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# Characterization of Networks Supporting Multi-dimensional Linear Interval Routing Schemes

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## Abstract

An *Interval routing scheme (IRS)* is a well-known, space efficient routing strategy for routing messages in a distributed network. In this scheme, each node of the network is assigned an integer label and each link at each node is labeled with an interval. The interval assigned to a link  $l$  at a node  $v$  indicates the set of destination addresses which should be forwarded through  $l$  from  $v$ . A Multi-dimensional Interval Routing Scheme (MIRS) is a generalization of IRS in which each node is assigned a multi-dimensional label (which is a list of  $d$  integers for the  $d$ -dimensional case). The labels assigned to the links of the network are also multi-dimensional (a list of  $d$  1-dimensional intervals). The class of networks supporting linear IRS (in which the intervals are not cyclic) is already known for the 1-dimensional case [FG94]. In this paper, we generalize this result and completely characterize the class of networks supporting linear MIRS (or MLIRS) for a given number of dimensions  $d$ . We show that by increasing  $d$ , the class of networks supporting MLIRS is strictly expanded. We also give a characterization of the class of networks supporting strict MLIRS, which is a modified version of MLIRS. **Key words:** Computer networks, interval routing schemes, graph theory, multi-dimensional, characterization.

# 1 Introduction

The classical method used for routing messages in a network of computers is to store a *routing table* at each node of the network. A routing table has one entry for each destination address which indicates which of the adjacent links should be used to forward the message towards that destination.

Each routing table needs  $O(n)$  space in an  $n$ -node network, which is not efficient for large networks of computers. The methods to reduce the amount of space needed at each node have been intensively studied and there are many techniques to compress the size of routing tables [FJ88, FJ89, ABNLP90, TvL95]. The general idea is to group the destination addresses that correspond to the same outgoing link (at a node), and to encode the group so that it is easy to verify if a given destination address is in the group or not. A well-known solution is to use intervals as groups of destination addresses.

In an *Interval Routing Scheme (IRS)*, which was originally introduced by Santoro and Khatib [SK85], each node of the network is assigned an integer label taken from  $\{1, 2, \dots, n\}$  and each link of the network at each node is assigned an interval. Routing messages is completed in a distributed way. At each intermediate node  $v$ , if the label of the node equals the destination address,  $dest$ , the routing process ends. Otherwise, the message is forwarded through a link labeled by an interval  $I$ , such that  $dest \in I$ . Clearly, this method requires  $O(l)$  space at each node ( $l$  is the number of links at the node), which is an efficient memory allocation. The IRS has been implemented in the INMOS C104 router [INM91, WMT93].

Santoro and Khatib have proved that there is an IRS for any directed network such that any message will eventually reach its destination [SK85].

A *Linear Interval Routing Scheme (LIRS)* is an IRS in which the intervals are not cyclic. Also, a *Strict Interval Routing Scheme (SIRS)* is an IRS in which the interval assigned to a link  $e$  at a node  $v$  does not contain the label of  $v$ . A *Strict and Linear Interval Routing Scheme (SLIRS)* is an IRS which is both linear and strict. If we assign  $k$  intervals to each link of the network we will have a  $k$ -IRS (respectively,  $k$ -LIRS,  $k$ -SIRS, and  $k$ -SLIRS).

It has been proved that any network supports an SIRS [vLT87]. The class of networks which support LIRS and SLIRS have also been characterized by Fraigniaud and Gavoille which excludes a large class of networks [FG94]. They define a class of graphs called *lithium graphs* and show that a network supports an LIRS if and only if its underlying graph is not a lithium graph. They also show that a network supports an SLIRS if and only if its underlying graph is not a *weak lithium graph*.

A very interesting extension of IRS is a *Multi-dimensional Interval Routing Scheme (MIRS)* in which the labels assigned to the nodes are elements from  $\mathbf{N}^d$  (in the  $d$ -dimensional case) and each link is labeled with a  $d$ -tuple  $([a_1, b_1], [a_2, b_2], \dots, [a_d, b_d])$  of intervals,  $a_i, b_i \in \mathbf{N}$ ,

for  $1 \leq i \leq d$  [FGNT98]. A network is said *to be in  $\langle k, d \rangle$ -MIRS* or *support  $\langle k, d \rangle$ -MIRS* if there is a  $d$ -dimensional MIRS with  $k$  intervals in each link such that for any pair of nodes  $s$  and  $t$ , the message originating from  $s$  eventually reaches  $t$ . The classes  $\langle k, d \rangle$ -MLIRS and  $\langle k, d \rangle$ -MSLIRS are defined similarly.

The only known classes of networks which support different variations of MIRS are specific interconnection networks such as rings, grids, tori, hypercubes and chordal rings or for the 1-dimensional case. In this paper, we will investigate the problem of characterizing classes of networks supporting MIRS. We give a complete characterization of the class of networks supporting  $\langle 1, d \rangle$ -MLIRS and  $\langle 1, d \rangle$ -MSLIRS. We show that the class of networks supporting  $\langle 1, d \rangle$ -MLIRS ( $\langle 1, d \rangle$ -MSLIRS) is a strict subset of the class of networks supporting  $\langle 1, d+1 \rangle$ -MLIRS ( $\langle 1, d+1 \rangle$ -MSLIRS) and therefore, increasing the number of dimensions in an MLIRS (MSLIRS) increases the power of the routing scheme.

The rest of this paper is organized as follows: first, we will introduce some definitions and preliminaries in Section 2. In Section 3 we will characterize the class of graphs supporting  $\langle 1, d \rangle$ -MLIRS. Then, in Section 4, based on the arguments of the previous section, we will give a characterization for graphs supporting  $\langle 1, d \rangle$ -MSLIRS. Finally, in Section 5 we will conclude and give a list of open problems.

## 2 Preliminaries

Throughout this paper, a network is modeled by a graph  $G = (V, E)$ . The set of vertices of the graph represents nodes in the network and the set of edges  $E$  represents the links between the nodes in the network. We assume that the graph is simple and does not have any self-loops. For any edge  $(u, v) \in E$  we will use both  $(u, v)$  and  $(v, u)$  in order to assign two unidirectional labels to the edge, but the edge is assumed to be undirected. We refer the reader to standard texts for basic graph theoretic definitions [BM76, Wes96].

