LINEAR REDUCTIONS OF MAXIMUM MATCHING*

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Abstract. Maximum matching is a classic problem in graph theory with many practical applications. The fastest currently known algorithm solves this problem in $O(m\sqrt{n})$ time for an n-vertex m-edge graph.

In this paper, we give linear reductions from maximum matching to maximum matching in a special graph class, such as 3-regular graphs, or biconnected graphs with maximum degree 3. We also reduce maximum matching in planar graphs to maximum matching in triangulations with maximum degree 9. Our results imply that rather than searching for faster maximum matching algorithms general, one should concentrate on maximum matching algorithms for these special graph classes.

Key words. maximum matching, linear reduction, planar graphs, bipartite graphs

AMS subject classifications. 68R10, 05C70, 68Q25

1. Introduction.

The maximum matching problem has many practical applications, for example in personnel assignments, determining chemical bonds and pattern recognition. It is also one of the first problems to be studied in graph theory, and characterizations such as Hall's theorem had wide-ranging implications in the development of combinatorial optimization and duality theory. See the book by Lovász and Plummer [LP86] for an overview of the history of the matching problem and algorithms to solve it.

Initial research focused on when a graph had a perfect matching (see Section 2 for precise definitions.) In particular, the famous theorems by Petersen [Pet91] and König [Kön16] state that any 3-regular biconnected graph and any k-regular bipartite graph has a perfect matching. Hall [Hal35] characterized when a bipartite graph has a perfect matching, and Tutte [Tut47] extended the result to general (not necessarily bipartite) graphs.

Hall's theorem can be used for a polynomial-time algorithm to find a perfect matching in a bipartite graph [Hal56], and this can easily be extended to finding a maximum matching. As opposed to that, Tutte's theorem does not easily lend itself to a polynomial-time algorithm. (However, some such algorithms do exist, see for example [Gee00].) In 1957, Berge [Ber57] provided another characterization of maximum matchings using augmenting paths, and this was used by Edmonds [Edm65] for the first polynomial-time maximum matching algorithm for general graphs.

From then on, research has focused on improving the time-complexity of maximum matching algorithms, both for the bipartite and the general case. For the bipartite maximum matching problem in an n-vertex m-edge graph, the time-complexity was improved to $O(n^{1/2}m)$ by Hopcroft and Karp [HK73]. For the general maximum matching problem, the time-complexity improved to $O(mn)$ [Gab76], then to $O(n^{5/2})$ or $O(n^{1/2}m\log n)$ [EK75], and finally to $O(n^{1/2}m)$ by Micali and Vazirani ([MV80], see also [PL88] and [Vaz94]).

So for both the bipartite and the general maximum matching problem, the currently best algorithm takes $O(m\sqrt{n})$ time, and no improvement has been made in 20 years. But it seems unlikely that this is the final answer, at least for graphs with a

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special structure such as planar graphs. These graphs have $O(n)$ edges, and hence the currently best algorithms take $O(n^{3/2})$ time.

Faster maximum matching algorithms are known for some special graph classes if we want to find a perfect matching if one exists. Examples of this include bipartite k-regular graphs [Sch99, COS99], 3-regular biconnected graphs [BBDL99], and various subgraphs of regular grids [Thu90, HZ93, KR96]. The last four algorithms all work in linear time for a subclass of planar graphs. We can also efficiently find a perfect matching if it is unique [GKT99].

1.1. Our results. Our research was motivated by the work in [BBDL99], where together with Bose, Demaine and Lubiw we showed how to find a perfect matching in a 3-regular biconnected graph in $O(n \log^4 n)$ time, and in $O(n)$ time for planar graphs. (A perfect matching always exists by Petersen's Theorem.) A naturally arising question is whether this extends to 3-regular graphs that are not biconnected. Such graphs need not have a perfect matching, but could a similar algorithm be used to find a maximum matching in a 3-regular graphs, or at least to test whether a 3-regular graph has a perfect matching?

In this paper, we show that whether a 3-regular graph is biconnected or not has significant impact for finding a matching. More precisely, we show that there is a *linear reduction* from the maximum matching problem to the maximum matching problem in 3-regular graphs. By this we mean that we can transform any graph G into a 3-regular graph G^* with $O(m(G))$ edges such that a maximum matching in G can be recovered from one in G^* in $O(m(G))$ time. We denote this linear reduction by MATCHING $\leq_l$ MATCHING IN 3-REGULAR GRAPHS,[1] where MATCHING stands for the problem of finding a maximum matching. Such a reduction implies that any $O(f(m))$ algorithm for maximum matching in 3-regular graphs yields an $O(f(m)+m)$ algorithm for maximum matching in arbitrary graphs.[2]

We provide a few more linear reductions of maximum matching, and study the special case when the graph is bipartite and/or planar. Our results are summarized in Table 1.1. Most reductions also hold for the problem PERFECT MATCHING, which is the problem of finding a perfect matching if one exists.

