HypergeometricSum: A Maple Package for Finding Closed Forms of Indefinite and Definite Sums of Hypergeometric Type

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Abstract

We describe the Maple module `HypergeometricSum` which provides various tools for finding closed forms of indefinite and definite sums of hypergeometric type, and for certifying a large class of combinatorial identities. The document is intended both as an elementary introduction to the subject and as a reference manual for the module.

1 Introduction

This document is about some standard algorithms which provide an automatic process for finding closed forms of indefinite and definite sums of hypergeometric type which occur in combinatorial mathematics. Even though there exist various methods for finding closed forms of sums, such as those where the summands involve binomial coefficients (see Chapter 5,[17]), these methods are often more like tricks than techniques. The algorithms to be discussed allow us to discover the answers in a systematic way. These algorithms also provide tools for proving and certifying identities that are known or conjectured.

We also describe various recent results relating to these algorithms. The main part of the document, however, is about describing the Maple module `HypergeometricSum`. Our focus is not only on the implementation of these algorithms, but also on their applications.

This document is organized in the following manner. In section 2, we give a brief overview on hypergeometric terms. In section 3, we describe various normal and canonical forms of rational functions and of hypergeometric terms. These forms are widely used in many applications. In section 4, we give description of algorithms for indefinite sums. They include Gosper's algorithm to solve the problem of indefinite hypergeometric summation, an algorithm to solve the decomposition problem of indefinite sums of hypergeometric terms, and an algorithm to solve the accurate integration problem. Section 5 is devoted to Zeilberger's

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algorithm which provides a useful tool for finding closed forms of definite hypergeometric sums, to the applicability of the algorithm, and to a direct algorithm to construct Zeilberger's recurrences for the case when the input function is a rational function. Section 6 shows various applications of these algorithms. In Section 7, we describe the module `HypergeometricSum`. The module is a Maple implementation of the algorithms and applications that we describe in sections 3, 4, 5, and 6. It consists of three main components. The first one includes the construction of the Polynomial Normal Forms (PNF), the Rational Canonical Forms (RNF) of rational functions, of the multiplicative decompositions of hypergeometric terms. The second one includes implementation of the algorithms as described in sections 4 and 5. The third component relates to applications. They include functions to compute closed forms of indefinite and definite sums of hypergeometric type, and the WZ method for certifying combinatorial identities.

2 Hypergeometric Terms

Let K be a field of characteristic zero, $n_1, \ldots, n_d$ integer-valued variables, and E_i the corresponding shift operators, acting on functions of $n_1, \ldots, n_d$ by $E_i f(n_1, \ldots, n_i, \ldots, n_d) = f(n_1, \ldots, n_i + 1, \ldots, n_d)$. A K-valued function $T(n_1, \ldots, n_d)$ is a *hypergeometric term* if there exist rational functions $R_i \in K(n_1, \ldots, n_d)$ such that $E_i T = R_i T$, for $i = 1, \ldots, d$. The rational function R_i is called the *certificate* of the hypergeometric term T w.r.t. the variable n_i.

The following gives the definition of proper hypergeometric terms in two variables. This concept is needed in subsequent sections.

Definition 1 *[27, 32] A function $T(n, k)$ is said to be a* proper hypergeometric term *if it can be written in the form*

$$T(n,k) = P(n,k) \frac{\prod_{i=1}^{l} (\alpha_i n + \beta_i k + \gamma_i)!}{\prod_{i=1}^{m} (\alpha'_i n + \beta'_i k + \gamma'_i)!} u^n v^k,$$

where $P(n,k) \in \mathbb{C}[n,k]$, $\alpha_i, \beta_i, \alpha'_i, \beta'_i \in \mathbb{Z}$, and $l, m \in \mathbb{N}$, $\gamma_i, \gamma'_i, u, v \in \mathbb{C}$.

3 Normal Forms

In this section we describe the polynomial normal forms, and rational normal forms of rational functions. These normal forms are then used as the basis for various normal forms (multiplicative, additive) of hypergeometric terms in one variable. They are also used in the construction of a canonical representation of hypergeometric terms in two variables. The diagram in Figure 1 shows various normal and canonical forms of rational functions and hypergeometric terms.

3.1 Normal Forms of Rational Functions

Multiplicative normal forms of rational functions which exhibit the shift structure of the factors are useful in the design of algorithms for summation and solution of difference equations in closed form. The *Polynomial Normal Form*, first introduced by R.W. Gosper, Jr. [16], is used in algorithms for hypergeometric summation [16], finding hypergeometric solutions of difference equations [26], and rational summation [29]. The *Rational Normal Forms*, introduced by S.A. Abramov and M. Petkovšek [7], are used to construct the minimal representations of hypergeometric terms in one variable, to construct a canonical representation of hypergeometric terms in two variables [7], and to solve the decomposition problem of hypergeometric terms [6].

Definition 2 *(Polynomial Normal Form). Let $R \in K(x)$ be a nonzero rational function. If there exist $z \in K$ and monic polynomials $a, b, c \in K[x]$ such that*

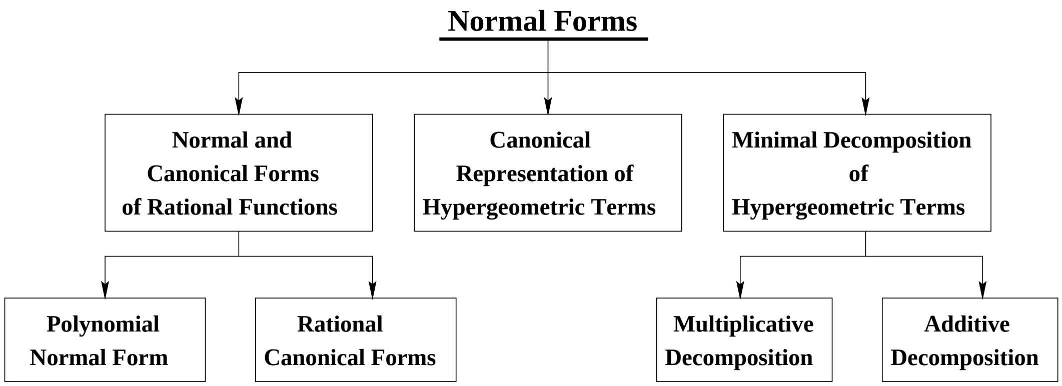

Figure 1: Normal and canonical forms.

(i) $R = z \cdot \frac{a}{b} \cdot \frac{Ec}{c}$,

(ii) $\gcd(a, E^k b) = 1$ *for all* $k \in \mathbb{N}$,

(iii) $\gcd(a, c) = 1$ *and* $\gcd(b, Ec) = 1$,

then (z, a, b, c) *is the* polynomial normal form (PNF) *of* R.

See [26, 27] for a description of how to construct the PNF. Note that the PNF of a rational function is unique.

Definition 3 *(Rational Normal Form). Let* $R \in K(x)$ *be a nonzero rational function. If there exist* $z \in K$ *and monic polynomials* $r, s, u, v \in K[x]$ *such that*

(i) $R = z \cdot \frac{r}{s} \cdot \frac{EF}{F}$ *where* $F = \frac{u}{v}$ *and* $\gcd(u, v) = 1$,

(ii) $\gcd(r, E^k s) = 1$ *for all* $k \in \mathbb{Z}$,

then (z, r, s, u, v) *is a* rational normal form (RNF) *of* R. *If, in addition,*

(iii) $\gcd(r, u \cdot Ev) = 1$ *and* $\gcd(s, Eu \cdot v) = 1$,

then (z, r, s, u, v) *is a* strict RNF *of* R.

Although a rational function can have several different RNF's, the degrees of the polynomials r and s are uniquely defined (Theorem 1, [7]). Among all possible RNF's for R, we can single out two canonical forms (not necessarily different). They are named the *first* and the *second rational canonical forms* (RCF_1 and RCF_2). The RCF_1 for R guarantees the minimality of the degree of the polynomial v, and the maximality of the degree of the polynomial u. When one considers the RCF_2 for R, the situation is reversed. Furthermore, if $\mathrm{RCF}_1(R) = \mathrm{RCF}_2(R)$, then there exists one unique RNF for R.

3.2 Normal Forms of Hypergeometric Terms

3.2.1 Minimal multiplicative decomposition of hypergeometric terms

Definition 4 *(Regular description of a hypergeometric term [8]). Let* $n_0 \in \mathbb{N}$ *and* $V, U \in K(n)$. *The triple* (V, U, n_0) regularly describes *a hypergeometric term* $T(n)$ *if for all integer* $n \geq n_0$, U, V *have neither a pole nor a zero, and* $T(n) = U(n) \prod_{k=n_0}^{n-1} V(k)$.