A graph  $G$  is said to be *connected* if for any pair of vertices,  $s$  and  $t$ , there is a path connecting  $s$  and  $t$ . In this paper, we always assume that the network is connected. If removing an edge  $e$  disconnects a graph  $G$ ,  $e$  is called a *bridge*. If a graph does not have a bridge, it is said to be *edge-biconnected*. *Edge-biconnected components of a graph  $G$*  are maximal subgraphs of  $G$  which are edge-biconnected.

**Observation 1.** *If  $G_1$  and  $G_2$  are two edge-biconnected components in a graph  $G$ , then any path  $P$  connecting  $G_1$  and  $G_2$  goes through a unique bridge, say  $b$ , connected to  $G_1$ .*

In the following section, we will give a characterization for the class of networks supporting a  $\langle 1, d \rangle$ -MLIRS.

### 3 Characterization of networks supporting $\langle 1, d \rangle$ -MLIRS

In this section we first give some examples of graphs which do not support  $\langle 1, d \rangle$ -MLIRS. Using the idea behind these examples, we introduce a class of graphs which do not support  $\langle 1, d \rangle$ -MLIRS. Finally, we show that for any graph that is not in this class, one can always construct a  $\langle 1, d \rangle$ -MLIRS.

Bakker et al. [BvLT91] have shown that the graph shown in Figure 1 (a) (known as the  $Y$  graph) does not have an LIRS (which is a  $\langle 1, 1 \rangle$ -MLIRS). Here, we prove a similar result in the  $d$ -dimensional case. First, we need to generalize the definition of a  $Y$  graph.

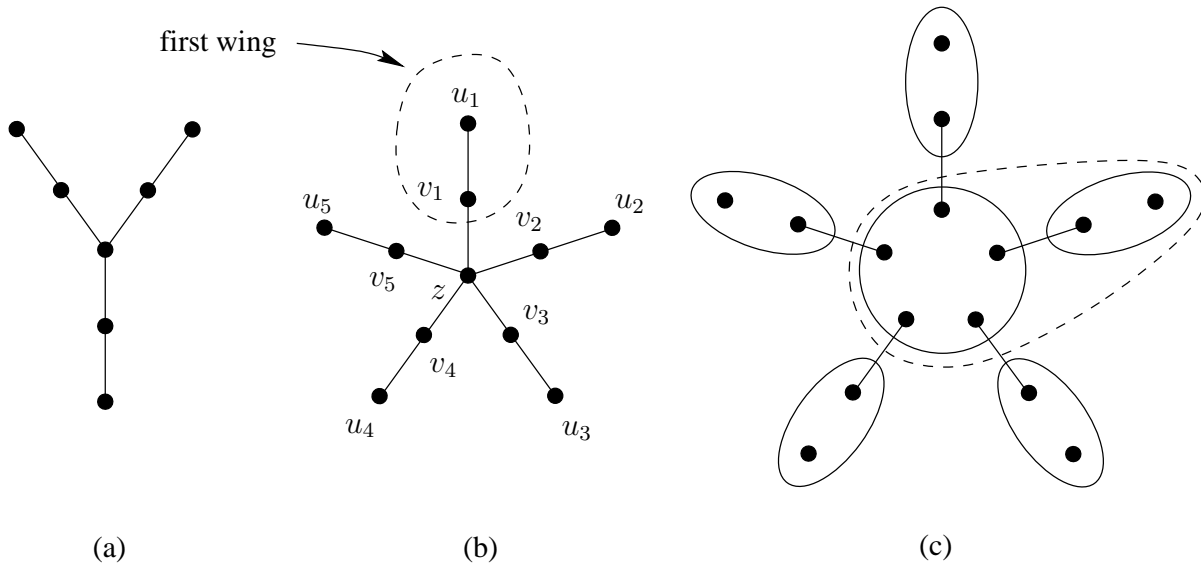


Figure 1: (a) The  $Y$  graph (b) The  $Y_5$  graph (c) A 5-windmill graph.

**Definition 1.** The  $Y_k$  graph is a graph having  $2k + 1$  vertices  $u_1, u_2, \dots, u_k, v_1, v_2, \dots, v_k$  and  $z$ . There is an edge connecting  $u_i$  to  $v_i$ , for every  $i, 1 \leq i \leq k$ , and another edge connecting each  $v_i$  to  $z$ ,  $1 \leq i \leq k$  (Figure 1 (b)). We call the subgraph consisting of  $u_i$  and  $v_i$  the  $i$ th wing of the graph.

The  $Y$  graph of Figure 1 (a) is a  $Y_3$  graph by our new definition. To prove that the  $Y_3$  graph does not have an LIRS let us assume it has an LIRS and the vertices of the graph are assigned integer labels taken from  $\{1, 2, \dots, 7\}$ . Since we have three wings, there is a wing, say the  $i$ th wing, which does not contain 1 or 7 (the minimum and the maximum label). Now, the interval assigned to the edge  $(v_i, z)$  at  $v_i$  must contain both 1 and 7. Therefore, this interval contains the label of  $u_i$  which is not possible. The following lemma generalizes this idea.

**Lemma 1.** The graph  $Y_{2d+1} \notin \langle 1, d \rangle$ -MLIRS.

In fact, we can immediately observe that if each wing of the  $Y_{2d+1}$  graph had more than just two vertices, as long as those vertices were not directly connected to the vertex  $z$  or to the vertices in other wings, Lemma 1 would still be valid. In order to prove this more general statement (which is also a proof for Lemma 1) we formally define a  $k$ -windmill graph as follows.

**Definition 2.** A  $k$ -windmill graph is a connected graph with  $k + 1$  connected components  $A_1, A_2, \dots, A_k$  (*arms of the  $k$ -windmill graph*) and  $R$  (*center of the  $k$ -windmill graph*) such that:

- (i) each component  $A_i, 1 \leq i \leq k$ , has at least two vertices;
- (ii) there is no edge connecting  $A_i$  to  $A_j$  for  $1 \leq i, j \leq k$  and  $i \neq j$ ; and
- (iii) each component  $A_i, 1 \leq i \leq k$ , is connected with  $R$  by exactly one bridge.

Figure 1 (c) illustrates a 5-windmill graph. Obviously, by this definition, a  $Y_k$  graph is also a  $k$ -windmill graph. Also, as Figure 1 (c) indicates, a  $k$ -windmill graph is a  $i$ -windmill graph for every  $i, 1 \leq i \leq k - 1$ . This can easily be shown by expanding  $R$  to include  $A_{i+1}, \dots, A_k$ .