Our results can be interpreted in two ways: A pessimist would take them as indication that there is no fast algorithm for 3-regular graphs (or any of the other special graph classes), for if there were, say, an $O(n \log^k n)$ algorithm for 3-regular graphs, then this would imply an $O(m \log^k m)$ algorithm for arbitrary graphs, and this seems unlikely since no progress has been made to this problem in so long.

We prefer an optimist's point of view, which would be to hope for improvement in the complexity of maximum matching by concentrating on much more structured graphs. In particular for planar graphs, we hope that an algorithm with time-complexity, say, $O(n \log^k n)$ for some $k \geq 0$, can be found. By our results (see Table 1.1) it would suffice to show such a time complexity for planar 3-regular graphs, planar biconnected graphs with maximum degree 3, or triangulations with maximum degree 9.

The paper is structured as follows: After giving definitions, we devote three sections to transformations: In Section 3, we develop some basic transformations, in

[1] The notation is borrowed from complexity theory where a similar concept exists, see for example [Sch78].

[2] However, the number of vertices in the 3-regular graph G^* is not necessarily proportional to the number of vertices in the original graph G. Hence, running times bounds containing n may or may not be preserved under this transformation. For example, $O(m\sqrt{n})$ is unfortunately not preserved.

MATCHING
$\leq_l$ MATCHING IN 3-REGULAR GRAPHS
$\leq_l$ MATCHING IN BICONNECTED MAX-DEG-3 GRAPHS

MATCHING IN BIPARTITE GRAPHS
$\leq_l$ MATCHING IN BICONNECTED BIPARTITE MAX-DEG-3 GRAPHS

MATCHING IN PLANAR GRAPHS
$\leq_l$ MATCHING IN PLANAR 3-REGULAR GRAPHS
$\leq_l$ MATCHING IN BICONNECTED PLANAR MAX-DEG-3 GRAPHS
$\leq_l$ MATCHING IN MAX-DEG-9 TRIANGULATIONS

MATCHING IN PLANAR BIPARTITE GRAPHS
$\leq_l$ MATCHING IN BICONNECTED PLANAR BIPARTITE MAX-DEG-3 GRAPHS
$\leq_l$ MATCHING IN MAX-DEG-6 QUADRANGULATIONS

TABLE 1.1
Overview of linear reductions presented in the paper.

Section 4, we study how to change vertex-degrees in a graph, and in Section 5, we study how to change the degrees of faces in a planar graph. These transformations, when combined suitably, give all results. We conclude with open problems in Section 6.

2. Definitions. Let $G = (V, E)$ be a graph with n vertices and m edges; we write $V(G), E(G), n(G)$ and $m(G)$ when the graph in question is not clear. The *degree* of a vertex v is the number of incident edges of v and denoted $\deg(v)$. G is called *k-regular* if all vertices in G have degree exactly k. G is called a *max-deg-k graph* if all vertices in G have degree at most k.

We assume throughout the paper that G is *simple*, i.e., there are no *loops* (edges (v, v) for some $v \in V$) or *multiple edges* (two or more edges (v, w) for some $v, w \in V$). We also assume that G is *connected*, i.e., there exists a path between any pair of vertices. A vertex v is called a *cutvertex* if deleting v results in a graph that is not connected. A graph is called *biconnected* if it has no cutvertex. The maximal biconnected subgraphs of the graph are called *biconnected components*.

A graph is called *bipartite* if it has a 2-coloring, i.e., if its vertices can be colored with two colors such that the endpoints of any edge have different color. A graph is called *planar* if it can be drawn in the plane without edges crossing each other. A planar drawing of a planar graph is characterized by giving the counter-clockwise cyclic order of edges around each vertex. For a given graph, testing whether it is planar, and finding such a cyclic order if it is can be done in linear time [HT74, MM96].

A *matching* in G is a subset M of the edges such that each vertex is incident to at most one edge of M. A *maximum matching of G* is a matching of maximum cardinality; its cardinality is denoted $\mu(G)$. A matching M is called a *perfect matching* if all vertices are incident to an edge of M.

