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Three-Dimensional Orthogonal Graph Drawing with Optimal Volume*

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Abstract An orthogonal drawing of a graph is an embedding of the graph in the rectangular grid, with vertices represented by axis-aligned boxes, and edges represented by paths in the grid which only possibly intersect at common endpoints. In this paper, we study three-dimensional orthogonal drawings and provide lower bounds for three scenarios: (1) drawings where vertices have bounded aspect ratio, (2) drawings where the surface of vertices is proportional to their degree, and (3) drawings without any such restrictions. Then we show that these lower bounds are asymptotically optimal, by providing constructions that match the lower bounds in all scenarios within an order of magnitude.

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1 Introduction

Graph drawing is a field with a wide range of applications, for example in network visualisation, data base design and telecommunications. See the recent book [10] for an overview of techniques in graph drawing. Orthogonal graph drawing, where edges are routed along a rectangular grid, is a popular drawing style which is also appropriate for VLSI circuit layout.

In this paper we study three-dimensional orthogonal graph drawings. Such drawings have application in three-dimensional VLSI; see [1,2,18,22,23]. We improve on previous results by generalising the existing lower bounds on the volume, and by giving new constructions with smaller volume. In fact, our upper and lower bounds are matching up to a constant factor, and hence asymptotically optimal. We give lower bounds and constructions for three different drawing scenarios, achieving matching upper and lower bounds in all of them. To state our results precisely, we first give formal definitions and notations.

The (*three-dimensional*) *rectangular grid* is the cubic lattice, consisting of *grid points* with integer coordinates, together with the axis-parallel *grid lines* determined by these points. We use the word *box* to mean a three-dimensional axis-parallel box with integral boundaries, i.e., a box B is a set of points $\{(p_X, p_Y, p_Z) : l_I \leq p_I \leq r_I, I \in \{X, Y, Z\}\}$ for some integers $l_I, r_I,\ I \in \{X, Y, Z\}$. At each grid point in a box B that is extremal in some direction $d \in \{\pm X, \pm Y \pm Z\}$, we say there is *port* on B in direction d. One grid point can thus define up to six incidents ports. For each dimension $I \in \{X, Y, Z\}$, an I-*line* is a line parallel to the I-axis, an I-*segment* is a line-segment within an I-line, and an I-*plane* is a plane perpendicular to the I-axis.

Let $G = (V, E)$ be a graph, which is allowed to have parallel edges but no self loops. We denote the number of vertices of G by $n = |V|$, the number of edges of G by $m = |E|$, and the maximum degree of G by $\Delta(G)$, or Δ if the graph in question is clear.

An *orthogonal* (*box-*)*drawing* of G represents vertices by pairwise non-intersecting boxes. Hence vertices are possibly degenerate, in the sense that they may be represented by a rectangle or even a line-segment or a point. This is the approach taken in [5,8,26,27], but not in [21]. (Enlarging vertices to remove this degeneracy increases the volume by a multiplicative constant.) An edge $vw \in E$ is represented by a sequence of contiguous segments of grid lines possibly bent at grid points, between ports on the boxes of v and w. The intermediate grid points along the path representing an edge do not intersect the box of any vertex or any other edge route.

An orthogonal drawing with a particular shape of box representing every vertex, e.g. point, line-segment, or cube, is called an orthogonal *shape*-drawing for each particular shape. Initial research in orthogonal drawing was mostly concerned with point-drawings, see for example [9,11–13,21,25]. However, three-dimensional orthogonal point-drawings can only exist for graphs with maximum degree at most six. Overcoming this restriction has motivated recent interest in orthogonal box-drawings [5,8,21, 26,27].

From now on, we use the term *drawing* to mean a three-dimensional orthogonal box-drawing. Furthermore, the graph-theoretic terms ‘vertex’ and ‘edge’ also refer to their representation in a drawing. The *size* of a vertex v in a drawing is denoted by $X(v) \times Y(v) \times Z(v)$, where for each $I \in \{X, Y, Z\}$, $I(v)$ is one more than the length of the side of (the box of) v parallel to the I-axis. Thus, if the boundaries of v are $r_I(v)$ and $l_I(v)$, then $I(v) = r_I(v) - l_I(v) + 1$. The number of ports of v is called its *surface*, denoted by *surface(*v*)*. The number of grid points in a box is called its *volume*.

The smallest box enclosing a drawing is called the *bounding box* of the drawing. The *volume* of a drawing is the volume of its bounding box. The volume and the maximum number of bends per edge are the most commonly proposed measures for determining the aesthetic quality of a drawing. For box-drawings the size and shape of a vertex with respect to its degree are also considered an important measure of aesthetic quality. We use the following two measures for the shape of a vertex:

Degree-restricted drawings: We say that a vertex v is *strictly* α*-degree-restricted* if

$$\text{surface}(v) \leq \alpha \cdot \deg(v)$$

for some constant α. If there exists a constant α such that every vertex in a drawing is strictly α-degree-restricted then we say the drawing is *strictly* α*-degree-restricted*.

For some drawing algorithms, the minimum α such that the drawings produced by the algorithm are strictly α-degree-restricted does not necessarily reflect the asymptotic relationship between the surface and the degree of the vertices. We therefore say that in a drawing, a vertex v is α*-degree-restricted* if

$$\text{surface}(v) \leq \alpha \cdot \deg(v) + o\left(\deg(v)\right) \quad .$$

If for some constant α, every vertex v is α-degree-restricted, then the drawing is said to be (α)-*degree-restricted*. This definition enables us to compare the asymptotic behaviour of α for various algorithms.

Clearly, if a drawing is strictly degree-restricted then it is also degree-restricted. Conversely, it is easily seen that the degree-restricted drawings produced by the algorithm presented in this paper (and all known algorithms) are also strictly degree-restricted, thus for our purposes the two notions coincide. However, one can contrive examples where this is not the case. It is necessary to distinguish the two terms as the lower bound in Theorem 2 is for strictly degree-restricted drawings.

Aspect ratio: The *aspect ratio* of a vertex v is the ratio between its largest and smallest side, i.e.,

$$\max\left\{X(v), Y(v), Z(v)\right\} / \min\left\{X(v), Y(v), Z(v)\right\} \quad .$$

In particular, a vertex with aspect ratio 1 is a cube. We say that a drawing *has bounded aspect ratios* if there exists a constant r such that all vertices have aspect ratio at most r.

There is no inherent relationship between whether a drawing is degree-restricted or has bounded aspect ratios. Previously, algorithms have been presented that give drawings that are degree-restricted, but do not have bounded aspect ratios [5,26,27]. It is conceivable that a drawing could have bounded aspect ratios, but not be degree-restricted (for example, by representing each vertex v with a $\deg(v) \times$

$\deg(v) \times \deg(v)$-box), though no algorithms to create such drawings have been presented, and as our lower bound results show, no improvement in volume is possible by doing so.

1.1 Lower bounds

For a graph G, denote by $vol(G, r, \alpha)$ the minimum volume, taken over all (orthogonal) drawings of G that have aspect ratios at most r and are strictly (α)-degree-restricted. Let $vol(n, m, r, \alpha)$ be the maximum, taken over all graphs G with n vertices and m edges, of $vol(G, r, \alpha)$. Thus, $vol(n, m, r, \alpha)$ describes a volume bound within which we can draw all graphs with n vertices and m edges such that each vertex v has aspect ratio at most r and surface at most $\alpha \cdot \deg(v)$.

The first lower bounds on the volume were due to Hagihara *et al.* [16][1]. They show that, in the above notation, $vol(n, m, 1, 1) = \Omega(\max\{n\Delta^2, (n\Delta/\log n)^{3/2}\})$. In fact, in their construction the graphs are Δ-regular, hence $m = \frac{1}{2}n\Delta$, which allows us to restate their result as

$$vol(n, m, 1, 1) = \Omega\left(\max\left\{m\Delta, (m/\log n)^{3/2}\right\}\right) \quad .$$

In this paper, we show that:

- $vol(n, m, \infty, \infty) = \Omega(m\sqrt{n})$
- $vol(n, m, r, \infty) = \Omega(m^{3/2}/\sqrt{r})$
- $vol(n, m, \infty, \alpha) = \Omega(m^{3/2}/\alpha)$

We thus improve the results of [16] in three ways: Firstly, we remove the log-factor, to establish $vol(n, m, 1, 1) = \Omega(m^{3/2})$ as the lower bound. Secondly, we show that the result holds even if only one of the conditions of having bounded aspect ratios and being strictly degree-restricted holds. Finally, we also study the case when neither of these two conditions hold. The lower bound here appears weaker, but as we prove by giving a construction later, it is asymptotically optimal.

[1] This paper seems widely unknown in the graph drawing community.

Our first result includes the lower bound of $\Omega(n^{5/2})$ for drawings of K_n established by Biedl *et al.* [8]. In fact, the proof of our lower bounds are based on techniques developed in this paper, generalised to graphs with fewer edges.

1.2 Algorithms

A trade-off between the maximum number of bends per edge route and the bounding box volume is apparent in algorithms for orthogonal graph drawing. Biedl *et al.* [8] construct drawings of K_n with $\mathcal{O}(n^{5/2})$ volume and three bends per edge, but these drawings are not degree-restricted. They also construct drawings of K_n with $\mathcal{O}(n^3)$ volume and one bend per edge, but these drawings are degree-restricted only for graphs where all vertices have degree $\theta(n)$. Recently, the first author showed that $\Omega(n^3)$ volume is required for K_n if only one bend per edge is allowed [4].

A drawing is said to be in *general position* if no two vertices are in a common grid plane. The algorithm of Papakostas and Tollis [21] produces general position drawings with $\mathcal{O}(m^3)$ volume. This bound has been improved to $\mathcal{O}((nm)^{3/2})$ for cube-drawings and $\mathcal{O}(n^2 m)$ for line-segment-drawings in general position by Biedl [5] and Wood [26].