**Lemma 2.** Any  $(2d + 1)$ -windmill graph  $\notin \langle 1, d \rangle$ -MLIRS.

**Proof.** Let us assume, by way of contradiction, that there is a  $\langle 1, d \rangle$ -MLIRS for a given  $(2d + 1)$ -windmill graph ( $d \geq 1$ ). We denote by  $m_i$  (respectively  $M_i$ ) the vertex (or one of the vertices, if there is more than one such vertex) having the minimum (maximum) value in the  $i$ th coordinate of its label. We have at most  $2d$  vertices in the set  $B = \{m_i, M_i | 1 \leq i \leq d\}$  which we call the *boundary set*. Since a  $(2d + 1)$ -windmill graph has  $2d + 1$  arms, there is an arm, say the  $j$ th arm, that does not contain any vertex in the boundary set  $B$ . Every  $d$ -dimensional interval containing all of the vertices in  $B$  contains all vertices of  $(2d + 1)$ -windmill graph as well. Thus, the interval assigned to the bridge connecting the  $j$ th arm to the center of the  $(2d + 1)$ -windmill graph, say  $(u, v)$  ( $u$  is in the  $j$ th arm and  $v$  is a vertex in the center of the graph) contains all vertices in the  $(2d + 1)$ -windmill graph. The  $j$ th wing has at least another vertex other than  $u$ , say  $u'$ . Thus, the interval assigned to the edge  $(u, v)$  includes  $u'$ . Obviously, there is no path going through  $(u, v)$  to reach  $u'$ , which is a contradiction. ■

Since a  $Y_{2d+1}$  graph is also a  $(2d + 1)$ -windmill graph, Lemma 2 also proves the Lemma 1.

Lemma 2 introduces a class of graphs which do not support  $\langle 1, d \rangle$ -MLIRS. In other words, it states a necessary condition for a graph to support a  $\langle 1, d \rangle$ -MLIRS. In the rest of this section we will show that this is also a sufficient condition.

Fraigniaud and Gavaille have proved that a graph supports LIRS if and only if it is not a lithium graph [FG94] (which is exactly the 3-windmill graph). We will use this result as the basis for an inductive construction of a  $\langle 1, d \rangle$ -MLIRS for a given graph  $G$ . We start with some new definitions.

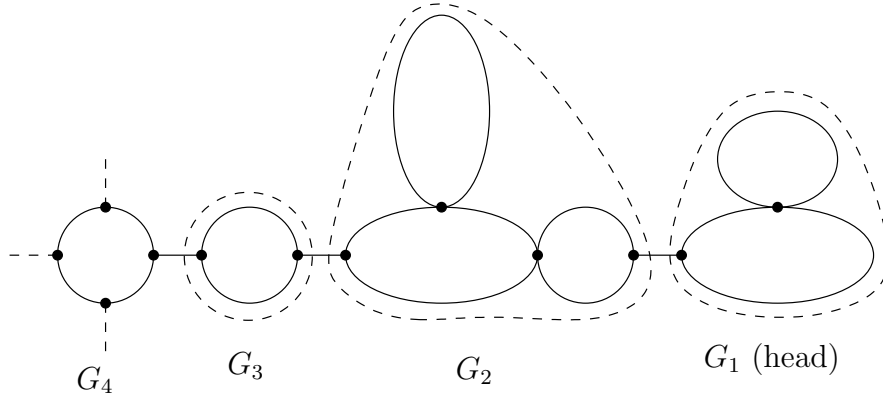


Figure 2: The dashed curves indicate edge-biconnected components in this figure. The edge-biconnected components  $G_1, G_2, \dots, G_4$  form a chain. The edge-biconnected components  $G_1, G_2$  and  $G_3$  form a perfect chain.

**Definition 3.** In a graph  $G$ , a *chain of edge-biconnected components*, or a *chain* for short, is a set of edge-biconnected components of  $G$  with a special ordering of these edge-biconnected components, say  $G_1, G_2, \dots, G_k$ , such that for each  $i, 1 \leq i \leq k-1$ , there is a bridge connecting  $G_i$  to  $G_{i+1}$ . A chain is said to be *perfect* if:

- (i)  $G_1$  is connected to exactly one bridge in  $G$ .
- (ii) Each edge-biconnected component  $G_i, 2 \leq i \leq k$  is connected to exactly two bridges in  $G$ .
- (iii) If  $G'$  is the edge-biconnected component in the graph  $G$  which is connected to  $G_k$  and  $G' \neq G_{k-1}$ , then  $G'$  is connected to more than two bridges.

We call  $G_1$  the *head* and  $G_k$  the *tail* of the chain. Trivially if  $k = 1$  then  $G_1$  is both the head and the tail of the chain.

### 3.1 Properties of chains and $k$ -windmill graphs

In this section we review some of the properties of chains and  $k$ -windmill graphs. The first observation follows directly from the definition of a chain.

**Observation 2.** *A perfect chain in a graph  $G$  is a proper induced subgraph of  $G$ , and the tail of a perfect chain (which is an edge-biconnected component) is connected to the rest of the graph by a bridge.*

The edge-biconnected components  $G_1, G_2, \dots, G_4$  in the graph depicted in Figure 2 and the bridges connecting them form a chain.  $G_1$  and  $G_4$  are the head and the tail of this chain, respectively. In this graph, if we just consider the subgraph containing the edge-biconnected components  $G_1, G_2$  and  $G_3$  and the bridges connecting them, then we have a perfect chain. The head of this perfect chain is  $G_1$  and the tail is  $G_3$ . As mentioned in Observation 2,  $G_3$  (which is the tail of the perfect chain) is connected to  $G_4$  by a bridge and  $G_4$  is connected to more than two bridges. Since,  $G_3$  is connected to exactly two bridges, the edge-biconnected components  $G_1$  and  $G_2$  does not form a perfect chain.