3. Two elementary transformations.

3.1. Transformation T_c: Making the graph biconnected. We first show that for maximum matching problems, we may assume that the graph is biconnected. For if it is not, repeatedly merge two biconnected components by adding a few vertices and edges. We denote this transformation (which is explained in detail below) by T_c, and the resulting graph by $T_c(G)$.

Assume that G has a cutvertex a, and let v_1 and v_2 be two neighbors of a in two different biconnected components. If G is planar, then v_1 and v_2 should be chosen such that (a, v_1) and (a, v_2) are consecutive in the cyclic order of edges around a.

We obtain $T_c(G)$ by adding a 4-cycle to the graph and connecting it to v_1 and v_2. More precisely, we add vertices b, c, d, e and edges (v_1, b), (b, c), (c, d), (d, e), (e, b) and (d, v_2). See Fig. 3.1.

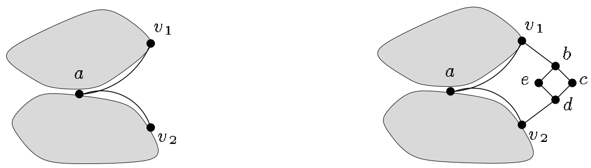

FIG. 3.1. *Merging two biconnected components.*

LEMMA 3.1. *$\mu(T_c(G)) = \mu(G) + 2$, i.e., G has a matching of cardinality k if and only if $T_c(G)$ has a matching of cardinality $k + 2$.*

Proof. If M is a matching in G, then $M \cup \{(b, e), (c, d)\}$ is a matching of cardinality $|M| + 2$ in $T_c(G)$.

Assume that M' is a matching in $T_c(G)$. At most two of the six added edges can belong to M', since they are all incident to vertices b and d. Removing them gives a matching of cardinality $\geq |M'| - 2$ in G. □

From the proof, it follows that a maximum matching of G can be recovered from a maximum matching of $T_c(G)$ in $O(1)$ time. The new graph has fewer biconnected components. So after k applications of this transformation (where $k \leq m(G)$ is the number of biconnected components), we have a biconnected graph, which we denote by $T_c^*(G)$.

From the definition of $T_c(G)$ and the proof of Lemma 3.1, we can observe the following properties of $T_c^*(G)$:

- $T_c^*(G)$ has a perfect matching if and only if G has a perfect matching.
- $T_c^*(G)$ is planar if and only if G is planar.
- $T_c^*(G)$ is bipartite if and only if G is bipartite.
- $T_c^*(G)$ has at most $m(G) + 6k \leq 7m(G)$ edges.

3.2. Transformation T_d: Doubling the graph. One very simple transformation, which we denote T_d, is to "double" the graph. More precisely, given a graph G, let $T_d(G)$ be the graph that consists of two copies of G.[3]

Observe the following properties of $T_d(G)$:

- $T_d(G)$ has a perfect matching if and only if G has a perfect matching.
- $T_d(G)$ is planar if and only if G is planar.
- $T_d(G)$ is bipartite if and only if G is bipartite.
- $T_d(G)$ has $2m(G)$ edges.

4. Changing degrees of vertices. In this section, we study transformations that decrease or increase degrees of vertices.

[3] At this point, we allow a graph that is not connected. In all applications of transformation T_d, the graph will be made connected again in the next step.

4.1. Transformation T_4: Remove vertices of degree ≥ 4. We first show how to remove vertices with degree ≥ 4 by repeatedly "splitting" them. We denote this transformation (which is explained in detail below) by T_4, and the resulting graph by $T_4(G)$.

Assume that G has a vertex a of degree ≥ 4, and let v_1 and v_2 be two neighbors of a. If G is planar, then v_1 and v_2 should be chosen such that (a, v_1) and (a, v_2) are consecutive in the cycle order of edges around a. We obtain $T_4(G)$ by adding two more vertices b and c and re-routing (a, v_1) and (a, v_2) to end at c instead of a. More precisely, we add the vertices b and c, and the edges (a, b), (b, c), (v_1, c), (v_2, c) and delete the edges (a, v_1) and (a, v_2). See Fig. 4.1.

