On the q-analogue of Zeilberger's algorithm to rational functions*

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Abstract

We consider the applicability (or terminating condition) of the q-analogue of Zeilberger's algorithm and give the complete solution to this problem for the case when the given q-hypergeometric term is a rational function.

1 Preliminaries

The well-known Zeilberger's algorithm [6, 12] is a useful tool for providing proof strategies for closed-form identities and for deriving new identities. It is shown in [5, 7, 11] that Zeilberger's algorithm can be carried over to the q-difference case (we name the algorithm hereafter as $q\mathcal{Z}$.) As in the case of its difference counterpart, $q\mathcal{Z}$ also has a wide range of applications [9].

Let q be an indeterminate parameter. For a given function $F(q^n, q^k)$, denote by Q_n, Q_k the q-shift operators on q^n and q^k, resp., defined by $Q_n F(q^n, q^k) = F(q^{n+1}, q^k)$, $Q_k F(q^n, q^k) = F(q^n, q^{k+1})$.

Let $F(q^n, q^k)$ be a q-hypergeometric term of q^n and q^k, i.e., the quotients $F(q^{n+1}, q^k)/F(q^n, q^k)$ and $F(q^n, q^{k+1})/F(q^n, q^k)$ belong to $\mathbb{C}(q)(q^n, q^k)$. For a given $F(q^n, q^k)$, $q\mathcal{Z}$ tries to construct a *qZ-pair* (L, G) consisting of a linear q-difference operator with coefficients which are polynomials of q^n over $\mathbb{C}(q)$

$$L = a_\rho(q^n)Q_n^\rho + \cdots + a_1(q^n)Q_n^1 + a_0(q^n)Q_n^0 \tag{1}$$

and a q-hypergeometric term $G(q^n, q^k)$ such that

$$LF(q^n, q^k) = G(q^n, q^{k+1}) - G(q^n, q^k). \tag{2}$$

Note that a qZ-pair does not exist for every q-hypergeometric term (see Example 1). The question for what q-hypergeometric terms qZ-pairs do exist is not conclusively answered although a sufficient condition is known. The "fundamental theorem" (see [6, 11]) states that a qZ-pair exists if $F(q^n, q^k)$ is a *q-proper term*, i.e., it can be written in the form

$$F(q^n, q^k) = \frac{\prod_s (c_s; q)_{a_s n + b_s k}}{\prod_s (w_s; q)_{u_s n + v_s k}} q^{an^2 + bnk + ck^2 + dk + en} \xi^k, \tag{3}$$

where $(c; q)_m = (1 - cq)(1 - cq^2)\dots(1 - cq^m)$, a_s, b_s, u_s, v_s are specific integers, c_s, w_s may depend on parameters, ξ is either a number or a parameter, and a, b, c, d, e are specific integers.

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In this paper we present the conclusive answer to the question of specifying the class of rational functions $F(q^n, q^k)$ that have qZ-pairs or, equivalently, the class of rational functions which, when given as input, allow $q\mathcal{Z}$ to terminate.

Note that the same problem for the difference case has been completely solved in [4]. In fact, after establishing two properties of the decomposition problem for the q-difference case [2] in Section 2, we are able to use the ideas as developed in [4] to derive an analogous theorem for the applicability of Zeilberger's algorithm to rational functions in the q-difference case.

2 Indefinite summation of rational functions

In this section, we describe the decomposition problem for the q-difference case [2], and prove a couple of properties that will be needed in subsequent sections.

For a given $F(q^k) \in \mathbb{C}(q)(q^k)$ in reduced form, an algorithm to solve the decomposition problem for the q-difference case constructs $R(q^k), H(q^k) \in \mathbb{C}(q)(q^k)$ such that

$$F(q^k) = (Q_k - 1)\, R(q^k) + H(q^k) \tag{4}$$

where the denominator of $H(q^k)$ has the lowest possible degree in q^k.