The *lifting half-edges* technique developed by Biedl [5] generates drawings of simple graphs starting with a two-dimensional general position drawing [7] (possibly with overlapping edges). The edge routes are partitioned into sub-drawings each consisting of X-segments or Y-segments. Each sub-drawing is then assigned its own Z-plane, vertices are extended to form lines passing through each layer, and vertical segments are added to the edges in such a way to avoid crossings. Using this technique an improved volume bound of $\mathcal{O}(n^2 \Delta)$ is attained. At the cost of an increase in volume, cube-drawings can also be produced in the lifting half-edges model.

The *plane drawing* technique consists of positioning vertices in the $(Z = 0)$-plane, and then routing edges above and possibly also below the $(Z = 0)$-plane. The two algorithms presented in this paper use the plane drawing technique. The first algorithm produces degree-restricted cube-drawings with $\mathcal{O}(m^{3/2})$

volume and at most six bends per edge. The technique used is a generalisation of the COMPACT algorithm of Eades *et al.* [13] for point-drawings, and is an improvement on the algorithms of Hagihara *et al.* [16] and Wood [27], who obtained upper bounds of $\mathcal{O}((n\Delta)^{3/2})$ and $\mathcal{O}(m^2/\sqrt{n})$, respectively. Our second algorithm produces box-drawings with $\mathcal{O}(m\sqrt{n})$ volume and at most four bends per edge; these drawings are not degree-restricted nor do the vertices have bounded aspect ratios [2].

Both upper bounds are therefore within an order of magnitude of the lower bound. We also present refinements of both our algorithms with one less bend per edge, at the cost of an increase in the volume. Table 1 summarises the known bounds for orthogonal graph drawing.

2 Lower Bounds

In this section we prove lower bounds on the volume of orthogonal graph drawings. Such lower bounds were previously only known for drawings of the complete graph K_n [8]. The crucial argument for K_n is that between any two disjoint vertex sets of size $\theta(n)$ in K_n, there are $\theta(n^2)$ edges. To generalise this to arbitrary graphs, we first exhibit graphs such that between any two disjoint vertex sets of size $\theta(n)$ there are $\theta(m)$ edges. Then we use an argument similar to that in [8] to obtain lower bounds on the volume.

2.1 Graphs with large cuts

Suppose $G = (V, E)$ is a graph and $S, T \subset V$ are disjoint sets of vertices. Let $e(S, T)$ denote the number of edges between S and T. For our lower bound proofs, we need graphs for which $e(S, T)$ is large under some conditions on S and T, and we prove the existence of such graphs in the following lemma.

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[2] These results assume that $m \geq n$, which one would expect in most applications. If $m < n$, then the volume of the drawings produced is $\mathcal{O}(n^{3/2})$ in both constructions.

Table 1 The tradeoff between volume and the maximum number of bends in orthogonal graph drawings for various aesthetic criteria. All lower bounds are proved in Theorem 2.

lower bound	volume	bends	model	graphs	reference
bounded aspect ratio / degree-restricted					
$\Omega(m^{3/2})$	$\mathcal{O}((nm)^{3/2})$	2	general position	simple	[5,26]
$\Omega(m^{3/2})$	$\mathcal{O}(nm\sqrt{\Delta})$	2	lifting $\frac{1}{2}$-edges	simple	[5]
$\Omega(m^{3/2})$	$\mathcal{O}(m^2)$	5	plane layout	multigraphs	Thm. 4
$\Omega(m^{3/2})$	$\mathcal{O}((n\Delta)^{3/2})$	10 [3]	plane layout	simple	[16]
$\Omega(m^{3/2})$	$\mathcal{O}(m^{3/2})$	6	plane layout	multigraphs	Thm. 3
no bounds on aspect ratio / degree-restricted					
$\Omega(m^{3/2})$	$\mathcal{O}(n^2 m)$	2	general position	simple	[5,26]
$\Omega(m^{3/2})$	$\mathcal{O}(n^2 \Delta)$	2	lifting $\frac{1}{2}$-edges	simple	[5]
$\Omega(m^{3/2})$	$\mathcal{O}(m^2)$	5	plane layout	multigraphs	Thm. 4
$\Omega(m^{3/2})$	$\mathcal{O}(m^{3/2})$	6	plane layout	multigraphs	Thm. 3
no bounds on aspect ratio / not necessarily degree-restricted					
$\Omega(m\sqrt{n})$	$\mathcal{O}(n^3)$	1	lifting edges	simple	[8]
$\Omega(m\sqrt{n})$	$\mathcal{O}(nm^{3/2})$	1	book embedding	multigraphs	[28]
$\Omega(m\sqrt{n})$	$\mathcal{O}(n^{5/2})$	3	lifting edges	simple	[8]
$\Omega(m\sqrt{n})$	$\mathcal{O}(mn)$	3	plane layout	multigraphs	Thm. 6
$\Omega(m\sqrt{n})$	$\mathcal{O}(m\sqrt{n})$	4	plane layout	simple	Thm. 5

Lemma 1 *If $p \neq q$ are primes, $p \equiv 1 \bmod 4$, $q \equiv 1 \bmod 4$, $144 \leq p < q(q-1)/2$, then there exists a simple n-vertex graph $G_{p,q} = (V, E)$ with the following properties:*

– *$G_{p,q}$ is d-regular with $d = p + 1$.*

[3] Hagihara *et al.* [16] did not consider the number of bends per edge; we deduce the bound of 10 from their construction.

– $q(q-1)/2 \leq n \leq q(q-1)$.

– *For any disjoint sets* $S, T \subset V$ *of vertices of* $G_{p,q}$ *with* $|S||T| \geq n^2/36$ *we have*

$$e(S,T) \geq C \cdot dn \,,$$

where $C > 0.00009$ *is a constant.*

Before proving this lemma, we need some background. For a graph G, denote by $\lambda(G)$ the second largest eigenvalue of its adjacency matrix. The following well-known inequality (see for example [3, pp. 119-125] and [24]) relates $\lambda(G)$ to the cut property we are interested in.

Lemma 2 *Let* $G = (V, E)$ *be a* d*-regular* n*-vertex graph with second largest eigenvalue* $\lambda(G)$*. Then, for all disjoint sets* $S, T \subset V$ *we have*

$$e(S,T) \geq \frac{d|S||T|}{n} - \lambda(G)\sqrt{|S||T|} \,.$$

This lemma suggest to look for graphs G with small $\lambda(G)$. Fortunately, such graphs (called *Ramanujan graphs*) have already been constructed.

Lemma 3 ([19,20]) *If* $p \neq q$ *are primes,* $p \equiv 1 \bmod 4$*,* $q \equiv 1 \bmod 4$*, then there exists a simple* n*-vertex graph* $G_{p,q} = (V, E)$ *with the following properties:*

– $G_{p,q}$ *is* d*-regular with* $d = p + 1$.

– $q(q-1)/2 \leq n \leq q(q-1)$.

– $\lambda(G_{p,q}) \leq 2\sqrt{d-1}$.

Remark. This second-largest eigenvalue is known to be asymptotically optimal, see [19].

Now we are in a position to prove Lemma 1.

Proof Given p and q, construct the graph $G_{p,q} = (V, E)$ as in Lemma 3. Let $S, T \subset V$ be two disjoint sets of vertices of $G_{p,q}$ with $|S||T| \geq n^2/36$. From Lemma 2, we know that

$$e(S,T) \geq \frac{d|S||T|}{n} - \lambda(G_{p,q})\sqrt{|S||T|} \,.$$

By $|S||T| \geq n^2/36$ and $d = p+1 \geq 145$ therefore

$$\lambda(G_{p,q})\sqrt{|S||T|} \leq 2\sqrt{d-1}\sqrt{|S||T|} \cdot \underbrace{6\sqrt{|S||T|}/n}_{\geq 1} \cdot \underbrace{\sqrt{d}/\sqrt{145}}_{\geq 1} \leq 12/\sqrt{145} \cdot d|S||T|/n\,.$$

Hence,

$$e(S,T) \geq (1 - 12/\sqrt{145}) \cdot d|S||T|/n \geq \frac{1}{36}(1 - 12/\sqrt{145})dn\,,$$

which proves the claim for $C = \frac{1}{36}(1 - 12/\sqrt{145}) \approx 0.000096$. □

For future reference, we will call the constant C the *Ramanujan-constant*.

2.2 Lower bounds on the volume of drawings

We start by proving the lower bound for those graphs that satisfy the conditions of Lemma 1. The proof is based on the technique developed in [8], which distinguishes three cases: either many vertices are intersected by one grid line, or many vertices are intersected by one grid plane, or neither of these is the case. Our approach is different in two ways: we use graphs with large cuts, rather than K_n, and we incorporate considerations of the aspect ratio and degree-restrictions.

Theorem 1 *Let $G = (V,E)$ be an d-regular simple graph with $n \geq 8$ vertices such that for any disjoint sets $S, T \subset V$ with $|S||T| \geq n^2/36$ we have $e(S,T) \geq C \cdot dn$. Then*

- $vol(G, \infty, \infty) \geq \frac{1}{3}C^{3/2} \cdot dn^{3/2}$
- $vol(G, r, \infty) \geq \frac{1}{3}C^{3/2} \cdot (dn)^{3/2}/\sqrt{r}$
- $vol(G, \infty, \alpha) \geq \frac{1}{3}C^{2} \cdot (dn)^{3/2}/\alpha$

Proof Consider an orthogonal drawing of G in a grid of dimensions $X \times Y \times Z$.

Case 1: A line intersects many vertices

Assume that there exists a Z-line that intersects at least $2\lceil \frac{1}{6}n \rceil$ vertices. Let $v_1, \ldots, v_t$ be the vertices intersected by the Z-line, listed in order of occurrence along the line. Let Z_0 be a not necessarily integer

Z-coordinate such that the ($Z = Z_0$)-plane intersects none of these t vertices and separates the first $\lceil \frac{1}{6}n \rceil$ of them from the remaining ones, of which there are at least $\lceil \frac{1}{6}n \rceil$.