Let $T(n)$ be a hypergeometric term such that $T(n)$ is defined for all $n \geq n_0$, and the certificate $R(n)$ of T has neither a pole nor a zero when $n \geq n_0$. Then

$$T(n) = T(n_0) \prod_{k=n_0}^{n-1} R(k), \ n \geq n_0, \tag{1}$$

and $(R, T(n_0), n_0)$ is a regular description of $T(n)$. Construct an RNF (z, r, s, u, v) of R and write

$$T(n) = T(n_0) \prod_{k=n_0}^{n-1} \left(z \cdot \frac{r(k)}{s(k)} \cdot \frac{F(k+1)}{F(k)} \right) = \frac{T(n_0)}{z^{n_0} F(n_0)} z^n F(n) \prod_{k=n_0}^{n-1} \frac{r(k)}{s(k)} \tag{2}$$

where $F = u/v$. The following theorem guarantees the minimality of the degrees of the numerator and of the denominator of the rational factor in the product in (2).

Theorem 1 *(Theorem 4, [8]). Let a hypergeometric term $T(n)$ have a regular description (R, α, n_0), $\alpha = T(n_0)$, and (z, r, s, u, v) be an RNF of R. Write $F = u/v$ and $G = r/s$. Then*

$$(zG, \ \frac{\alpha}{F(n_0)} F, \ n_0) \tag{3}$$

is a regular description of $T(n)$. If (A, B, n_1) is another regular description of $T(n)$, $A = r'/s'$, $r', s' \in K[n]$, then $\deg r \leq \deg r'$, $\deg s \leq \deg s'$. Additionally, if one uses the RCF_1 or the RCF_2 as (z, r, s, u, v) then the denominator (resp., the numerator) of $\alpha F/F(n_0)$ is of minimal possible degree.

A regular description (V, U, n_0) of $T(n)$ with the minimal possible degrees of the numerator and denominator of V is a *minimal regular description* of $T(n)$.

Suppose that a term $T(n)$ has a minimal regular description (V, U, n_0) and $z = \operatorname{lc} V$ (the *leading coefficient* of a rational function is the quotient of the leading coefficients of its numerator and denominator.) Write $V = zW$. Then we have for all $n \geq n_0$:

$$T(n) = z^{n-n_0} U(n) \prod_{k=n_0}^{n-1} W(k), \tag{4}$$

where W is a monic rational function with the numerator and denominator of minimal possible degrees. In this case (4) is a *minimal multiplicative decomposition* of $T(n)$.

The two minimal multiplicative decompositions of a hypergeometric term $T(n)$, which are constructed from the two rational canonical forms RCF_1 and RCF_2 of the certificate of T, are named the first and the second minimal multiplicative decompositions of T, respectively.

3.3 Canonical representations of hypergeometric terms

Using the concept of RNF, one can construct a canonical representation of hypergeometric terms in two variables. The following theorem summarizes the main result.

Theorem 2 *[7]. Every hypergeometric term in two variables is the quotient of a proper hypergeometric term by a polynomial.*

It is worth noting that this opens the way to a proof of a conjecture of Wilf and Zeilberger [32, p. 585] which states that a hypergeometric term is proper if and only if it is holonomic (see [9]).

4 Algorithms for Indefinite Sums

We describe in this section three algorithms for indefinite sums: Gosper's algorithm, and two algorithms to solve the decomposition problem for indefinite sums of hypergeometric terms, and the accurate integration problem.

4.1 Gosper's Algorithm

Gosper's algorithm [16] (we name it hereafter as $\mathcal{G}$) solves the *problem of indefinite hypergeometric summation*: given a hypergeometric term $T(n)$, $\mathcal{G}$ either constructs another hypergeometric term $T_1(n)$ such that

$$T(n) = T_1(n+1) - T_1(n), \tag{5}$$

provided that such a term exists, or proves that (5) has no hypergeometric solution. If it does, and if $T(n)$ and $T_1(n)$ are defined for all $n \geq n_0$, then we obtain from (5) the summation identity

$$\sum_{k=n_0}^{n-1} T(k) = T_1(n) - T_1(n_0).$$

A hypergeometric term $T(n)$ is summable if there exists a hypergeometric term $T_1(n)$ such that relation (5) is satisfied.

4.2 The Decomposition Problem for Indefinite Sums of Hypergeometric Terms

Let R be a rational function which has no poles at nonnegative arguments. The well-known algorithms to solve the decomposition problems for indefinite integrals [28, 20] and indefinite sums [1, 2, 29, 24] of rational functions construct the representations

$$\int_0^x R(t)\,dt = F(x) + \int_0^x H(t)\,dt, \quad \sum_{k=0}^{n-1} R(k) = S(n) + \sum_{k=0}^{n-1} T(k),$$

where F, H and S, T are rational functions such that H, T have denominators of minimal possible degrees.

If no hypergeometric term $T_1(n)$ satisfies (5), it is shown in [6] that one can construct *two* hypergeometric terms $T_1(n)$ and $T_2(n)$ such that

$$T(n) = T_1(n+1) - T_1(n) + T_2(n), \tag{6}$$

and the certificate ET_2/T_2 has an RNF

$$(z, r, s, u, v) \tag{7}$$

with v of minimal possible degree.

This formulation agrees with the decomposition problem for indefinite sums of rational functions [1, 2, 29] because if $T_2 \in K(n)$ then $r = s = 1$ and v is the denominator of T_2. In other words, the minimal additive decomposition of hypergeometric terms is a generalization of the additive decomposition of rational functions. It can be shown that the algorithm also covers $\mathcal{G}$, i.e., if $T(n)$ is summable, then the constructed hypergeometric term $T_2(n)$ in (6) will vanish. It follows from (6) that

$$\sum_{k=n_0}^{n-1} T(k) = T_1(n) - T_1(n_0) + \sum_{k=n_0}^{n-1} T_2(k). \tag{8}$$

4.3 Accurate Integration

The following is a description of the *accurate integration* problem in the context of *shift* algebra. See [3] for a discussion of the problem in the general Ore algebra setting.

For a given function $f(n)$ where the minimal annihilating operator $L \in K(n)[E_n]$ for $f(n)$ is given. This means $n = \text{ord } L$ is minimal with the property that $L(f) = 0$. Decide whether there exists a primitive g of f, i.e., $(E_n - 1)\, g = f$ such that the minimal annihilating operator $\tilde{L}$ for g has order n. If so, then construct all such g together with their minimal annihilating operators.

Note that the algorithm to solve the accurate integration problem generalizes Gosper's algorithm in the sense that it works for any order n instead of only $n = 1$ as in the case of Gosper's algorithm.

5 Algorithms for Definite Sums

We describe in this section Zeilberger's algorithm (which is named hereafter as $\mathcal{Z}$). The algorithm has many applications [32] one of which relates to finding closed forms of definite sums of hypergeometric type. We also discuss the applicability of $\mathcal{Z}$, and present some recent results for the case when the given hypergeometric term is a rational function.

5.1 Zeilberger's Algorithm

Given a *hypergeometric term* $T(n, k)$ of both n and k, i.e., the quotients $T(n+1, k)/T(n, k)$ and $T(n, k+1)/T(n, k)$ are rational functions of n and k, Zeilberger's algorithm [33, 34] tries to construct for $T(n, k)$ a *Z-pair* (L, G) which consists of a linear difference operator with coefficients which are polynomials in n over $\mathbb{C}$

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \tag{9}$$

and a hypergeometric term $G(n, k)$ such that

$$LT(n, k) = (E_k - 1)G(n, k). \tag{10}$$

The operator L, which we call a *telescoper,* is *k-free.* It is proven in [34] that if there exist Z-pairs for $T(n, k)$, then $\mathcal{Z}$ terminates with one of the Z-pairs and the telescoper L in the returned Z-pair is of minimal possible order. Note that L is unique up to a left-hand factor $P(n) \in \mathbb{C}[n]$, and we name it *the minimal telescoper.* We call a Z-pair (L, G) where L is the minimal telescoper the *minimal* Z-pair.

$\mathcal{Z}$ uses an *item-by-item examination* of the order ρ of the operator L in (10). It starts with the value of 0 for ρ and increases ρ until it is successful in finding a Z-pair (L, G) for T, provided that such a pair exists.

5.2 Applicability of Zeilberger's Algorithm

The following fundamental theorem provides a sufficient condition for the termination of $\mathcal{Z}$ on a hypergeometric term.