By representing $H(q^k)$ in the reduced form

$$H(q^k) = \frac{f(q^k)}{g(q^k)}, \;\; f(q^k), g(q^k) \in \mathbb{C}(q)[q^k], \; \gcd(f(q^k), g(q^k)) = 1,$$

one can derive the following property:

P1. If $p_1(q^k), p_2(q^k)$ are factors of $g(q^k)$ irreducible over $\overline{\mathbb{C}(q)}$, then there does not exist a non-zero integer h such that $p_1(q^{h+k}) = p_2(q^k)$.

Proof : Let $\prod_{i=1}^{s} p_i^{\gamma_i}(q^k)$ be the complete factorization of $g(q^k)$ over $\overline{\mathbb{C}(q)}$. Assume there exist factors $p_1(q^k), p_2(q^k)$ irreducible over $\overline{\mathbb{C}(q)}$ and a non-zero integer h such that $p_1(q^{h+k}) = p_2(q^k)$. Without loss of generality we can assume that $h > 0$. Rewrite $H(q^k)$ as

$$\frac{f_1(q^k)}{p_1^{\gamma_1}(q^k)} + \frac{f_2(q^k)}{p_1^{\gamma_2}(q^{h+k})} + H_1(q^k) \tag{5}$$

where $H_1(q^k) = \sum_{i=3}^{n} f_i(q^k)/p_i^{\gamma_i}(q^k)$, $f_i(q^k) \in \overline{\mathbb{C}(q)}[q^k]$.

Let $r_1(q^k) = f_2(q^{-1+k})/p_1^{\gamma_2}(q^{h-1+k})$. Then

$$H(q^k) = (Q_k - 1) r_1(q^k) + \frac{f_1(q^k)}{p_1^{\gamma_1}(q^k)} + \frac{f_2(q^{-1+k})}{p_1^{\gamma_2}(q^{h-1+k})} + H_1(q^k).$$

By continuing this process, we obtain the relation:

$$H(q^k) = (Q_k - 1) \sum_{i=1}^{h} r_i(q^k) + \frac{f_1(q^k)}{p_1^{\gamma_1}(q^k)} + \frac{f_2(q^{-h+k})}{p_1^{\gamma_2}(q^k)} + H_1(q^k), \text{ where} \tag{6}$$

$$r_i(q^k) = \frac{f_2(q^{-i+k})}{p_1^{\gamma_2}(q^{h-i+k})}, 1 \le i \le h.$$

This shows that we are able to extract a rational part from $H(q^k)$ and the degree of the denominator of the remaining part is less than that of $g(q^k)$. Contradiction. □

On the other hand, let us consider $F(q^k)$ to be rational summable, i.e., the term $H(q^k)$ in (4) vanishes. Suppose that $F(q^k)$ is a proper function, and does not have a pole at zero, then $b(q^k)$ has the following property:

P2. If $p_1(q^k)$ is a monic factor of $b(q^k)$ irreducible over an algebraic closure $\overline{\mathbb{C}(q)}$, then there exist an irreducible factor $p_2(q^k)$ of $b(q^k)$ and a non-zero integer h such that

$$q^{-h}p_1(q^{h+k}) = p_2(q^k).$$

Proof : Consider the partial fraction decomposition of $F(q^k)$ w.r.t. the complete factorization of $b(q^k)$ over $\overline{\mathbb{C}(q)}$:

$$F(q^k) = \sum_{i=1}^{m}\sum_{j=1}^{t_i} \frac{\beta_{ij}}{(q^k-\alpha_i)^j},\ \beta_{ij}, \alpha_i \in \overline{\mathbb{C}(q)}. \tag{7}$$

Define $\alpha_i \sim \alpha_j$ iff $\alpha_i = q^l\alpha_j$ where l is an integer. $\sim$ is an equivalence relation on the set $\{\alpha_1, \ldots, \alpha_m\}$. For each equivalence class, let α_i be the *largest* element in the sense that for all elements α_j in the same class, $\alpha_j = q^l\alpha_i$ where l is a non-positive integer. It is shown in [2] that

$$F(q^k) = \sum_{i=1}^{s}\sum_{j=1}^{l_i} M_{ij}(Q_k)\frac{1}{(q^k-\alpha_i)^j},\ M_{ij} \in \overline{\mathbb{C}(q)}[Q_k], s \le m.$$

