ are said to be *q-shift equivalent.* For each equivalence class, let α_i be the *largest* element in the sense that for all elements α_j in the same class, $\alpha_j = q^l \alpha_i$ where l is a non-positive integer. It is shown in [2] that $T(q^k)$ can be written as

$$\sum_{i=1}^{s} \sum_{j=1}^{l_i} M_{ij} \frac{1}{(q^k - \alpha_i)^j}$$

where $M_{ij} \in \overline{\mathbb{C}(q)}[Q_k], 0 < s \le m, l_i > 0$; α_i is the representative, which is the largest element, for the i-th q-shift equivalence class; and $T(q^k)$ is rational summable iff

$$M_{ij} = L_{ij} \circ (Q_k - 1),\ L_{ij} \in \overline{\mathbb{C}(q)}[Q_k],\ i = 1, \ldots, s,\ j = 1, \ldots, l_i. \tag{4}$$

Let us represent $F(q^k)$ in the reduced form $F(q^k) = f(q^k)/g(q^k)$, where $f(q^k), g(q^k) \in \mathbb{C}(q)[q^k]$. By [7], $g(q^k)$ has the following property:

P1. If $p_1(q^k), p_2(q^k)$ are factors of $g(q^k)$ irreducible over $\overline{\mathbb{C}(q)}$ then there does not exist a non-zero integer h such that $p_1(q^{k+h}) = p_2(q^k)$.

On the other hand, suppose that $T(q^k)$ is rational summable. Let $T(q^k) = a(q^k)/b(q^k)$ where $a(q^k), b(q^k)$ are relatively prime elements of $\mathbb{C}(q)[q^k]$. Then $b(q^k)$ has the following property [7]:

P2. If $p_1(q^k)$ is a factor of $b(q^k)$ irreducible over $\overline{\mathbb{C}(q)}$, then there exist a factor $p_2(q^k)$ irreducible over $\overline{\mathbb{C}(q)}$ of $b(q^k)$ and a non-zero integer h such that $q^{-h}\, p_1(q^{k+h}) = p_2(q^k)$.

2.2. **The Existence of a qZ-pair.** Let $T(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$. Apply an algorithm to solve the decomposition problem w.r.t. q^k to represent $T(q^n, q^k)$ in the form

$$T(q^n, q^k) = (Q_k - 1)S(q^n, q^k) + F(q^n, q^k),\ \ S(q^n, q^k), F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k), \tag{5}$$

and the denominator $g(q^n, q^k)$ of $F(q^n, q^k)$ is of minimal possible degree w.r.t. q^k. It is shown in [7] that a qZ-pair for $T(q^n, q^k)$ exists iff $g(q^n, q^k)$ can be written in the form

$$g(q^n, q^k) = \alpha q^{cn} \prod_i (q^k - \gamma_i q^{(a_i/b_i)n}),\ c,\ a_i,\ b_i \in \mathbb{Z},\ \gcd(a_i, b_i) = 1,\ b_i > 0,\ \gamma_i, \alpha \in \overline{\mathbb{C}(q)}. \tag{6}$$

An algorithm for using the above criterion as well as the verification of the algorithm are presented in [7].

3. On the Minimal Telescoper of a Sum of Rational Functions

We introduce in this section the concept of *similarity* among q-hypergeometric terms. It is the q-analogue of the same concept for hypergeometric terms as discussed in Section 5.6 [8]. The main result of this section is Theorem 1 which provides a sufficient condition for the construction of the minimal telescoper for a sum of rational functions based on the minimal telescopers for each rational function of the sum. This theorem is the q-analogue of Theorem 2 in [6] for the difference case.

Definition 1. *Two* q-*hypergeometric terms $s(q^n)$ and $t(q^n)$ are similar if their ratio is an element from $\mathbb{C}(q)(q^n)$.*

It is easy to see that similarity is an equivalence relation on the set of all q-hypergeometric terms. The following proposition is the q-analogue of Proposition 5.6.2 [8].

Proposition 1. *Let $s(q^n)$ and $t(q^n)$ be two* q-*hypergeometric terms such that $s(q^n) + t(q^n) \neq 0$. Then $s(q^n) + t(q^n)$ is* q-*hypergeometric if and only if $s(q^n) \sim t(q^n)$.* □

Lemma 1. *Let (L, G) be the minimal* qZ-*pair for a* q-*hypergeometric term $F(q^n, q^k)$. If L_1 is a telescoper for F, then L_1 is right divisible by L in $\mathbb{C}(q)(q^n)[Q_n]$.*

Proof. We have

$$LF = (Q_k - 1)G. \tag{7}$$

Since L_1 is a telescoper for F, there exists a q-hypergeometric term $G_1(q^n, q^k)$ such that

$$L_1 F = (Q_k - 1)G_1. \tag{8}$$

Applying the right Euclidean division of L_1 by L yields

$$L_1 = S \circ L + R,\ \ S, R \in \mathbb{C}(q)(q^n)[Q_n],\ R = 0 \text{ or } \operatorname{ord} R < \operatorname{ord} L. \tag{9}$$

From (9), one obtains

$$L_1 F = S(LF) + RF. \tag{10}$$

The substitution of (7) and (8) into (10) yields

$$RF = (Q_k - 1)(G_1 - SG). \tag{11}$$

Since $T(q^n, q^k)$ is a q-hypergeometric term, $T(q^{n+1}, q^{k+1})/T(q^n, q^k)$ is a rational function of q^n and q^k. One can show by induction that for any pair of integers s_1, s_2, $T(q^{n+s_1}, q^{k+s_2})/T(q^n, q^k)$ is also a rational function of q^n and q^k. In general,

$$\frac{M\,T(q^n, q^k)}{T(q^n, q^k)} = R(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k) \text{ for any operator } M \in \mathbb{C}(q)(q^n)[Q_n, Q_k]. \tag{12}$$