By assumption at least $C \cdot dn$ edges connect these two groups. These edges cross the ($Z = Z_0$)-plane, which thus must contain at least $C \cdot dn$ points having integer X- and Y-coordinates. Hence $XY \geq C \cdot dn$. Since the Z-line intersects at least $2\lceil \frac{1}{6}n \rceil \geq \frac{1}{3}n$ vertices, we have $Z \geq \frac{1}{3}n$; thus $XYZ \geq \frac{1}{3}C \cdot dn^2 \geq \frac{1}{3}C^{3/2} \cdot (dn)^{3/2}$ by $C \leq 1$ and $d \leq n$. This proves all claims.

Similarly, one proves all claims if any X-line or any Y-line intersects at least $2\lceil \frac{1}{6}n \rceil$ vertices.

Case 2: No plane intersects many vertices

Assume that any X-plane, Y-plane or Z-plane intersects at most $n - 2\lceil \frac{1}{6}n \rceil + 1$ vertices. A vertex is *left* of an ($X = X_0$)-plane if all the points in its grid box have X-coordinates less than X_0. The notion of *right* of an ($X = X_0$)-plane is analogous. As an ($X = X_0$)-plane is swept from smaller to larger values of X_0, it intersects at most $n - 2\lceil \frac{1}{6}n \rceil + 1$ vertices at any time by assumption.

During the sweep by the ($X = X_0$)-plane, an integer X' is encountered where, for the last time, there at most $\lceil \frac{1}{6}n \rceil - 1$ vertices left of the ($X = X'$)-plane. Since the ($X = X'$)-plane intersects at most $n - 2\lceil \frac{1}{6}n \rceil + 1$ vertices, there are at least $\lceil \frac{1}{6}n \rceil$ vertices right of the ($X = X'$)-plane. All these vertices also lie to the right of ($X = X'+\frac{1}{2}$)-plane.

By definition of X', the number of vertices that lie left of the ($X = X'+1$)-plane is at least $\lceil \frac{1}{6}n \rceil$. All these vertices also lie to the left of ($X = X'+\frac{1}{2}$)-plane.

By assumption there are at least $C \cdot dn$ edges between the vertices on the left and the vertices on the right of the ($X = X'+\frac{1}{2}$)-plane, thus $YZ \geq C \cdot dn$. Since the same argument holds for the other two directions, $XYZ = \sqrt{XY \cdot YZ \cdot XZ} \geq (C \cdot dn)^{3/2}$, which proves all claims.

Case 3: A plane intersects many vertices

Assume now that none of the previous cases are true. Therefore any X-line, Y-line or Z-line intersects at most $2\lceil \frac{1}{6}n \rceil - 1$ vertices, but there exists, say, a ($Z = Z_0$)-plane that intersects at least $n - 2\lceil \frac{1}{6}n \rceil + 2 \geq \frac{2}{3}n$ vertices. As an ($X = X_0$)-plane is swept from smaller to larger values of X_0, the Y-line determined by the intersection of this ($X = X_0$)-plane with the ($Z = Z_0$)-plane sweeps the ($Z = Z_0$)-plane. At any time, this Y-line intersects at most $2\lceil \frac{1}{6}n \rceil - 1$ vertices by assumption.

During the sweep by the ($X = X_0$)-plane, an integer X' is encountered where, for the last time, there are at most $\lceil \frac{1}{6}n \rceil - 2$ vertices left of the ($X = X'$)-plane and intersecting the ($Z = Z_0$)-plane. Since the Y-line determined by the intersection of the ($X = X'$)-plane and the ($Z = Z_0$)-plane intersects at most $2\lceil \frac{1}{6}n \rceil - 1$ vertices, and the ($Z = Z_0$)-plane intersects at least $n - 2\lceil \frac{1}{6}n \rceil + 2$ vertices, at least $n - 5\lceil \frac{1}{6}n \rceil + 5$ vertices intersect the ($Z = Z_0$)-plane and lie right of the ($X = X'$)-plane. All these vertices, which we denote by S, also lie to the right of the ($X = X' + \frac{1}{2}$)-plane.

By definition of X', the number of vertices that intersect the ($Z = Z_0$)-plane and that lie left of the ($X = X' + 1$)-plane is at least $\lceil \frac{1}{6}n \rceil - 1$. All these vertices, which we denote by T, also lie to the left of the ($X = X' + \frac{1}{2}$)-plane.

Note that $|S||T| \geq (n - 5\lceil \frac{1}{6}n \rceil + 5)(\lceil \frac{1}{6}n \rceil - 1)$, and we claim that this is at least $\frac{1}{36}n^2$. If $n \equiv 0 \bmod 6$ then $|S||T| \geq (n - \frac{5}{6}n + 5)(\frac{1}{6}n - 1) = \frac{1}{36}n^2 + \frac{4}{6}n - 5 \geq \frac{1}{36}n^2$ (since $n \geq 8$). If $n \equiv k \bmod 6$ for some k, $1 \leq k \leq 5$, then $\lceil \frac{1}{6}n \rceil = \frac{1}{6}(n + 6 - k)$ and $|S||T| \geq (n - \frac{5}{6}(n + 6 - k) + 5)((\frac{1}{6}n + 6 - k) - 1) = \frac{1}{36}(n^2 + 4kn - 5k^2) \geq \frac{1}{36}n^2$ by $n \geq 8$ and $k \leq 5$.

By assumption there are at least $C \cdot dn$ edges between T and S, thus $YZ \geq C \cdot dn$. Apply exactly the same argument in the Y-direction to obtain $XZ \geq C \cdot dn$.

Now we obtain the three lower bounds as follows:

– The ($Z = Z_0$)-plane intersects at least $\frac{2}{3}n$ vertices, thus $XY \geq \frac{2}{3}n$. This implies $XYZ = \sqrt{XY \cdot YZ \cdot XZ} \geq \sqrt{\frac{2}{3}n \cdot (C \cdot dn)^2} = \sqrt{\frac{2}{3}}C \cdot dn^{3/2}$, which proves the first lower bound.
– Assume that every vertex has aspect ratio at most r ($r \geq 1$). In particular therefore, $Z(v) \leq rX(v)$ and $Z(v) \leq rY(v)$ for every vertex v. Since the surface of v is at least $\deg(v)$, we have $\deg(v) \leq 2(X(v)Y(v) + Y(v)Z(v) + X(v)Z(v)) \leq 6rX(v)Y(v)$.

Thus, every vertex v that intersects the $(Z = Z_0)$-plane has $X(v)Y(v) \geq \deg(v)/6r = d/6r$. Since the $(Z = Z_0)$-plane intersects at least $\frac{2}{3}n$ vertices, and these intersections are disjoint, there must be at least $\frac{2}{3}n \cdot d/6r$ grid points in the $(Z = Z_0)$-plane, hence $XY \geq \frac{1}{9} \cdot dn/r$.

Therefore $XYZ = \sqrt{XY \cdot YZ \cdot XZ} \geq \sqrt{\frac{1}{9} \cdot dn/r \cdot (C \cdot dn)^2} = \frac{1}{3}C \cdot (dn)^{3/2}/\sqrt{r}$, which proves the second lower bound.

– Assume that the surface of every vertex v is at most $\alpha \cdot \deg(v) = \alpha d$ ($\alpha \geq 1$), which in particular implies that $Z(v) < \alpha d/4$. Define $Z_- = Z_0 - \alpha d/4$ and $Z_+ = Z_0 + \alpha d/4$. We say that a point is *inside* if its Z-coordinate z satisfies $Z_- < z < Z_+$, and *outside* otherwise. Note that all vertices in T and S cross the $(Z = Z_0)$-plane, hence they can cross neither the $(Z = Z_-)$-plane nor the $(Z = Z_+)$-plane, and all ports of all vertices in T and S are inside.

  Define $X^* = X' + \frac{1}{2}$; we have shown above that at least $C \cdot dn$ edges cross the $(X = X^*)$-plane. Of these, at least $C \cdot dn - Y \cdot \alpha d/2$ edges cross the $(X = X^*)$-plane at an outside point, because there are at most $Y \cdot \alpha d/2$ inside points with integer Y- and Z-coordinate on the $(X = X^*)$-plane.

  Each of these $C \cdot dn - Y \cdot \alpha d/2$ edges starts at a vertex in T (therefore at an inside point), crosses the $(X = X^*)$-plane at an outside point, and ends at a vertex in S (therefore at an inside point). This implies that each edge crosses either the $(Z = Z_-)$-plane or the $(Z = Z_+)$-plane at least twice. These two planes together therefore must have at least $2(C \cdot dn - Y \cdot \alpha d/2)$ points with integral X- and Y-coordinate, therefore $XY \geq C \cdot dn - Y \cdot \alpha d/2$. Applying the exact same argument in the Y-direction, we obtain $XY \geq C \cdot dn - X \cdot \alpha d/2$, so $XY \geq C \cdot dn - \min\{X, Y\}\alpha d/2$.

  If $\min\{X, Y\} \leq Cn/\alpha$, then this implies $XY \geq C \cdot dn/2$, therefore $XYZ = \sqrt{XY \cdot YZ \cdot XZ} \geq \sqrt{\frac{1}{2}C \cdot dn \cdot (C \cdot dn)^2} = \frac{1}{\sqrt{2}}C^{3/2} \cdot (dn)^{3/2}$. If $X > Cn/\alpha$ and $Y > Cn/\alpha$, then $XY > (Cn)^2/\alpha^2$, and $XYZ = \sqrt{XY \cdot YZ \cdot XZ} \geq \sqrt{(C^2 \cdot n^2/\alpha^2) \cdot (C \cdot dn)^2} = C^2 \cdot dn^2/\alpha \geq C^2 \cdot (dn)^{3/2}/\alpha$ by $d \leq n$. Either way, the third claim is proved.

