A DIRECT ALGORITHM TO CONSTRUCT THE MINIMAL TELESCOPERS 
FOR RATIONAL FUNCTIONS (Q-DIFFERENCE CASE)

H. Q. LE, S. A. ABRAMOV, AND K. O. GEDDES

Abstract. In this paper we present a direct algorithm to construct the minimal telescopers for rational functions for the q-difference case.

1. Preliminaries

The well-known Zeilberger’s algorithm [4, 11] is a useful tool for finding closed forms of definite hypergeometric sums, and for certifying the truth for large classes of identities that occur in combinatorics and in the theory of special functions. It is shown in [3, 5, 10] that Zeilberger’s algorithm can be carried over to the q-difference case; and similar to the case of its difference counterpart, the q-analogue version of Zeilberger’s algorithm (we name it hereafter as qZ) also has a wide range of applications [9].

Let \( q \) be an indeterminate parameter. Denote by \( Q_n, Q_k \) the q-shift operators w.r.t. \( q^n \) and \( q^k \), respectively, defined by \( Q_n T(q^n, q^k) = T(q^{n+1}, q^k), Q_k T(q^n, q^k) = T(q^n, q^{k+1}) \). For a given \( q \)-hypergeometric term \( T(q^n, q^k) \) of \( q^n \) and \( q^k \), i.e., the consecutive-term ratios \( T(q^{n+1}, q^k)/T(q^n, q^k) \) and \( T(q^n, q^{k+1})/T(q^n, q^k) \) are elements from \( \mathbb{C}(q)(q^n, q^k) \), \( qZ \) tries to construct for \( T(q^n, q^k) \) a \( qZ \)-pair \((L, G)\) which consists of a linear \( q \)-difference operator \( L \) with coefficients which are polynomials of \( q^n \) over \( \mathbb{C}(q) \)

\[
L = a_0(q^n)Q_n^n + \cdots + a_1(q^n)Q_n + a_0(q^n)Q_n^0, \quad a_i(q^n) \in \mathbb{C}(q)[q^n],
\]

and a \( q \)-hypergeometric term \( G(q^n, q^k) \) such that

\[
LT(q^n, q^k) = (Q_k - 1) G(q^n, q^k).
\]

It can be shown that if there exist \( qZ \)-pairs for \( T(q^n, q^k) \), then \( qZ \) terminates with one of the \( qZ \)-pairs and the telescopers \( L \) in the returned \( qZ \)-pair is of minimal possible order. Note that \( L \) is unique up to a factor \( P(q^n) \in \mathbb{C}(q)[q^n] \), and we name it the minimal telescopers. We also name the \( qZ \)-pair \((L, G)\) where \( L \) is the minimal telescopers the minimal \( qZ \)-pair.

\( qZ \) uses an item-by-item examination of the order \( \rho \) of \( L \). It starts with the value of 0 for \( \rho \) and increases \( \rho \) until it is successful in finding the minimal \( qZ \)-pair \((L, G)\). For a given \( q \)-hypergeometric term \( T(q^n, q^k) \), the question whether there exists a \( qZ \)-pair for \( T(q^n, q^k) \) is not conclusively answered although a sufficient condition is known via the fundamental theorem (see [4, 8, 10]). If \( T(q^n, q^k) \) is a proper \( q \)-hypergeometric term [8, 10], then \( qZ \) is applicable to \( T \), and an upper bound for \( \rho \) can be computed [8]. This upper bound, however, is much too large in practice. This is the main reason for applying item-by-item examination strategy. As a consequence, when applying \( qZ \) to a \( q \)-hypergeometric term, we waste resources either trying to compute a \( qZ \)-pair in the case when no such \( qZ \)-pair exists or trying to compute without success a telescopers with \( \text{ord} L < \rho \) in the case when the \( qZ \)-pairs exist and the order of the minimal telescopers is \( \rho \).

Let \( T(q^n, q^k) \) be a rational function of \( q^n \) and \( q^k \). The problem of determining a necessary and sufficient condition for the termination of \( qZ \) on \( T(q^n, q^k) \) is solved and presented in [7]. First,
it provides a decision procedure as to whether or not one should apply \( qZ \) to the input \( T(q^n, q^k) \). Secondly, in the case when the termination of \( qZ \) is guaranteed, it helps to speed up the construction of the minimal \( qZ \)-pair by reducing the size of the problem to be solved. Note that an item-by-item examination technique is still used in this case.

In this paper we present a direct algorithm to construct the minimal \( qZ \)-pair \((L, G)\) for a given rational function \( T(q^n, q^k) \). By direct algorithm, we mean an algorithm which computes the minimal \( qZ \)-pair directly, without using an item-by-item examination. The algorithm is based on a special form of representation of \( T(q^n, q^k) \), on direct construction of the minimal telescoper for each member of this representation. The minimal \( qZ \)-pair for \( T(q^n, q^k) \) can then be obtained using Least Common Left Multiple (LCLM) computation.

Note that the same problem for the difference case is solved and presented in [6]. In fact, even though the details are different, we essentially use the same approach as the one that we used for the difference case in solving the problem for the \( q \)-difference case.

2. The Decomposition Problem and the Existence of a \( qZ \)-Pair

In this section, we recall some known results relating to the decomposition problem [2] and the existence of a \( qZ \)-pair [7] for rational functions. They are needed in subsequent sections.

2.1. The Decomposition Problem. Let \( T(q^k) \) be an element from \( \mathcal{C}(q)(q^k) \). An algorithm to solve the decomposition problem constructs a rational function \( S(q^k) \) and a proper rational function \( F(q^k) \) such that

\[
T(q^k) = (Q_k - 1)S(q^k) + F(q^k), \quad S(q^k), \quad F(q^k) \in \mathcal{C}(q)(q^k),
\]

where the denominator \( g(q^k) \) of \( F(q^k) \) has the lowest possible degree in \( q^k \).

If \( F(q^k) = 0 \), then \( T(q^k) \) is said to be rational summable.