FIG. 4.1. *Decreasing the degree of a vertex of degree ≥ 4.*

LEMMA 4.1. *$\mu(T_4(G)) = \mu(G) + 1$, i.e., G has a matching of cardinality k if and only if $T_4(G)$ has a matching of cardinality $k + 1$.*

Proof. Assume that M is a matching in G. We have three cases: If (a, v_1) belongs to M, then $M' = M - (a, v_1) \cup \{(a, b), (c, v_1)\}$ is a matching of cardinality $|M| + 1$ in $T_4(G)$. If (a, v_2) belongs to M, then $M' = M - (a, v_2) \cup \{(a, b), (c, v_2)\}$ is a matching of cardinality $|M| + 1$ in $T_4(G)$. If neither of these edges belongs to M, then $M \cup \{(b, c)\}$ is a matching of cardinality $|M| + 1$ in $T_4(G)$.

Assume that M' is a matching in $T_4(G)$. At most two edges in M' are incident to a, b or c, since b is only incident to a and c. If there are two such edges, then remove one of them from M' to obtain a matching M of cardinality $\geq |M'| - 1$. Now contract the edges (a, b) and (b, c). Since at most one edge in M is incident to a, b or c, M is also a matching in the resulting graph, which is G. Thus we obtain a matching of cardinality $\geq |M'| - 1$ in G. $\square$

Following the steps of this proof, one sees that a maximum matching of G can be recovered from a maximum matching of $T_4(G)$ in $O(1)$ time.

The new graph $T_4(G)$ is in some sense "closer" to being a max-deg-3 graph. More precisely, define $x(G) = \sum_{v \in V(G)} \max\{0, \deg(v) - 3\}$, then $x(G) = 0$ if and only if G has maximum degree 3. Since in $T_4(G)$ vertex a contributes one unit less to the sum, we have $x(T_4(G)) = x(G) - 1$.

After applying transformation T_4 $x(G)$ times, we therefore have a max-deg-3 graph, which we denote by $T_4^*(G)$. From the definition of $T_4(G)$, the proof of Lemma 4.1 and $x(G) \leq 2m(G)$, we can observe the following properties of $T_4^*(G)$:

- $T_4^*(G)$ has a perfect matching if and only if G has a perfect matching.
- $T_4^*(G)$ is planar if and only if G is planar.
- $T_4^*(G)$ is bipartite if and only if G is bipartite.
- $T_4^*(G)$ is biconnected if and only if G is biconnected.
- $T_4^*(G)$ has at most $m(G) + 2x(G) \leq 5m(G)$ edges.

Using graph $T_4^*(T_c^*(G)$ we thus obtain the following linear reductions:

- MATCHING $\leq_l$ MATCHING IN BICONNECTED MAX-DEG-3 GRAPHS
- MATCHING IN PLANAR GRAPHS $\leq_l$ MATCHING IN BICONNECTED PLANAR MAX-DEG-3 GRAPHS

- MATCHING IN BIPARTITE GRAPHS $\leq_l$ MATCHING IN BICONNECTED BIPARTITE MAX-DEG-3 GRAPHS
- MATCHING IN BIPARTITE PLANAR GRAPHS $\leq_l$ MATCHING IN BICONNECTED BIPARTITE PLANAR MAX-DEG-3 GRAPHS

Since $T_4^*(T_c^*(G))$ has a perfect matching if and only if G has one, the same reductions also hold for PERFECT MATCHING.

4.2. Removing vertices of degree 2. Now we study how to remove vertices of degree 2. We give not one, but two transformations to achieve this. The first one maintains planarity, but does not maintain whether the graph has a perfect matching. The second one has opposite properties: it maintains whether the graph has a perfect matching, but need not maintain planarity.

4.2.1. Transformation T_{2a}. Assume that G has a vertex a of degree 2. We obtain $T_{2a}(G)$ by adding vertices b, c, d and edges (a, b), (b, c) and (b, d). See Fig. 4.2.

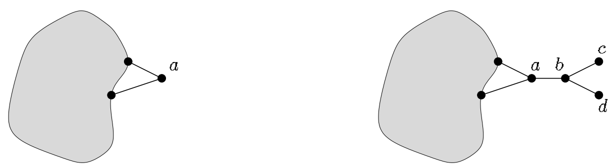

FIG. 4.2. *Removing a vertex of degree 2.*

LEMMA 4.2. $\mu(T_{2a}(G)) = \mu(G) + 1$, *i.e., G has a matching of cardinality k if and only if $T_{2a}(G)$ has a matching of cardinality $k+1$.*

Proof. If M is a matching in G, then $M \cup \{(b, c)\}$ is a matching of cardinality $|M| + 1$ in $T_{2a}(G)$.