□

Using this theorem, we construct the lower bound for arbitrary values of m and n, as long as both are large enough. Before proving this, we need a result that shows that primes $\equiv 1 \bmod 4$ are frequent.

Lemma 4 *There exist constants $k \geq 2$ and $x_1 \geq k$ such that for all $x \geq x_1$, the interval $[\frac{1}{k}x, x]$ contains a prime number p with $p \equiv 1 \bmod 4$.*

Proof Denote by $\pi_{4,1}(x)$ the number of primes $p \leq x$ that satisfy $p \equiv 1 \bmod 4$. A famous theorem by de la Vallée Poussin establishes that

$$\pi_{4,1}(x) = \theta\left(\frac{1}{\phi(4)} \cdot \frac{x}{\log x}\right) ,$$

where ϕ is Euler's function, in particular $\phi(4) = 2$. See for example [15] for a proof.

Let c_1, c_2, x_0 be constants such that $c_2 \geq c_1$ and

$$c_1 \cdot \frac{x}{\log x} \leq \pi_{4,1}(x) \leq c_2 \cdot \frac{x}{\log x}$$

for all $x \geq x_0$. Let $k = 2c_2/c_1$. If $x_1 \geq x_0$ is so big that also $\log(k) \leq \frac{1}{4}\log(x)$ and $x/\log x \geq 3/c_1$ for all $x \geq x_1$, then

$$\pi_{4,1}(x) - \pi_{4,1}\left(\frac{1}{k}x\right) \geq c_1 \frac{x}{\log x} - c_2 \frac{\frac{c_1}{2c_2}x}{\log(\frac{1}{k}) + \log x} \geq \frac{c_1 x}{\log x} - \frac{\frac{1}{2}c_1 x}{\frac{3}{4}\log x} \geq \frac{1}{3}\frac{c_1 x}{\log x} \geq 1 ,$$

hence there is at least one prime number $\equiv 1 \bmod 4$ between $\frac{1}{k}x$ and x. $\square$

To establish this lower bound, we prove that for sufficiently large n and sufficiently large m there exist a graph G with n vertices and m edges that has an induced subgraph G' that satisfies the conditions of Theorem 1. Moreover, G' is asymptotically as big as G, i.e., G' has $\theta(n)$ vertices and $\theta(m)$ edges. G' is also d'-regular, and vertices in G' have degree $\theta(d')$ in G. The precise conditions are as follows:

Lemma 5 *Let $n \geq \max\{32k^4, 2x_1^2\}$ and $m \geq \max\{\frac{1}{4}(144x_1 + 1)n, \frac{145}{4}kn\}$, where $k \geq 2$ and $x_1 \geq k$ are the constants of Lemma 4. Then there exists a simple graph G with n vertices and m edges that has a d'-regular subgraph G' with n' vertices and m' edges such that*

– *G' satisfies the conditions of Theorem 1.*

– $n \geq n' \geq n/8k^2$.

– $m \geq m' = d'n'/2 \geq m/64k^4$.

– *Every vertex of G' has degree at most $8k^2d'$ in G.*

Proof The proof splits into two cases, depending on the size of m. If m is big enough, then G' can be a complete graph; if m is small, then we use a Ramanujan graph for G'. In both cases, we “pad” G' with additional vertices and edges to achieve the desired number of vertices and edges.

Case 1: $m \geq n^2/64k^4$. In this case, let G' be the complete graph on $n' = \lceil n/8k^2 \rceil + 1$ vertices.

Clearly G' as a complete graph satisfies the conditions of Theorem 1. Also, $n/8k^2 + 1 \leq n' \leq n/8k^2 + 2 \leq n$. G' has $m' = \binom{n'}{2}$ edges, which is not too many edges since $m' \leq (n')^2/2 \leq (n/8k^2 + 2)^2/2 = n^2/128k^4 + n/4k^2 + 2 \leq n^2/64k^4 \leq m$ by $n \geq 32k^4$. Also, $m' = \binom{n'}{2} \geq (n/8k^2)^2/2 = 1/64k^4 \cdot n^2/2 \geq 1/64k^4 \cdot m$ by $m \leq \binom{n}{2}$.

To obtain G, add $n - n'$ vertices to G', and add $m - m'$ arbitrary edges such that the resulting graph is simple; this is possible since $m \leq \binom{n}{2}$. The maximum degree of G is $\leq n - 1$. G' is d'-regular with $d' = n' - 1 \geq n/8k^2$, hence the degree (in G) of any vertex in G' is at most $n - 1 \leq n \leq 8k^2d'$.

Case 2: $m < n^2/64k^4$. In this case, G' will be a Ramanujan-graph $G_{p,q}$ for some carefully chosen primes p and q.

Let $q' = \frac{1}{2} + \sqrt{\frac{n}{2} + \frac{1}{4}}$, which implies that $q'(q' - 1) = n/2$, $q' \leq \sqrt{n/2} + 1$ and $q' \geq \sqrt{n/2} \geq x_1$. Find a prime q with $q \equiv 1 \bmod 4$ such that $\frac{1}{k}q' \leq q \leq q'$; this exists by Lemma 4. Note that $q(q - 1) \leq q'(q' - 1) = n/2$. Also, since $n \geq 32k^4$ we have $\sqrt{n/2} \leq n/8k^2$ and $1 \leq n/8k^2$, hence $\sqrt{n/2} + 1 \leq n/4k^2$ and $q(q - 1) \geq q^2 - q \geq (q')^2/k^2 - q' \geq n/2k^2 - (\sqrt{n/2} + 1) \geq n/4k^2$.

Let $p' = 2m/q(q - 1) - 1$; since $m \geq \frac{1}{4}(144x_1 + 1)n$ and $q(q - 1) \leq n/2$ this implies that $p' \geq 144x_1$. Find a prime p with $p \equiv 1 \bmod 4$ such that $\frac{1}{k}p' \leq p \leq p'$; this exists by Lemma 4. We have

$p \geq \frac{1}{k}p' \geq \frac{1}{k}144x_1 \geq 144$. Also, by $m \leq n^2/64k^4$ and $q(q-1) \geq n/4k^2$ we have

$$p \leq p' < \frac{2n^2}{64k^2}\frac{1}{q(q-1)}\left(\frac{q(q-1)}{n/4k^2}\right)^2 = \frac{q(q-1)}{2}\,.$$

Let G' be the graph $G_{p,q}$ for primes p and q as defined in Lemma 1 and suppose it has n' vertices and m' edges. By Lemma 1 G' satisfies the conditions of Theorem 1. Also,

$$n \geq \frac{n}{2} \geq q(q-1) \geq n' \geq \frac{1}{2}q(q-1) \geq \frac{n}{8k^2}\,.$$

By definition G' is d'-regular with $d' = p+1 \leq p'+1$, thus m' satisfies

$$m' = \frac{1}{2}(p+1)n' \leq \frac{1}{2}(p'+1)q(q-1) = m\,,$$

and

$$m' = \frac{1}{2}(p+1)n' \geq \frac{1}{2k}(p'+k)\frac{1}{2}q(q-1) \geq \frac{1}{2k}m\,.$$

Create G by adding $n-n'$ vertices to G'. Since $m < n^2/64k^4$ we have $2m \leq n^2/32 \leq (n/2-1)^2 \leq (n-q(q-1)-1)^2 \leq (n-n'-1)^2$, hence $m \leq \binom{n-n'}{2}$ and we can add $m-m'$ edges between the $n-n'$ added vertices in such a way that G is simple. Note that no incident edges were added to any vertex in G'; in particular the degree of vertices in G' remains d'. This proves the claim in both cases. $\square$

Now we use these graphs to prove the lower bounds for almost any value of n and m.

Lemma 6 *Let $C \leq 1$ be the Ramanujan-constant and let $k \geq 2$ and x_1 be the constants of Lemma 4. Then for any $n \geq \max\{32k^4, 2x_1^2\}$ and $m \geq \max\{\frac{1}{4}(144x_1+1)n, \frac{145}{4}kn\}$, there exists a graph G with n vertices and m edges such that:*

– $vol(G,\infty,\infty) \geq \sqrt{2}/(384k^5) \cdot C^{3/2} \cdot m\sqrt{n}$

– $vol(G,r,\infty) \geq \sqrt{2}/(768k^6) \cdot C^{3/2} \cdot m^{3/2}/\sqrt{r}$

– $vol(G,\infty,\alpha) \geq \sqrt{2}/(6144k^8) \cdot C^2 \cdot m^{3/2}/\alpha$

Proof Let G be the graph of Lemma 5. Any drawing of G contains a drawing of G', and thus has volume at least $\frac{1}{3}C^{3/2}d'(n')^{3/2}$ by Theorem 1. Using the known bounds on n' and m' from Lemma 5, we obtain a lower bound on the volume of at least

$$\frac{1}{3}C^{3/2} \cdot 2m' \cdot \sqrt{n'} \geq \frac{2}{3}C^{3/2} \cdot m/64k^4 \cdot \sqrt{n/8k^2} = \sqrt{2}/(384k^5) \cdot C^{3/2} \cdot m\sqrt{n}\,,$$

which proves the first claim.

If the drawing of G has aspect ratios at most r, then so does the drawing of G', which thus has volume at least $\frac{1}{3}C^{3/2}(d'n')^{3/2}/\sqrt{r}$, which is at least

$$\frac{1}{3}C^{3/2} \cdot (2m')^{3/2}/\sqrt{r} \geq \sqrt{8}/3 \cdot C^{3/2} \cdot (m/64k^4)^{3/2}/\sqrt{r} = \sqrt{2}/(768k^6) \cdot C^{3/2} \cdot m^{3/2}/\sqrt{r}\,;$$

this proves the second claim.