It follows from (7), (8), (11) and (12) that $F \sim G \sim G_1$, and $SG \sim G$. Since $\sim$ is an equivalence relation, $G_1 \sim SG$, and $G_1 - SG$ is a q-hypergeometric term (Proposition 1). Since $\operatorname{ord} R < \operatorname{ord} L$ and L is the minimal telescoper for F, it follows from (11) that $R = 0$. □

Theorem 1. *Let $F_1, \dots, F_s$ be rational functions such that if L^* is a telescoper for $F = F_1 + \cdots + F_s$ then L^* is a telescoper for every $F_i, i = 1, \dots, s$. Let $L_1, \dots, L_s$ be the minimal telescopers for $F_1, \dots, F_s$, respectively (assuming that they all exist). Then $\operatorname{lclm}(L_1, \dots, L_s)$ is the minimal telescoper for $F_1 + \cdots + F_s$.*

Proof. First we show that $\operatorname{lclm}(L_1, \dots, L_s)$ is a telescoper for F. Since L_i is the minimal telescoper for F_i, there exists $G_i \in \mathbb{C}(q)(q^n, q^k)$ such that

$$L_i F_i = (Q_k - 1) G_i, \; 1 \le i \le s. \tag{13}$$

Set $L = \operatorname{lclm}(L_1, \dots, L_s)$. This implies there exist $L'_1, \dots, L'_s \in \mathbb{C}(q)(q^n)[Q_n]$ such that

$$L = \operatorname{lclm}(L_1, \dots, L_s) = L'_1 \circ L_1 = \cdots = L'_s \circ L_s.$$

Then

$$LF = LF_1 + \cdots + LF_s = L'_1(L_1 F_1) + \cdots + L'_s(L_s F_s) = (Q_k - 1)(L'_1 G_1 + \cdots + L'_s G_s).$$

Since $L'_1 G_1 + \cdots + L'_s G_s \in \mathbb{C}(q)(q^n, q^k)$, $(L, L'_1 G_1 + \cdots + L'_s G_s)$ is a qZ-pair for F.

Now let L^* be any telescoper for F. From the hypothesis, L^* is also a telescoper for each $F_i, 1 \le i \le s$. Since L_i is the minimal telescoper for F_i and since every rational function is a q-hypergeometric term, one can apply Lemma 1 and deduces that L^* is right divisible by $L_i, 1 \le i \le s$. Consequently, L^* is right divisible by $\operatorname{lclm}(L_1, \dots, L_s)$. □

4. The Basic Case

We show in this section how to compute directly the minimal qZ-pair (L, G) for VF where $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$, and

$$F(q^n, q^k) = \frac{1}{\left(q^k - \gamma\, q^{(a/b)n}\right)^m}, \; a,\, b \in \mathbb{Z},\; \gcd(a, b) = 1,\; b > 0,\; \gamma \in \overline{\mathbb{C}(q)},\; m \in \mathbb{N} \setminus \{0\}. \tag{14}$$

Lemma 2. *Let $F(q^n, q^k)$ be of the form (14). Then $L = q^{am} Q_n^b - 1$ is the minimal telescoper for $F(q^n, q^k)$.*

Proof. Set $L = q^{am} Q_n^b - 1$, $M = Q_k^{-a} - 1 = (Q_k - 1) \circ U$, where

$$U = \begin{cases} \sum_{i=0}^{-a-1} Q_k^i & a < 0 \\ \sum_{i=1}^{a} -Q_k^{-i} & \text{otherwise}. \end{cases}$$

It is easy to check that

$$LF = MF = (Q_k - 1)(UF),\; UF \in \overline{\mathbb{C}(q, q^n)}(q^k).$$

Therefore, (L, UF) is a qZ-pair for $F(q^n, q^k)$. Now suppose there exists a qZ-pair (L_1, G_1) such that $\operatorname{ord} L_1 < \operatorname{ord} L = b$. Without loss of generality we can assume that the coefficient of Q_n^0 in L_1 is a non-zero element of $\overline{\mathbb{C}(q)}[q^n]$. Otherwise, choose a new qZ-pair for F

$$\left(Q_n^{-\lambda} \circ L_1, G_1(q^{n-\lambda}, q^k)\right)$$

where λ is the minimal positive integer such that the coefficient of Q_n^λ in L_1 is not zero. Set

$$H = L_1\, F = \frac{s(q^n, q^k)}{t(q^n, q^k)},\; s(q^n, q^k),\; t(q^n, q^k) \in \overline{\mathbb{C}(q, q^n)}[q^k].$$

Since H is rational summable w.r.t. q^k, $t(q^n, q^k)$ has property **P2**, i.e., for the irreducible factor $q^k - \gamma\, q^{(a/b)n}$ of $t(q^n, q^k)$, there exists a non-zero integer h such that

$$q^{-h}\left(q^{k+h} - \gamma\, q^{(a/b)n}\right) = q^k - \gamma\, q^{(a/b)n}\, q^{-h}$$

is also an irreducible factor of $t(q^n, q^k)$. Since all the irreducible factors of $t(q^n, q^k)$ have the form $q^k - \gamma\, q^{(a/b)(n+i)}$, for $i = 0, 1, \ldots, \operatorname{ord} L_1$, this means

$$\gamma\, q^{(a/b)n}\left(q^{(a/b)i} - q^{-h}\right) = 0 \text{ for some } i. \tag{15}$$

Case 1. $\gamma \neq 0$ and $a \neq 0$. If follows from (15) that

$$h = -\frac{a}{b} i.$$

If $i = 0$, then $h = 0$. This contradicts the assumption on h. Otherwise, since $\gcd(a, b) = 1$, $b|i$. This is impossible since $0 < i < b$.