Assume that M' is a matching in $T_{2a}(G)$. At most one of the three added edges can belong to M', since all three edges are incident to b. Removing it from M' gives a matching of cardinality $\geq |M'| - 1$ in G. ☐

Denote by $T_{2a}^*(G)$ the graph that results from G by applying this transformation to all vertices of degree 2 in G. Observe that $T_{2a}^*(G)$ is planar if and only if G is planar. Also, $T_{2a}^*(G)$ has at most $m(G) + 3n(G) \leq 4m(G)$ edges.

Unfortunately, this transformation does not preserve biconnectivity of the graph. It also does not preserve whether the graph has a perfect matching.

4.2.2. Transformation T_{2b}. Assume that G has two vertices a and b of degree 2. If G is bipartite, then furthermore assume that a and b have the same color in a 2-vertex-coloring. We obtain $T_{2b}(G)$ by adding vertices c, d and edges (a, c), (b, c) and (c, d). See Fig. 4.3.

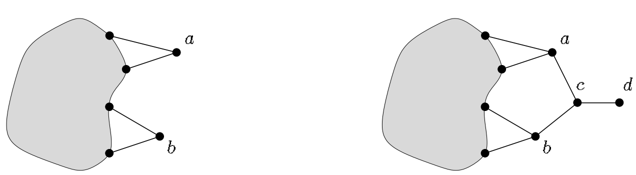

FIG. 4.3. *Pairing two vertices of degree 2.*

LEMMA 4.3. $\mu(T_{2b}(G)) = \mu(G) + 1$, *i.e., G has a matching of cardinality k if and only if $T_{2b}(G)$ has a matching of cardinality $k+1$.*

Proof. If M is a matching in G, then $M \cup \{(c,d)\}$ is a matching of cardinality $|M|+1$ in $T_{2b}(G)$.

Assume that M' is a matching in $T_{2b}(G)$. At most one of the three added edges can belong to M', since all three edges are incident to c. Removing it gives a matching of cardinality $\geq |M'|-1$ in G. □

If we double graph G (i.e., apply transformation T_d) beforehand, then we can always use a vertex of degree 2 in G and its copy in the copy of G for transformation T_{2b}. Hence by applying this transformation repeatedly to all vertices of degree 2 in $T_d(G)$, we can remove all vertices of degree 2. Observe the following properties of the resulting graph $T^*_{2b}(T_d(G))$:

- $T^*_{2b}(T_d(G))$ has a perfect matching if and only if G has a perfect matching.
- $T^*_{2b}(T_d(G))$ is bipartite if and only if G is bipartite.
- $T^*_{2b}(T_d(G))$ has at most $m(T_d(G)) + 3n(G) = 2m(G) + 3n(G) \leq 5m(G)$ edges.

Unfortunately, this transformation does not necessarily preserve planarity or biconnectivity of the graph.

4.3. Transformation T_1: Remove vertices of degree 1. Assume that G has a vertex a of degree 1. We obtain $T_1(G)$ by adding vertices b, c, d, e and edges $(a,b), (b,c), (c,d), (d,a), (b,e), (c,e)$, and (d,e). See Fig. 4.4.

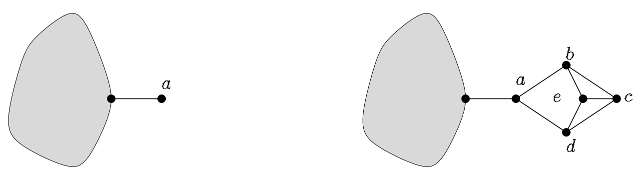

FIG. 4.4. *Removing a vertex of degree 1.*

LEMMA 4.4. *$\mu(T_1(G)) = \mu(G) + 2$, i.e., G has a matching of cardinality k if and only if $T_1(G)$ has a matching of cardinality $k+2$.*

Proof. If M is a matching in G, then $M \cup \{(b,e),(c,d)\}$ is a matching of cardinality $|M|+2$ in $T_1(G)$.

Assume that M' is a matching in $T_1(G)$. At most two of the seven added edges can belong to M', since they are incident to only five different vertices. Removing them gives matching of cardinality $\geq |M'|-2$ in G. □

Denote by $T^*_1(G)$ the graph that results from applying this transformation to all vertices of degree 1 in G. Observe the following properties of $T^*_1(G)$:

- $T^*_1(G)$ has a perfect matching if and only if G has a perfect matching.
- $T^*_1(G)$ is planar if and only if G is planar.
- $T^*_1(G)$ has at most $m(G) + 7n(G) \leq 8m(G)$ edges.