Consider the partial fraction decomposition of \( T(q^k) \) w.r.t. the complete factorization of its denominator over \( \mathcal{C}(q) \):

\[
T(q^k) = \sum_{i=1}^{m} \sum_{j=1}^{t_i} \frac{\beta_{ij}}{(q^k - \alpha_i)^j}, \quad \alpha_i, \beta_{ij} \in \mathcal{C}(q).
\]

Define \( \alpha_i \sim \alpha_j \) iff \( \alpha_i = q^l \alpha_j \) where \( l \) is an integer. It is easy to check that \( \sim \) is an equivalence relation on the set \( \{\alpha_1, \ldots, \alpha_m\} \). If \( \alpha_i \) and \( \alpha_j \) are in the same equivalence class, then the two monic irreducibles \( q^k - \alpha_i \) and \( q^k - \alpha_j \) are said to be \( q \)-shift equivalent. For each equivalence class, let \( \alpha_i \) be the largest element in the sense that for all elements \( \alpha_j \) in the same class, \( \alpha_j = q^l \alpha_i \) where \( l \) is a non-positive integer. It is shown in [2] that \( T(q^k) \) can be written as

\[
\sum_{i=1}^{s} \sum_{j=1}^{t_i} M_{ij} \frac{1}{(q^k - \alpha_i)^j}
\]

where \( M_{ij} \in \mathcal{C}(q)[Q_k], 0 < s \leq m, t_i > 0; \alpha_i \) is the representative, which is the largest element, for the \( i \)-th \( q \)-shift equivalence class; and \( T(q^k) \) is rational summable iff

\[
M_{ij} = L_{ij} \circ (Q_k - 1), \quad L_{ij} \in \mathcal{C}(q)[Q_k], \quad i = 1, \ldots, s, \quad j = 1, \ldots, t_i.
\]

Let us represent \( F(q^k) \) in the reduced form \( F(q^k) = f(q^k)/g(q^k) \), where \( f(q^k), g(q^k) \in \mathcal{C}(q)[q^k] \).

By [7], \( g(q^k) \) has the following property:

**P1.** If \( p_1(q^k), p_2(q^k) \) are factors of \( g(q^k) \) irreducible over \( \mathcal{C}(q) \) then there does not exist a non-zero integer \( h \) such that \( p_1(q^k + h) = p_2(q^k) \).

On the other hand, suppose that \( T(q^k) \) is rational summable. Let \( T(q^k) = a(q^k)/b(q^k) \) where \( a(q^k), b(q^k) \) are relatively prime elements of \( \mathcal{C}(q)[q^k] \). Then \( b(q^k) \) has the following property [7]:

**P2.** If \( p_1(q^k) \) is a factor of \( b(q^k) \) irreducible over \( \mathcal{C}(q) \), then there exist a factor \( p_2(q^k) \) irreducible over \( \mathcal{C}(q) \) of \( b(q^k) \) and a non-zero integer \( h \) such that \( q^{-h} p_1(q^k + h) = p_2(q^k) \).
2.2. The Existence of a qZ-pair. Let $T(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$. Apply an algorithm to solve the decomposition problem w.r.t. $q^k$ to represent $T(q^n, q^k)$ in the form

$$T(q^n, q^k) = (Q_k - 1) S(q^n, q^k) + F(q^n, q^k),$$

where $S(q^n, q^k), F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$, and the denominator $g(q^n, q^k)$ of $F(q^n, q^k)$ is of minimal possible degree w.r.t. $q^k$. It is shown in [7] that a $qZ$-pair for $T(q^n, q^k)$ exists if $g(q^n, q^k)$ can be written in the form

$$g(q^n, q^k) = \alpha q^n \prod_{i=1}^{n} (q^k - \gamma_i q^{(a_i/b_i)n}), \quad c, a_i, b_i \in \mathbb{Z}, \gcd(a_i, b_i) = 1, b_i > 0, \gamma_i, \alpha \in \mathbb{C}(q).$$

An algorithm for using the above criterion as well as the verification of the algorithm are presented in [7].

3. On the Minimal Telescope of a Sum of Rational Functions

We introduce in this section the concept of similarity among $q$-hypergeometric terms. It is the $q$-analogue of the same concept for hypergeometric terms as discussed in Section 5.6 [8]. The main result of this section is Theorem 1 which provides a sufficient condition for the construction of the minimal telescope for a sum of rational functions based on the minimal telescopes for each rational function of the sum. This theorem is the $q$-analogue of Theorem 2 in [6] for the difference case.

**Definition 1.** Two $q$-hypergeometric terms $s(q^n)$ and $t(q^n)$ are similar if their ratio is an element from $\mathbb{C}(q)(q^n)$.

It is easy to see that similarity is an equivalence relation on the set of all $q$-hypergeometric terms. The following proposition is the $q$-analogue of Proposition 5.6.2 [8].

**Proposition 1.** Let $s(q^n)$ and $t(q^n)$ be two $q$-hypergeometric terms such that $s(q^n) + t(q^n) \neq 0$. Then $s(q^n) + t(q^n)$ is $q$-hypergeometric if and only if $s(q^n) \sim t(q^n)$.

**Lemma 1.** Let $(L, G)$ be the minimal $qZ$-pair for a $q$-hypergeometric term $F(q^n, q^k)$. If $L_1$ is a telescope for $F$, then $L_1$ is right divisible by $L$ in $\mathbb{C}(q)(q^n)[Q_n]$.

**Proof.** We have

$$LF = (Q_k - 1)G.$$

Since $L_1$ is a telescope for $F$, there exists a $q$-hypergeometric term $G_1(q^n, q^k)$ such that

$$L_1 F = (Q_k - 1)G_1.$$

Applying the right Euclidean division of $L_1$ by $L$ yields

$$L_1 = S \circ L + R, \quad S, R \in \mathbb{C}(q)(q^n)[Q_n], \quad R = 0 \text{ or ord } R < \text{ord } L.$$

From (9), one obtains

$$L_1 F = S(LF) + RF.$$

The substitution of (7) and (8) into (10) yields

$$RF = (Q_k - 1)(G_1 - SG).$$

Since $T(q^n, q^k)$ is a $q$-hypergeometric term, $T(q^{n+1}, q^{k+1})/T(q^n, q^k)$ is a rational function of $q^n$ and $q^k$. One can show by induction that for any pair of integers $s_1, s_2$, $T(q^{n+s_1}, q^{k+s_2})/T(q^n, q^k)$ is also a rational function of $q^n$ and $q^k$. In general,

$$\frac{MT(q^n, q^k)}{T(q^n, q^k)} = R(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k) \text{ for any operator } M \in \mathbb{C}(q)(q^n)[Q_n, Q_k].$$

It follows from (7), (8), (11) and (12) that $F \sim G \sim G_1$, and $SG \sim G$. Since $\sim$ is an equivalence relation, $G_1 \sim SG$, and $G_1 - SG$ is a $q$-hypergeometric term (Proposition 1). Since ord $R < \text{ord } L$ and $L$ is the minimal telescope for $F$, it follows from (11) that $R = 0$. \qed
Theorem 1. Let $F_1, \ldots, F_s$ be rational functions such that if $L^*$ is a telescopor for $F = F_1 + \cdots + F_s$, then $L^*$ is a telescopor for every $F_i, i = 1, \ldots, s$. Let $L_1, \ldots, L_s$ be the minimal telescopors for $F_1, \ldots, F_s$, respectively (assuming that they all exist). Then $\text{lcm}(L_1, \ldots, L_s)$ is the minimal telescopor for $F_1 + \cdots + F_s$.