Case 2. $\gamma = 0$ or $a = 0$. It is easy to see that $(L, G) = (Q_n - 1, 0)$ is a qZ-pair for F. Suppose that the minimal telescoper for F is of order 0, i.e., $L = 1$. Then $LF = F = \{1/q^{km}, 1/(q^k - \gamma)^m\}$. In both cases, F is not rational summable w.r.t. q^k (Property **P2**). □

Lemma 3. *Let F be of the form (14), (L_F, G_F) be the minimal* qZ*-pair for F, $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$ and*

$$\operatorname{lclm}(V, L_F) = L_1 \circ V = L_2 \circ L_F,\ \ L_1, L_2 \in \overline{\mathbb{C}(q)}(q^n)[Q_n]. \tag{16}$$

Then $(L_1, L_2 G_F)$ is the minimal qZ*-pair for VF.*

Proof. It follows from Lemma 2 that the existence of the minimal qZ-pair (L_F, G_F) for F is guaranteed. Since

$$L_1(VF) = L_2(L_F F) = L_2 \circ (Q_k - 1) G_F = (Q_k - 1)(L_2 G_F),$$

$(L_1, L_2 G_F)$ is a qZ-pair for VF.

Let L^* be any telescoper for VF, i.e., there exists $G^* \in \overline{\mathbb{C}(q, q^n)}(q^k)$ such that $L^*(VF) = (Q_k - 1)G^*$. Therefore, $(L^* \circ V, G^*)$ is a qZ-pair for F. Since L_F is the minimal telescoper for F, one deduces from Lemma 1 that $L^* \circ V$ is right divisible by L_F. Additionally, $L^* \circ V$ is right divisible by V and $L_1 \circ V = \operatorname{lclm}(V, L_F)$. Therefore, $L^* \circ V$ is right divisible by $L_1 \circ V$. Consequently, L_1 is the minimal telescoper for VF. □

Theorem 2. *One can directly compute the minimal* qZ*-pair (L, G) for a rational function $VF \in \overline{\mathbb{C}(q, q^n)}(q^k)$, where F is of the form (14) and $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$.*

Proof. The proof follows from Lemmas 2 and 3. □

Example 1 Consider the rational function

$$VF \text{ where } V = (q^2 - 1)\, Q_n^3 + (q + 1),\ \ F = \frac{1}{q^k - q^{-3n}}.$$

Applying Lemma 2 results in the minimal qZ-pair (L_F, G_F) for F :

$$(L_F, G_F) = \left(q^{-3}\, Q_n - 1, \frac{1}{q^{k+2} - q^{-3n}} + \frac{1}{q^{k+1} - q^{-3n}} + \frac{1}{q^k - q^{-3n}}\right).$$

Computing the $\operatorname{lclm}(V, L_F)$ results in the operators L_1, L_2 that satisfy (16):

$$L_1 = \frac{1}{(q+1)(q^{10} - q^9 + 1)\, q^3)}\, Q_n - \frac{1}{(q+1)(q^{10} - q^9 + 1)};$$

$$L_2 = \frac{q - 1}{q^{10} - q^9 + 1}\, Q_n^3 + \frac{1}{q^{10} - q^9 + 1}.$$

It follows from Lemma 3 that the minimal qZ-pair (L, G) for VF is

$$(L, G) = (L_1, L_2\, G_F) \text{ where}$$

$$G = \frac{1}{q^{10} - q^9 + 1}\left(\frac{1}{q^{k+2} - q^{-3n}} + \frac{1}{q^{k+1} - q^{-3n}} + \frac{1}{q^k - q^{-3n}}\right) +$$

$$+\frac{q-1}{q^{10}-q^9+1}\left(\frac{1}{q^{k+2}-q^{-3n-9}}+\frac{1}{q^{k+1}-q^{-3n-9}}+\frac{1}{q^k-q^{-3n-9}}\right).$$

We conclude this section with a description of the algorithm $qZpairVF$ to construct the minimal qZ-pair for VF where $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$, and F is of the form (14).

`algorithm` $qZpairVF$;
`input:` $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$,
$F = 1/\left(q^k - \gamma\, q^{(a/b)n}\right)^m$, $a, b \in \mathbb{Z}$, $\gcd(a,b) = 1$, $b > 0$, $\gamma \in \overline{\mathbb{C}(q)}$, $m \in \mathbb{N} \setminus \{0\}$;
`output:` the minimal qZ-pair for VF;

if $a < 0$ **then**
$(L_F, G_F) := \left(q^{am}\, Q_n^b - 1, \left(\sum_{i=0}^{-a-1} Q_k^i\right) F\right);$
else
$(L_F, G_F) := \left(q^{am}\, Q_n^b - 1, \left(\sum_{i=1}^{a} - Q_k^{-i}\right) F\right);$
fi;
apply lclm computation to construct $L_1, L_2 \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$ such that
$L_1 \circ V = L_2 \circ L_F = \operatorname{lclm}(V, L_F)$;
return $(L_1, L_2 G_F)$.

5. The Generic Case

Lemma 4. *Let* $F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$, *and*

(17)
$$F = \sum_{i=1}^{t}\sum_{j=1}^{m_i} \frac{r_{ij}(q^n)}{\left(q^k - \gamma_i\, q^{(a_i/b_i)n}\right)^j},\ r_{ij}(q^n) \in \overline{\mathbb{C}(q)}(q^n),\ \gamma_i \in \overline{\mathbb{C}(q)},\ a_i,\ b_i \in \mathbb{Z},\ b_i > 0,\ \gcd(a_i, b_i) = 1.$$

Then $F(q^n, q^k)$ *can be represented in the form*

$$V_1\, F_1 + \cdots + V_s\, F_s, \tag{18}$$

where $V_i \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$, *and* $F_i = 1/\left(q^k - \gamma_i\, q^{(a_i/b_i)n}\right)^{m_i}$ *is such that*

$$a_i,\ b_i \in \mathbb{Z},\ b_i > 0,\ \gamma_i \in \overline{\mathbb{C}(q)},\ \gcd(a_i, b_i) = 1,\ m_i \in \mathbb{N} \setminus \{0\}, \tag{19}$$

and for all $i \neq j$, *at least one of the following four relations is not valid:*

$$m_i = m_j,\ a_i = a_j,\ b_i = b_j,\ \gamma_i = q^{(a/b)h}\, \gamma_j,\ h \in \mathbb{Z} \setminus \{0\}. \tag{20}$$

Proof. Let R_1, R_2 be two simple fractions in (17). Define $R_1 \sim R_2$ iff all four relations in (20) hold. It is easy to check that $\sim$ is an equivalence relation on the set of simple fractions in (17).