Unfortunately, this transformation does not preserve bipartiteness.

4.4. 3-regular graphs. To reduce matching to matching in 3-regular graphs, we combine the transformations in this section. First, remove all vertices of degree ≥ 4 with transformation T_4. Then, remove all vertices of degree 2 with transformation T_{2a} or T_{2b}. Finally, remove all vertices of degree 3 with transformation T_1.

Studying the properties of the individual transformations, we can observe that graph $T^*_1(T^*_{2a}(T^*_4(G)))$ proves the following linear reductions:

- MATCHING $\leq_l$ MATCHING IN 3-REGULAR GRAPHS
- MATCHING IN PLANAR GRAPHS $\leq_l$ MATCHING IN 3-REGULAR PLANAR GRAPHS

On the other hand, graph $T^*_1(T^*_{2b}(T_d(T_4(G))))$ proves the following reductions:

- MATCHING $\leq_l$ MATCHING IN 3-REGULAR GRAPHS
- PERFECT MATCHING $\leq_l$ PERFECT MATCHING IN 3-REGULAR GRAPHS

Sadly, neither bipartiteness nor biconnectivity is maintained in these reductions, since transformations T_{2a} and T_{2b} add vertices of degree 1, which destroys biconnectivity. To remove the vertices of degree 1 we apply transformation T_1, which destroys bipartiteness.

Indeed, this is to be expected, because every 3-regular bipartite graph has a perfect matching by König's Theorem [Kön16], and hence a reduction MATCHING IN BIPARTITE GRAPHS $\leq_l$ MATCHING IN 3-REGULAR BIPARTITE GRAPHS seems unlikely. For the same reason, Petersen's Theorem makes a reduction MATCHING $\leq_l$ MATCHING IN 3-REGULAR BICONNECTED GRAPHS unlikely.

5. Changing degrees of faces. Now we study matching in planar graphs. We first give a few more definitions. A planar drawing of a graph defines connected components of the plane; these components are called *faces*. The number of vertices incident to a face is called the *degree* of the face, where vertices that are encountered repeatedly while walking around the face are counted repeatedly to the degree.

A planar graph is called a *triangulation* if it is simple and all faces have degree 3. Much is known about triangulations. For example, a triangulation is 3-connected (it cannot become disconnected by removing 2 vertices), its edges can be split into three edge-disjoint spanning trees [Sch90], and its vertices can be ordered such that any prefix of the order induces a biconnected graph [FPP90]. Recently, we showed that any triangulation has a matching with at least $(n+4)/3$ edges [BCD$^+$00].

For many combinatorial problems on planar graphs, for example straight-line drawings and vertex-coloring, it suffices to study triangulations, since any planar graph can be triangulated by adding vertices and/or edges without changing the problem. We show now that this also holds for MAXIMUM MATCHING and PERFECT MATCHING.

Triangulated graphs are by nature not bipartite, but a related concept for bipartite graphs is a *quadrangulation*, where every face has degree 4. We also show that maximum matching in bipartite planar graphs reduces to maximum matching in quadrangulations.

5.1. Transformation T_t: Triangulating. Transformation T_t converts an arbitrary planar graph into a triangulation by inserting a subgraph into a face with degree ≥ 4. So assume that f is a face with degree $\ell \geq 4$. We insert ℓ vertices into f, and connect them as an ℓ-cycle C_1. Next we add a 2ℓ-cycle that connects alternatingly a vertex in C_1 with a vertex on the cycle C_0 that forms the boundary of f. (This does not add multiple edges since G has no loops.) Next we insert ℓ vertices inside C_1, and connect them as an ℓ-cycle C_2. Next we add a 2ℓ-cycle that connects alternatingly a vertex in C_2 with a vertex in C_1. This 2ℓ-cycle creates 2ℓ triangular faces, and inside each of them, we add one more vertex connected to the three vertices on the face. See Fig. 5.1. Finally we triangulate the inside of C_2 by adding edges in a zig-zag line, such that all degree on C_2 increase by at most 2. The resulting graph $T_t(G)$ has 4ℓ new vertices.