Theorem 3 *(Fundamental Theorem [17, 27, 32]). If $T(n, k)$ is a proper hypergeometric term, then a* Z-pair *for $T(n, k)$ exists.*

Consider the two hypergeometric terms

$$T_1(n, k) = (E_k - 1)\frac{1}{nk+1}, \; T_2(n, k) = \frac{1}{nk+1}.$$

It is shown in [5] that T_1 is not a proper term but $\mathcal{Z}$ terminates on T_1 and returns a Z-pair; while T_2 is not a proper term either, and $\mathcal{Z}$ never terminates. Therefore the set T of hypergeometric terms on which

$\mathcal{Z}$ terminates is a proper subset of the set of all hypergeometric terms, but a super-set of the set of proper terms. The complete explicit description of T, we reiterate, is unknown.

For the case when the hypergeometric term $T(n,k)$ is also a rational function, then the problem of establishing a necessary and sufficient condition for the termination of $\mathcal{Z}$ is solved and presented in [4, 5]. The result can be summarized as follows.

Definition 5 *(Integer-linear polynomial [7]). A polynomial $p(n,k) \in \mathbb{C}[n,k]$ is* integer-linear *if it has the form $an + bk + c$ where $a, b \in \mathbb{Z}$ and $c \in \mathbb{C}$.*

Theorem 4 *(Criterion for the existence of a Z-pair for a rational function [4].) Let $F(n,k)$ be a rational function of n and k. Apply to $F(n,k)$ an algorithm to solve the decomposition problem [2] w.r.t. k to construct two rational functions $S(n,k)$ and $R(n,k)$ such that*

$$F(n,k) = (E_k - 1)S(n,k) + R(n,k), \tag{11}$$

and the denominator $g(n,k)$ of $R(n,k)$ has the minimal possible degree w.r.t. k. Then a Z-pair for $F(n,k)$ exists iff each factor of $g(n,k)$ irreducible in $\mathbb{C}[n,k]$ is an integer-linear polynomial.

Note that the algorithm to determine the applicability of $\mathcal{Z}$ to a rational function does not require a complete factorization of the denominator $g(n,k)$ into integer-linear factors.

5.3 A Direct Algorithm to Construct Zeilberger's Recurrences for Rational Functions

Once a *Z*-pair for a rational function $F(n,k)$ is guaranteed to exist, one can use a *direct algorithm* to construct the minimal *Z*-pair (L, G) for $F(n,k)$ [22]. By direct algorithm, we mean an algorithm which computes the minimal *Z*-pair directly, without using an item-by-item examination. The algorithm is based on a special form of representation of $R(n,k)$ in (11), on a direct construction of the minimal telescopers for each member of this representation. The minimal *Z*-pair for $R(n,k)$, and subsequently for $F(n,k)$, can then be obtained using Least Common Left Multiple (LCLM) computation. This direct algorithm is in general much more efficient than the original $\mathcal{Z}$ (see Section 7, [22]).

5.4 Open Problems

Note that the results mentioned in the last two subsections only apply to rational functions (recall that the class of rational functions is a proper subset of the class of hypergeometric terms). It is natural to try to obtain similar results for the general case. We believe that with the recent work as presented in [6, 7], it is possible to establish a necessary and sufficient condition for the termination of Zeilberger's algorithm to non-rational hypergeometric terms. Another open problem is whether it is possible to compute a good lower bound for the order of the minimal telescoper L to avoid redundant computation trying to find a telescoper of lower order (than the order of the minimal telescoper).

6 Applications

The applications we describe in this section include methods for finding closed forms of indefinite and definite sums of hypergeometric type, and for certifying combinatorial identities.

6.1 Indefinite Hypergeometric Summation

As mentioned in subsection 4.1, $\mathcal{G}$ answers the following question: Given a hypergeometric term t_n, does there exist a hypergeometric term z_n satisfying the relation $z_{n+1} - z_n = t_n$? If the answer is positive, then the indefinite sum

$$s_n = \sum_{k=0}^{n-1} t_k \tag{12}$$

can be expressed as a hypergeometric term plus a constant, and the algorithm outputs such a term. On the other hand, if $\mathcal{G}$ returns a negative answer, then that *proves* that (5) has no hypergeometric solutions.

Example 1 *Consider the hypergeometric term*

$$t_k = \frac{k^4 4^k}{\binom{2k}{k}}.$$

Applying $\mathcal{G}$ results in the hypergeometric term

$$z_k = \frac{1}{693} \frac{(2k-1)\left(63k^4 - 140k^3 + 60k^2 + 26k - 6\right)4^k}{\binom{2k}{k}}.$$

Therefore,

$$s_n = \sum_{k=0}^{n-1} \frac{k^4 4^k}{\binom{2k}{k}} = \frac{1}{693} \frac{(2n-1)\left(63n^4 - 140n^3 + 60n^2 + 26n - 6\right)4^n}{\binom{2n}{n}} - \frac{2}{231}.$$

Note that there exists an extended $\mathcal{G}$ [27] which answers the following question: Given a linear combination of hypergeometric terms $\sum_{i=1}^{m_1} T_i(n)$, does there exist a linear combination of hypergeometric terms $\sum_{i=1}^{m_2} G_i(n)$ satisfying the relation

$$(E_n - 1)\sum_{i=1}^{m_2} G_i(n) = \sum_{i=1}^{m_1} T_i(n) \ ?$$

It follows from subsection (4.3) that the problem of indefinite summation can also be solved using *accurate integration.* Not only does the algorithm handle summands of hypergeometric type, it also handles a wider class of summands where the minimal annihilators for them can be constructed, e.g., d'Alembertian terms [12, 10]. Note that a hypergeometric term is also a d'Alembertian term, but not vice versa.

Example 2 *(Example 3, [3]). Consider the function*

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Applying the algorithm as described in [11] results in the minimal annihilator L for $F(n)^2$ where

$$L = E_n^3 - 2E_n^2 - 2E_n + 1.$$

We now apply the algorithm to solve the accurate integration problem [3] to L. This results in the pair of operators

$$(\tilde{L}, r) = \left(-\frac{1}{2}E_n^3 + E_n^2 + E_n - \frac{1}{2}, \frac{1}{2}E_n^2 - \frac{1}{2}E_n - \frac{3}{2}\right).$$

Therefore,

$$\sum_n F(n)^2 = r(F(n)^2) = \frac{1}{5}(-1)^n - \frac{1}{10}(1+\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^{2n} - \frac{1}{10}(1-\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^{n}.$$

Since $F(n)$ is not a hypergeometric term, Gosper's algorithm is not applicable. Also note that $F(n)$ is a formula to compute the n^{th} Fibonacci number.

6.2 Definite Hypergeometric Summation

The combination of $\mathcal{Z}$ and Petkovšek's algorithm [26] (which we name hereafter as $\mathcal{P}$) plays an important role in the study of *definite* sums. For a given hypergeometric term $T(n,k)$, we are interested in knowing if there exists a *closed form* for $\sum_k T(n,k)$. By closed form, we mean that the sum can be expressed as a linear combination of a fixed number of hypergeometric terms. First, the application of $\mathcal{Z}$ to $T(n,k)$ yields a linear recurrence operator $L \in \mathbb{C}[n, E_n]$ of the form (9) and a hypergeometric term $G(n,k)$ such that relation (10) holds. By summing both sides of (10) over k, we obtain in general an inhomogeneous linear recurrence equation with polynomial coefficients of the form

$$\sum_{i=0}^{\rho} a_i(n) f(n+i) = b(n), \;\; a_i(n) \in \mathbb{C}[n]. \tag{13}$$

As an example, let

$$f(n) = \sum_{k=rn+s}^{un+v} T(n,k), \;\; r,s,u,v \in \mathbb{Z}.$$

Then (13) becomes

$$\sum_{i=0}^{\rho} a_i(n) f(n+i) = G(n, un+v+1) - G(n, rn+s) + \sum_{i=0}^{\rho} a_i(n)\left(\sum_{k=rn+s+ri}^{rn+s-1} T(n+i,k) + \sum_{k=un+v+1}^{un+v+ui} T(n+i,k)\right). \tag{14}$$

$\mathcal{P}$ now comes into play (also see [19] for an efficient algorithm designed by M.v. Hoeij). If the recurrence (13) has a solution $f(n)$ which is a linear combination of a fixed number of hypergeometric terms in n, then $\mathcal{P}$ will find the solutions, otherwise it returns the message "No such solution exists." It is not surprising that closed forms of many sums with binomial coefficients as summands in [18] can be obtained by using $\mathcal{Z}$ and then $\mathcal{P}$.