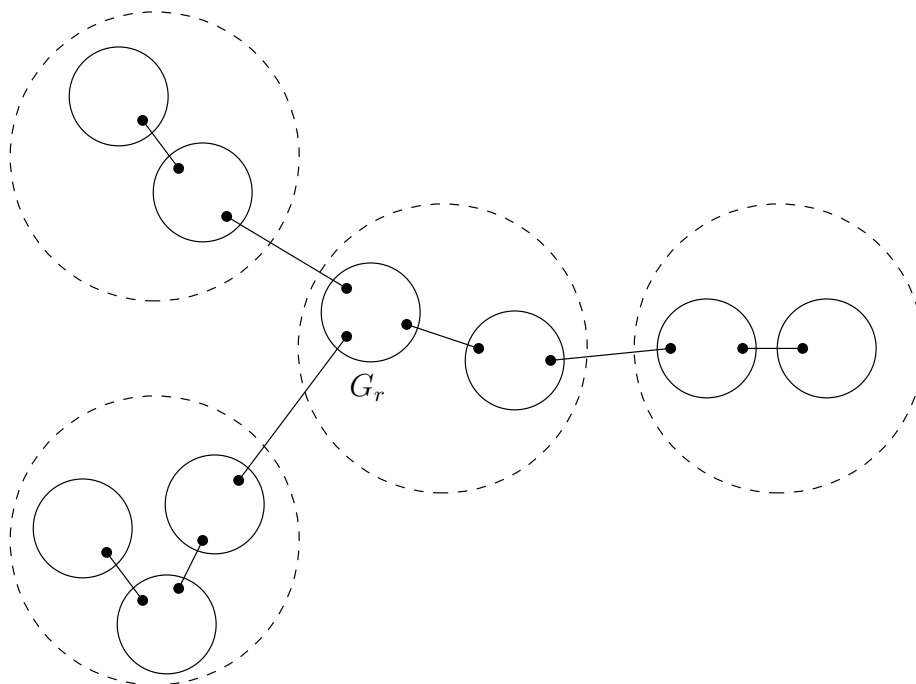


Figure 3: Edge-biconnected components in a 3-windmill graph.

**Lemma 3.** If a graph  $G$  is a  $k$ -windmill graph for  $k \geq 3$  then it is not a chain.

**Proof.** We consider each edge-biconnected component of  $G$  as a super-node. Clearly, the resulting graph is a tree. Since,  $G$  is a  $k$ -windmill graph ( $k \geq 3$ ), there is a node  $v$  in this tree such that the degree of  $v$  is at least 3 (the super-node  $G_r$  in Figure 3). In any chain, each edge-biconnected component is connected to at most 2 other edge-biconnected components. Therefore,  $G$  is not a chain. ■

**Lemma 4.** Any non-trivial (having at least one vertex) graph  $G$  which is not a chain contains a perfect chain as a proper induced subgraph.

**Proof.** Since  $G$  is not a chain and is non-trivial, it has more than one edge-biconnected component. It also has an edge-biconnected component which is connected to exactly one bridge (otherwise the edge-biconnected components form a cycle and this forces the bridges connecting edge-biconnected components to be contained in a cycle, a contradiction to the definition of a bridge). We denote this component by  $G_1$ , which is a chain consisting of just one edge-biconnected component. Let us denote this chain by  $C_1$ .  $G_1$  is both the head and the tail of  $C_1$ . In the  $i$ th iteration, we expand  $C_i$  by adding to it a new edge-biconnected component. If  $G_i$  denotes the tail of  $C_i$ , since  $C_i \neq G$  ( $G$  is not a chain) and  $G$  is connected,  $G_i$  is connected to an edge-biconnected component, say  $G_{i+1}$ , which is not in  $C_i$ . If  $G_{i+1}$  is connected to more than two bridges, then  $C_i$  is a perfect chain and is a proper induced subgraph of  $G$ . This completes the proof. Otherwise, we expand  $C_i$  to  $C_{i+1}$  by adding  $G_{i+1}$  and the bridge connecting  $G_i$  to  $G_{i+1}$ . The tail of the new chain  $C_{i+1}$  is now  $G_{i+1}$ . If we repeat this step, the algorithm will eventually terminate since  $G$  is finite and not a chain. ■

For example, in the graph depicted in Figure 2 we start with the edge-biconnected component  $G_1$ , which is connected to exactly one bridge ( $C_1 = G_1$ ).  $G_1$  is connected to the edge-biconnected component  $G_2$  which has exactly two bridges so we let  $C_2$  be the chain consisting of  $G_1$  and  $G_2$ . Similarly we add  $G_3$  to  $C_2$  to obtain  $C_3$ . Now,  $G_3$  is connected to the edge-biconnected component  $G_4$  which is connected to more than two bridges. This terminates our algorithm and the chain  $C_3$ , which is the subgraph consisting of  $G_1, G_2$  and  $G_3$  and the bridges connecting them, is a perfect chain.

In constructing a  $\langle 1, d \rangle$ -MLIRS, we will use this lemma in the induction step to reduce the size of the graph. This reduction has a very nice property that is the heart of the main proof, which is stated in the following lemma.

**Lemma 5.** If a graph  $G$  is not a chain and is not a  $k$ -windmill graph ( $k > 3$ ), we can remove any perfect chain from  $G$  and the resulting graph is not a  $(k - 1)$ -windmill graph.



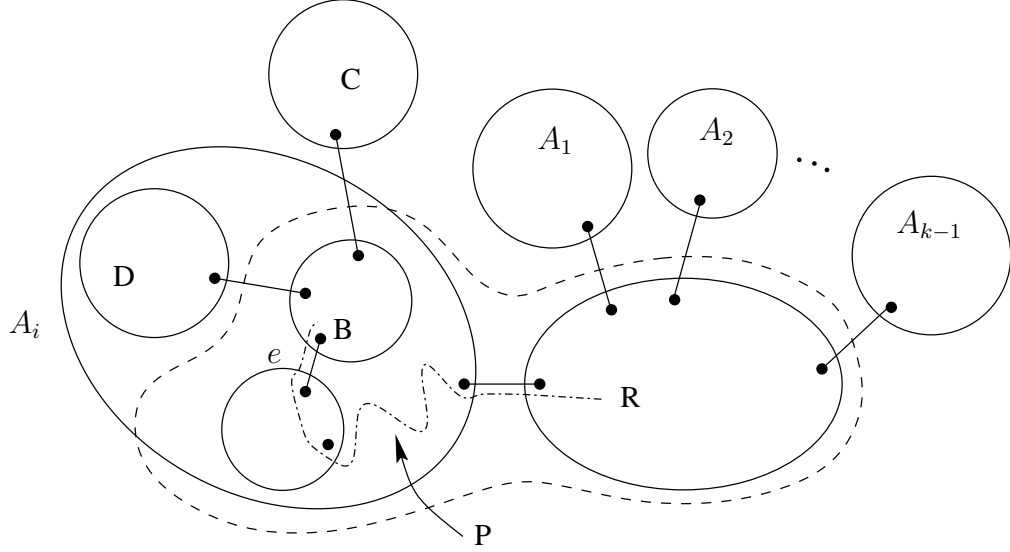


Figure 4:  $C$  and  $D$  will become arms in the  $k$ -windmill graph.