For any pair of simple fractions R_1 and R_2 in the same equivalence class, i.e., R_1 and R_2 can be written as

$$R_1 = \frac{r_1(q^n)}{\left(q^k - \gamma_1\, q^{(a/b)n}\right)^m},\ R_2 = \frac{r_2(q^n)}{\left(q^k - \gamma_2\, q^{(a/b)n}\right)^m},\ \gamma_1 = q^{(a/b)h}\, \gamma_2,\ h \in \mathbb{Z} \setminus \{0\}. \tag{21}$$

If $h > 0$, then

$$R_1 + R_2 = V\, \frac{1}{\left(q^k - \gamma_2\, q^{(a/b)n}\right)^m} \text{ where } V = \left(r_2(q^n) + r_1(q^n)\, Q_n^h\right) \in \overline{\mathbb{C}(q)}(q^n)[Q_n].$$

Otherwise, $h < 0$ and

$$R_1 + R_2 = V\, \frac{1}{\left(q^k - \gamma_1\, q^{(a/b)n}\right)^m} \text{ where } V = \left(r_1(q^n) + r_2(q^n)\, Q_n^{-h}\right) \in \overline{\mathbb{C}(q)}(q^n)[Q_n].$$

By using induction on the number of elements in each equivalence class, the sum of the elements in each class can be represented as

$$V_i \, \frac{1}{\left(q^k - \gamma_i \, q^{(a_i/b_i)n}\right)^{m_i}}, \; V_i \in \overline{\mathbb{C}(q)}(q^n)[Q_n].$$

□

Lemma 5. *Let $F(q^n, q^k)$ be written in the form (18) where $V_i \in \overline{\mathbb{C}(q)}[Q_n]$,*

$$F_i = \frac{1}{\left(q^k - \gamma_i \, q^{(a_i/b_i)n}\right)^{m_i}}$$

is such that (19) takes place and if $i \neq j$ then at least one of the four relations in (20) is not valid. Let L be a telescoper for $F(q^n, q^k)$. Then L is a telescoper for each $V_i \, F_i$, $1 \leq i \leq s$.

Proof. Let $L \in \overline{\mathbb{C}(q)}[q^n, Q_n]$ be any telescoper for $F(q^n, q^k)$. Consider

$$V_i \, F_i = V_i \, \frac{1}{\left(q^k - \gamma_i \, q^{(a_i/b_i)n}\right)^{m_i}}, \; 1 \leq i \leq s.$$

Let $R_i(q^n, q^k) = r_i(q^n)/\left(q^k - \gamma_i \, q^{(a_i/b_i)(n+h_1)}\right)^{m_i}$ be any simple fraction of $V_i \, F_i$, $h_1 \in \mathbb{N}$. Then for any non-negative integer h_2, $R_i(q^{n+h_2}, q^k) = r_i(q^{n+h_2})/\left(q^k - \gamma_i \, q^{(a_i/b_i)h_2} \, q^{(a_i/b_i)(n+h_1)}\right)^{m_i}$. Consequently, it is easy to see that the application of $L \in \overline{\mathbb{C}(q)}[q^n, Q_n]$ to a simple fraction of $V_i \, F_i$ yields a sum of simple fractions such that each of them is in the same equivalence class (w.r.t. the relation $\sim$ as defined in the proof of Lemma 4) as that of F_i.

Let R_1, R_2 be any two simple fractions of $LF(q^n, q^k)$. Define $R_1 \sim' R_2$ iff in addition to the four relations in (20), i.e., R_1 and R_2 can be written as

$$R_1 = \frac{r_1(q^n)}{\left(q^k - \gamma_1 \, q^{(a/b)n}\right)^m}, \; R_2 = \frac{r_2(q^n)}{\left(q^k - \gamma_2 \, q^{(a/b)n}\right)^m}, \; \gamma_1 = q^{(a/b)h} \, \gamma_2, \; h \in \mathbb{Z} \setminus \{0\}, \tag{22}$$

the relation $b|h$ also holds. It is easy to check that $\sim'$ is an equivalence relation, and if $R_1 \sim' R_2$ then $R_1 \sim R_2$. Consequently, it is impossible that $L \circ V_i \, F_i$ and $L \circ V_j \, F_j$ both have simple fractions from the same class w.r.t. $\sim'$ for $i \neq j$.

Considering any $L \circ V_i \, F_i$, we distribute the corresponding simple fractions to the classes w.r.t. $\sim'$. For every pair of simple fractions R_1, R_2 as defined in (22), since $b|h$, there exists $s \in \mathbb{Z} \setminus \{0\}$ such that $h = s\,b$. Set $h_1 = -a\,s$, $p_1(q^n, q^k) = q^k - \gamma_1 \, q^{(a/b)n}$, $p_2(q^n, q^k) = q^k - \gamma_2 \, q^{(a/b)n}$. Then

$$q^{-h_1} \, p_2(q^n, q^{k+h_1}) = q^h - \gamma_2 \, q^{(a/b)(n+h)} = p_1(q^n, q^k).$$

This means the monic irreducible factors of the denominator of the sum of simple fractions in each class are q-shift-equivalent. Since $LF(q^n, q^k)$ is rational summable w.r.t. q^k, from the necessary and sufficient condition as defined in (4), Subsection 2.1, the sum of all fractions from such a class is rational summable w.r.t. q^k. This implies that $L \circ V_i F_i$ is rational summable w.r.t. q^k. □