Example 3 *Find a closed form of*

$$s(n) = \sum_{k=0}^{n} T(n,k) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}\binom{2k}{k}}{2^{2k}}.$$

Applying $\mathcal{Z}$ to the hypergeometric term $T(n,k)$ results in a Z-pair (L,G) such that

$$(L,G) = \left((2n+2)E_n - (2n+1),\, 2\,\frac{\left(\binom{n}{k} + 2\binom{n}{k}n - 2\binom{n+1}{k}n - 2\binom{n+1}{k}\right)\binom{2k}{k}(-1)^k k^2}{2^{2k}(n+k+2kn+1)}\right).$$

Summing both sides of (10) for k from 0 to n results in the homogeneous recurrence equation

$$2(n+1)f(n+1) - (2n+1)f(n) = 0, \tag{15}$$

A closed form of $f(n)$ can now be obtained by applying $\mathcal{P}$ to (15) with the initial condition $s(0) = 1$:

$$s(n) = \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k}\binom{2k}{k}}{2^{2k}} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+1/2)}{\Gamma(n+1)}.$$

Note that we can *enlarge* the domain of closed forms by including d'Alembertian terms (a d'Alembertian term can be described as nested indefinite sums of hypergeometric terms, or equivalently, as a term which is annihilated by a product of first-order difference operators (see [12, 10])).

Example 4 *Find a closed form of*

$$f(n) = \sum_{k=1}^{n-1} T(n,k) = \sum_{k=1}^{n-1} \frac{(-1)^k\, 2^{(2n-2k)} \binom{2n-k+1}{k}}{k+1}.$$

Applying $\mathcal{Z}$ to the hypergeometric term $T(n,k)$ results in the minimal Z*-pair (L, G) where*

$$L = (2n+5)\,E_n - (2n+3), \quad \textit{and}$$

$$G = -\frac{(-1)^k\,(2^n)^2\,\Gamma(2n-k)\,(6n+9-4k)\,(2n-k)\,(2n-k+1)\,(2n+2-k)}{(2^k)^2\,\Gamma(2n-2k)\,(2n-2k+1)\,(n-k)\,(2n+3-2k)\,(n-k+1)\,\Gamma(k)\,k}.$$

Applying formula (14) results in the inhomogeneous linear recurrence equation

$$(2n+5)\,f(n+1) - (2n+3)\,f(n) = 4\,(-1)^n\,n - 6\,2^{2n}\,n - 5\,2^{2n} + 8\,(-1)^n. \qquad (16)$$

Since the right-hand side is a d'Alembertian term, i.e., it is annihilated by the completely factored operator

$$\left(E_n + \frac{180\,n^3 + 1200\,n^2 + 2363\,n + 1240}{(6\,n+11)\,(30\,n^2+115\,n+103)}\right) \circ \left(E_n - 4\,\frac{6\,n+11}{6\,n+5}\right),$$

we first apply the algorithm as described in [12] to get a particular d'Alembertian solution of (16) which is

$$\frac{-2\,n4^n + 2\,(-1)^{1+n}\,n + 4^n + 3\,(-1)^{1+n}}{2\,n+3}.$$

Note that the general solution of the homogeneous recurrence equation $(2n+5)\,f(n+1) - (2n+3)\,f(n) = 0$ is

$$\frac{3}{2}\,\frac{f(0)\,\Gamma(n+3/2)}{\Gamma(n+5/2)}.$$

Given the initial condition $f(0) = -1/3$, a closed form of $f(n)$ is

$$f(n) = -\frac{2\,n\,4^n + 2\,(-1)^n\,n + 3\,(-1)^n - 4^n + 1}{2\,n+3}.$$

6.2.1 Verification of Combinatorial Identities

We describe two methods that use $\mathcal{G}$ and $\mathcal{Z}$ to verify combinatorial identities.

1. **Canonical Form**
   In order to find out whether two objects A and B are equal to each other. One way is to find their canonical forms and check if the two canonical forms are the same. Suppose we would like to know if two sums represent the same function, one way is to show that they satisfy the same recurrence equation and the same initial values. This can be attained by using $\mathcal{Z}$.

Example 5 *Let*

$$T_1 = \frac{(-1)^k \binom{n+1}{k+1}}{\binom{(ak+1)/a}{k}}, \; T_2 = \frac{1}{ak+1}.$$

We will now prove formula (1.44) in [18] which states that

$$f_n = \sum_{k=0}^{n} T_1 = \sum_{k=0}^{n} T_2. \tag{17}$$

Applying $\mathcal{Z}$ to T_1 and T_2 results in the same inhomogeneous recurrence equation

$$f(n+1) - f(n) = \frac{1}{an+a+1}.$$

The two sums also satisfy the initial value $f(0) = 1$. Hence, the validity of relation (17) follows.

2. **WZ Method**

WZ method [31] provides a very short way for certifying the truth of combinatorial identities. Suppose we want to prove an identity of the form $\sum_k F(n,k) = r(n)$. If $r(n)$ is non-zero, divide through by that right-hand side, and obtain

$$\sum_k \frac{F(n,k)}{r(n)} = 1.$$

In general, we can assume that the identity to be certified is of the form

$$f(n) = \sum_k F(n,k) = const. \tag{18}$$

It suffices to show that $f(n+1) - f(n) = 0$ for all n. Suppose there exists a function $G(n,k)$ such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k). \tag{19}$$

If $G(n,k)$ has finite, compact support, i.e., $\lim_{k\to\pm\infty} G(n,k) = 0$, then by summing (19) over all integers k, the right-hand side of (19) telescopes to 0, while the left-hand side becomes $f(n+1) - f(n)$. To construct the function $G(n,k)$ using WZ method, we simply need to call $\mathcal{G}$ w.r.t. k on the hypergeometric term $F(n+1,k) - F(n,k)$. If such a G exists, then the method succeeds in verifying (18); otherwise, it fails. Note that since $G(n,k)$ and $F(n,k)$ are *similar* (Proposition 5.6.1, [27]),

$$R(n,k) = \frac{G(n,k)}{F(n,k)}$$

is a rational function of n and k. R is called the WZ certificate of the identity (18).

Example 6 *To prove Saalschütz's ${}_3F_2$ identity in the form $\sum_k F(n,k) = 1$ where*

$$F(n,k) = \frac{(a+k-1)!\,(b+k-1)!\,n!\,(n+c-a-b-k-1)!\,(n+c-1)!}{k!\,(n-k)!\,(k+c-1)!\,(n+c-a-1)!\,(n+c-b-1)!},$$

we first construct

$$F(n+1,k) - F(n,k) = \frac{(a+k-1)!\,(b+k-1)!\,(n+1)!\,(n+c-a-b-k)!\,(n+c)!}{k!\,(n+1-k)!\,(k+c-1)!\,(n+c-a)!\,(n+c-b)!} - \frac{(a+k-1)!\,(b+k-1)!\,n!\,(n+c-a-b-k-1)!\,(n+c-1)!}{k!\,(n-k)!\,(k+c-1)!\,(n+c-a-1)!\,(n+c-b-1)!}. \tag{20}$$

The application of $\mathcal{G}$ w.r.t. k to (20) yields the WZ certificate

$$R(n,k) = -\frac{k\,(k-n+a+b-c)(k+c-1)}{(k-n-1)(n-a+c)(n-b+c)}.$$

Note that WZ Method also allows one to discover *new identities* such as companion and dual identities whenever it succeeds in finding a proof certificate for a known identity (see Chapter 7, [27]).