**Proof.** Since  $G$  is not a chain, by Lemma 4, there is a perfect chain  $C$  which is a proper induced subgraph of  $G$ . We let  $G'$  denote the graph  $G - C$ . We assume, to the contrary, that  $G'$  is a  $(k - 1)$ -windmill graph. By the definition of a  $(k - 1)$ -windmill graph,  $G'$  has  $k$  disjoint sets of vertices  $A_1, A_2, \dots, A_{k-1}$  and  $R$ . Since  $C$  is a perfect chain, by Observation 2 its tail is connected to  $G'$  by a bridge.  $C$  cannot be connected to  $R$ , otherwise  $G$  must be a  $k$ -windmill graph. Let us assume that  $C$  is connected to an edge-biconnected component,  $B$ , which is in the arm  $A_i$  for some  $i$ ,  $1 \leq i \leq k - 1$  (Figure 4).

By part (iii) of the definition of a perfect chain, the edge-biconnected component  $B$  is connected to at least three bridges, one connecting  $B$  to  $C$  and at least two other bridges connecting  $B$  to some other edge-biconnected components in  $G'$ . By Observation 1 all the paths connecting  $B$  and  $R$  go through one of the bridges connected to  $B$ , say  $e$ . We let  $D$  be the edge-biconnected component which is connected to  $B$  and is not connected to  $e$ .

Now, we expand  $R$  to contain  $B$  and all the edge-biconnected components in the arm  $A_i$  except  $D$ . Since  $G$  is a  $(k - 1)$ -windmill graph it has  $k - 2$  arms other than  $A_i$ . We can also consider  $C$  and  $D$  as two new arms. Hence,  $G$  has  $k$  arms and is a  $k$ -windmill graph, a contradiction. ■

### 3.2 Characterization

In this section we will prove the main result of this paper. First, we need to show how to convert a  $d$ -dimensional IRS into a  $(d + 1)$ -dimensional IRS.

If a graph  $G$  supports a  $\langle 1, d \rangle$ -MLIRS ( $\langle 1, d \rangle$ -MSLIRS), we can convert the  $d$ -dimensional

to a  $(d + 1)$ -dimensional one, by adding a new coordinate to the labels of vertices. The label of this coordinate is set to zero for all vertices. We also set the newly added coordinate of each interval to be  $[0..0]$ . It is a trivial task to verify that this IRS routes the messages exactly like the  $d$ -dimensional IRS. In other words, we can expand a  $d$ -dimensional IRS to a  $(d + 1)$ -dimensional IRS.

**Lemma 6.** If a graph  $G$  supports a  $\langle 1, d \rangle$ -MLIRS ( $\langle 1, d \rangle$ -MSLIRS) it also supports a  $\langle 1, d + 1 \rangle$ -MLIRS ( $\langle 1, d + 1 \rangle$ -MSLIRS).

Now, we have all the tools we need to prove the main theorem of this section.

**Theorem 1.** *A graph  $G$  has a  $\langle 1, d \rangle$ -MLIRS if and only if it is not a  $(2d + 1)$ -windmill graph.*

**Proof.** First, we show that if a graph is not in the class of  $(2d + 1)$ -windmill graphs, then it has a  $\langle 1, d \rangle$ -MLIRS. We use induction on  $d$ , the number of dimensions. Fraigniaud and Gavoille [FG94] have proved that if a graph  $G$  is not a lithium graph, which is exactly a 3-windmill graph, then there is a 1-LIRS for  $G$  (a  $\langle 1, 1 \rangle$ -MLIRS). This is the basis of our induction.

Let us suppose that for any  $i \leq d - 1$ , if a graph is not a  $(2i + 1)$ -windmill graph, it has a  $\langle 1, i \rangle$ -MLIRS. Now, we want to show that if a graph  $G$  is not a  $(2d + 1)$ -windmill graph,  $d > 1$ , then it has a  $\langle 1, d \rangle$ -MLIRS. We first show how to label the vertices of  $G$ . Then, we describe how we can update intervals in each step of the induction. Finally, we prove the correctness of such vertex and link labeling.

**Labeling vertices:**

Although  $G$  is not a  $(2d + 1)$ -windmill graph it can be a  $(2d - 1)$ -windmill graph. If  $G$  is not a  $(2d - 1)$ -windmill graph, by the induction hypothesis it has a  $\langle 1, d - 1 \rangle$ -MLIRS and by Lemma 6,  $G$  also has a  $\langle 1, d \rangle$ -MLIRS, completing the proof. Hence, we can assume that  $G$  is a  $(2d - 1)$ -windmill graph and by recalling Lemma 3, we can assume that  $G$  is not a chain. Therefore, by Lemma 4,  $G$  has a perfect chain, say  $C_1$ , as a proper induced subgraph. Since  $G$  is not a  $(2d + 1)$ -windmill graph and  $d > 1$ , by applying Lemma 5 we can remove  $C_1$  and the resulting graph will not be a  $2d$ -windmill graph. Since  $2d > 3$ , we can repeat these steps and remove another perfect chain,  $C_2$ , so that the resulting graph,  $G'$ , is not a  $(2d - 1)$ -windmill graph.

By the induction hypothesis,  $G'$  has a  $\langle 1, d - 1 \rangle$ -MLIRS. We just need to expand this labeling to a  $\langle 1, d \rangle$ -MLIRS for  $G$ .