Theorem 3. *For any rational function $F(q^n, q^k)$ which can be written in the form*

$$F = \sum_{i=1}^{t} \sum_{j=1}^{m_i} \frac{r_{ij}(q^n)}{\left(q^k - \gamma_i \, q^{(a_i/b_i)n}\right)^j}, \; r_{ij}(q^n) \in \overline{\mathbb{C}(q)}(q^n), \; \gamma_i \in \overline{\mathbb{C}(q)}, \; a_i, \, b_i \in \mathbb{Z}, \, b_i > 0, \, \gcd(a_i, b_i) = 1. \tag{23}$$

one can directly compute the minimal qZ*-pair (L, G) for $F(q^n, q^k)$.*

Proof. Lemma 4 allows one to rewrite $F(q^n, q^k)$ in the form (18). Theorem 2 allows one to compute the minimal qZ-pair (L_i, G_i) for each term $V_i F_i$, $1 \leq i \leq s$. One then uses Theorem 1 to compute a qZ-pair (L, G) for $F(q^n, q^k)$ where $L = \mathrm{lclm}(L_1, \ldots, L_s)$. Lemma 5 and Theorem 1 guarantee that L is indeed the minimal telescoper for $F(q^n, q^k)$. □

The following is a description of the algorithm $qZpairF$ which constructs the minimal qZ-pair for F of the form (17).

algorithm $qZpairF$;
input: $F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$ and is of the form (17);
output: the minimal qZ-pair (L_F, G_F) for F;

let $R_1, \ldots, R_s$ be the equivalence classes of the set of simple fractions of F as defined by the relation $\sim$;
for $i = 1, 2, \ldots, s$ **do**
  let $\{F_{i_1}, \ldots, F_{i_d}\}$ be the members of the equivalence class R_i;
  let $F_{i_j} = r_{i_j}(q^n)/(q^k - \gamma_{i_j}\, q^{(a_i/b_i)n})^{m_i}$, $1 \le j \le d$;
  $F_i := F_{i_1}(q^n, q^k)$; $V_i := r_{i_1}(q^n)$; $\gamma_i := \gamma_{i_1}$;
  for $j = 2, 3, \ldots, d$ **do**
    let $h \in \mathbb{Z} \setminus \{0\}$ be such that $\gamma_i = q^{(a_i/b_i)h}\, \gamma_{i_j}$;
    if $h > 0$ **then**
      $V_i := r_{i_j}(q^n) + V_i\, Q_n^h$;
      $\gamma_i := \gamma_{i_j}$;
    else
      $V_i := V_i + r_{i_j}(q^n)\, Q_n^{-h}$;
    fi;
  od;
  $F_i := 1/(q^k - \gamma_i\, q^{(a_i/b_i)n})^{m_i}$;
  $(L_{V_iF_i}, G_{V_iF_i}) := qZpairVF(V_i, F_i)$;
od;
apply lclm computation to construct $L'_1, \ldots, L'_s \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$ such that
  $L_F = \mathrm{lclm}(L_{V_1F_1}, \ldots, L_{V_sF_s}) = L'_1 \circ L_{V_1F_1} = \cdots = L'_s \circ L_{V_sF_s}$;
return $(L_F, L'_1 G_{V_1F_1} + \cdots + L'_s G_{V_sF_s})$.

6. A Direct Algorithm to Construct the Minimal qZ-pairs for Rational Functions

For a given rational function $T(q^n, q^k)$, a direct algorithm to construct the minimal qZ-pair for $T(q^n, q^k)$ can be decomposed into two steps. In the first step, check for the existence of a qZ-pair and if its existence is guaranteed, rewrite $F(q^n, q^k)$ in (5) such that the denominator of $F(q^n, q^k)$ is of the form (6). In the second step, apply the results shown in Sections 3, 4 and 5 to obtain the minimal qZ-pair (L_1, G_1) for $F(n, k)$. Then the minimal qZ-pair (L, G) for the given rational function $T(q^n, q^k)$ is $(L_1, L_1 S(q^n, q^k) + G_1(q^n, q^k))$ where $S(q^n, q^k)$ is defined in (5).

An algorithm to perform the first step is presented in [7], and we will make a very minor addition to this algorithm to complete the step.

Denote by A_c the transformation

$$q^n \to q^{n+1},\ q^k \to q^{k+c},\ c \in \mathbb{Q}.$$

Let $t_0(q^n, q^k) = \gcd(A_c w, w)$. Define the sequence of computation

$$t_i(q^n, q^k) = \gcd(A_c t_{i-1}, t_{i-1}),\quad i = 1, 2, \ldots \tag{24}$$

where the termination condition in (24) takes place when $\deg_{q^k} t_i(q^n, q^k) = \deg_{q^k} t_{i-1}(q^n, q^k)$, i.e., the degree w.r.t. q^k stops decreasing. (Note that the number of gcd computation in (24) is guaranteed to be finite.) Set $w_c(q^n, q^k) = t_{i-1}(q^n, q^k)$.

Lemma 6. *(Lemma 7, [7]). Let $w(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k]$. Let $t_0(q^n, q^k) = \gcd(A_c w, w)$. Then the maximal factor of $t_0(q^n, q^k)$ which can be written in the form*

$$\prod_{i=1}^{s}(q^k - \gamma_i q^{cn}), \quad \gamma_i \in \overline{\mathbb{C}(q)} \tag{25}$$

is $w_c(q^n, q^k)$.