If the drawing of G is strictly (α)-degree-restricted, then every vertex in G' has surface at most $\alpha d' \leq \alpha 8k^2 d'$. Thus the drawing of G' is strictly $(\alpha 8k^2)$-degree-restricted, and has volume at least $\frac{1}{3}C^2(d'n')^{3/2}/\alpha 8k^2$, which is at least

$$1/24k^2 \cdot C^2 \cdot (2m')^{3/2}/\alpha \geq \sqrt{2}/12k^2 \cdot C^2 \cdot (m/64k^4)^{3/2}/\alpha = \sqrt{2}/(6144k^8) \cdot C^2 \cdot m^{3/2}/\alpha\,;$$

this proves the third claim. □

From Lemma 6 we can conclude the main result of this section.

Theorem 2 *We have the following lower bounds:*

- $vol(n, m, \infty, \infty) = \Omega(m\sqrt{n})$
- $vol(n, m, r, \infty) = \Omega(m^{3/2}/\sqrt{r})$
- $vol(n, m, \infty, \alpha) = \Omega(m^{3/2}/\alpha)$

3 Constructions

In the following, we give two constructions. The first creates degree-restricted cube-drawings with asymptotically optimal volume. The second creates drawings without restrictions on vertex boxes; again the drawings produced have asymptotically optimal volume.

3.1 Cube-drawings

In the following algorithm for producing orthogonal drawings, each vertex v is initially represented by a square of size $\mathcal{O}(\sqrt{\deg(v)}) \times \mathcal{O}(\sqrt{\deg(v)})$ in the ($Z = 0$)-plane. (The algorithm by Hagihara *et al.* [16] is similar in spirit, but uses squares of size $\mathcal{O}(\sqrt{\Delta})$ for each vertex, hence resulting in a drawing with $\mathcal{O}((n\Delta)^{3/2})$ volume.) Edges are then routed either above or below the ($Z = 0$)-plane in a similar manner to the COMPACT point-drawing algorithm of Eades *et al.* [13]. Finally, the vertices are extended in the Z-dimension to form cubes.

Algorithm OPTIMAL VOLUME CUBE-DRAWING

Input: graph $G = (V, E)$.

Output: orthogonal drawing of G.

---

1. Initially represent each vertex $v \in V$ by a square S_v with side length $2\left\lceil\sqrt{\lceil \deg(v)/2 \rceil + 1}\right\rceil$.
2. Position the squares $\{S_v : v \in V\}$ in the ($Z = 0$)-plane with the square-packing algorithm of Kleitman and Krieger [17]. Note that squares may touch, and since all squares have even side length, we may assume that all corners of the squares have even coordinates.
3. For each vertex $v \in V$, remove the top two rows from S_v and the two rightmost columns from S_v. Vertices are now disjoint; see Fig. 1.

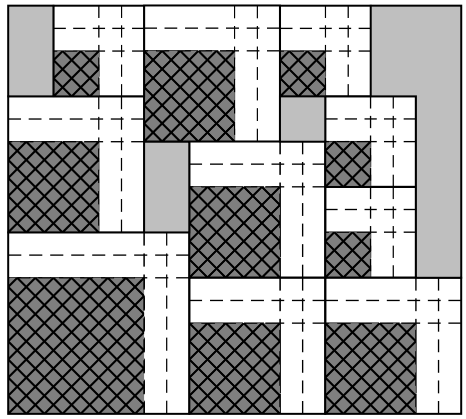

Fig. 1 Positioning the vertices with a square-packing.

4. Pair the odd degree vertices in G, and add an edge between the paired vertices. All vertices now have even degree. Orient the edges of G and alternately label the edges '+' and '$-$' by following an Eulerian tour of G. Remove the added edges.
5. Assign each edge $vw \in E$ labelled '+' unique Z^+ ports at v and w both with an even X-coordinate and an even Y-coordinate.
6. Construct a graph $H = (V_H, E_H)$ with vertex set V_H corresponding to the edges of G labelled '+'. For oriented edges $vw, xy \in E$, add the edge $\{vw, xy\}$ to E_H if the port assigned to vw at v is in the same column as the port assigned to xy at x, or the port assigned to vw at w is in the same row as the port assigned to xy at y.
7. Determine a proper vertex-colouring of H using the sequential greedy algorithm (with colours $\{1, 2, \ldots, \Delta(H)+1\}$). For each vertex of H coloured i corresponding to an edge vw in E, set the *height* $h(vw) \leftarrow i$.
8. For each oriented edge $vw \in E$ labelled '+', construct an edge route for vw as follows. Suppose the ports on v and w assigned to vw have coordinates $(v_X, v_Y, 0)$ and $(w_X, w_Y, 0)$, respectively. Route the edge vw with one of the following four or six bend routes, as illustrated in Fig. 2.

– $\boldsymbol{v_X = w_X}$**:** $(v_X, v_Y, 0) \to (v_X, v_Y, 2h(vw)) \to (v_X + 1, v_Y, 2h(vw)) \to$
$(v_X + 1, w_Y, 2h(vw)) \to (v_X, w_Y, 2h(vw)) \to (v_X, w_Y, 0) = (w_X, w_Y, 0)$

– $\boldsymbol{v_Y = w_Y}$**:** $(v_X, v_Y, 0) \to (v_X, v_Y, 2h(vw) + 1) \to (v_X, v_Y + 1, 2h(vw) + 1) \to$
$(w_X, v_Y + 1, 2h(vw) + 1) \to (w_X, v_Y, 2h(vw) + 1) \to (w_X, v_Y, 0) = (w_X, w_Y, 0)$

– $\boldsymbol{v_X \neq w_X}$ and $\boldsymbol{v_Y \neq w_Y}$**:** $(v_X, v_Y, 0) \to (v_X, v_Y, 2h(vw)) \to (v_X + 1, v_Y, 2h(vw)) \to$
$(v_X + 1, w_Y + 1, 2h(vw)) \to (v_X + 1, w_Y + 1, 2h(vw) + 1) \to$
$(w_X, w_Y + 1, 2h(vw) + 1) \to (w_X, w_Y, 2h(vw) + 1) \to (w_X, w_Y, 0)$

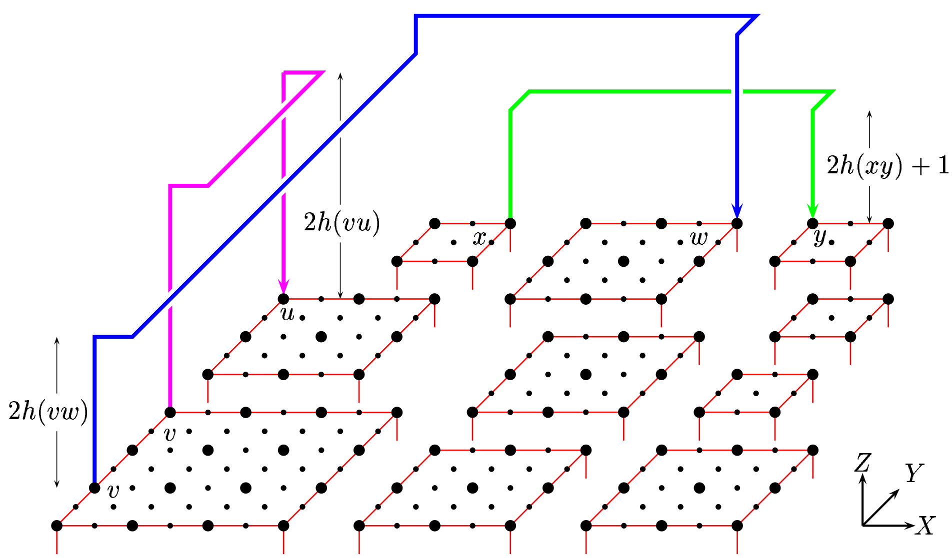

Fig. 2 Routing edges above the $(Z = 0)$-plane.

9. Repeat Steps 5 - 8 for the edges labelled '$-$' assigning Z^- ports and constructing edge routes below the $(Z = 0)$-plane in an analogous manner.
10. So that each vertex is enlarged into a cube, insert enough Z-planes at $Z = 0$, extend the side of each vertex parallel to the Z-axis, and possibly lengthen incident edges labelled '$-$'.

Theorem 3 *The algorithm* Optimal Volume Cube-Drawing *determines a 12-degree-restricted cube-drawing of any loopless graph G in $\mathcal{O}(m^{3/2})$ time, with $\mathcal{O}(m^{3/2})$ volume (assuming $m \geq n$), and at most six bends per edge route.*

Proof After Step 3 the square S_v has side length $2\left\lceil\sqrt{\lceil\deg(v)/2\rceil + 1}\right\rceil - 2$. Since the corners of the square have even coordinates, the number of Z^+ ports on S_v with even X- and Y-coordinates is at least

$$\left\lceil\sqrt{\lceil\deg(v)/2\rceil + 1}\right\rceil^2 \geq \lceil\deg(v)/2\rceil + 1 \ .$$

At most $\lceil\deg(v)/2\rceil + 1$ edges incident to v are labelled '+'. (In fact, all vertices v, except the starting vertex in the Eulerian tour, have at most $\lceil\deg(v)/2\rceil$ incident edges labelled '+'.) Similarly, at most $\lceil\deg(v)/2\rceil + 1$ edges incident to v are labelled '-'. Thus there are enough ports on v.

If a unit-length edge segment intersects another edge route then so does one of the adjacent non-unit-length edge segments. Therefore, to show that the drawing is crossing-free, we need only show that non-unit-length edge segments do not intersect, and consider only such segments in the following. Vertical segments cannot intersect because ports are assigned to unique edges. X-segments have odd Z-coordinate and Y-segments have even Z-coordinate, thus an X-segment cannot intersect a Y-segment. A vertical segment has even X- and Y-coordinate, an X-segment has odd Y-coordinate, and a Y-segment has odd X-coordinate, hence a vertical segment cannot intersect a X-segment or Y-segment. Two Y-segments can only intersect if they overlap. Since edge routes originating in the same column have different heights, two Y-segments cannot intersect. Similarly, two X-segments can only intersect if originating in the same row, and in this case they have different heights, thus they cannot intersect. Hence no two edge routes can intersect.