$C_1$  and  $C_2$  are chains and therefore, by Lemma 3, they are not 3-windmill graphs. Therefore, by the induction hypothesis, there is a  $\langle 1, 1 \rangle$ -MLIRS for each of them. In fact, in

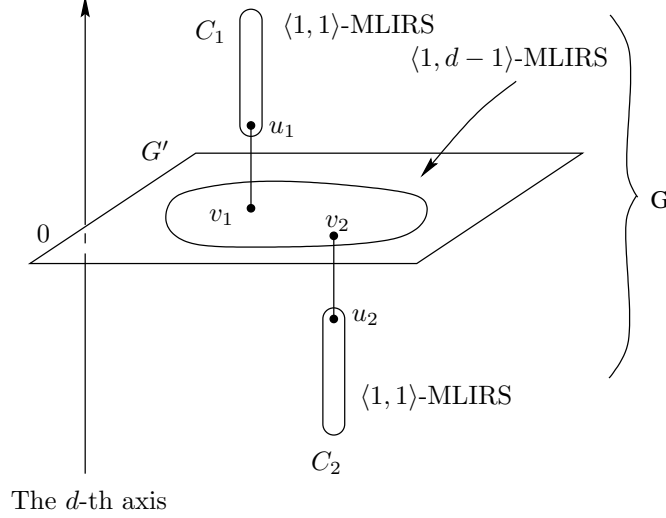


Figure 5: Expanding the labels of vertices in  $G'$  to labels for vertices in  $G$ .

[FG94], it has been proved that if a given graph is not a 3-windmill (lithium) graph, we can specify a vertex and find a labeling for the vertices such that the label of the specified vertex is 1. We find such a  $\langle 1, 1 \rangle$ -MLIRS for  $C_1$  ( $C_2$ ) such that the label for the vertex in  $C_1$  ( $C_2$ ) joining  $C_1$  ( $C_2$ ) to the rest of the graph  $G$ , say  $u_1$  ( $u_2$ ), is 1 (Figure 5).

To construct the new labeling for  $G$ , each vertex in  $G'$  is assigned a  $d$ -dimensional label in which the first  $d - 1$  coordinates are the same as the labels in the linear  $\langle 1, d - 1 \rangle$ -MIRS corresponding to  $G'$  and the  $d$ th coordinate is 0. Figure 5 illustrates an example in which  $d = 3$ . The third coordinates of the labels assigned to the vertices of  $G'$  are all 0, so  $G'$  lies in the plane passing through the first and the second axes. For now, we assume that the labels assigned to the vertices can have any integer value (including 0 and negative integers) as their  $d$ th coordinates. We can shift all the labels such that the  $d$ th coordinates of all labels becomes positive later.

Let  $(v_1, u_1)$  and  $(v_2, u_2)$  respectively denote the bridges connecting  $G'$  to  $C_1$  and  $C_2$  and let  $v_1$  and  $v_2$  be vertices of  $G'$ . We will set the first  $d - 1$  coordinates of each vertex in  $C_1$  to be equal to the first  $d - 1$  coordinates of  $v_1$ . The  $d$ th coordinates of vertex labels in  $C_1$  are the labels assigned to vertices in the previously mentioned  $\langle 1, 1 \rangle$ -MLIRS. In Figure 5 the vertices in  $C_1$  all lie on the line passing through  $v_1$  and parallel to the  $d$ th axis.

For the vertices in  $C_2$ , we will similarly set the first  $d - 1$  coordinates of each vertex equal to the first  $d - 1$  coordinates of  $v$ . If the label of a vertex  $v$  in the previously mentioned  $\langle 1, 1 \rangle$ -MLIRS is  $l(v)$ , we assign  $-l(v)$  as the  $d$ th coordinate of the new labeling (Figure 5). Now as mentioned before, we can shift the  $d$ th coordinate of all the labels such that the  $d$ th coordinate of the vertex with minimum value becomes 1. We let  $s$  denote the amount of this shifting and  $M$  denote the maximum value in the  $d$ th coordinate of all new labels.

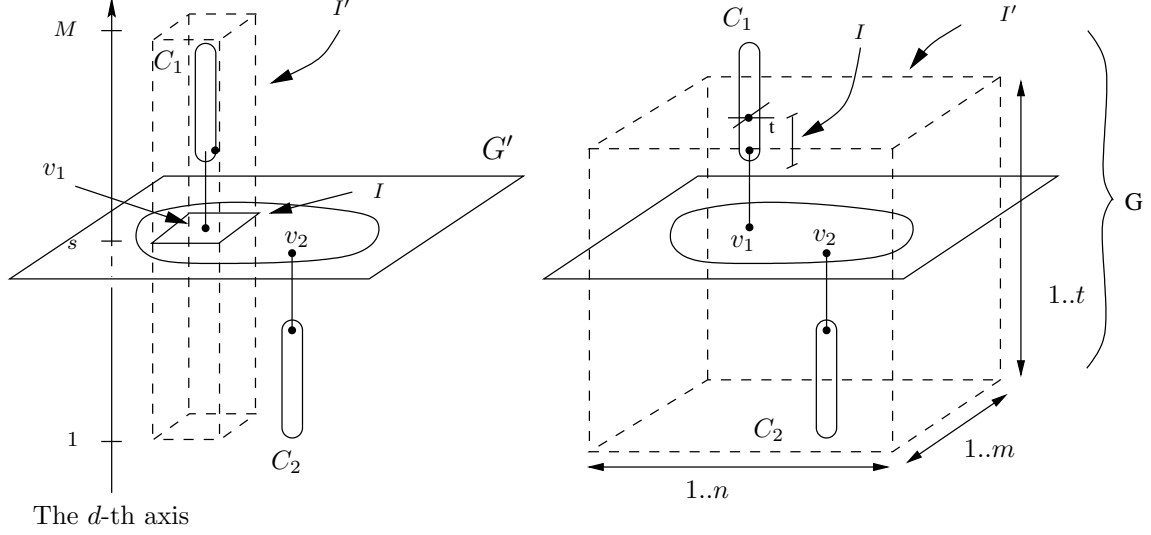


Figure 6: (a) Updating an interval in  $G'$  (b) Updating an interval, which includes  $u_1$ , in  $C_1$  ( $I$  is the old interval,  $I'$  is the new one in both (a) and (b))

### Updating Intervals:

We update intervals as follows: the first  $d - 1$  coordinates of each interval assigned to a link in  $G'$  is the same as the  $(d - 1)$ -dimensional interval associated with that edge in the  $\langle 1, d - 1 \rangle$ -MLIRS defined on  $G'$ . The  $d$ th coordinate of all intervals is set to be  $[1..M]$ . Any  $(d - 1)$ -dimensional interval in  $G'$  that does not contain  $v_1$  or  $v_2$  will still contain the same set of vertices and any interval containing  $v_1$  (respectively  $v_2$ ) will also contain all the vertices in  $C_1$  ( $C_2$ ). For example the two dimensional interval  $I$ , shown in Figure 6 (a), contains  $v_1$ , so the new three-dimensional interval  $I'$  contains all the vertices in  $C_1$ . Since  $I$  does not contain  $v_2$ ,  $I'$  does not contain any of the vertices in  $C_2$ .