The following simple algorithm *wc* is used to compute $w_c(q^n, q^k)$ for a given $w(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k]$, and $c \in \mathbb{Q}$.

```
algorithm wc;
input:   w(q^n, q^k) ∈ C(q)[q^n, q^k],
         c ∈ Q;
output:  w_c(q^n, q^k);

  t_0(q^n, q^k) := gcd(w(q^{n+1}, q^{k+c}), w(q^n, q^k));
  i := 1;
  do
     t_i(q^n, q^k) := gcd(t_{i-1}(q^{n+1}, q^{k+c}), t_{i-1}(q^n, q^k));
     if deg_{q^k} t_i(q^n, q^k) = deg_{q^k} t_{i-1}(q^n, q^k) then
        return t_{i-1}(q^n, q^k);
     fi;
     i := i + 1;
  od;
```

Before describing the direct algorithm to construct the minimal *qZ*-pairs for rational functions, we need to develop the following auxiliary function. Let $G(x) \in \mathbb{C}[q][x]$, the algorithm *ratsol* computes the set of non-zero rational numbers c such that $G(q^c) = 0$ [7].

```
algorithm ratsol;
input:   F(q, x) ∈ C[q][x] which can be written as F(q, x) = a_m(q)x^m + ··· + a_0(q) where
         a_i(q) = c_{i_j} q^{i_j} + ··· + c_{i_0},  0 ≤ i ≤ m;
output:  the set of non-zero values c ∈ Q such that F(q, q^c) = 0;

  cands := {};
  m := deg_x F(q, x);  j := deg_q lc (F(q, x), x);
  for d = 0, 1, ... , m − 1 do
     for l = 0, 1, ... , deg_q a_i(q) do
        if (c_{d_l} = 0) or (l = j) then next; fi;
        cands := cands ∪ {(l − j)/(m − d)};
     od;
  od;
  sol := {};
  let {x_0, x_1, ... , x_ρ} be the members of cands;
  for i = 0, 1, ... , ρ do
     if F(q, q^{x_i}) = 0 then
        sol := sol ∪ {x_i};
     fi;
  od;
  return sol.
```

Now for a given $T(q^n,q^k) \in \mathbb{C}(q)(q^n,q^k)$, rewrite T in the form (5) where

$$F(q^n,q^k) = \frac{f(q^n,q^k)}{g(q^n,q^k)}$$

is in reduced form. Extract from $g(q^n,q^k)$ the maximal factor $v_1(q^n) \in \mathbb{C}(q)[q^n], v_2(q^k) \in \mathbb{C}(q)[q^k]$. We know that $v_2(q^k)$ can be written as the product of factors of the form

$$q^k - \gamma, \gamma \in \overline{\mathbb{C}(q)}.$$

As for $v_1(q^n)$, if it cannot be written in the form $q^{an}, a \in \mathbb{Z}$, then it follows from (6) that $q\mathcal{Z}$ is not applicable to $F(q^n,q^k)$, and hence to $T(q^n,q^k)$. Otherwise, set

$$w(q^n,q^k) = \frac{g(q^n,q^k)}{v_1(q^n)v_2(q^k)}.$$

It remains to investigate whether $w(q^n,q^k)$ can be decomposed into factors of the form

$$k + \gamma\, q^{cn},\ c \in \mathbb{Q} \setminus \{0\},\ \gamma \in \overline{\mathbb{C}(q)}.$$

Let $\tilde{w}$ be an element from $\mathbb{C}(q)[q^x,q^n,q^k]$ obtained from w by substituting k by $k+x$ and n by $n+1$. Let

$$S(q^x,q^n) = \mathrm{Resultant}_{q^k}(w,\tilde{w}),\ S(q^x,q^n) \in \mathbb{C}(q)[q^x][q^n].$$

Find all rational values of x such that $S=0$, i.e., w and $\tilde{w}$ have a non-trivial common factor. To attain this goal, consider S as a polynomial in q^n with coefficients which are polynomials in q^x. Let $G(q^x)$ be the gcd of all these coefficients. Now we need to find all values of $x \in \mathbb{Q} \setminus \{0\}$ such that $G(q^x) = 0$. This is accomplished by applying the algorithm *ratsol* to $G(X)$. Let $x_0,\ldots,x_m$ be the set of all non-zero rational numbers such that $G(q^{x_i}) = 0, 0 \le i \le m$. We now apply algorithm *wc* to find $c_0,\ldots,c_d$ from the set $\{x_0,\ldots,x_m\}$ such that $deg_{q^k} w_{c_i}(q^n,q^k) \neq 0$. Set $\delta_i = \deg_{q^k} w_{c_i}(q^n,q^k)$. To check whether there exists a $q\mathcal{Z}$-pair for $F(q^n,q^k)$, it is sufficient to check if the relation

$$\delta_0 + \cdots + \delta_d = \deg_{q^k} w(q^n,q^k)$$

is satisfied. If it is not, we conclude that $q\mathcal{Z}$ is not applicable to $F(q^n,q^k)$, and subsequently, to $T(q^n,q^k)$. Otherwise, it follows from Lemma 6 that

$$w_{c_i}(q^n,q^k) = \left(q^k - \gamma_{i_1}\, q^{c_i n}\right) \cdots \left(q^k - \gamma_{i_s}\, q^{c_i n}\right),\ 0 \le i \le d. \tag{26}$$

We can then equate the coefficients of like powers of q^k in (26) and solve the corresponding system of equations to obtain the values of $\gamma_{i_1},\ldots,\gamma_{i_s}$ (note that the $\gamma_{i_j} \in \overline{\mathbb{C}(q)}$, $1 \le j \le s, 0 \le i \le d$ are guaranteed to exist).

At this point, we complete step 1. As for the second step, we apply the partial fraction decomposition to $F(q^n,q^k)$. Note that the denominator of $F(q^n,q^k)$ is already in the desired factored form, and no factorization needs to be done. This gives a representation of $F(q^n,q^k)$ in the form (17). We then apply algorithm *qZpairF* to construct the minimal $q\mathcal{Z}$-pair (L_1,G_1) for $F(q^n,q^k)$.

We now describe the algorithm *qZpairDirect* which constructs the minimal $q\mathcal{Z}$-pair for $T(q^n,q^k) \in \mathbb{C}(q)(q^n,q^k)$ provided that it exists.