7 Implementation

We have developed various tools for finding closed forms of indefinite and definite sums of hypergeometric type, and for certifying the truth of combinatorial identities in the computer algebra system Maple [23]. They are grouped together in the module `HypergeometricSum`.

```
> eval(HypergeometricSum);
```

module HypergeometricSum ()
export IsHypergeometricTerm, AreSimilar, PolynomialNormalForm, RationalCanonicalForm,
MultiplicativeDecomposition, SumDecomposition, Gosper, ExtendedGosper,
AccurateIntegration, WZMethod, Zeilberger, IsZApplicable, ZpairDirect,
ZeilbergerRecurrence, IndefinitSum, DefiniteSum;
option package;
description "Tools for finding closed forms of indefinite and definite sums of hypergeometric type";
end module

The exported local variables indicate the functions that are available. They are

- `IsHypergeometricTerm`: check if a given expression is a hypergeometric term of one particular variable;
- `AreSimilar`: check if two given hypergeometric terms are similar;
- `PolynomialNormalForm`: construct the polynomial normal form of a given rational function;
- `RationalCanonicalForm`: construct the first and the second rational canonical forms of a rational function;
- `MultiplicativeDecomposition`: construct the first and the second minimal multiplicative decompositions of a hypergeometric term;
- `SumDecomposition:` solve the sum decomposition problem for hypergeometric terms;
- `Gosper`: implement Gosper's algorithm;
- `ExtendedGosper`: implement the extended Gosper's algorithm;
- `AccurateIntegration`: implement the algorithm to solve the accurate integration problem;
- `WZMethod`: implement the WZ method;
- `Zeilberger`: implement Zeilberger's algorithm;
- `IsZApplicable`: determine the applicability of Zeilberger's algorithm to rational functions;
- `ZpairDirect:` implement a direct algorithm to construct the minimal Z-pairs for rational functions;
- `ZeilbergerRecurrence:` construct the Zeilberger's recurrences for definite sums of hypergeometric terms;
- `IndefiniteSum:` find closed forms of indefinite sums of hypergeometric terms and d'Alembertian terms;

- `DefiniteSum`: find closed forms of definite sums of hypergeometric terms.

IsHypergeometricTerm(T, n) checks if the input T is a hypergeometric term of n. AreSimilar(T_1, T_2, n) check if the two hypergeometric terms of n T_1 and T_2 are similar (two hypergeometric terms are similar if their ratio is a rational function of n. Theorem 5.6.2 [27] shows how to decide the rationality of a hypergeometric term given its consecutive-term ratio). We now describe, for the remainder of this section, the general use of the module and the specifications of the main functions.

7.1 Introduction to the HypergeometricSum module

Calling Sequence
function($args$)
HypergeometricSum[function]($args$)

Description

- The HypergeometricSum module provides various tools for finding closed forms of indefinite and definite sums of hypergeometric type. It can also be used for proving and certifying combinatorial identities. The module consists of three main components:
  1. normal forms of rational functions and of hypergeometric terms,
  2. algorithms for indefinite and definite sums of hypergeometric type,
  3. applications.
- To use a HypergeometricSum function, either use one of the long forms HypergeometricSum[function] and HypergeometricSum:-function or define a short form for that function using the command with(HypergeometricSum, function), or define a short form for all the functions using the command with(HypergeometricSum).

Examples

```
> with(HypergeometricSum):
```

1. Find a closed form of $\sum_{k=0}^{n} \binom{2\,n}{2\,k}^2$ (this example is provided by R.W. Gosper, Jr.).

```
> T := binomial(2*n,2*k)^2;
```

$$T := \binom{2\,n}{2\,k}^2$$

```
> s := DefiniteSum(T,n,k,0..n);
```

$$s := \frac{1}{2}\,\frac{4^n\left(\sqrt{\pi}\,\Gamma\left(2\,n+1/2\right)+(-1)^n\,\Gamma\left(n+1/2\right)^2\right)}{\sqrt{\pi}\,\Gamma\left(n+1/2\right)\Gamma\left(n+1\right)}$$

2. Construct the Apéry's recurrence:

```
> T := binomial(n,k)^2 * binomial(n+k,k)^2;
```

$$T := \binom{n}{k}^2\binom{n+k}{k}^2$$

```
> lre := ZeilbergerRecurrence(T,n,k,a,0..n);
```

$$lre := (n^3 + 6\,n^2 + 12\,n + 8)\,a(n+2) - (34\,n^3 + 153\,n^2 + 231\,n + 117)\,a(n+1) + (n^3 + 3\,n^2 + 3\,n + 1) = 0$$

Replace n by $n-1$ in lre :

```
> collect(subs(n=n-1,lre),[a(n+1),a(n),a(n-1)],'factor');
```

$$(n+1)^3\,a(n+1) - (2\,n+1)(17\,n^2 + 17\,n + 5)\,a(n) + n^3 a(n-1) = 0$$

Notice that the above recurrence equation is required in the proof of the irrationality of $\zeta(3)$ [13].

See Also LREtools, rsolve

7.2 HypergeometricSum[PolynomialNormalForm] – construct the polynomial normal form of a rational function

Calling Sequence

PolynomialNormalForm(F, n)

Parameters

F - a rational function of n

n - a variable

Description

- Let F be a rational function of n over a field K of characteristic 0. PolynomialNormalForm(F, n) constructs the polynomial normal form for F.
- The output is a sequence of 4 elements (z, a, b, c) where $z \in K$, and a, b, c are monic polynomials over K such that
$$F(n) = z\,\frac{a(n)}{b(n)}\,\frac{c(n+1)}{c(n)},$$
and all three properties (i), (ii), and (iii) in Definition 2 hold.

Reference

- R. W. Gosper, Jr., Decision procedure for indefinite hypergeometric summation. *Proc. Natl. Acad. Sci. USA* **75**, 1977, 40–42.
- M. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, *J. Symb. Comput.* **14**, 1992, 243–264.

Examples

```
> with(HypergeometricSum):
> F := 3/2*n*(n+2)*(3*n+2)*(3*n+4)/((n-1)*(2*n+9)*(n+4)^2);
```

$$\frac{3}{2}\,\frac{n(n+2)(3n+2)(3n+4)}{(n-1)(2n+9)(n+4)^2}$$

```
> (z,a,b,c) := PolynomialNormalForm(F,n);
```

$$z, a, b, c := \frac{27}{4}, (n+\frac{2}{3})(n+\frac{4}{3})(n+2), (n+\frac{9}{2})(n+4)^2, n-1$$

See Also Gosper, RationalCanonicalForm

7.3 HypergeometricSum[RationalCanonicalForm] – construct the first and the second rational canonical forms of a rational function

Calling Sequence

RationalCanonicalForm[1](F, n)
RationalCanonicalForm[2](F, n)

Parameters

F - a rational function of n
n - a variable

Description

- Let F be a rational function of n over a field K of characteristic 0. RationalCanonicalForm[1](F, n), RationalCanonicalForm[2](F, n) construct the first and the second rational canonical forms for F, respectively.
- The output is a sequence of 5 elements (z, r, s, u, v) where $z \in K$, and r, s, u, v are monic polynomials over K such that
$$F(n) = z\,\frac{r(n)}{s(n)}\,\frac{E_n(u(n)/v(n))}{(u(n)/v(n))},$$
and all three properties (i), (ii), and (iii) in Definition 3 hold.

Reference

- S.A. Abramov, M. Petkovšek, Canonical representations of hypergeometric terms. *Proc. FPSAC'2001*, 2001, 1–10.

Examples

```
> with(HypergeometricSum):
> F := 3/2*n*(n+2)*(3*n+2)*(3*n+4)/((n-1)*(2*n+9)*(n+4)^2);
```

$$F := \frac{3}{2}\,\frac{n(n+2)(3n+2)(3n+4)}{(n-1)(2n+9)(n+4)^2}$$

```
> (z1,r1,s1,u1,v1) := RationalCanonicalForm[1](F,n);
```

$$z1, r1, s1, u1, v1 := \frac{27}{4}, (n+\frac{2}{3})(n+\frac{4}{3}), (n+4)(n+\frac{9}{2}), n-1, (n+2)(n+3)$$

```
> (z2,r2,s2,u2,v2) := RationalCanonicalForm[2](F,n);
```

$$z2, r2, s2, u2, v2 := \frac{27}{4}, (n+\frac{2}{3})(n+\frac{4}{3}), (n-1)(n+\frac{9}{2}), 1, n\,(n+1)(n+2)^2(n+3)^2$$

See Also PolynomialNormalForm, SumDecomposition, MultiplicativeDecomposition

7.4 HypergeometricSum[MultiplicativeDecomposition] – construct the first and the second minimal multiplicative decompositions of a hypergeometric term

Calling Sequence

MultiplicativeDecomposition[1](H, n, k)
MultiplicativeDecomposition[2](H, n, k)

Parameters

H - a hypergeometric term of n
n - a variable
k - a name

Description

- Let H be a hypergeometric term of n. MultiplicativeDecomposition[1](H, n, k), MultiplicativeDecomposition[2](H, n, k) construct the first and the second minimal multiplicative decompositions for H, respectively. See subsection 3.2.1 for a description of how to construct the minimal multiplicative decompositions.

Reference

- S.A. Abramov, M. Petkovšek, Canonical representations of hypergeometric terms. *Proc. FPSAC'2001*, 2001, 1–10.