Proof. First we show that $\text{lcm}(L_1, \ldots, L_s)$ is a telescopor for $F$. Since $L_i$ is the minimal telescopor for $F_i$, there exists $G_i \in \mathbb{C}(q)(q^n, q^k)$ such that

$$L_i F_i = (Q_k - 1) G_i, \quad 1 \leq i \leq s.$$ 

Set $L = \text{lcm}(L_1, \ldots, L_s)$. This implies there exist $L'_1, \ldots, L'_s \in \mathbb{C}(q)(q^n)[Q_n]$ such that

$$L = \text{lcm}(L_1, \ldots, L_s) = L'_1 \circ L_1 = \cdots = L'_s \circ L_s.$$ 

Then

$$LF = LF_1 + \cdots + LF_s = L'_1 (L_1 F_1) + \cdots + L'_s (L_s F_s) = (Q_k - 1)(L'_1 G_1 + \cdots + L'_s G_s).$$

Since $L'_1 G_1 + \cdots + L'_s G_s \in \mathbb{C}(q)(q^n, q^k)$, $(L, L'_1 G_1 + \cdots + L'_s G_s)$ is a qZ-pair for $F$.

Now let $L^*$ be any telescopor for $F$. From the hypothesis, $L^*$ is also a telescopor for each $F_i, 1 \leq i \leq s$. Since $L_i$ is the minimal telescopor for $F_i$ and since every rational function is a q-hypergeometric term, one can apply Lemma 1 and deduces that $L^*$ is right divisible by $L_i, 1 \leq i \leq s$. Consequently, $L^*$ is right divisible by lcm($L_1, \ldots, L_s$).

\[\square\]

4. The Basic Case

We show in this section how to compute directly the minimal qZ-pair $(L, G)$ for $VF$ where $V \in \overline{\mathbb{C}(q)(q^n)[Q_n]}$, and

$$F(q^n, q^k) = \frac{1}{(q^b - \gamma q^{a/b} q^n)}, \quad a, b \in \mathbb{Z}, \gcd(a, b) = 1, \quad b > 0, \quad \gamma \in \overline{\mathbb{C}(q)}, \quad m \in \mathbb{N} \setminus \{0\}.$$ 

Lemma 2. Let $F(q^n, q^k)$ be of the form (14). Then $L = q^a M Q_n^b - 1$ is the minimal telescopor for $F(q^n, q^k)$.

Proof. Set $L = q^a M Q_n^b - 1, M = Q_n^{-a} - 1 = (Q_k - 1) \circ U$, where

$$U = \left\{ \begin{array}{ll} \sum_{i=0}^{a-1} Q_k^i & a < 0 \\ \sum_{i=a}^{m} -Q_n^k & \text{otherwise} \end{array} \right.$$ 

It is easy to check that

$$LF = MF = (Q_k - 1)(UF), \quad UF \in \overline{\mathbb{C}(q, q^n)[q^k]}.$$ 

Therefore, $(L, UF)$ is a qZ-pair for $F(q^n, q^k)$. Now suppose there exists a qZ-pair $(L_1, G_1)$ such that $\text{ord } L_1 < \text{ord } L = b$. Without loss of generality we can assume that the coefficient of $Q_n^0$ in $L_1$ is a non-zero element of $\overline{\mathbb{C}(q)[q^n]}$. Otherwise, choose a new qZ-pair for $F$

$$\left( Q_n^{-\lambda} \circ L_1, G_1(q^{-\lambda}, q^k) \right)$$

where $\lambda$ is the minimal positive integer such that the coefficient of $Q_n^\lambda$ in $L_1$ is not zero. Set

$$H = L_1 F = \frac{s(q^n, q^k)}{t(q^n, q^k)} s(q^n, q^k), \quad t(q^n, q^k) \in \overline{\mathbb{C}(q, q^n)[q^k]}.$$ 

Since $H$ is rational summable w.r.t. $q^k, t(q^n, q^k)$ has property $\textbf{P2}$, i.e., for the irreducible factor $q^k - \gamma q^{(a/b)n}$ of $t(q^n, q^k)$, there exists a non-zero integer $h$ such that

$$q^{-h} \left( q^k + h - \gamma q^{(a/b)n} \right) = q^k - \gamma q^{(a/b)n} q^{-h}.$$
is also an irreducible factor of $t(q^n, q^k)$. Since all the irreducible factors of $t(q^n, q^k)$ have the form $q^k - \gamma q^{(a/b)(n+i)}$, for $i = 0, 1, \ldots, \text{ord} L_1$, this means
\begin{equation}
\gamma q^{(a/b)n} \left( q^{(a/b)i} - q^{-h} \right) = 0 \text{ for some } i.
\end{equation}

\textbf{Case 1.} $\gamma \neq 0$ and $a \neq 0$. If follows from (15) that
\[h = -\frac{a}{b}i.\]
If $i = 0$, then $h = 0$. This contradicts the assumption on $h$. Otherwise, since $\gcd(a, b) = 1$, $b|i$. This is impossible since $0 < i < b$.

\textbf{Case 2.} $\gamma = 0$ or $a = 0$. It is easy to see that $(L, G) = (Q_n - 1, 0)$ is a qZ-pair for $F$. Suppose that the minimal telescopers for $F$ is of order 0, i.e., $L = 1$. Then $LF = F = \{1/q^i k, 1/(q^k - \gamma)^m\}$. In both cases, $F$ is not rational summable w.r.t. $q^k$ (Property P2). \hfill \square

\textbf{Lemma 3.} Let $F$ be of the form (14), $(L_F, G_F)$ be the minimal qZ-pair for $F$, $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$ and
\begin{equation}
lcdm(V, L_F) = L_1 \circ V = L_2 \circ L_F, \quad L_1, L_2 \in \overline{\mathbb{C}(q)}(q^n)[Q_n].
\end{equation}
Then $(L_1, L_2 G_F)$ is the minimal qZ-pair for $F$.

\textbf{Proof.} It follows from Lemma 2 that the existence of the minimal qZ-pair $(L_F, G_F)$ for $F$ is guaranteed. Since
\[L_1(VF) = L_2(L_F F) = L_2 \circ (Q_n - 1)G_F = (Q_n - 1)(L_2 G_F), \quad (L_1, L_2 G_F) \text{ is a qZ-pair for } V F.
\]
Let $L^*$ be any telescopers for $VF$, i.e., there exists $G^* \in \overline{\mathbb{C}(q, q^n)}(q^k)$ such that $L^*(VF) = (Q_n - 1)G^*$. Therefore, $(L^* \circ V, G^*)$ is a qZ-pair for $F$. Since $L_F$ is the minimal telescopers for $F$, one deduces from Lemma 1 that $L^* \circ V$ is right divisible by $L_F$. Additionally, $L^* \circ V$ is right divisible by $V$ and $L_1 \circ V = \text{lcdm}(V, L_F)$. Therefore, $L^* \circ V$ is right divisible by $L_1 \circ V$. Consequently, $L_1$ is the minimal telescopers for $VF$. \hfill \square

\textbf{Theorem 2.} One can directly compute the minimal qZ-pair $(L, G)$ for a rational function $VF \in \overline{\mathbb{C}(q, q^n)}(q^k)$, where $F$ is of the form (14) and $V \in \overline{\mathbb{C}(q)}(q^n)[Q_n]$.