The total area of the squares $\{S_v : v \in V\}$ (before Step 3) is

$$\sum_v \left(2\left\lceil\sqrt{\lceil\deg(v)/2\rceil + 1}\right\rceil\right)^2 \leq \sum_v \left(2\deg(v) + 4\sqrt{2(\deg(v)+3)} + 10\right) \ ,$$

which is

$$4m + \mathcal{O}\left(n + \sum_v \sqrt{\deg(v)}\right) \ .$$

By the Cauchy-Schwarz inequality, $\sum_v \sqrt{\deg(v)} \leq \sqrt{2nm}$. Since we assume that $m \geq n$ it follows that $n \leq \sqrt{nm}$ and the total area of squares is therefore $4m + \mathcal{O}(\sqrt{nm})$.

The algorithm of [17] packs squares with a total area of 1 in a $2/\sqrt{3} \times \sqrt{2}$-rectangle. Thus the squares $\{S_v : v \in V\}$ can be packed in a rectangle with size

$$\left(4\sqrt{\frac{m}{3}} + \mathcal{O}\left((nm)^{1/4}\right)\right) \times \left(2\sqrt{2m} + \mathcal{O}\left((nm)^{1/4}\right)\right) \quad .$$

Hence the maximum degree of H is

$$\Delta(H) \leq \left(\frac{4}{\sqrt{3}} + 2\sqrt{2}\right)\sqrt{m} + \mathcal{O}\left((nm)^{1/4}\right) \quad .$$

A greedy vertex-colouring of H requires at most $\Delta(H) + 1$ colours, thus the height of the drawing above the $(Z = 0)$-plane, and the height below the vertices, is at most

$$\left(\frac{8}{\sqrt{3}} + 4\sqrt{2}\right)\sqrt{m} + \mathcal{O}\left((nm)^{1/4}\right) \quad .$$

The height of the vertices is $\max_v 2\left\lceil\sqrt{(\deg(v)+1)/2}\right\rceil \leq \sqrt{2\Delta(G)} + \mathcal{O}(1) \leq \sqrt{2m} + \mathcal{O}(1)$, and thus the height of the drawing is at most

$$\left(\frac{16}{\sqrt{3}} + 9\sqrt{2}\right)\sqrt{m} + \mathcal{O}\left((nm)^{1/4}\right) \quad .$$

The bounding box is therefore at most

$$\left(4\sqrt{\frac{m}{3}} + \mathcal{O}\left((nm)^{1/4}\right)\right) \times \left(2\sqrt{2m} + \mathcal{O}\left((nm)^{1/4}\right)\right) \times \left(\left(\frac{16}{\sqrt{3}} + 9\sqrt{2}\right)\sqrt{m} + \mathcal{O}\left((nm)^{1/4}\right)\right) \quad .$$

A simple calculation establishes that the bounding box volume is at most

$$\left(\frac{144}{\sqrt{3}} + \frac{128\sqrt{2}}{3}\right)m^{3/2} + \mathcal{O}\left(m(nm)^{1/4}\right) \leq 144m^{3/2} + \mathcal{O}\left(m(nm)^{1/4}\right) = \mathcal{O}\left(m^{3/2}\right) \quad .$$

The time-consuming stage of the algorithm is the vertex colouring of H. This can be computed in $\mathcal{O}(|E_H|) = \mathcal{O}(|V_H|\Delta(H)) = \mathcal{O}(m\sqrt{m}) = \mathcal{O}(m^{3/2})$ time. The surface of a vertex v is

$$6\left(2\left\lceil\sqrt{(\deg(v)+1)/2}\right\rceil\right)^2 \leq 12 \cdot \deg(v) + \mathcal{O}\left(\sqrt{\deg(v)}\right) \quad .$$

Thus each vertex is 12-degree-restricted. By construction, there are at most six bends per edge route. □

Very recently Biedl and Chan [6] developed a technique based on edge-colouring a certain bipartite graph to more efficiently implement Steps 6 and 7 of Algorithm OPTIMAL VOLUME CUBE-DRAWING. With this technique, the time complexity reduces to $\mathcal{O}(m \log m)$, and the volume of the produced drawings decreases to $\approx 83m^{3/2} + \mathcal{O}\left(m(nm)^{1/4}\right)$.

If we remove the middle segment from each 6-bend edge route and route each edge with unique height, then the overall height is $\mathcal{O}(m)$ and we obtain the following result.

Theorem 4 *Every loopless graph has a 12-degree-restricted cube-drawing, which can be computed in $\mathcal{O}(m)$ time, with $\mathcal{O}(m^2)$ volume, and at most five bends per edge route.*

3.2 Drawings with unbounded aspect ratio

We now show how to create drawings of a simple graph that have volume $\mathcal{O}(m\sqrt{n})$, which is optimal. The vertices have unbounded aspect ratio and are not necessarily degree-restricted.

The following algorithm initially represents vertices by points or line segments in the ($Z = 0$)-plane, and edges are routed above this plane. The vertices are then extended in the Z-dimension to form lines or rectangles.

Algorithm OPTIMAL VOLUME BOX-DRAWING

Input: simple graph $G = (V, E)$.

Output: orthogonal drawing of G.

---

1. Let $N = \lceil\sqrt{n}\rceil$, and define $V_{big} = \{v \in V : \deg(v) > 4m/N\}$ and $V_{small} = V \setminus V_{big}$.

   Define E_{big} to be the set of edges with both endpoints in V_{big}, E_{cut} to be the set of edges with exactly one endpoint in V_{big}, and E_{small} to be edges with both endpoints in V_{small}.

2. Let $V_{big} = \{w_1, \dots, w_l\}$. For each vertex $w_j \in V_{big}$, define $k(w_j) = \lceil \deg(w_j)N/4m \rceil$, and initially represent w_j by the line segment with endpoints at

$$\left(2\sum_{i=1}^{j-1} k(w_i), -2, 0\right) \text{ and } \left(2\left(\sum_{i=1}^{j} k(w_i) - 1\right), -2, 0\right) .$$

Note that w_j has $k(w_j) \geq 2$ grid points with an even X-coordinate.

3. Add extra vertices of degree 0 to V_{small} so that V_{small} has exactly N^2 vertices (simply to ease the description). Let $V_{small} = (v_0, v_1, v_2, \dots, v_{N^2-1})$ sorted by non-increasing degree. For each original vertex $v_k \in V_{small}$, $0 \leq k \leq N^2 - 1$, initially represent v_k by the point $(2(i+j) \bmod 2N, 2j, 0)$, where $k = iN + j$ for the unique pair i, j with $0 \leq i, j \leq N - 1$. See Fig. 3.

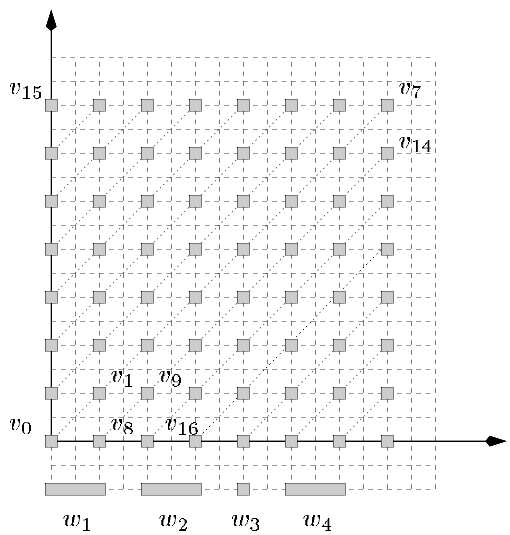

Fig. 3 Placing points and segments of vertices. Vertices w_1, w_2, w_3 belong to V_{big}.

4. Orient each edge in E_{cut} from the endpoint in V_{big} to the other endpoint. Orient all edges in E_{small} arbitrarily.

5. For each oriented edge $e = vw \in E_{cut}$, assign to e a Z^+-port at v with even coordinates, such that at most $\lceil 4m/N \rceil$ edges are assigned to any Z^+-port of any vertex in V_{big}. Assign the unique Z^+-port at w to e. For each edge $e = vw \in E_{small}$, assign to e the unique Z^+-ports at v and w.

6. Construct a graph $H = (V_H, E_H)$ with vertex set V_H corresponding to the edges of $E_{cut} \cup E_{small}$. For oriented edges $vw, xy \in E_{cut} \cup E_{small}$, add the edge $\{vw, xy\}$ to E_H if the port assigned to vw at v is in the same column as the port assigned to xy at x, or the port assigned to vw at w is in the same row as the port assigned to xy at y.
7. Determine a proper vertex-colouring of H using the sequential greedy algorithm (with colours $\{1, 2, \ldots, \Delta(H) + 1\}$). For each vertex of H coloured i corresponding to an edge $vw \in E_{cut} \cup E_{small}$, set the *height* $h(vw) \leftarrow i$.
8. For each oriented edge $vw \in E_{cut} \cup E_{small}$ construct an edge route for vw exactly as described in Step 8 Algorithm OPTIMAL VOLUME CUBE-DRAWING.
9. Construct edge routes for edges in E_{big} by copying the 2-bend layout of the complete graph developed in [8]. More precisely, recall that $|V_{big}| = l$. It is possible to partition the edges of the complete graph K_l, and therefore E_{big}, into l matchings $M_1, M_2, \ldots, M_l$ such that if the edges (w_i, w_j), $i < j$, and (w_a, w_b), $a < b$ are in the same matching, and (say) $i \leq a$, then $i < a < b < j$ or $i < j < a < b$. If the edge (w_i, w_j) with $i < j$ belongs to matching M_k, and if the leftmost points of w_i and w_j have coordinates $(x_i, -2, 0)$ and $(x_j, -2, 0)$, respectively, then route (w_i, w_j) as:
$(x_i, -2, 0) \to (x_i, -2, k) \to (x_i, -2 - \lceil (j-i)/2 \rceil, k) \to$
$(x_j, -2 - \lceil (j-i)/2 \rceil, k) \to (x_j, -2, k) \to (x_j, -2, 0)$
10. Enlarge points/lines representing vertices into lines/rectangles by extending their sides parallel to the Z-axis from the minimum to the maximum Z-coordinate of the drawing. For each edge $vw \in E$, clip the segment of vw incident to v if it is contained in the box representing v (and similarly at w); see Fig. 4.