`algorithm` *qZpairDirect*;
`input:` a rational function $T(q^n,q^k) \in \mathbb{C}(q)(q^n,q^k)$;
`output:` the minimal $q\mathcal{Z}$-pair (L,G) for T, if it exists;
the message "There does not exist a $q\mathcal{Z}$-pair", otherwise;

apply an algorithm to solve the rational sum decomposition
problem w.r.t. q^k to obtain $S(q^n,q^k)$, $F(q^n,q^k)$ in (5);
if $F(q^n,q^k) = 0$ **then**
return $(1, S(q^n,q^k))$;
fi;

```
f(q^n, q^k) := numerator(F(q^n, q^k)); g(q^n, q^k) := denominator(F(q^n, q^k));
v_1(q^n) := content_{q^k}(g(q^n, q^k));
if v_1(q^n) ≠ q^{an}, a ∈ ℤ then
    return  "There does not exist a qZ-pair";
fi;
w(q^n, q^k) := g(q^n, q^k)/v_1(q^n);
v_2(q^k) := content_{q^n}(w(q^n, q^k));
w(q^n, q^k) := w(q^n, q^k)/v_2(q^k);
if w(q^n, q^k) = 1 then
    rewrite v_2(q^k) as v_2(q^k) = (q^k − γ_1) ··· (q^k − γ_s);
    g̃(q^n, q^k) := v_1(q^n) v_2(q^k);
else
    w̃(q^x, q^n, q^k) := w(q^{n+1}, q^{k+x});
    S(q^x, q^n) := resultant_{q^k}(w(q^n, q^k), w̃(q^x, q^n, q^k));
    G(q^x) := content_{q^n}(S(q^x, q^n));
    cands := ratsol(G(X));
    if cands = {} then
        return  "There does not exist a qZ-pair";
    fi;
    let x_0, ..., x_m be the members of cands;
    c := {};  s := 1;
    for j = 0, 1, ..., m do
        cand := wc(w(q^n, q^k), x_j);
        δ_s := deg_{q^k} cand;
        if δ_s > 0 then
            c := c ∪ {x_j};
            w_{x_j}(q^n, q^k) := cand;
            s := s + 1;
        fi;
    od;
    if deg_{q^k} w(q^n, q^k) ≠ (δ_1 + ··· + δ_{s−1}) then
        return  "There does not exist a qZ-pair";
    fi;
    rewrite v_2(q^k) as v_2(q^k) = (q^k − γ_1) ··· (q^k − γ_s);
    g̃(q^n, q^k) := v_1(q^n) v_2(q^k);
    let c_0, ..., c_d be the members of c;
    for t = 1, 2, ..., d do
        use the method of undetermined coefficients to rewrite w_{c_t}(q^n, q^k) as
           w_{c_t}(q^n, q^k) = (q^k − γ_{t_1} q^{c_t n}) ··· (q^k − γ_{t_s} q^{c_t n});
        g̃(q^n, q^k) := g̃(q^n, q^k) w_{c_t}(q^n, q^k);
    od;
fi;
F̃(q^n, q^k) := f(q^n, q^k)/g̃(q^n, q^k);
assign to F̃(q^n, q^k) the partial fraction decomposition w.r.t. q^k of F̃(q^n, q^k);
(L_F, G_F) := qZpairF(F̃(q^n, q^k));
return  (L_F, L_F S(q^n, q^k) + G_F);
```

Example 2 Consider the rational function

$$T(q^n, q^k) = \frac{q^{k+1} - q^n}{q^{k+1} + q^n + 1} - \frac{q^k - q^n}{q^k + q^n + 1} + \frac{1}{q^{2k} + (1 + q^n - q^{2n})q^k - q^{3n} - q^{2n}}.$$

Applying *ZpairDirect* yields

$$S(q^n,q^k)=\frac{q^k-q^n}{q^k+q^n+1},\ \ F(q^n,q^k)=\frac{1}{q^{2k}+(1+q^n-q^{2n})q^k-q^{3n}-q^{2n}},$$
$$v_1(q^n)=v_2(q^k)=1,\ \ w(q^n,q^k)=q^{2k}+(1+q^n-q^{2n})q^k-q^{3n}-q^{2n},$$

and $G(q^x)=-q^x+q^2$. Applying *ratsol* to $G(X)$ results in $c=\{2\}$, and the corresponding $w_2(q^n,q^k)$ obtained from applying *wc* to $w(q^n,q^k)$ and 2 is $w_2(q^n,q^k)=q^k-q^{2n}$. Hence, $\delta_1=1$. Since $\deg_{q^k} w(q^n,q^k)=2>1=\delta_1$, $q\mathcal{Z}$ is not applicable to $T(q^n,q^k)$. (Note that the denominator of $F(q^n,q^k)$ can be written as $(q^k-q^{2n})(q^k+q^n+1)$.)

Example 3 Consider the rational function

$$T(q^n,q^k)=\frac{1}{q^k-q\,q^{2n}}+\frac{q+1}{q^k-q^{-3n}}+\frac{q^2-1}{q^k-q^{-9}q^{-3n}}. \tag{27}$$

Applying *ZpairDirect* yields

$$S(q^n,q^k)=0,\ \ F(q^n,q^k)=T(q^n,q^k),$$
$$v_1(q^n)=v_2(q^k)=1,\ \ w(q^n,q^k)=(q^k-q\,q^{2n})(q^k-q^{-3n})(q^k-q^{-9}q^{-3n}),$$

and

$$\begin{aligned}G(q^x)\ =\ & q^{31}q^{5x}-(q^{37}+q^{33}+2\,q^{28}+q^{19})q^{4x}+(q^{39}+2\,q^{34}+2\,q^{30}+2\,q^{25}+q^{21}+2\,q^{16})q^{3x}-\\ &(2\,q^{36}+q^{31}+2\,q^{27}+2\,q^{22}+2\,q^{18}+q^{13})q^{2x}+(q^{33}+2\,q^{24}+q^{19}+q^{15})q^x-q^{21}.\end{aligned}$$