\textbf{Proof.} The proof follows from Lemmas 2 and 3. \hfill \square

\textbf{Example 1} Consider the rational function
\[VF \text{ where } V = (q^2 - 1)Q_n^3 + (q + 1), \quad F = \frac{1}{q^k - q^{-3n}}.
\]
Applying Lemma 2 results in the minimal qZ-pair $(L_F, G_F)$ for $F$:
\[(L_F, G_F) = \left( q^{-3} Q_n - 1, \frac{1}{q^k + 2 - q^{-3n}} + \frac{1}{q^k + 1 - q^{-3n}} + \frac{1}{q^k - q^{-3n}} \right). \]
Computing the lcldm$(V, L_F)$ results in the operators $L_1, L_2$ that satisfy (16):
\[L_1 = \frac{1}{(q + 1)(q^{10} - q^9 + 1)Q_n^3} \quad \text{and} \quad L_2 = \frac{q - 1}{q^{10} - q^9 + 1} Q_n^3 + \frac{1}{q^k - q^{-3n}}.
\]
It follows from Lemma 3 that the minimal qZ-pair $(L, G)$ for $VF$ is $(L, G) = (L_1, L_2 G_F)$ where
\[G = \frac{1}{q^{10} - q^9 + 1} \left( q^{k+2} - q^{-3n} + \frac{1}{q^k + 1 - q^{-3n}} + \frac{1}{q^k - q^{-3n}} \right). \]
\[
\frac{q-1}{q^{10}-q^5+1} \left( \frac{1}{q^{k+2} - q^{3n-9}} + \frac{1}{q^{k+1} - q^{3n-9}} + \frac{1}{q^k - q^{3n-9}} \right).
\]

We conclude this section with a description of the algorithm \( qZpairVF \) to construct the minimal \( qZ \)-pair for \( VF \) where \( V \in \overline{\mathbb{C}(q)}(q^n)[Q_n] \), and \( F \) is of the form (14).

**Algorithm \( qZpairVF \):**

**Input:** \( V \in \overline{\mathbb{C}(q)}(q^n)[Q_n] \),
\( F = 1/(q^k - \gamma q^{(a/b)n})^m \), \( a, b \in \mathbb{Z}, \gcd(a, b) = 1, b > 0, \gamma \in \overline{\mathbb{C}(q)} \), \( m \in \mathbb{N} \setminus \{0\} \);

**Output:** the minimal \( qZ \)-pair for \( VF \);

if \( a < 0 \) then
\[(L_F, G_F) := \left(q^m Q_n^b - 1, \left(\sum_{i=0}^{-a-1} Q_i^b\right) F\right) ;\]
else
\[(L_F, G_F) := \left(q^m Q_n^b - 1, \left(\sum_{i=1}^a - Q_i^b\right) F\right) ;\]
fi;
apply lclm computation to construct \( L_1, L_2 \in \overline{\mathbb{C}(q)}(q^n)[Q_n] \) such that \( L_1 \circ V = L_2 \circ L_F = \text{lclm}(V, L_F) \);
return \( (L_1, L_2G_F) \).

5. **The Generic Case**

**Lemma 4.** Let \( F(q^n, q^k) \in \overline{\mathbb{C}(q)}(q^n, q^k) \), and
\[
F = \sum_{i=1}^t \sum_{j=1}^{m_i} r_{ij}(q^n) \quad r_{ij}(q^n) \in \overline{\mathbb{C}(q)}(q^n), \gamma_i \in \overline{\mathbb{C}(q)} \), \( a_i, b_i \in \mathbb{Z}, b_i > 0, \gcd(a_i, b_i) = 1 \).
\]
Then \( F(q^n, q^k) \) can be represented in the form
\[
V_1 F_1 + \cdots + V_s F_s ,
\]
where \( V_i \in \overline{\mathbb{C}(q)}(q^n)[Q_n] \), and \( F_i = 1/(q^k - \gamma_i q^{(a_i/b_i)n})^{m_i} \) is such that
\[
a_i, b_i \in \mathbb{Z}, b_i > 0, \gamma_i \in \overline{\mathbb{C}(q)} \), \( \gcd(a_i, b_i) = 1 \), \( m_i \in \mathbb{N} \setminus \{0\} \),
\]
and for all \( i \neq j \), at least one of the following four relations is not valid:
\[
m_i = m_j, a_i = a_j, b_i = b_j, \gamma_i = q^{(a_i/b_i)h} \gamma_j, h \in \mathbb{Z} \setminus \{0\} \).
\]

**Proof.** Let \( R_1, R_2 \) be two simple fractions in (17). Define \( R_1 \sim R_2 \) iff all four relations in (20) hold.
It is easy to check that \( \sim \) is an equivalence relation on the set of simple fractions in (17).

For any pair of simple fractions \( R_1 \) and \( R_2 \) in the same equivalence class, i.e., \( R_1 \) and \( R_2 \) can be written as
\[
R_1 = \frac{r_1(q^n)}{(q^k - \gamma_1 q^{(a/b)n})^m}, \quad R_2 = \frac{r_2(q^n)}{(q^k - \gamma_2 q^{(a/b)n})^m}, \quad \gamma_1 = q^{(a/b)h} \gamma_2, h \in \mathbb{Z} \setminus \{0\} .
\]
If \( h > 0 \), then
\[
R_1 + R_2 = V \frac{1}{(q^k - \gamma_1 q^{(a/b)n})^m} \quad \text{where } V = (r_2(q^n) + r_1(q^n) Q_n^h) \in \overline{\mathbb{C}(q)}(q^n)[Q_n] .
\]
Otherwise, \( h < 0 \) and
\[
R_1 + R_2 = V \frac{1}{(q^k - \gamma_1 q^{(a/b)n})^m} \quad \text{where } V = (r_1(q^n) + r_2(q^n) Q_n^{-h}) \in \overline{\mathbb{C}(q)}(q^n)[Q_n] .
\]
By using induction on the number of elements in each equivalence class, the sum of the elements in each class can be represented as

\[ V_i \left( \frac{1}{q^k - \gamma_i q^{(a_i/b_i)n}} \right)^{m_i}, \quad V_i \in \overline{C(q)}(q^n)[Q_n]. \]

**Lemma 5.** Let \( F(q^n, q^k) \) be written in the form (18) where \( V_i \in \overline{C(q)}(q^n)[Q_n] \),

\[ F_i = \frac{1}{(q^k - \gamma_i q^{(a_i/b_i)n})^{m_i}} \]

is such that (19) takes place and if \( i \neq j \) then at least one of the four relations in (20) is not valid. Let \( L \) be a telescoper for \( F(q^n, q^k) \). Then \( L \) is a telescoper for each \( V_i F_i, 1 \leq i \leq s \).