For the intervals associated with the links in  $C_1$  or  $C_2$ , the first  $d - 1$  coordinates are set to  $[1..n]$ . To set the  $d$ th coordinate of each interval we will use the previously mentioned  $\langle 1, 1 \rangle$ -MLIRS. Let us assume that in the  $\langle 1, 1 \rangle$ -MLIRS defined on  $C_1$  the interval assigned to a link  $e$  is  $I_e = [a..b]$ . If  $I_e$  does not contain  $u_1$ , the  $d$ th coordinate of the newly assigned  $d$ -dimensional interval will be  $[a + s..b + s]$  (we shift the  $d$ th coordinate by  $s$  units because we have already shifted the vertices in this dimension). If  $I_e$  contains  $u_1$ , i.e.  $I_e = [1..b]$  for some  $b$ , the  $d$ th coordinate of the newly assigned interval will be  $I_e = [1..b + s]$ . This means that any 1-dimensional interval defined in  $C_1$  will be transformed into a  $d$ -dimensional interval containing the same set of vertices in  $C_1$  and if it contains  $u_1$ , it will also contain all the vertices in  $G'$  and  $C_2$ . The interval  $I$  depicted in Figure 6 (b) contains  $u_1$ , so the new interval  $I'$  contains the set of vertices in  $C_1$  that where in  $I$  and also all the vertices in  $C_2$  and  $G'$ . We will analogously assign intervals to the links in  $C_2$ .

The only remaining labels to update are labels of the links  $(v_1, u_1)$ ,  $(u_1, v_1)$ ,  $(v_2, u_2)$  and  $(u_2, v_2)$ . The first  $d - 1$  coordinates of intervals associated with  $(v_1, u_1)$ ,  $(u_1, v_1)$ ,  $(v_2, u_2)$  and  $(u_2, v_2)$  are set to  $[1..n]$  and the  $d$ th coordinates will respectively be  $[s + 1..n]$ ,  $[1..s]$ ,  $[1..s - 1]$  and  $[s..n]$ .

**Correctness:**

Now, let us consider a message originating from vertex  $w_s$  and with destination  $w_t$ . If both  $w_s$  and  $w_t$  are in  $C_1$  (similarly  $C_2$  or  $G'$ ) one can easily check that the newly defined  $\langle 1, d \rangle$ -MLIRS will route the messages on the same path as the  $\langle 1, 1 \rangle$ -MLIRS defined on  $C_1$  ( $C_2$  or the  $\langle 1, d - 1 \rangle$ -MLIRS defined on  $G'$ ). This is because if we just considering the set of vertices in  $C_1$  ( $C_2$  or  $G'$ ) each interval assigned to a link contains the same set of vertices as it contained before expanding the labels to  $d$  dimensions. If  $w_s$  is in  $C_1$  and  $w_t$  in  $G'$ , the message must go through the link  $(u_1, v_1)$  because this is the only link connecting  $C_1$  to  $G'$ . The intervals in  $C_1$  which contain  $w_t$  are exactly the intervals containing  $u_1$ . Therefore, this message will be forwarded through the same links as the links through which a message towards  $u_1$  would be forwarded. When the message reaches  $u_1$ , the bridge  $(u_1, v_1)$  forwards the message to  $v_1$ , because the interval assigned to  $(u_1, v_1)$  contains all the vertices in  $G'$  and  $C_2$ . The rest of the routing will be the same as the  $\langle 1, d - 1 \rangle$ -MLIRS defined on  $G'$ .

We can show that if there is a message in node  $x$  ( $x = u_2, v_1$  or  $v_2$ ) which is supposed to be forwarded the bridge connected to  $x$ , say  $e_x$  ( $e_x = (u_2, v_2)$ ,  $(v_1, u_1)$  or  $(v_2, u_2)$  respectively), will be sent to the other end of  $e_x$ . Verifying the cases in which  $w_s$  is in  $C_2$  or  $G'$  is similar. Hence, any message originating at any vertex and going to an arbitrary destination will eventually reach the destination, and the  $\langle 1, d \rangle$ -MLIRS routes messages on  $G$  properly.

We now have shown that if a graph is not in the class of  $(2d + 1)$ -windmill graphs it has a  $\langle 1, d \rangle$ -MLIRS. Lemma 2 shows that no graph in this class can support a  $\langle 1, d \rangle$ -MLIRS. Combining these two results completes the proof of the theorem. ■

Since for each  $d > 1$ , we have a  $(2d + 1)$ -windmill graph which is not a  $(2d + 3)$ -windmill graph (for example the  $Y_{2d+1}$  graph), we can state the following corollary:

**Corollary 1.** *The class of graphs supporting  $\langle 1, d \rangle$ -MLIRS is a strict subset of the class of graphs supporting  $\langle 1, d + 1 \rangle$ -MLIRS.*

In other words, increasing the number of dimensions increases the power of the routing scheme.

## 4 Characterization of networks supporting $\langle 1, d \rangle$ -MSLIRS

In this section we will give a characterization of the class of graphs supporting  $\langle 1, d \rangle$ -MSLIRS. We will just give some new definitions and will show that with slight changes in some steps in proofs, we can use the same ideas used to characterize the class of graphs supporting  $\langle 1, d \rangle$ -MLIRS.

In proving the Lemma 2, we needed to have at least two vertices in each arm of a  $(2d+1)$ -windmill graph. Otherwise, if the arm which did not have any vertex in the boundary set, say  $A_i$ , had just one vertex, say  $x$ , the interval assigned to the edge connecting  $A_i$  to  $R$  could contain  $x$  and this was not a contradiction. On the other hand, if the intervals assigned to the links are supposed to be strict, we could prove a similar lemma, even if we had an arm having just one vertex. This is the main difference between the proofs of this section and the previous one. More formally, let us start with a new definition.