Examples

```
> with(HypergeometricSum):
> F := (n^2+5)*(n-3)!/(n-4)!/(n+7)!/(n+6)!;
```

$$F := \frac{(n^2+5)(n-3)!}{(n-4)!(n+6)!(n+7)!}$$

```
> MultiplicativeDecomposition[1](F,n,k);
```

$$\frac{1}{144850083840000}(n-3)(n^2+5)\prod_{k=4}^{n-1}\frac{1}{(k+7)(k+8)}$$

```
> MultiplicativeDecomposition[2](F,n,k);
```

$$\frac{1}{39916800}\frac{(n^2+5)}{(n+6)(n+5)(n+4)(n+3)(n+2)(n+1)n(n-1)(n-2)}\prod_{k=4}^{n-1}\frac{1}{(k+8)(k-3)}$$

See Also RationalCanonicalForm, SumDecomposition

7.5 HypergeometricSum[SumDecomposition] – construct the minimal additive decomposition of a hypergeometric term

Calling Sequence

SumDecomposition($T, n, k, newT$)

Parameters

T - a hypergeometric term of n
n - a variable
k - a name
$newT$ - (optional) a name

Description

- For a given hypergeometric term T of n, SumDecomposition constructs two hypergeometric terms T_1 and T_2 such that $T(n) = T_1(n+1) - T_1(n) + T_2(n)$ and the certificate $E\,T_2/T_2$ has a rational normal form (z, r, s, u, v) with v of minimal degree.
- The output from SumDecomposition is a list of two elements $[T_1, T_2]$. Both are written in the form (2). If the fourth optional argument $newT$ which is an unassigned name is given, $newT$ will be assigned to an equivalence of T also written in the form (2).

Reference

- S.A. Abramov, M. Petkovšek, Minimal Decomposition of Indefinite Hypergeometric Sums, *Proc. ISSAC'2001*, 2001, 7–14.
- S.A. Abramov, Indefinite sums of rational functions, *Proc. ISSAC'95*, 303–308.

Examples

```
> T := (n^2-2*n-1)*2^n/((n+1)*n^2*(n+3)!);
```

$$T := \frac{n^2 - 2\,n - 1}{(n+1)\,n^2}\,\frac{2^n}{(n+3)!}$$

```
> SumDecomposition(T,n,k,'newT');
```

$$\left[\frac{1}{12}\,\frac{n+1}{n^2}\prod_{k=1}^{n-1}\frac{2}{k+4},\ \frac{1}{10}\,\frac{n^2+4\,n+2}{3\,n^2+6\,n+3}\prod_{k=1}^{n-1}\frac{2}{k+5}\right]$$

```
> newT;
```

$$\frac{1}{12}\,\frac{n^2 - 2\,n - 1}{(n+1)\,n^2}\prod_{k=1}^{n-1}\frac{2}{k+4}$$

```
> T := n^3*2^n;
```

$$T := n^3 2^n$$

```
> SumDecomposition(T,n,k);
```

$$\left[\frac{1}{2}\left(2\,n^3 - 12\,n^2 + 36\,n - 52\right)2^n,\ 0\right]$$

The above result shows that the input hypergeometric term T is summable.

See Also Gosper, RationalCanonicalForm, MultiplicativeDecomposition

7.6 HypergeometricSum[Gosper] – indefinite hypergeometric summation

Calling Sequence
Gosper(T, n)

Parameters
T - a hypergeometric term of n
n - a variable

Description

- The function Gosper(T, n) solves the problem of indefinite hypergeometric summation, i.e., for the input hypergeometric term T, it constructs another hypergeometric term G such that $T(n) = G(n+1) - G(n)$, provided that such a term exists. Otherwise, the function returns the error message "no solution found".

Reference

- R.W. Gosper, Jr., Decision procedure for indefinite hypergeometric summation, *Proc. Natl. Acad. Sci. USA*, **75**, 1977, 40–42.

Examples

```
> with(HypergeometricSum):
> T := 4^n*n^4/binomial(2*n,n);
```

$$T := \frac{4^n n^4}{\binom{2n}{n}}$$

```
> Gosper(T,n);
```

$$\frac{1}{693} \frac{(2n-1)\left(63n^4 - 140n^3 + 60n^2 + 26n - 6\right) 4^n}{\binom{2n}{n}}$$

See Also ExtendedGosper, PolynomialNormalForm, AccurateIntegration, Zeilberger, SumDecomposition

7.7 HypergeometricSum[ExtendedGosper] – extended Gosper's algorithm

Calling Sequence
ExtendedGosper(T, n)

Parameters
T - a list/set of hypergeometric terms of n
n - a variable

Description

- For the given list/set

$$T = \{t_1(n), \ldots, t_p(n)\}$$

where the $t_i(n)$'s are hypergeometric terms of n, the function ExtendedGosper(T, n) returns a list/set

$$S = \{s_1(n), \ldots, s_q(n)\}$$

of hypergeometric terms $s_i(n)$ such that

$$(E_n - 1) \sum_{i=1}^{q} s_i(n) = \sum_{n=1}^{p} t_i(n)$$

if each of the hypergeometric term $s_i(n)$ exists; otherwise, ExtendedGosper returns the error message "no solution found".

Reference

- M. Petkovšek, H. Wilf, D. Zeilberger, *A=B,* Wellesley, Massachusetts: A. K. Peters Ltd., 1996.

Examples

```
> with(HypergeometricSum):
> T := [n^2*4^n/(n+1)/(n+2), 2^(2*n-1)/n/(2*n+1)/binomial(2*n,n),
> -n^2*4^n/(n+1)/(n+2)+(n+1)^2*4^(n+1)/(n+2)/(n+3)];
```

$$T := \left[\frac{n^2 4^n}{(n+1)(n+2)}, \frac{2^{2n-1}}{n(2n+1)\binom{2n}{n}}, -\frac{n^2 4^n}{(n+1)(n+2)} + \frac{(n+1)^2 4^{n+1}}{(n+2)(n+3)} \right]$$

```
> ExtendedGosper(T,n);
```

$$\left[-\frac{(2n+1)\, 2^{2n-1}}{n(2n+1)\binom{2n}{n}}, \frac{1}{3}\frac{(n-1)\, 4^{n+1}}{n+2} \right]$$

See Also Gosper

7.8 HypergeometricSum[AccurateIntegration] – indefinite summation using accurate integration

Calling Sequence
AccurateIntegration(T, n)

Parameters
T - a function of n
n - a variable

Description

- The function AccurateIntegration(T, n) solves the problem of indefinite summation using accurate integration as described in subsection 4.3.
- The output from AccurateIntegration is a function G such that $T(n) = G(n+1) - G(n)$ if the algorithm succeeds in constructing one; otherwise, it returns the error message "no solution found".

Reference

- S.A. Abramov, M.v. Hoeij, Integration of solutions of linear functional equations. *Integral Transformations and Special Functions,* 1999, Vol.8, No. 1-2, 1999, 3–12.

Examples

```
> with(HypergeometricSum):
> T := GAMMA(n+1)-GAMMA(n)-Psi(n);
```

$$T := \Gamma(n+1) - \Gamma(n) - \Psi(n)$$

```
> AccurateIntegration(T,n);
```

$$\frac{n^4 - n^3 - 6\,n^2 - 6\,n - 5}{n^2 + n + 3}(\Gamma(n+1) - \Gamma(n) - \Psi(n)) - \frac{n^5 - n^4 - 10\,n^3 - 9\,n^2 - 2}{n\,(n^2 + n + 3)}(\Gamma(n+2) - \Gamma(n+1)\Psi(n+1))$$
$$- \frac{n^4 - 4\,n^3 - n^2 + 2\,n - 2}{n\,(n^2 + n + 3)}(\Gamma(n+3) - \Gamma(n+2) - \Psi(n+2))$$

Try the Maple `sum` command:

```
> sum(T,n);
```

$$\sum_n (\Gamma(n+1) - \Gamma(n) - \Psi(n))$$

See Also Gosper, IndefiniteSum

7.9 HypergeometricSum[Zeilberger] – Zeilberger's algorithm HypergeometricSum[ZeilbergerRecurrence] – construct the Zeilberger's recurrence

Calling Sequence

Zeilberger(T, n, k, E_n)
ZeilbergerRecurrence$(T, n, k, f, l\,..\,u)$

Parameters

T - a hypergeometric term of n and k
n - a variable
k - a variable
E_n - a name denoting the shift operator w.r.t. n
f - the name of the recurrence function
$l\,..\,u$ - the range for k

Description

- For a given hypergeometric term $T(n, k)$ of n and k, the function Zeilberger(T, n, k, E_n) tries to construct for $T(n, k)$ a Z-pair (L, G) which consists of a linear difference operator with coefficients which are polynomials in n over the complex number field $\mathbb{C}$

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n + a_0(n),$$

and a hypergeometric term $G(n, k)$ such that

$$LT(n, k) = G(n, k + 1) - G(n, k). \tag{21}$$

- The algorithm uses an *item-by-item examination* of the order ρ of L. It starts with the value of 0 for ρ and increases ρ until it is successful in finding a Z-pair (L, G). Since a terminating condition that allows a hypergeometric term to have a Z-pair is unknown, a maximum value of the order of the difference operator L in the Z-pair (L, G) needs to be specified. This is done by assigning a value to the global variable `_MAXORDER` (the default value of `_MAXORDER` is 6). Additionally, by assigning values to the global variables `_MINORDER` and `_MAXORDER`, the algorithm will restrict to finding a Z-pair (L, G) for $T(n, k)$ where the guessed order of L is from `_MINORDER` to `_MAXORDER`.