**Proof.** Let \( L \in \overline{C(q)}(q^n, Q_n) \) be any telescoper for \( F(q^n, q^k) \). Consider

\[ V_i F_i = V_i \left( \frac{1}{q^k - \gamma_i q^{(a_i/b_i)n}} \right)^{m_i}, \quad 1 \leq i \leq s. \]

Let \( R_i(q^n, q^k) = r_i(q^n)/(q^k - \gamma_i q^{(a_i/b_i)(n+h)})^{m_i} \) be any simple fraction of \( V_i F_i, h_1 \in \mathbb{N} \). Then for any non-negative integer \( h \), \( R_i(q^{n+h}, q^k) = r_i(q^{n+h})(q^k - \gamma_i q^{(a_i/b_i)(n+h)})^{m_i} \). Consequently, it is easy to see that the application of \( L \in \overline{C(q)}[q^n, Q_n] \) to a simple fraction of \( V_i F_i \) yields a sum of simple fractions such that each of them is in the same equivalence class (w.r.t. the relation \( \sim \) as defined in the proof of Lemma 4) as that of \( F_i \).

Let \( R_1, R_2 \) be any two simple fractions of \( LF(q^n, q^k) \). Define \( R_1 \sim R_2 \) iff in addition to the four relations in (20), i.e., \( R_1 \) and \( R_2 \) can be written as

\[ R_1 = \frac{r_1(q^n)}{(q^k - \gamma_1 q^{(a_1/b_1)n})^{m_1}}, \quad R_2 = \frac{r_2(q^n)}{(q^k - \gamma_2 q^{(a_2/b_2)n})^{m_2}}, \quad \gamma_1 = q^{(a_1/b_1)h}, \quad h \in \mathbb{Z} \setminus \{0\}, \]

the relation \( b|h \) also holds. It is easy to check that \( \sim \) is an equivalence relation, and if \( R_1 \sim R_2 \) then \( R_1 \sim R_2 \). Consequently, it is impossible that \( L \circ V_i F_i \) and \( L \circ V_j F_j \) both have simple fractions from the same class w.r.t. \( \sim \) for \( i \neq j \).

Considering any \( L \circ V_i F_i \), we distribute the corresponding simple fractions to the classes w.r.t. \( \sim \). For every pair of simple fractions \( R_1, R_2 \) as defined in (22), since \( b|h \), there exists \( s \in \mathbb{Z} \setminus \{0\} \) such that \( h = sb \). Set \( h_1 = -a s \), \( p_1(q^n, q^k) = q^n - \gamma_1 q^{(a_1/b_1)n}, p_2(q^n, q^k) = q^n - \gamma_2 q^{(a_2/b_2)n} \). Then

\[ q^{n+h_1}, p_2(q^n, q^{n+h_1}) = q^n - \gamma_2 q^{(a_2/b_2)(n+h)} = p_1(q^n, q^k). \]

This means the monic irreducible factors of the denominator of the sum of simple fractions in each class are \( q \)-shift-equivalent. Since \( LF(q^n, q^k) \) is rational summable w.r.t. \( q^k \), from the necessary and sufficient condition as defined in (4), Subsection 2.1, the sum of all fractions from such a class is rational summable w.r.t. \( q^k \). This implies that \( L \circ V_i F_i \) is rational summable w.r.t. \( q^k \).

**Theorem 3.** For any rational function \( F(q^n, q^k) \) which can be written in the form

\[ F = \sum_{i=1}^{t} \sum_{j=1}^{m_i} \frac{r_{ij}(q^n)}{(q^k - \gamma_i q^{(a_i/b_i)n})^{m_i}}, \quad r_{ij}(q^n) \in \overline{C(q)}(q^n), \quad \gamma_i \in \overline{C(q)}, \quad a_i, b_i \in \mathbb{Z}, \quad b_i > 0, \quad \gcd(a_i, b_i) = 1. \]

one can directly compute the minimal \( q \)-LZ-pair \( (L, G) \) for \( F(q^n, q^k) \).

**Proof.** Lemma 4 allows one to rewrite \( F(q^n, q^k) \) in the form (18). Theorem 2 allows one to compute the minimal \( q \)-LZ-pair \( (L_i, G_i) \) for each term \( V_i F_i, 1 \leq i \leq s \). One then uses Theorem 1 to compute a \( q \)-pair \( (L, G) \) for \( F(q^n, q^k) \) where \( L = \text{lcm}(L_1, \ldots, L_s) \). Lemma 5 and Theorem 1 guarantee that \( L \) is indeed the minimal telescoper for \( F(q^n, q^k) \).
The following is a description of the algorithm $qZpair F$ which constructs the minimal $qZ$-pair for $F$ of the form (17).

**Algorithm $qZpair F$:**

**Input:** $F(q^n, q^k) \in \mathbb{C}(q^n, q^k)$ and is of the form (17);

**Output:** the minimal $qZ$-pair $(L_F, G_F)$ for $F$;

let $R_1, \ldots, R_s$ be the equivalence classes of the set of simple fractions of $F$ as defined by the relation $\sim$;

for $i = 1, 2, \ldots, s$ do

let $\{F_{i1}, \ldots, F_{im}\}$ be the members of the equivalence class $R_i$;

let $F_{ij} = r_{ij}(q^n)/(q^k - \gamma_{ij} q^{a_i/b_i} m_i), 1 \leq j \leq d$;

$F_i := F_{i1}(q^n, q^k); V_i := r_{i1}(q^n); \gamma_i := \gamma_{i1};$

for $j = 2, 3, \ldots, d$ do

let $h \in \mathbb{Z} \setminus \{0\}$ be such that $\gamma_i = q^{(a_i/b_i)h} \gamma_{ij}$;

if $h > 0$ then

$V_i := r_{ij}(q^n) + V_i Q_n^h$;

$\gamma_i := \gamma_{ij}$;

else

$V_i := V_i + r_{ij}(q^n) Q_n^{-h}$;

fi;

od;

$F_i := 1/(q^k - \gamma_i q^{a_i/b_i} m_i$;

$(LV_i F_i, GV_i F_i) := qZpairVF(V_i, F_i)$;

od;

apply idm computation to construct $L_1', \ldots, L_s' \in \mathbb{C}(q^n)[Q_n]$ such that $L_F = ldm(LV_1 F_1, \ldots, LV_s F_s) = L_1' \circ LV_1 F_1 = \cdots = L_s' \circ LV_s F_s$;

return $(L_F, L_1' G_1 F_1, \ldots, L_s' G_s F_s$).