LEMMA 5.1. *$\mu(T_t(G)) = \mu(G) + 2\ell$, i.e., G has a matching of cardinality k if and only if $T_t(G)$ has a matching of cardinality $k + 2\ell$.*

Proof. For convenience of notation, we call the 2ℓ vertices of the two added cycles the "black" vertices, and the 2ℓ vertices added inside the triangles the "white" vertices. If M is a matching in G, then we can take the same matching in $T_t(G)$ and

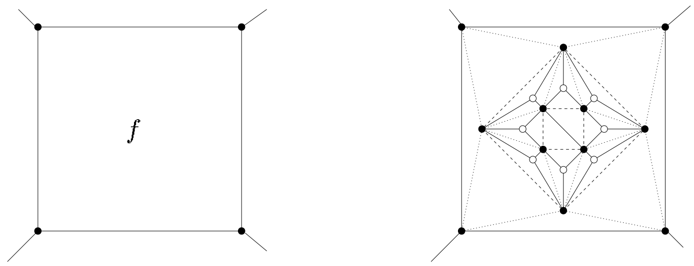

FIG. 5.1. *Triangulating a face of degree ℓ; shown here is $\ell = 4$. The ℓ-cycles are dashed and the 2ℓ-cycles are dotted.*

add 2ℓ edges, one from each white vertex to a black vertex, to get a matching of size $|M| + 2\ell$.

Assume that M' is a matching in $T_t(G)$. Note that any new edge is incident to one of the $2l$ black vertices, so at most 2ℓ of the added edges can belong to M'. Removing them gives a matching of size $\geq |M'| - 2\ell$ in G. □

Denote by $T_t^*(G)$ the graph that results from G by applying transformation T_t to all faces of degree ≥ 4. Then $T_t^*(G)$ is a triangulation. Also, $T_t^*(G)$ has a perfect matching if and only if G has one. Since the sum of the degrees of the faces in G is $2m(G) \leq 6n(G)$, the number of vertices of $T_t^*(G)$ is at most $n(G) + 24n(G) \in O(m(G))$.[4]

All added vertices have degree ≤ 9. If a vertex v has degree d in G, then it is incident to d faces of G, and at most $2d$ incident edges have been added to v in $T_t^*(G)$. In particular, if G has maximum degree 3, then $T_t^*(G)$ has maximum degree 9. Therefore, graph $T_t^*(T_4(G))$ proves that

- MATCHING IN PLANAR GRAPHS $\leq_l$ MATCHING IN MAX-DEG-9 TRIANGULATIONS and
- PERFECT MATCHING IN PLANAR GRAPHS $\leq_l$ PERFECT MATCHING IN MAX-DEG-9 TRIANGULATIONS.

5.2. Transformation T_q: Quadrangulating. Transformation T_q converts a bipartite planar graph into a quadrangulation by inserting a subgraph into each face with degree ≥ 6. So assume that G has a face f with degree $\ell \geq 6$. We insert an ℓ-cycle C_1 in f, and add a perfect matching between the vertices of C_1 and the vertices of the ℓ-cycle C_0 that forms the boundary of f. We then add another ℓ-cycle C_2 inside C_1, and add a perfect matching between C_2 and C_1.

Inside each of the ℓ resulting quadrangular faces between C_1 and C_2, we add two more vertices connected to two diagonally opposite vertices on the face. Finally we quadrangulate the inside of C_2 by adding a matching inside C_2; this is possible because G is bipartite and hence ℓ is even. See Fig. 5.2. The resulting graph $T_q(G)$ has 4ℓ new vertices.

LEMMA 5.2. *$\mu(T_q(G)) = \mu(G) + 2\ell$, i.e., G has a matching of cardinality k if and only if $T_q(G)$ has a matching of cardinality $k + 2\ell$.*

Proof. For convenience of notation, we call the 2ℓ vertices of the two added cycles the "black" vertices, and the 2ℓ vertices added inside the quadrangular faces the "white" vertices. If M is a matching in G, then we can take the same matching

[4] With a more careful analysis, this bound could be improved to $n(G) + 16n(G)$, since faces of degree 3 do not add new vertices.

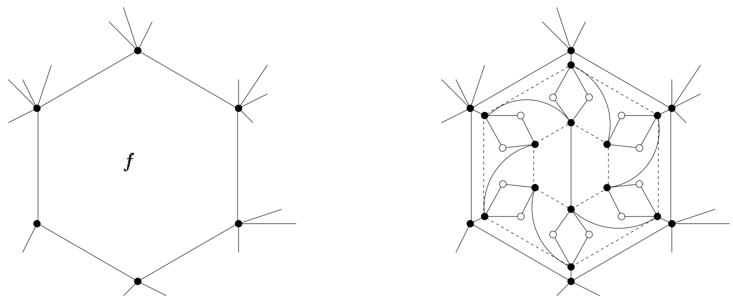

FIG. 5.2. *Quadrangulating a face of degree ℓ; shown here is $\ell = 6$. The ℓ-cycles are dashed.*

in $T_q(G)$ and add 2ℓ edges, one from each white vertex to a black vertex, to get a matching of size $|M| + 2\ell$.