When $F(q^k)$ is rational summable, the operators $M_{ij}(Q_k)$ can be written in the form

$$M_{ij}(Q_k) = L_{ij}(Q_k)(Q_k - 1),\ i = 1, \ldots, s, j = 1, \ldots, l_i, L_{ij} \in \overline{\mathbb{C}(q)}[Q_k]. \tag{8}$$

Note that the right hand side of (8) is of order at least 1. As a consequence, each equivalence class has at least 2 elements. Let $p_1(q^k) = (q^k - \alpha_1)$ be any monic irreducible factor of $g(q^k)$ over $\overline{\mathbb{C}(q)}$, where α_1 uniquely belongs to the equivalence class represented by α_i, $1 \le i \le s$. If $\alpha_1 \ne \alpha_i$, then there exists a negative integer h such that $\alpha_1 = q^h\alpha_i$, and hence there exists an irreducible factor $p_2(q^k) = (q^k - \alpha_i)$ such that

$$p_2(q^k) = (q^k - q^{-h}\alpha_1) = q^{-h}p_1(q^{h+k}).$$

If $\alpha_1 = \alpha_i$, let σ be the order of $M_{ij}(Q_k)$, λ be the minimum positive integer such that the coefficient Q_k^λ in M_{ij} is a non-zero element of $\overline{\mathbb{C}(q)}$. Let $h = \sigma - \lambda$. Then there exists an α_2 in the equivalence class represented by α_i such that $p_2(q^k) = (q^k - \alpha_2) = (q^k - q^{-h}\alpha_i) = q^{-h}p_1(q^{h+k})$. □

3 $q\mathcal{Z}$ on sum of rational functions

The following Lemma is a generalization of the q-analogue of Lemma 1, Section 2 in [4]:

Lemma 1 *Suppose there exist qZ-pairs for $F_1, F_2, \ldots, F_m \in \mathbb{C}(q)(q^n, q^k)$. Then there exists a qZ-pair for $F = \sum_{i=1}^m F_i$.*

Proof : Except for replacing the shift operator by the q-shift operator, the proof is exactly the same as that for Lemma 1, Section 2 in [4] for the case when $m = 2$. The generalization can be easily obtained by induction. □

4 Criterion for the existence of a qZ-pair for a rational function

The goal of this section is to present a criterion for a given rational function $F(q^n, q^k)$ to have a qZ-pair. For $F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$, denote $F(q^n; q^k)$ as an element of $\mathbb{C}(q)(q^n)(q^k)$ (sometimes, when fitting, as an element of the ring $\mathbb{C}(q)(q^n)[q^k]$). We also consider polynomials of q^k whose coefficients are algebraic functions of q^n, i.e. they are elements of the ring $\overline{\mathbb{C}(q)(q^n)}[q^k]$. Denote these polynomials by $p_1(q^n; q^k), p_2(q^n; q^k)$, and so on.

Lemma 2 *Let $F(q^n, q^k)$ be a polynomial in q^k, i.e., $F(q^n, q^k) \in \mathbb{C}(q)(q^n)[q^k]$. Then there exists a qZ-pair for F.*

Proof : Let $F(q^n; q^k)$ be represented as

$$F(q^n; q^k) = \sum_{i=0}^{m} a_i(q^n)\left(q^k\right)^i, \ a_i(q^n) \in \mathbb{C}(q)(q^n). \tag{9}$$

For each term $F_i = a_i(q^n)\left(q^k\right)^i$, $1 \le i \le m$ in (9), there exists a qZ-pair $(L_i, G_i) = (1, a_i(q^n)\left(q^k\right)^i/(q^i - 1))$ such that $L_i F_i(q^n, q^k) = (Q_k - 1)G_i(q^n, q^k)$. By Lemma 1 a qZ-pair for F exists iff a qZ-pair for the trailing coefficient $a_0(q^n)$ of F exists.