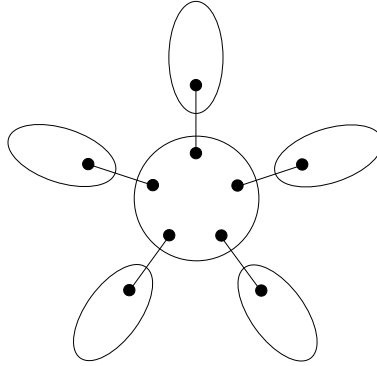


Figure 7: A weak 5-windmill graph.

**Definition 4.** A *weak  $k$ -windmill graph* is a connected graph  $G$  with  $k + 1$  connected components  $A_1, A_2, \dots, A_k$  (arms) and  $R$  such that:

- (i) there is no edge in  $G$  connecting  $A_i$  to  $A_j$  for  $1 \leq i, j \leq k$  and  $i \neq j$ ;
- (ii) each component  $A_i, 1 \leq i \leq k$  is connected with  $R$  by exactly one bridge (Figure 7).

As mentioned above, if the IRS is strict, then with even one vertex in each arm the proof of Lemma 2 will still be valid, because a vertex which is not in the boundary set is contained in an edge connected to it. Therefore, any weak  $(2d + 1)$ -windmill graph does not have a  $\langle 1, d \rangle$ -MSLIRS. We can also verify, with the same argument as the proof of Lemma 5, that removing any perfect chain from a graph  $G$  which is not a weak  $k$ -windmill graph will produce a graph which is not a weak  $(k - 1)$ -windmill graph.

The only remaining step is to show that the induction basis and step are also valid in constructing a  $\langle 1, d \rangle$ -MSLIRS for any graph that is not a weak  $(2d + 1)$ -windmill graph. We already know that any graph which is not weak 3-windmill graph (a weak lithium graph as defined in [FG94]) has a  $\langle 1, 1 \rangle$ -MSLIRS, so the induction basis is true. Since we have lemmas similar to Lemmas 4 and 5 one can verify that a similar induction step still works here. This give us the complete characterization of graph supporting  $\langle 1, d \rangle$ -MSLIRS as follows:

**Theorem 2.** *A graph  $G$  has a  $\langle 1, d \rangle$ -MSLIRS if and only if it is not a weak  $(2d + 1)$ -windmill graph.*

**Corollary 2.** *The class of graphs supporting  $\langle 1, d \rangle$ -MSLIRS is a strict subset of the class of graphs supporting  $\langle 1, d + 1 \rangle$ -MSLIRS.*

## 5 Conclusions and open problems

In this paper we completely characterized the class of networks supporting  $\langle 1, d \rangle$ -MLIRS and the class of networks supporting  $\langle 1, d \rangle$ -MSLIRS. We showed that increasing the number of dimensions makes the routing scheme more powerful. One natural extension to this problem is to characterize the networks having a  $\langle 1, d \rangle$ -MLIRS or  $\langle 1, d \rangle$ -MSLIRS when the network has weighted links with dynamic costs. If the routing paths are supposed to be shortest paths, and we can relabel the edges after each change in the cost of links, there is a complete characterization for  $\langle 1, d \rangle$ -MSLIRS [Gan01]. If the intervals are the same for any costs of links, the characterization problem is open even except for the 1-dimensional case [BvLT91]. There is a partial characterization for the class of networks supporting optimum LIRS in 1-dimension [NS98]. Another interesting problem is to characterize networks supporting  $\langle k, d \rangle$ -MLIRS for a given  $k$  and  $d$ . Finally, one can consider the problem of finding bounds on the length of routing paths for each of these classes.

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## References

[ABNLP90] Baruch Awerbuch, Amotz Bar-Noy, Nathan Linial, and David Peleg. Improved routing strategies with succinct tables. *J. Algorithms*, 11(3):307–341, 1990.

- [BM76] John A. Bondy and U. S. R. Murty. *Graph theory with applications*. American Elsevier Publishing Co., Inc., New York, 1976.
- [BvLT91] Erwin M. Bakker, Jan van Leeuwen, and Richard Tan. Linear interval routing. *ALCOM: Algorithms Review, Newsletter of the ESPRIT II Basic Research Actions Program Project no. 3075 (ALCOM)*, 2, 1991.
- [FG94] Pierre Fraigniaud and Cyril Gavoille. A characterization of networks supporting linear interval routing. In *13<sup>th</sup> Annual ACM Symposium on Principles of Distributed Computing (PODC)*, pages 216–224. ACM PRESS, August 1994.
- [FGNT98] Michele Flammini, Giorgio Gambosi, Umberto Nanni, and Richard B. Tan. Multidimensional interval routing schemes. *Theoret. Comput. Sci.*, 205(1-2):115–133, 1998.
- [FJ88] Greg N. Frederickson and Ravi Janardan. Designing networks with compact routing tables. *Algorithmica*, 3(1):171–190, 1988.
- [FJ89] Greg N. Frederickson and Ravi Janardan. Efficient message routing in planar networks. *SIAM J. Comput.*, 18(4):843–857, 1989.
- [Gan01] Yashar Ganjali. Optimum multi-dimensional interval routing schemes on networks with dynamic cost links. *Technical Report CS-2001-04, Department of Computer Science, University of Waterloo, Waterloo, Canada*, 2001.
- [INM91] INMOS. The T9000 Transputer overview manual. 1991.
- [NS98] Lata Narayanan and Sunil Shende. Partial characterizations of networks supporting shortest path interval labeling schemes. *Networks*, 32(2):103–113, 1998.
- [SK85] Nicola Santoro and Ramez Khatib. Labelling and implicit routing in networks. *The Comput. J.*, 28(1):5–8, 1985.
- [TvL95] Richard B. Tan and Jan van Leeuwen. Compact routing methods: A survey. In *Proceedings of Colloquium on Structural Information and Communication Complexity (SICC'94), SCS, Carleton University, Ottawa*, pages 99–109, 1995.
- [vLT87] Jan van Leeuwen and Richard B. Tan. Interval routing. *Comput. J.*, 30(4):298–307, 1987.
- [Wes96] Douglas B. West. *Introduction to graph theory*. Prentice Hall Inc., Upper Saddle River, NJ, 1996.



- [WMT93] P.H. Welch, M.D. May, and P.W. Thompson. *Networks, Routers and Transputers: Function, Performance and Application*. IOS Press, Netherlands, February 1993. ISBN 90-5199-129-0.