Assume that M' is a matching in $T_q(G)$. Note that any new edge is incident to one of the $2l$ black vertices, so at most 2ℓ of the added edges can belong to M'. Removing them gives a matching of size $\geq |M'| - 2\ell$ in G. □

Denote by $T_q^*(G)$ the graph that results from G by applying transformation T_q to all faces of degree ≥ 6. Then $T_q^*(G)$ is a quadrangulation. Also, $T_q^*(G)$ has a perfect matching if and only if G has one. Since the sum of the degrees of the faces in G is $2m(G) \leq 4n(G)$ (recall that G is planar and bipartite), the number of vertices of $T_q^*(G)$ is at most $n(G) + 16n(G) \in O(m(G))$.[5]

All added vertices have degree ≤ 6. If a vertex v has degree d in G, then it is incident to d faces of G, and hence at most d incident edges have been added to v in $T_q^*(G)$. In particular, if G has maximum degree 3, then $T_q^*(G)$ has maximum degree 6. Therefore, graph $T_q^*(T_4(T_c(G)))$ proves that

- MATCHING IN PLANAR BIPARTITE GRAPHS $\leq_l$ MATCHING IN MAX-DEG-6 QUADRANGULATIONS and
- PERFECT MATCHING IN PLANAR BIPARTITE GRAPHS $\leq_l$ PERFECT MATCHING IN MAX-DEG-6 QUADRANGULATIONS.

6. Conclusion and open problems. In this paper, we studied how to reduce matching in an arbitrary graph to matching in a more structured graph, such as a 3-regular graph or a biconnected max-deg-3 graph. For planar graphs, we also reduced matching to matching in triangulations with maximum degree 9. Our result should have implications for future research into simplified/improved matching algorithms, since such research can restrict the these special graph classes.

Many other linear reductions are possible. In particular, one can reduce matching in 3-regular graphs to matching in 4-regular graphs, by doubling the graph and doing a transformation similar as Transformation T_{2b}. Another intriguing reduction is to reduce matching to weighted matching in biconnected 3-regular graphs (which have a perfect matching by Petersen's theorem). This is possible by first reducing the problem to a biconnected max-deg 3 graph (which has only vertices of degree 3 and 2), then doubling the graph, and finally adding a new edge between the two copies of any vertex of degree 2. If all new edges have weight 1, while all other edges have a weight of 0, then a maximum matching in the original graph can be found by computing a minimum-weight perfect matching in the new graph.

[5] With a more careful analysis, this bound could be improved to $n(G) + 12n(G)$, since faces of degree 4 do not add new vertices.

Our results highlight the sensitivity of Petersen's theorem: Every 3-regular biconnected graph has a perfect matching, but as soon as we are allowed to have (a) vertices of degree 2 or (b) cutvertices or (c) weights on the edges, such graphs can be used to encode *any* matching problem.

As for open problems, we are mainly interested in other simplifications of matching. In particular:

- Do we have MATCHING $\leq_l$ MATCHING IN BIPARTITE GRAPHS? Note that the time-complexities of the best currently known algorithms are equally high in both cases ($O(m\sqrt{n})$, see [HK73] and [MV80]), though the algorithm for the non-bipartite case is significantly more complicated.
  While such a reduction would not directly simplify the algorithm by Micali and Vazirani (because the time-complexity of $O(m\sqrt{n})$ is not maintained under a linear reduction), one would expect such a reduction to give enough insight into the structure of non-bipartite graphs to allow for a simpler algorithm.
  Note that for the *maximal* matching problem, a reduction that adds a log-factor is possible and has been used for parallel algorithms [KS00].
- Can matching be reduced to matching in planar graphs? That is, given a graph G drawn with c crossings, is there a planar graph G^* with $O(m(G)+c)$ edges such that a maximum matching in G can be recovered in $O(m(G)+c)$ time from a maximum matching in G^*? (This would not be a linear reduction, since $c \in \Omega(m^2)$ might be necessary.)

Finally, can the time-complexity of maximum matching be improved, especially for planar graphs?

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