Applying *ratsol* to $G(X)$ results in $c=\{-12,-3,2,6\}$, and the corresponding $w_c(q^n,q^k)$ obtained from *wc* are

$$w_{-12}=1,\ \ w_{-3}=q^{2k}-(q^{-3n-9}+q^{-3n})q^k+q^{-6n-9},\ \ w_2=q^k-q\,q^{2n},\ \ w_6=1.$$

Hence, $\delta_1=2$, $\delta_2=1$. Since $\deg_{q^k} w(q^n,q^k)=3=\delta_1+\delta_2$, $q\mathcal{Z}$ is applicable to $T(q^n,q^k)$. Note that $w_2(q^n,q^k)$ is already in the desired form. As for $w_{-3}(q^n,q^k)$, we have

$$q^{2k}-(q^{-3n-9}+q^{-3n})q^k+q^{-6n-9}=(q^k-\gamma_1\,q^{-3n})(q^k-\gamma_2\,q^{-3n}). \tag{28}$$

Equating the coefficients of like powers in (28) yields the system of equations

$$\{\gamma_1+\gamma_2=q^{-9}+1,\ \ \gamma_1\,\gamma_2=q^{-9}\}$$

whose solution is $\{\gamma_1=1,\ \gamma_2=q^{-9}\}$. Therefore,

$$w_{-3}(q^n,q^k)=(q^k-q^{-3n})(q^k-q^{-9}\,q^{-3n}),$$

and the denominator $g(q^n,q^k)$ of $F(q^n,q^k)$ can be written as

$$g(q^n,q^k)=(q^k-q\,q^{2n})(q^k-q^{-3n})(q^k-q^{-9}\,q^{-3n}).$$

Applying the partial fraction decomposition w.r.t. q^k to $F(q^n,q^k)$ yields (27). We now start step 2 of *ZpairDirect*. The algorithm *qZpairF* allows us to decompose F into

$$F=V_1F_1+V_2F_2,\ \ V_1=1,\ F_1=\frac{1}{q^k-q\,q^{2n}},\ V_2=(q^2-1)\,Q_n^3+(q+1),\ F_2=\frac{1}{q^k-q^{-3n}}.$$

Applying *qZpairVF* to V_1F_1 yields the minimal qZ-pair (L_1,G_1) for $V_1F_1=F_1$ where

$$(L_1,G_1)=\left(q^2\,Q_n-1,\ -\frac{1}{q^{k-1}-q\,q^{2n}}-\frac{1}{q^{k-2}-q\,q^{2n}}\right).$$

The minimal qZ-pair (L_2,G_2) for V_2F_2 is computed in Example 1. Therefore, the minimal qZ-pair (L,G) for $F(q^n,q^k)$, and also for $T(q^n,q^k)$ is

$$(L,G)=(\mathrm{lclm}(L_1,L_2),\ L_1'G_1+L_2'G_2)\ \text{where}$$
$$L=\mathrm{lclm}(L_1,L_2)=\frac{q^4}{q^5-1}\,Q_n^2-\frac{q^2\,(1+q^5)}{q^5-1}\,Q_n+\frac{q^5}{q^5-1},$$

$$L_1' = \frac{q^7\,(q+1)\left(q^{10}-q^9+1\right)}{q^5-1}\,Q_n - \frac{(q+1)\left(q^{10}-q^9+1\right)q^5}{q^5-1}, \text{ and } L_2' = \frac{q^2}{q^5-1}\,Q_n - \frac{q^5}{q^5-1}.$$

References

[1] S.A. Abramov, Rational component of the solutions of a first-order linear recurrence relation with a rational right-hand side. *USSR Comput. Math. Phys.* Transl. from *Zh. vychisl. mat. mat. fyz.* **14** 1035–1039, 1975.

[2] S.A. Abramov, Indefinite sums of rational functions. *Proceedings ISSAC'95*, 303–308, 1995.

[3] H. Böing, W. Koepf, Algorithms for q-Hypergeometric Summation in Computer Algebra, *J. Symb. Comput.* **11**, 1999, 1–23.

[4] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics.* Addison Wesley, Reading, MA, 1994, Second Edition.

[5] T.H. Koornwinder, On Zeilberger's algorithm and its q-analogue, *J. Comp. Appl. Math* **48**, 1993, 91–111.

[6] H.Q. Le, A Direct Algorithm to Construct Zeilberger's Recurrences for Rational Functions, *Proc. FPSAC'2001*, 2001, 303–312.

[7] H.Q. Le, On the q-analogue of Zeilberger's Algorithm to Rational Functions, *Programming and Comput. Software (Programmirovanie)* **27**, 2001, 49–58.

[8] M. Petkovšek, H. Wilf, D. Zeilberger, *A=B.* A.K. Peters, Wellesley, Massachusetts, 1996.

[9] A. Riese (1995): A Mathematica q-analogue of Zeilberger's algorithm for proving q-hypergeometric identities, Diploma Thesis, University of Linz, Linz.

[10] H. Wilf, D. Zeilberger, An algorithmic proof theory for hypergeometric (ordinary and "q") multisum/integral identities. *Inventiones Mathematicae* **108**, 1992, 575–633.

[11] D. Zeilberger, The method of creative telescoping. *J. Symb. Comput.* **11**, 1991, 195–204.

E-mail address: hqle@scg.math.uwaterloo.ca, abramov@ccas.ru, kogeddes@scg.math.uwaterloo.ca

Symbolic Computation Group, Department of Computer Science, University of Waterloo, Waterloo, N2L 3G1, Canada; Computer Center of the Russian Academy of Science, Vavilova 40, Moscow 117967, Russia; Symbolic Computation Group, Department of Computer Science, University of Waterloo, Waterloo, N2L 3G1, Canada