6. A Direct Algorithm to Construct the Minimal $qZ$-pairs for Rational Functions

For a given rational function $T(q^n, q^k)$, a direct algorithm to construct the minimal $qZ$-pair for $T(q^n, q^k)$ can be decomposed into two steps. In the first step, check for the existence of a $qZ$-pair and if its existence is guaranteed, rewrite $F(q^n, q^k)$ in (5) such that the denominator of $F(q^n, q^k)$ is of the form (6). In the second step, apply the results shown in Sections 3, 4 and 5 to obtain the minimal $qZ$-pair $(L_1, G_1)$ for $F(n, k)$. Then the minimal $qZ$-pair $(L, G)$ for the given rational function $T(q^n, q^k)$ is $(L_1, L_1 S(q^n, q^k) + G_1(q^n, q^k))$ where $S(q^n, q^k)$ is defined in (5).

An algorithm to perform the first step is presented in [7], and we will make a very minor addition to this algorithm to complete the step.

Denote by $A_c$ the transformation

$q^n \rightarrow q^{n+1}, q^k \rightarrow q^{k+c}, c \in \mathbb{Q}$.

Let $t_0(q^n, q^k) = \gcd(A_c, w, w)$. Define the sequence of computation

$$t_i(q^n, q^k) = \gcd(A_c t_{i-1}, t_{i-1}), \quad i = 1, 2, \ldots$$

where the termination condition in (24) takes place when $\deg_q t_i(q^n, q^k) = \deg_q t_{i-1}(q^n, q^k)$, i.e., the degree w.r.t. $q^k$ stops decreasing. (Note that the number of gcd computation in (24) is guaranteed to be finite.) Set $w_c(q^n, q^k) = t_{i-1}(q^n, q^k)$. 
Lemma 6. (Lemma 7, [7]). Let \( w(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k] \). Let \( t_0(q^n, q^k) = \gcd(A_c w, w) \). Then the maximal factor of \( t_0(q^n, q^k) \) which can be written in the form

\[
\prod_{i=1}^{s} (q^k - \gamma_i q^c n), \quad \gamma_i \in \mathbb{C}(q)
\]

is \( w_c(q^n, q^k) \).

The following simple algorithm \( w_c \) is used to compute \( w_c(q^n, q^k) \) for a given \( w(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k] \), and \( c \in \mathbb{Q} \).

\textbf{algorithm} \( w_c \):
\begin{itemize}
  \item \textbf{input}: \( w(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k] \),
  \( c \in \mathbb{Q} \);
  \item \textbf{output}: \( w_c(q^n, q^k) \);
\end{itemize}

\[
t_0(q^n, q^k) := \gcd(w(q^{n+1}, q^{k+c}), w(q^n, q^k));
\]
\[
i := 1;
\]
\begin{itemize}
  \item \textbf{do}
  \begin{itemize}
    \item \( t_i(q^n, q^k) := \gcd(t_{i-1}(q^{n+1}, q^{k+c}), t_{i-1}(q^n, q^k)) \);
    \item \textbf{if} \( \deg_q t_i(q^n, q^k) = \deg_q t_{i-1}(q^n, q^k) \) \textbf{then}
      \begin{itemize}
        \item \textbf{return} \( t_{i-1}(q^n, q^k) \);
      \end{itemize}
    \item \( i := i + 1 \);
  \end{itemize}
\end{itemize}
\end{itemize}

Before describing the direct algorithm to construct the minimal \( q \)-Z-pairs for rational functions, we need to develop the following auxiliary function. Let \( G(x) \in \mathbb{C}[q][x] \), the algorithm \texttt{ratsol} computes the set of non-zero rational numbers \( c \) such that \( G(q^c) = 0 \).

\textbf{algorithm} \texttt{ratsol}:
\begin{itemize}
  \item \textbf{input}: \( F(q, x) \in \mathbb{C}[q][x] \) which can be written as \( F(q, x) = a_m(q)x^m + \cdots + a_0(q) \) where \( a_i(q) = c_{i1}q^{j1} + \cdots + c_{in}q^{jn}, \ 0 \leq i \leq m \);
  \item \textbf{output}: the set of non-zero values \( c \in \mathbb{Q} \) such that \( F(q, q^c) = 0 \);
\end{itemize}

\begin{verbatim}
cands := \{ \};
\end{verbatim}
\begin{verbatim}
m := \text{deg}_x F(q, x); j := \text{deg}_q \text{lc}(F(q, x), x);
\end{verbatim}
\begin{verbatim}
for d = 0, 1, \ldots, m - 1 do
  for l = 0, 1, \ldots, \text{deg}_q a_l(q) do
    if (c_{d_l} = 0) or (l = j) then next; fi;
    \begin{verbatim}
    cands := cands \cup \{(l - j)/(m - d)\};
    \end{verbatim}
  \end{verbatim}
\begin{verbatim}
od;
\end{verbatim}
\begin{verbatim}
od;
\end{verbatim}
\begin{verbatim}
sol := \{ \};
\end{verbatim}
\begin{verbatim}
let \{x_0, x_1, \ldots, x_\rho\} be the members of \texttt{cands};
\end{verbatim}
\begin{verbatim}
for i = 0, 1, \ldots, \rho do
  \begin{verbatim}
  if F(q, q^{x_i}) = 0 then
    sol := sol \cup \{x_i\};
  fi;
  \end{verbatim}
\end{verbatim}
\begin{verbatim}
od;
\end{verbatim}
\begin{verbatim}
return sol.
\end{verbatim}
Now for a given $T(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$, rewrite $T$ in the form (5) where

$$F(q^n, q^k) = \frac{f(q^n, q^k)}{g(q^n, q^k)}$$

is in reduced form. Extract from $g(q^n, q^k)$ the maximal factor $v_1(q^n) \in \mathbb{C}(q)[q^n], v_2(q^k) \in \mathbb{C}(q)[q^k]$. We know that $v_2(q^k)$ can be written as the product of factors of the form

$$q^k - \gamma, \gamma \in \overline{\mathbb{C}(q)}.$$

As for $v_1(q^n)$, if it cannot be written in the form $q^a, a \in \mathbb{Z}$, then it follows from (6) that $qZ$ is not applicable to $F(q^n, q^k)$, and hence to $T(q^n, q^k)$. Otherwise, set

$$w(q^n, q^k) = \frac{g(q^n, q^k)}{v_1(q^n)v_2(q^k)}.$$

It remains to investigate whether $w(q^n, q^k)$ can be decomposed into factors of the form

$$k + \gamma q^c, c \in \mathbb{Q} \setminus \{0\}, \gamma \in \overline{\mathbb{C}(q)}.$$

Let $w$ be an element from $\mathbb{C}(q)[q^7, q^n, q^k]$ obtained from $w$ by substituting $k$ by $k+x$ and $n$ by $n+1$. Let $S(q^n, q^k) = \text{Resultant}_{q^k}(w, \hat{w}), S(q^n, q^k) \in \mathbb{C}(q)[q^n][q^k]$.