To analyse the size of the resulting drawing, we need a bound on the maximum degree of H.

Lemma 7 *The maximum degree of H in the algorithm* OPTIMAL VOLUME BOX-DRAWING *is at most* $16m/N - 1$.

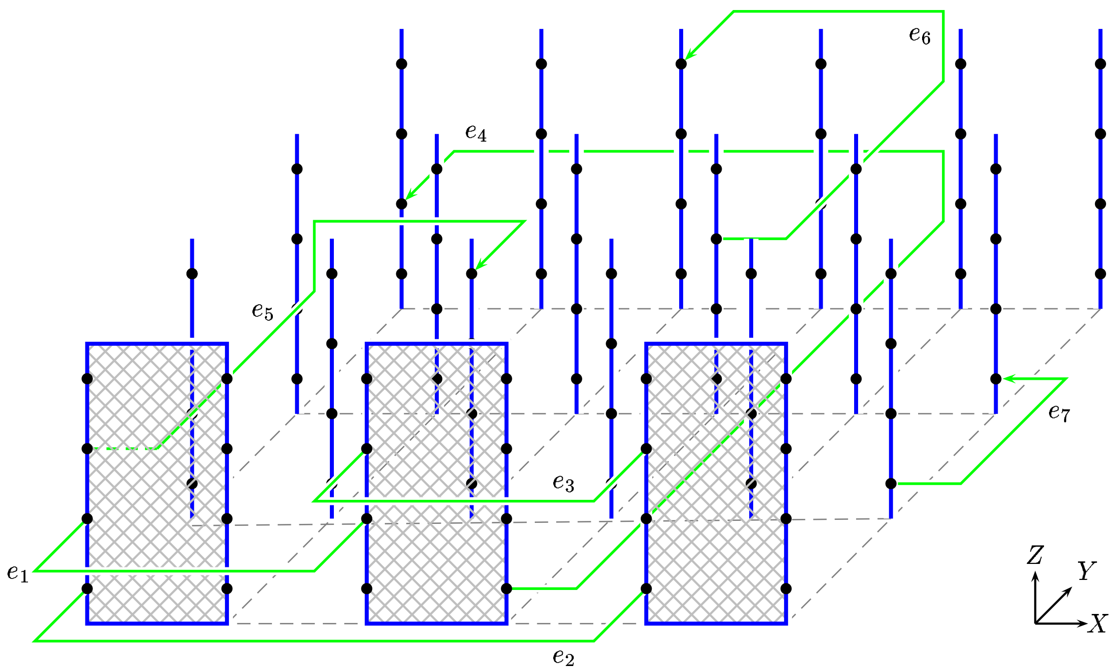

Fig. 4 A selection of edge routes in Algorithm OPTIMAL VOLUME BOX-DRAWING; e_1, e_2 and e_3 belong to E_{big}, e_4 and e_5 belong to E_{cut}, and e_6 and e_7 belongs to E_{small}.

Proof Let xy be an arbitrary oriented edge in $E_{cut} \cup E_{small}$. Any neighbour of xy in H must have a port in the same column as x or in the same row as y. Thus we want to study the number of edges that have a port in a specific row/column.

Assume that $xy \in E_{cut}$. By the direction of edges, we know that $x \in V_{big}$ and $y \in V_{small}$. If vw is a neighbour of xy in H, because w and y have ports in the same row r, then row r contains vertices in V_{small}, and therefore this is *not* the row with Y-coordinate -2. If $xy \in E_{small}$, then both ports must be in a row that does not have Y-coordinate -2 because both endpoints are in V_{small}. Hence no edge in H is added because of the row with Y-coordinate -2, which we can therefore ignore for considerations of the maximum degree of H.

Also, notice that by the way we assign ports to edges, there are at most $\lceil 4m/N \rceil$ edges in E_{cut} assigned to the same Z^+-port of a vertex in V_{big} (i.e., to a Z^+-port with Y-coordinate -2). To obtain a bound on the maximum degree, we thus need a bound on the number of edges in H resulting from a Z^+-port with non-negative Y-coordinate. For future reference, let a *proper row/column* be the intersection of a row/column with the first quadrant, i.e., with the range of non-negative X- and Y-coordinates.

Claim: For any proper row and any proper column, there are at most $6m/N$ edges that use a Z^+-port in that row/column.

We prove this by showing that the sum of the degrees of vertices placed in any one proper row/column is at most 6m/N. Define for $i = 0, \ldots, N-1$ the *ith diagonal* to be the grid-points $\{(2(i+k) \bmod 2N, 2k) : k = 0, \ldots, N-1\}$. In Step 3, vertices in V_{small} are positioned, in order of non-increasing degree, in the 0th diagonal from bottom to top, then in the 1st diagonal from bottom to top, and so on. This implies the following properties for each i, $0 \le i \le N-1$:

- The vertex with largest degree in the ith diagonal is at $(2i, 0)$, while the vertex with the smallest degree in this diagonal is at $(2(i+N-1) \bmod 2N, 2(N-1))$.
- The last vertex in the ith diagonal (i.e., the vertex at $(2(i+N-1) \bmod 2N, 2(N-1))$) has degree no larger than the first vertex in the $(i+1)$th diagonal (i.e., the vertex at $(2(i+1), 0)$).

For any proper row/column, define the *degree-sum* as the sum of the degrees of the vertices placed in this row/column. Denote the degree-sum of the row with Y-coordinate $2i$ by $R(i)$.

Notice that each proper column and each proper row contains exactly one vertex from each diagonal, and no other vertices. Thus the degree-sum of each proper row/column can be at most the sum of the maximal degrees in each diagonal, and is at least the sum of the minimal degrees in each diagonal. By the first observation above, we know that the degree-sum of each proper row/column is at most $R(0)$ and at least $R(N-1)$.

Also, by the second observation above, we know that $R(0) \le R(N-1) + \Delta_{small}$, where Δ_{small} is the maximum degree among all vertices in V_{small}, in particular $\Delta_{small} \le 4m/N$ by the definition of V_{small}. This follows, because each entry in $R(0)$ is the first entry in some diagonal, and can be upper-bounded by the last entry in the previous diagonal. The only exception is the first entry in the 0th diagonal, which has degree Δ_{small}.

Now we can estimate

$$2m \geq \sum_{v \in V_{small}} \deg(v) = \sum_{i=0}^{N-1} R(i) \geq N \cdot R(N-1) \geq N \cdot R(0) - N\Delta_{small} \, ,$$

and therefore $R(0) \leq 2m/N + \Delta_{small} \leq 6m/N$. Since the degree-sum of each proper row/column is at most $R(0)$, the claim follows.

So any vertex in H (which corresponds to an edge in $E_{cut} \cup E_{small}$) has at most $6m/N - 1$ neighbours that use a Z^+-port in the same row, and at most $6m/N - 1 + \lceil 4m/N \rceil$ neighbours that use a Z^+-port in the same column, hence the maximum degree of H is at most $16m/N - 1$. □

Theorem 5 *The algorithm* OPTIMAL VOLUME BOX-DRAWING *determines a drawing of any simple graph G in $\mathcal{O}(m^2/\sqrt{n})$ time, with $\mathcal{O}(m\sqrt{n})$ volume, and at most four bends per edge route.*

Proof First, we show that no edges overlap or intersect. Exactly as in Theorem 3, one shows that if two edges in $E_{cut} \cup E_{small}$ have different ports at both endpoints, then the edges neither overlap nor cross. The same holds for any two edges in E_{big} as discussed in [8]. No edge of E_{big} can overlap or intersect an edge in $E_{cut} \cup E_{small}$, because they are separated by the ($Y = 0$)-plane.

If two edges xy and vw have a common port at one endpoint, say at $x = v$, then the edges do overlap, but only at the segment incident to $x = v$, which is parallel to the Z-axis. This segment will be clipped when extending the vertices in Step 10; hence there is no overlap in the final drawing.

Since V_{small} contains at most $n \leq N^2$ vertices, the vertices in V_{small} can be placed in a $2N \times 2N$-rectangle. Since $\deg(v) > 4m/N$ for all vertices $v \in V_{big}$, we have

$$|V_{big}| \leq \sum_{v \in V_{big}} \frac{\deg(v)N}{4m} \leq \frac{N}{2} \, ,$$

and

$$\sum_{v \in V_{big}} k(v) = \sum_{v \in V_{big}} \left\lceil \frac{\deg(v)N}{4m} \right\rceil \leq \sum_{v \in V_{big}} \frac{\deg(v)N}{4m} + |V_{big}| \leq N \, ,$$

thus the maximum X-coordinate of a vertex in V_{big} is $2N$, and we need at most $2N$ X-planes.

By $|V_{big}| \leq N/2$, the edges in E_{big} have Y-coordinate $\geq -2 - \lceil (N/2-1)/2 \rceil$, hence we need at most $\frac{9}{4}N + 3$ Y-planes.

In Step 5, a vertex $v \in V_{big}$ has at most $\deg(v)$ incident edges that need a Z^{+}-port in v, and there are $k(v)$ Z^{+}-ports at v with even coordinates. Thus, we can assign edges to ports such that each port has at most $\lceil \deg(v)/k(v) \rceil \leq \lceil 4m/N \rceil$ edges assigned to it. Hence there are enough ports for the edges.