Let $a_0(q^n) = a_{01}(q^n)/a_{02}(q^n)$ where $a_{01}(q^n), a_{02}(q^n) \in \mathbb{C}(q)[q^n]$. Set

$$L_0 = a_{02}(q^{n+1})a_{01}(q^n)Q_n - a_{01}(q^{n+1})a_{02}(q^n).$$

We have $L_0 a_0(q^n) = 0 = (Q_k - 1)G_0(q^n)$ for all $G_0(q^n) \in \mathbb{C}(q)(q^n)$. Hence, there exists a qZ-pair (L_0, G_0) such that $L_0 a_0(q^n) = (Q_k - 1)G_0$. □

Now let $F(q^n; q^k)$ be an element from $\mathbb{C}(q)(q^n, q^k)$. One can then extract the polynomial part $p(q^n; q^k) \in \mathbb{C}(q)(q^n)[q^k]$ from F such that

$$F(q^n; q^k) = p(q^n; q^k) + F^*(q^n; q^k)$$

where $F^*(q^n; q^k)$ is a proper rational function. By Lemmas 1 and 2, F has a qZ-pair iff there exists a qZ-pair for $F^*(q^n; q^k)$.

By applying to $F^*(q^n; q^k)$ an algorithm to solve the decomposition problem [2], we can represent $F^*(q^n; q^k)$ in the form

$$F^*(q^n; q^k) = (Q_k - 1)S(q^n; q^k) + T(q^n; q^k),$$

where $S, T \in \mathbb{C}(q)(q^n)(q^k)$ are such that the denominator of $T(q^n; q^k)$ has the minimal possible degree w.r.t. q^k. For $(Q_k - 1)S(q^n, q^k)$ we have a qZ-pair

$$(1, S(q^n, q^k)).$$

By Lemma 1 a qZ-pair for F^* exists iff there exists a qZ-pair for $T(q^n, q^k)$, which can be represented in the reduced form

$$T(q^n, q^k) = \frac{f(q^n, q^k)}{g(q^n, q^k)}, \tag{10}$$

where $f(q^n, q^k), g(q^n, q^k) \in \mathbb{C}(q)[q^n, q^k]$.

Lemma 3 *Let a rational function $T(q^n; q^k)$ of the form (10) be such that $g(q^n; q^k)$ has property* **P1**. *Let $L \in \mathbb{C}(q)[q^n, Q_n]$ be such that $LT(q^n; q^k)$ is of the form*

$$LT(q^n; q^k) = V(q^n; q^k) = \frac{a(q^n; q^k)}{b(q^n; q^k)} \tag{11}$$

where $a(q^n;q^k), b(q^n;q^k)$ are relatively prime elements of $\mathbb{C}(q)[q^n,q^k]$, and $b(q^n;q^k)$ has property **P2**. *Then for any monic factor $u(q^n;q^k)$ of the polynomial $g(q^n;q^k)$ irreducible over $\overline{\mathbb{C}(q)(q^n)}$ there exists an irreducible factor $v(q^n;q^k)$ of $g(q^n;q^k)$ (it is possible that $u(q^n;q^k)=v(q^n;q^k)$) and $j,h\in\mathbb{Z}$, $j>0$, such that $u(q^n;q^k)=q^{-h}v(q^{n+j};q^{k+h})$.*

Proof : Note that properties **P1** and **P2** described in Section (2) are the q-analogue of properties **P1** and **P2** in [4]. Consequently, a proof of this Lemma can be directly obtained from the proof of Lemma 2, Section 3 in [4] by simply replacing the shift operator by the q-shift operator. □

Lemma 4 *Let a rational function $T(q^n;q^k)$ of the form (10) be such that $g(q^n;q^k)$ has property* **P1**. *Let $L\in\overline{\mathbb{C}(q)}[q^n,Q_n]$ be such that $LT(q^n;q^k)$ is of the form (11) and $b(q^n,q^k)$ has property* **P2**. *Then all monic factors of $g(q^n;q^k)$ irreducible over $\overline{\mathbb{C}(q)(q^n)}$ are of the form*

$$q^k-\gamma q^{cn}\quad c\in\mathbf{Q},\gamma\in\overline{\mathbb{C}(q)}. \tag{12}$$