- The output from Zeilberger is a list of two elements $[L, G]$ representing the computed Z-pair (L, G).

- ZeilbergerRecurrence$(T, n, k, f, l..u)$ is used to construct the Zeilberger's recurrence for the definite sum $f(n) = \sum_{k=l}^{u} T(n, k)$. The function first computes the Z-pair (L, G) for T and sums both sides of (21) over k in the specified range $l..u$. This yields in general an inhomogeneous recurrence equation of the form (13).

- The possible ranges of $l\,..\,u$ include $rn + s\,..\,un + v$, $rn + s\,..\,\infty$, $-\infty\,..\,un + v$, and $-\infty\,..\,\infty$ where $r,\ s,\ u,\ v\ \in \mathbb{N}$.

Reference

- D. Zeilberger, The method of creative telescoping, *J. Symb. Comput.*, **11**, 1991, 195–204.

- M. Petkovšek, H. Wilf, D. Zeilberger, *A=B,* Wellesley, Massachusetts: A. K. Peters Ltd., 1996.

Examples

```
> with(HypergeometricSum):
> T := (-1)^k*binomial(2*n-2*k,n-k)*binomial(2*k,k);
```

$$T := (-1)^k \binom{2\,n-2\,k}{n-k} \binom{2\,k}{k}$$

```
> Zpair := Zeilberger(T,n,k,En):
> L := Zpair[1];
```

$$L := (n+2)\,En^2 - (16\,n+16)$$

```
> G := Zpair[2];
```

$$G := (-1)^k \frac{k\,(2\,n-2\,k+1)\binom{2\,k}{k}\left(-16\binom{2\,n-2\,k}{n-k}n - 16\binom{2\,n-2\,k}{n-k} + \binom{2\,n+4-2\,k}{n+2-k}n + 2\binom{2\,n+4-2\,k}{n+2-k}\right)}{4\,k^2 - 4\,nk - 4\,k - n - 2}$$

```
> T := (2*n+3)/(n^2-1)*(n+8*k+1)!;
```

$$T := \frac{(2\,n+3)\,(n+8\,k+1)!}{n^2-1}$$

```
> Zeilberger(T,n,k,En);
```

Error, (in Zeilberger) No recurrence of order 6 was found

There does not exist a Z-pair (L, G) where $\operatorname{ord} L \leq 6$. Now try $\mathcal{Z}$ with order of L being from 7 to 9. Also use infolevel to print out some diagnostics:

```
> _MINORDER := 7: _MAXORDER := 9:
> infolevel[HypergeometricSum] := 3:
```

```
> Zpair := Zeilberger(T,n,k,En):
Zeilberger:   "applying Zeilberger's algorithm for order 7"
Zeilberger:   "applying Zeilberger's algorithm for order 8"
> L := Zpair[1];
```

$$L := \left(2\,n^3 + 35\,n^2 + 174\,n + 189\right) En^8 - \left(2\,n^3 + 19\,n^2 - 2\,n - 19\right)$$

Construct the Zeilberger's recurrence in Example 4.

```
> T := (-1)^k*binomial(n+1,k+1)/binomial((a*k+1)/a,k);
```

$$\frac{(-1)^k \binom{n+1}{k+1}}{\binom{(ak+1)/a}{k}}$$

```
> ZeilbergerRecurrence(T,n,k,f,0..n);
```

$$f(n+1) - f(n) = \frac{1}{an + a + 1}$$

See Also Gosper, IsZApplicable, ZpairDirect, LREtools[hypergeomsols], LinearOperators[dAlembertianSolver], DefiniteSum

7.10 HypergeometricSum[WZMethod] – Wilf-Zeilberger's algorithm

Calling Sequence

WZMethod($f, r, n, k, cert$)

Parameters

f - a function of n and k
r - a function of n
n - a variable
k - a variable
$cert$ - (optional) a name

Description

- The function WZMethod($f, r, n, k, cert$) is used to certify identities of the form $\sum_k f(n,k) = r(n)$.
- Let $F(n,k) = f(n,k)/r(n)$ if $r(n) \neq 0$ and $F(n,k) = f(n,k)$, otherwise. If the method succeeds in certifying the given identity, the output is a list of two elements $[F, G]$ representing the WZ-pair (F, G) such that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$. Otherwise, it returns the error message "WZ method fails".
- If the method is successful and if the fifth optional argument $cert$, which can be any name, is given, $cert$ will be assigned to the WZ certificate $R(n,k) = G(n,k)/F(n,k)$.
- It is assumed that for each integer $n \geq 0$, we have $\lim_{k\to\pm\infty} G(n,k) = 0$.

Reference

- H. Wilf, D. Zeilberger, Rational functions certify combinatorial identities, *J. Amer. Math. Soc.* **3**, 1990, 147–158.

Examples

```
> with(HypergeometricSum):
```

Proof of Gauss's ${}_2F_1$ identity [27]:

```
> f := (n+k)!*(b+k)!*(c-n-1)!*(c-b-1)!/(c+k)!/(n-1)!/(c-n-b-1)!/(k+1)!/(b- 1)!;
```

$$f := \frac{(n+k)!\ (b+k)!\ (c-n-1)!\ (c-b-1)!}{(c+k)!\ (n-1)!\ (c-n-b-1)!\ (k+1)!\ (b-1)!}$$

```
> r := 1:
> WZpair := WZMethod(f,r,n,k,'cert'):
> F := WZpair[1];
```

$$F := \frac{(n+k)!\ (b+k)!\ (c-n-1)!\ (c-b-1)!}{(c+k)!\ (n-1)!\ (c-n-b-1)!\ (k+1)!\ (b-1)!}$$

```
> G := WZpair[2];
```

$$G := -\frac{(k+1)\,(c-n-1)!\ (n+1)\,(n+1+k)!\ (c+k)\,(b+k+1)!\,b\,(c-n-b)\,(c-b)!}{(k+1)!\ (c-n-1)\,(n+1)!\ (n+1+k)\,(c+k)!\ (b+k+1)\,b!\ (c-n-b)!\ (c-b)}$$

The WZ certificate:

```
> cert;
```

$$\frac{(k+1)\,(c+k)}{n\ (n-c+1)}$$

See Also Gosper

7.11 HypergeometricSum[IsZApplicable] – Applicability of Zeilberger's algorithm to rational functions

Calling Sequence

IsZApplicable($F, n, k, E_n, Zpair$)

Parameters

F - a rational function of n and k
n - a variable
k - a variable
E_n - (optional) a name denoting the shift operator w.r.t. n
$Zpair$ - (optional) a name

Description

- Let F be a rational function of n and k. The function IsZApplicable(F, n, k) determines the applicability of Zeilberger's algorithm to F. It returns *false* if $\mathcal{Z}$ is not applicable to F; and *true* if it is. In this case, if the fourth and the fifth optional arguments (each of which can be any name) are given, the fifth argument $Zpair$ will be assigned to the computed $\mathcal{Z}$-pair (L, G) for F.

- If the input F is not a rational function of n and k, then IsZApplicable returns FAIL. In this case, if the optional arguments are given, then the function Zeilberger will be called, and the computed $\mathcal{Z}$-pair is assigned to $Zpair$ if the function succeeds in finding one.

Reference

- S.A. Abramov, H.Q. Le, Applicability of Zeilberger's algorithm to rational functions, *Proc. FPSAC'2000*, Springer–Verlag LNCS, 2000, 91–102.