Find all rational values of $x$ such that $S = 0$, i.e., $w$ and $\hat{w}$ have a non-trivial common factor. To attain this goal, consider $S$ as a polynomial in $q^n$ with coefficients which are polynomials in $q^k$. Let $G(q^n)$ be the gcd of all these coefficients. Now we need to find all values of $x \in \mathbb{Q} \setminus \{0\}$ such that $G(q^n) = 0$. This is accomplished by applying the algorithm ratsol to $G(X)$. Let $x_0, \ldots, x_m$ be the set of all non-zero rational numbers such that $G(q^{x_i}) = 0, 0 \leq i \leq m$. We now apply algorithm wc to find $c_0, \ldots, c_d$ from the set of $\{x_0, \ldots, x_m\}$ such that $\deg_q w_i(q^n, q^k) \neq 0$. Set $\delta_i = \deg_q w_i(q^n, q^k)$.

To check whether there exists a $qZ$-pair for $F(q^n, q^k)$, it is sufficient to check if the relation

$$\delta_0 + \cdots + \delta_d = \deg_q w(q^n, q^k)$$

is satisfied. If it is not, we conclude that $qZ$ is not applicable to $F(q^n, q^k)$, and subsequently, to $T(q^n, q^k)$. Otherwise, it follows from Lemma 6 that

$$w_i(q^n, q^k) = (q^k - \gamma_i q^{c_i}) \cdots (q^k - \gamma_i q^{c_i})$$

(26)

We can then equate the coefficients of like powers of $q^k$ in (26) and solve the corresponding system of equations to obtain the values of $\gamma_i, \ldots, \gamma_i$ (note that the $\gamma_i \in \overline{\mathbb{C}(q)}, 1 \leq j \leq s, 0 \leq i \leq d$ are guaranteed to exist).

At this point, we complete step 1. As for the second step, we apply the partial fraction decomposition to $F(q^n, q^k)$. Note that the denominator of $F(q^n, q^k)$ is already in the desired factored form, and no factorization needs to be done. This gives a representation of $F(q^n, q^k)$ in the form (17). We then apply algorithm $qZpairF$ to construct the minimal $qZ$-pair $(L_1, G_1)$ for $F(q^n, q^k)$.

We now describe the algorithm $qZpairDirect$ which constructs the minimal $qZ$-pair for $T(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k]$ provided that it exists.

**algorithm qZpairDirect:**

**input:** a rational function $T(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$;

**output:** the minimal $qZ$-pair $(L, G)$ for $T$, if it exists;

the message “There does not exist a $qZ$-pair”, otherwise;

apply an algorithm to solve the rational sum decomposition

problem w.r.t. $q^k$ to obtain $S(q^n, q^k), F(q^n, q^k)$ in (5);

if $F(q^n, q^k) = 0$ then

return $(1, S(q^n, q^k))$;

fi;
\( f(q^n, q^k) := \text{numerator}(F(q^n, q^k)); g(q^n, q^k) := \text{denominator}(F(q^n, q^k)); \)
\( v_1(q^n) := \text{content}_{q^n}(g(q^n, q^k)); \)
\[ \text{if } v_1(q^n) \neq q^n, \ a \in \mathbb{Z} \text{ then} \]
\[ \text{return } \text{"There does not exist a } q\text{-Z-pair";} \]
fi;
\[ w(q^n, q^k) := g(q^n, q^k)/v_1(q^n); \]
\[ v_2(q^n) := \text{content}_{q^n}(w(q^n, q^k)); \]
\[ w(q^n, q^k) := w(q^n, q^k)/v_2(q^n); \]
\[ \text{if } w(q^n, q^k) = 1 \text{ then} \]
\[ \text{rewrite } v_2(q^n) \text{ as } v_2(q^n) = (q^k - \gamma_1) \cdots (q^k - \gamma_s); \]
\[ \hat{g}(q^n, q^k) := v_1(q^n) v_2(q^n); \]
else
\[ \hat{w}(q^n, q^k) := w(q^n+1, q^{k+x}); \]
\[ \hat{S}(q^n, q^k) := \text{resultant}_{q^n}(w(q^n, q^k), \hat{w}(q^n, q^k)); \]
\[ \hat{G}(q^n) := \text{content}_{q^n}(\hat{S}(q^n, q^n)); \]
\[ \text{cands := } \text{ratsol}(\hat{G}(X)); \]
\[ \text{if cands is } \{ \} \text{ then} \]
\[ \text{return } \text{"There does not exist a } q\text{-Z-pair";} \]
fi;
let \( x_0, \ldots, x_m \) be the members of cands;
\[ c := \{ \}; \ s := 1; \]
for \( j = 0, 1, \ldots, m \) do
\[ \text{cand := } \text{wdeg}(w(q^n, q^k), x_j); \]
\[ \delta_s := \text{deg}_{q^n} \text{ cand}; \]
\[ \text{if } \delta_s > 0 \text{ then} \]
\[ c := c \cup \{ x_j \}; \]
\[ w_{x_j}(q^n, q^k) := \text{cand}; \]
\[ s := s + 1; \]
fi;
od;
\[ \text{if deg}_{q^n} w(q^n, q^k) \neq (\delta_1 + \cdots + \delta_{s-1}) \text{ then} \]
\[ \text{return } \text{"There does not exist a } q\text{-Z-pair";} \]
fi;
\[ \text{rewrite } v_2(q^n) \text{ as } v_2(q^n) = (q^k - \gamma_1) \cdots (q^k - \gamma_s); \]
\[ \hat{g}(q^n, q^k) := v_1(q^n) v_2(q^n); \]
let \( c_1, \ldots, c_d \) be the members of \( c; \)
for \( t = 1, 2, \ldots, d \) do
\[ \text{use the method of undetermined coefficients to rewrite } w_{c_t}(q^n, q^k) \text{ as} \]
\[ w_{c_t}(q^n, q^k) = (q^k - \gamma_t, q^{c_t+n}) \cdots (q^k - \gamma_t, q^{c_t+n}); \]
\[ \hat{g}(q^n, q^k) := \hat{g}(q^n, q^k) w_{c_t}(q^n, q^k); \]
od;
fi;
\[ F(q^n, q^k) := f(q^n, q^k)/\hat{g}(q^n, q^k); \]
assign to \( F(q^n, q^k) \) the partial fraction decomposition w.r.t. \( q^k \) of \( \hat{F}(q^n, q^k); \)
\( (L_F, G_F) := qZpair(F(q^n, q^k)); \)
\[ \text{return } (L_F, L_F S(q^n, q^k) + G_F); \]