Proof : Note that the direction of this proof is the same as that used for Lemma 3 in [4]. Take any factor $p_1(q^n;q^k)$ of $g(q^n;q^k)$ irreducible over $\overline{\mathbb{C}(q)(q^n)}$. By Lemma 3 there exist $j_1,h_1\in\mathbb{Z}$, $j_1>0$, such that

$$p_1(q^n;q^k)=q^{-h_1}p_2(q^{n+j_1};q^{k+h_1}),$$

where $p_2(q^n;q^k)$ is a factor of $g(q^n;q^k)$ irreducible over $\overline{\mathbb{C}(q)(q^n)}$. We can therefore construct a sequence

$$p_1(q^n;q^k),p_2(q^n;q^k),p_3(q^n;q^k),\ldots$$

of factors of $g(q^n;q^k)$ irreducible over $\overline{\mathbb{C}(q)(q^n)}$ such that for any $l\geq 1$,

$$p_l(q^n;q^k)=q^{-h_l}p_{l+1}(q^{n+j_l};q^{k+h_l}),\ j_l,h_l\in\mathbb{Z},\ j_l>0.$$

Since $g(q^n;q^k)$ has only a finite number of irreducible factors, there exists an irreducible factor $p(q^n;q^k)$ such that for some $1\leq\alpha<\beta$, and for $J=j_\alpha+\cdots+j_{\beta-1}$, $H=h_\alpha+\cdots+h_{\beta-1}$ we have

$$p_\alpha(q^n;q^k)=p(q^n;q^k)=q^{-H}p(q^{n+J};q^{k+H})=q^{-H}p_\beta(q^n;q^k) \tag{13}$$

with $J>0$. Note that $p(q^n;q^k)$ is linear in q^k because the coefficient fields $\overline{\mathbb{C}(q)(q^n)}$ is algebraically closed. One can suppose $p(q^n;q^k)$ to be monic. Let $p(q^n;q^k)=q^k-\varphi(q^n)$, where $\varphi(q^n)$ is an algebraic function of q^n, then from (13) we have

$$q^k-\varphi(q^n)=q^k-q^{-H}\varphi(q^{n+J}),$$

which leads to

$$\varphi(q^n)=q^{-mH}\varphi(q^{n+mJ})\text{ for all }m\in\mathbb{Z}.$$

Let h be a non-negative integer such that φ does not have a pole at q^h. (Since the number of poles is finite, such an h exists.) Setting $n=h$, we have

$$\varphi(q^h)=q^{-mH}\varphi(q^{h+mJ}).$$

Setting $\varphi(q^h)=\gamma'=const\in\overline{\mathbb{C}(q)}$, we obtain

$$\varphi(q^{h+mJ})=\gamma' q^{mH}\text{ for all }m\in\mathbb{Z}.$$

Therefore,

$$\varphi(q^n)=\gamma q^{n\frac{H}{J}},\ \text{where }\gamma=\gamma' q^{-h\frac{H}{J}},$$

and $p(q^n;q^k)=q^k-\gamma q^{\frac{H}{J}n}$, i.e., a factor of the form (12).

For $J'=j_1+\cdots+j_{\alpha-1}, H'=h_1+\cdots+h_{\alpha-1}$, we have

$$p_1(q^n;q^k)=q^{-H'}p\left(q^{n+J'};q^{k+H'}\right).$$

This implies that $p_1(q^n;q^k)$ is of the form (12). □

Theorem 1 *(Criterion for the existence of a qZ-pair for a rational function.) Let* $F(q^n,q^k) \in \mathbb{C}(q)(q^n,q^k)$ *be such that*