Examples

```
> with(HypergeometricSum):
> F := 1/(n^2+3*n*k-2*n-10*k^2+11*k-3);
```

$$F := \frac{1}{n^2 + 3\,nk - 2\,n - 10\,k^2 + 11\,k - 3}$$

```
> IsZApplicable(F,n,k,En,'Zpair');
```

$$true$$

```
> L := Zpair[1];
```

$$L := (7\,n + 41)\,En^6 + (7\,n + 34)\,En^5 - (7\,n + 6)\,En - (7\,n - 1)$$

```
> F := 1/(n*k+1);
```

$$F := \frac{1}{nk + 1}$$

```
> IsZApplicable(F,n,k);
```

$$false$$

See Also Zeilberger, ZpairDirect

7.12 HypergeometricSum[ZpairDirect] – A direct algorithm to construct Zeilberger's recurrences for rational functions

Calling Sequence

ZpairDirect(F, n, k, E_n)

Parameters

F - a rational function of n and k
n - a variable
k - a variable
E_n - a name denoting the shift operator w.r.t. n

Description

- Let F be a rational function of n and k. The function ZpairDirect(F, n, k, E_n) computes a Z-pair (L, G) such that
$$LF(n, k) = G(n, k + 1) - G(n, k).$$
- The output from ZpairDirect is a list of two elements $[L, G]$ representing the computed Z-pair (L, G) provided that such a pair exists.

- The main distinction between ZpairDirect and Zeilberger is that Zeilberger uses an *item-by-item examination technique* for the order of the computed difference operator L. The function ZpairDirect, on the other hand, uses a direct algorithm to construct a Z-pair (L, G) for F. It first determines if there exists a Z-pair for F or not. If the answer is positive, then it computes a Z-pair directly; otherwise, it gives the conclusive error message "there does not exist a Z-pair for F" where F is the input rational function. ZpairDirect in general is much more efficient than Zeilberger .
- Note that ZpairDirect only works when the input F is a rational function.

Reference

- H.Q. Le, A Direct Algorithm to Construct Zeilberger's Recurrences for Rational Functions, *Proc. FPSAC'2001*, 2001, 303–312.

Examples

```
> with(HypergeometricSum):
> F := 1/(3*n+20*k+2)^3;
```

$$F := \frac{1}{(3\,n + 20\,k + 2)^2}$$

```
> ZpairDirect(F,n,k,En);
```

$$\left[{E_n}^{20} - 1, \frac{1}{(3\,n + 20\,k + 2)^3} + \frac{1}{(3\,n + 20\,k + 22)^3} + \frac{1}{(3\,n + 20\,k + 42)^3}\right]$$

```
> F := 1/(k^5+k^3*n+3*k^3-5*n*k^2-2*k^2-5*n^2-17*n-6);
```

$$F := \frac{1}{k^5 + k^3 n + 3\,k^3 - 5\,nk^2 - 2\,k^2 - 5\,n^2 - 17\,n - 6}$$

```
> ZpairDirect(F,n,k,En);
```

Error, (in ZpairDirect) there does not exist a Z-pair for
`1/(k^5+k^3*n+3*k^3-5*k^2*n-2*k^2-5*n^2-17*n-6)`

The function Zeilberger, on the other hand, wastes time trying to compute a Z-pair for F, and returns an inconclusive answer.

```
> Zeilberger(F,n,k,En);
```

Error, (in Zeilberger) No recurrence of order 6 was found

See Also Zeilberger, IsZApplicable

7.13 HypergeometricSum[IndefiniteSum] – indefinite sum

Calling Sequence

IndefiniteSum(T, n)

Parameters

T - a function of n

n - a variable

Description

- For a given function T, IndefiniteSum(T, n) computes a function G such that $T(n) = (E_n - 1)\, G(n)$, if it exists.
- The classes of functions T supported are rational functions, hypergeometric terms, and those for which the minimal annihilator in $K(n)[E_n]$ for T can be computed.

Reference

- S.A. Abramov, Indefinite sums of rational functions, *Proc. ISSAC'95*, 1995, 303–308.
- R.W. Gosper, Jr., Decision procedure for indefinite hypergeometric summation. *Proc. Natl. Acad. Sci. USA* **75**, 1977, 40–42.
- S.A. Abramov, M.v. Hoeij, Integration of solutions of linear functional equations. *Integral Transformations and Special Functions,* 1999, Vol.8, No. 1-2, 1999, 3–12.

Examples

```
> with(HypergeometricSum):
> T := 1/(n^2+sqrt(5)*n-1);
```

$$T := \frac{1}{n^2 + \sqrt{5}\, n - 1}$$

```
> Sum(T,n) = IndefiniteSum(T,n);
```

$$\sum_n \frac{1}{n^2 + \sqrt{5} n - 1} = -4/3\, \frac{7 - 3\sqrt{5} - 6\,n + 6\sqrt{5} n + 6\,n^2}{\left(2\,n + 1 + \sqrt{5}\right)\left(2\,n - 1 + \sqrt{5}\right)\left(2\,n - 3 + \sqrt{5}\right)}$$

```
> T := n^3*2^n;
```

$$T := n^3\, 2^n$$

```
> Sum(T,n) = IndefiniteSum(T,n);
```

$$\sum_n n^3\, 2^n = (n^3 - 6\,n^2 + 18\,n - 26)\, 2^n$$

See Also Gosper, AccurateIntegration, DefiniteSum, sum

7.14 HypergeometricSum[DefiniteSum] – definite sum of hypergeometric terms

Calling Sequence

DefiniteSum$(T, n, k, l\,..\,u)$

Parameters

T - a hypergeometric term of n and k
n - a variable
k - a variable
$l\,..\,u$ - the range for k

Description

- For a given hypergeometric term T of n and k. DefiniteSum$(T, n, k, l..u)$ computes a closed form for the definite sum $f(n) = \sum_{k=l}^{u} T$, if it exists.

- Let $r, s, u, v \in \mathbb{N}$. DefiniteSum tries to compute closed forms for four types of definite sums. They are

$$\sum_{k=rn+s}^{un+v} T(n,k), \quad \sum_{k=rn+s}^{\infty} T(n,k), \quad \sum_{k=-\infty}^{un+v} T(n,k), \quad \sum_{k=-\infty}^{\infty} T(n,k).$$

- A closed form is defined as one which can be represented either as a sum of hypergeometric terms or as a d'Alembertian term.

Reference

- D. Zeilberger, The method of creative telescoping, *J. Symb. Comput.* **11**, 1991, 195–204.
- M. Petkovšek, Hypergeometric solutions of linear recurrences with polynomial coefficients, *J. Symb. Comput.* **14**, 1992, 243–264.
- M. van Hoeij, Finite Singularities and Hypergeometric Solutions of Linear Recurrence Equations. *J. Pure Appl. Algebra*, **139**, 1999, 109-131.
- S.A. Abramov, E.V. Zima, D'Alembertian Solutions of Inhomogeneous Linear Equations (differential, difference, and some other). *Proc. ISSAC'96*, 1996, 232–240.

Examples

```
> with(HypergeometricSum):
> T := (-1)^k*binomial(2*n,k)*binomial(2*n-k,n)^2*(2*n+1)/(2*n+1+k);
```

$$T := (-1)^k \frac{(2n+1)\binom{2n}{k}\binom{2n-k}{n}^2}{2n+k+1}$$

```
> Sum(T,k=0..n) = DefiniteSum(T,n,k,0..n);
```

$$\sum_{k=0}^{n} (-1)^k \frac{(2n+1)\binom{2n}{k}\binom{2n-k}{n}^2}{2n+k+1} = \frac{64}{27} \frac{729^{-n}\, 1024^n\, \Gamma(2n+3/2)\Gamma(n+3/2)^3}{(2n+1)^2\, \Gamma(n+1)\Gamma(n+2/3)^2\Gamma(n+4/3)^2}$$

```
> T := (-1)^k/(k+1)/binomial(2*n,k);
```

$$T := \frac{(-1)^k}{(k+1)\binom{2n}{k}}$$

```
> Sum(T,k=0..2*n-1) = DefiniteSum(T,n,k,0..2*n-1);
```

$$\sum_{k=0}^{2n-1} \frac{(-1)^k}{(k+1)\binom{2n}{k}} = \left(\frac{3}{2}n + \frac{3}{4}\right)\Psi(1, n+1) - \left(\frac{1}{2}n + \frac{1}{4}\right)\Psi\left(1, n+\frac{1}{2}\right)$$

See Also Zeilberger, LREtools[hypergeomsols], LinearOperators[dAlembertianSolver], IndefiniteSum, sum

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