**Example 2** Consider the rational function

\[ T(q^n, q^k) = \frac{q^{k+1} - q^n}{q^{k+1} + q^n + 1} - \frac{q^k - q^n}{q^k + q^n + 1} + \frac{1}{q^{2k} + (1 + q^n - q^{2n})q^k - q^{3n} - q^{2n}}. \]
Applying $ZpairDirect$ yields
\[ S(q^n, q^k) = \frac{q^k - q^n}{q^k + q^n + 1}, \quad F(q^n, q^k) = \frac{1}{q^{2k} + (1 + q^n - q^{2n})q^k - q^{3n} - q^{2n}}, \]
\[ v_1(q^n) = v_2(q^k) = 1, \quad w(q^n, q^k) = \frac{q^{2k} + (1 + q^n - q^{2n})q^k - q^{3n} - q^{2n}}, \]
and $G(q^e) = -q^e + q^2$. Applying ratsol to $G(X)$ results in $c = \{2\}$, and the corresponding $w_2(q^n, q^k)$ obtained from applying $w$ to $w(q^n, q^k)$ and $2$ is $w_2(q^n, q^k) = q^k - q^{2n}$. Hence, $\delta_1 = 1$. Since $\deg u_w(q^n, q^k) = 2 > 1 = \delta_1$, $qZ$ is not applicable to $T(q^n, q^k)$. (Note that the denominator of $F(q^n, q^k)$ can be written as $(q^k - q^{2n})((q^k + q^n + 1)$.)

**Example 3** Consider the rational function
\[ (27) \quad T(q^n, q^k) = \frac{1}{q^k - q^{2n}} + \frac{q + 1}{q^k - q^{3n}} + \frac{q^2 - 1}{q^k - q^{3n}q^{3n}}. \]
Applying $ZpairDirect$ yields
\[ S(q^n, q^k) = 0, \quad F(q^n, q^k) = T(q^n, q^k), \]
\[ v_1(q^n) = v_2(q^k) = 1, \quad w(q^n, q^k) = (q^k - q^{2n})(q^k - q^{3n})(q^k - q^{3n} - q^{3n}), \]
and
\[ \begin{align*}
G(q^e) &= q^{31}q^{5e} - (3q^{37} + 3q^{33} + 2q^{28} + q^{19})q^{4x} + (q^{39} + 2q^{34} + 2q^{30} + 2q^{25} + q^{11} + 2q^{16})q^{5x} - \\
&- (2q^{36} + q^{31} + 2q^{27} + 2q^{22} + 2q^{18} + q^{13})q^{2x} + (q^{33} + 2q^{24} + q^{20} + q^{15})q^{x} - q^{21}.
\end{align*} \]
Applying ratsol to $G(X)$ results in $c = \{-12, -3, 2, 6\}$, and the corresponding $w_2(q^n, q^k)$ obtained from $w$ are
\[ w_{-12} = 1, \quad w_{-3} = q^{2k} - (q^{3n-9} + q^{3n-9})q^k + q^{6n-9}, \quad w_2 = q^k - q^{2n}, \quad w_6 = 1. \]
Hence, $\delta_1 = 2$, $\delta_2 = 1$. Since $\deg u_w(q^n, q^k) = 3 = \delta_1 + \delta_2$, $qZ$ is applicable to $T(q^n, q^k)$. Note that $w_2(q^n, q^k)$ is already in the desired form. As for $w_{-3}(q^n, q^k)$, we have
\[ (28) \quad q^{2k} - (q^{3n-9} + q^{3n-9})q^k + q^{6n-9} = (q^k - q^{3n})(q^k - q^{3n}). \]
Equating the coefficients of like powers in (28) yields the system of equations
\[ \{ \gamma_1 + \gamma_2 = q^{-9} + 1, \quad \gamma_1 \gamma_2 = q^{-9} \} \]
whose solution is $\{ \gamma_1 = 1, \quad \gamma_2 = q^{-9} \}$. Therefore,
\[ w_{-3}(q^n, q^k) = (q^k - q^{-3n})(q^k - q^{-9}q^{-3n}), \]
and the denominator $g(q^n, q^k)$ of $F(q^n, q^k)$ can be written as
\[ g(q^n, q^k) = (q^k - q^{2n})(q^k - q^{3n})(q^k - q^{-9}q^{-3n}). \]
Applying the partial fraction decomposition w.r.t. $q^k$ to $F(q^n, q^k)$ yields (27). We now start step 2 of $ZpairDirect$. The algorithm $qZpairF$ allows us to decompose $F$ into
\[ F = V_1F_1 + V_2F_2, \quad V_1 = 1, \quad F_1 = \frac{1}{q^{k} - q^{2n}}, \quad V_2 = (q^2 - 1) Q_n^2 + (q + 1), \quad F_2 = \frac{1}{q^{k} - q^{3n}}. \]
Applying $qZpairVF$ to $V_1F_1$ yields the minimal $qZ$-pair $(L_1, G_1)$ for $V_1F_1 = F_1$ where
\[ (L_1, G_1) = \left( q^2 Q_n - 1, -\frac{1}{q^{k-1} - q^{2n}} - \frac{1}{q^{k-2} - q^{2n}} \right). \]
The minimal $qZ$-pair $(L_2, G_2)$ for $V_2F_2$ is computed in Example 1. Therefore, the minimal $qZ$-pair $(L, G)$ for $F(q^n, q^k)$, and also for $T(q^n, q^k)$ is
\[ (L, G) = (\text{lcm}(L_1, L_2), L'_1G_1 + L'_2G_2) \text{ where} \]
\[ L = \text{lcm}(L_1, L_2) = \frac{q^4}{q^5 - 1} Q_n^2 - \frac{q^2 (1 + q^5)}{q^5 - 1} Q_n + \frac{q^5}{q^5 - 1}. \]
\[ L_1 = \frac{q^7 (q + 1) (q^{10} - q^9 + 1)}{q^5 - 1} Q_n - \frac{(q + 1) (q^{10} - q^9 + 1) q^5}{q^5 - 1}, \text{ and } L'_2 = \frac{q^2}{q^5 - 1} Q_n - \frac{q^5}{q^5 - 1}. \]

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E-mail address: hqle@csog.math.uwaterloo.ca, abramov@ccas.ru, kogodden@csog.math.uwaterloo.ca

SYMBOLIC COMPUTATION GROUP, DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF WATERLOO, WATERLOO, N2L 3G1, CANADA; COMPUTER CENTER OF THE RUSSIAN ACADEMY OF SCIENCE, VAVILOVA 40, MOSCOW 117967, RUSSIA; SYMBOLIC COMPUTATION GROUP, DEPARTMENT OF COMPUTER SCIENCE, UNIVERSITY OF WATERLOO, WATERLOO, N2L 3G1, CANADA