A greedy vertex colouring of H requires at most $\Delta(H) + 1 \leq 16m/N$ colours, thus to route the edges in $E_{cut} \cup E_{small}$, we need at most $2 \cdot 16m/N$ Z-planes. To route the edges in E_{big}, we need at most $|V_{big}| \leq N/2$ Z-planes. Since $m \geq n$, we have $32m/N \geq N/2$, and the height of the drawing above the $Z = 0$ plane is at most $32m/N$. The bounding box therefore has volume at most $2N \times (\frac{9}{4}N + 3) \times 32m/N = 144mN + \mathcal{O}(m) = 144m\sqrt{n} + \mathcal{O}(m) = \mathcal{O}(m\sqrt{n})$.

The time-consuming stage of the algorithm is the vertex colouring of H. This can be computed in $\mathcal{O}(|E_H|) = \mathcal{O}(|V_H|\Delta(H)) = \mathcal{O}(m \cdot m/N) = \mathcal{O}(m^2/\sqrt{n})$ time.

Originally, there are at most six bends per edge route, and at most four bends per edge route for edges in V_{big}. During the clipping step, the first and last segment of each edge gets clipped, hence every edge has at most four bends. □

The technique of Biedl and Chan [6] works for Steps 6 and 7 of Algorithm OPTIMAL VOLUME BOX-DRAWING as well, reducing the time complexity to $\mathcal{O}(m \log n)$, and decreasing the volume of the produced drawings to $\approx 90m\sqrt{n} + \mathcal{O}(m)$.

Note that we required G to be simple; this is necessary for the routing of the edges in E_{big}. It is not hard to show that if each edge in E_{big} has multiplicity $\leq k$, then all edges in E_{big} can be routed with $\leq kN$ Z-planes. Thus, as long as $m = \Omega(kn)$, the drawing still has $\mathcal{O}(m\sqrt{n})$ volume.

We can decrease the number of bends per edge to three if we allow an increase in volume. In fact, the construction greatly simplifies, since there is now no need for E_{big}. Place all n vertices as points in a $2N \times 2N$-grid. Assign to each edge a unique height, and route each edge as before, but omitting the middle segment. The height is then m, and the width and depth are both $2N$.

Theorem 6 *Every loopless graph has a drawing, which can be computed in $\mathcal{O}(m)$ time, with $\mathcal{O}(mn)$ volume, and three bends per edge route.*

This algorithm is particularly appropriate for multilayer VLSI as there are no vertical edge segments or ‘cross-cuts’; see [2].

4 Conclusions and open problems

In this paper, we provided matching upper and lower bounds for the volume of three-dimensional orthogonal box-drawings, under various restrictions on the shape of vertex boxes.

In particular, we showed that any algorithm to create three-dimensional orthogonal drawings that have bounded aspect ratios or are degree-restricted cannot do better than $\Omega(m^{3/2})$ volume. Then we gave an algorithm that matches this bound, i.e., constructs three-dimensional degree-restricted orthogonal cube-drawings with $\mathcal{O}(m^{3/2})$ volume.

If there are no restrictions on the drawing, then we showed that no algorithm can do better than $\Omega(m\sqrt{n})$ volume. We gave a second algorithm that matches this bound, i.e., constructs three-dimensional orthogonal drawings with $\mathcal{O}(m\sqrt{n})$ volume.

Thus, no more order-of-magnitude improvements are possible for the volume of drawings. We do see room for improvement with respect to the number of bends per edge. In particular, (a) does every graph have a 5-bend degree-restricted cube-drawing with $\mathcal{O}(m^{3/2})$ volume, and (b) does every graph have a 3-bend drawing with $\mathcal{O}(m\sqrt{n})$ volume? Note that K_n *does* have a $\mathcal{O}(n^{5/2}) = \mathcal{O}(m\sqrt{n})$ volume 3-bend drawing [8].

Table 1 suggests a trade-off between the number of bends per edge and the bounding box volume. Can such a trade-off be proved? What are lower bounds for drawings where edges are allowed to have at most k bends per edge, for $k = 1, \ldots, 5$? (Note that some graphs do not have a drawing without bends [8, 14], and lower bounds for drawings with one bend per edge were given in [4].)

References

1. M. A. Aboelaze and B. W. Wah, Complexities of layouts in three-dimensional VLSI circuits. *Inform. Sci.* **55(1-3)** (1991), 167–188.
2. A. Aggarwal, M. Klawe, and P. Shor, Multilayer grid embeddings for VLSI. *Algorithmica* **6(1)** (1991), 129–151.
3. N. Alon and J. Spencer, *The Probabilistic Method*. John Wiley & Sons, New York, 1992.
4. T. Biedl. 1-bend 3-D orthogonal drawings: two open problems solved. In *Proc. of 12th Canadian Conf. on Computational Geometry* (CCCG'00), pp. 173-180, 2000.
5. T. Biedl, Three approaches to 3D-orthogonal box-drawings, in *Proc. Graph Drawing: 6th International Symp.* (GD'98), pp. 30–43, Lecture Notes in Comput. Sci., Vol. 1547, Springer-Verlag, Berlin, 1998.
6. T. Biedl and T. Chan, Cross-coloring: improving the technique by Kolmogorov and Barzdin. Technical Report CS-2000-13. Department of Computer Science, University of Waterloo, Canada, 2000.
7. T. Biedl and M. Kaufmann, Area-efficient static and incremental graph drawings, in *Proc. 5th European Symp. on Algorithms* (ESA'97), pp. 37–52, Lecture Notes in Computer Science, Vol. 1284, Springer-Verlag, Berlin, 1997.
8. T. Biedl, T. Shermer, S. Whitesides, and S. Wismath, Bounds for orthogonal 3-D graph drawing. *J. Graph Alg. Appl.* **3(4)** (1999), 63–79.
9. M. Closson, S. Gartshore, J. Johansen, and S. K. Wismath, Fully dynamic 3-dimensional orthogonal graph drawing, in *Proc. Graph Drawing: 7th International Symp.* (GD'99), pp. 49–58, Lecture Notes in Comput. Sci., Vol. 1731, Springer-Verlag, Berlin, 1999.
10. G. Di Battista, P. Eades, R. Tamassia, and I. Tollis. *Graph Drawing: Algorithms for Geometric Representations of Graphs*. Prentice-Hall, 1998.
11. G. Di Battista, M. Patrignani, and F. Vargiu, A split&push approach to 3D orthogonal drawing, in *Proc. Graph Drawing: 6th International Symp.* (GD'98), pp. 87–101, Lecture Notes in Comput. Sci., Vol. 1547, Springer-Verlag, Berlin, 1998.

12. P. Eades, C. Stirk, and S. Whitesides, The techniques of Kolmogorov and Bardzin for three-dimensional orthogonal graph drawings. *Inform. Proc. Lett.* **60(2)** (1996), 97–103.

13. P. Eades, A. Symvonis, and S. Whitesides, Three dimensional orthogonal graph drawing algorithms. *Discrete Applied Math.* **103** (2000), 55–87.

14. S. Fekete and H. Meijer. Rectangle and box visibility graphs in 3D. *Internat. J. Comput. Geom. Appl.*, **9(1)** (1999), 1–27.

15. É. K. Fogel, An elementary proof of formulae of de la Vallée Poussin (In Russian). *Latvijas PSR Zinātņu Akad. Vestis* **11(40)** (1950), 123–130.

16. K. Hagihara, N. Tokura, and N. Suzuki, Graph embedding on a three-dimensional model. *Systems-Comput.-Controls* **14(6)** (1983), 58–66.

17. D. Kleitman and M. Krieger, An optimal bound for two dimensional bin packing, in *Proc. 16th Annual Symp. on Foundations of Computer Science* (FOCS'75), pp. 163–168, IEEE, 1975.

18. F. T. Leighton and A. L. Rosenberg, Three-dimensional circuit layouts. *SIAM J. Comput.* **15(3)** (1986), 793–813.

19. A. Lubotzky, R. Phillips, and P. Sarnak, Ramanujan graphs. *Combinatorica* **8** (1988), 261–277.

20. A. Margulis, Explicit group-theoretical constructions of combinatorial schemes and their applications to the design of expanders and superconcentrators. *Problems of Information Transmission* **24** (1988), 39–64.

21. A. Papakostas and I. Tollis, Incremental orthogonal graph drawing in three dimensions. *J. Graph Algs. Appl.* **3(4)** (1999), 81–115.

22. F. P. Preparata, Optimal three-dimensional VLSI layouts. *Math. Systems Theory* **16** (1983), 1–8.

23. A. L. Rosenberg, Three-dimensional VLSI: A case study. *J. ACM* **30(2)** (1983), 397–416.

24. A. Widgerson and D. Zuckerman, Expanders that beat the eigenvalue bound: explicit construction and applications, in *Proc. 25th Annual ACM Symp. on Theory of Computing* (STOC'93), pp. 245–251, ACM, 1993.

25. D. R. Wood, An algorithm for three-dimensional orthogonal graph drawing, in *Proc. Graph Drawing: 6th International Symp.* (GD'98), pp. 332–346, Lecture Notes in Comput. Sci., Vol. 1547, Springer-Verlag, Berlin, 1998.

26. D. R. Wood, Multi-dimensional orthogonal graph drawing with small boxes, in *Proc. Graph Drawing: 7th International Symp.* (GD'99), pp. 311–322, Lecture Notes in Comput. Sci., Vol. 1731, Springer-Verlag, Berlin, 1999.

27. D. R. Wood, A new algorithm and open problems in three-dimensional orthogonal graph drawing, in *Proc. Australasian Workshop on Combinatorial Algorithms* (AWOCA'99), pp. 157–167, Curtin University of Technology, Perth, Australia, 1999.

28. D. R. Wood, Three-Dimensional Orthogonal Graph Drawing, Ph.D. thesis, School of Computer Science and Software Engineering, Monash University, Melbourne, Australia, 2000.