$$F(q^n,q^k) = (Q_k - 1)S(q^n,q^k) + T(q^n,q^k), \tag{14}$$

$S(q^n,q^k), T(q^n,q^k) \in \mathbb{C}(q)(q^n,q^k)$, *and the denominator* $g(q^n,q^k)$ *of* $T(q^n,q^k)$ *is such that* $\deg_{q^k} g(q^n,q^k)$ *has the minimal possible value. Then a qZ-pair for* $F(q^n,q^k)$ *exists iff*

$$g(q^n,q^k) = \alpha q^{an} \prod_i (q^k - \gamma_i q^{c_i n}), \quad c_i \in \mathbb{Q}, \gamma_i, \alpha \in \overline{\mathbb{C}(q)}, a \in \mathbb{Z}. \tag{15}$$

Proof: The factor q^{an} in $g(q^n,q^k)$ can be moved to the numerator of $T(q^n,q^k)$, and the necessity of the theorem follows from Lemma 4. Since $(Q_k-1)S(q^n,q^k)$ has a qZ-pair, the sufficient condition is proven if we can show that $T(q^n,q^k)$ also has a qZ-pair.

Let

$$T(q^n,q^k) = \frac{f(q^n,q^k)}{g(q^n,q^k)} = \frac{f(q^n,q^k)}{\alpha q^{an}\prod_{i=1}^{u}\left(q^k - \gamma_i q^{\frac{H_i}{J_i}n}\right)}, \gamma_i \in \overline{\mathbb{C}(q)}, H_i, J_i \in \mathbb{Z}, J_i > 0.$$

Each factor $q^k - \gamma_s q^{\frac{H_s}{J_s}n}$ in the denominator can be written as

$$\frac{u(q^n,q^k)}{v(q^n,q^k)},$$

where

$$u(q^n,q^k) = q^{J_s k} - \gamma_s q^{H_s n}, v(q^n,q^k) = \prod_{m=1}^{J_s-1}\left(q^k - e^{\frac{2\pi i m}{J_s}}\gamma_s q^{\frac{H_s}{J_s}n}\right).$$

Hence

$$T(q^n,q^k) = \frac{f^*(q^n,q^k)\alpha^{-1}q^{-an}}{\prod_{u=1}^{m_1}(q^{J_u k} - \gamma_u^{J_u} q^{H_u n})\prod_{v=1}^{m_2}(1-\gamma_v^{-J_v} q^{J_v k - H_v n})}, \tag{16}$$

where $J_u, J_v > 0, H_u \geq 0, H_v < 0, \gamma_u, \gamma_v \in \overline{\mathbb{C}(q)}, f^*(q^n,q^k) \in \mathbb{C}(q)[q^n,q^k]$. Note that each factor of the denominator of $T(q^n,q^k)$ can be written in the form

$$\frac{(c_s;q)_{a_s n + b_s k}}{(w_s;q)_{u_s n + v_s k}} q^{dk}$$

where the assumptions on the parameters are the same as in (3). Additionally, $f^*(q^n,q^k) \in \mathbb{C}(q)[q^n,q^k]$. Hence $T(q^n,q^k)$ is a sum of q-proper terms. By using the "fundamental theorem" and Lemma 1, we can conclude that there exists a qZ-pair for $T(q^n,q^k)$. □

Example 1 Let

$$F(q^n,q^k) = \frac{1}{q^{2k} + (1+q^n-q^{2n})q^k - q^{3n} - q^{2n}}.$$

In this example, the rational function $S(q^n,q^k)$ in (14) vanishes. The denominator of $T(q^n,q^k)$ can be written in the form

$$(q^k - q^{2n})(q^k + q^n + 1),$$

and hence does not satisfy the criterion. We ran the program `qsumrecursion` [5] which is a Maple implementation of $q\mathcal{Z}$ on $F(q^n,q^k)$. Not realizing that no qZ-pair exists, the program tried to compute one, and returned the inconclusive answer "Found no recursion of order smaller than 6".

Example 2 Let

$$F_1(q^n,q^k) = \frac{1}{q^{3k} - q^n},$$

$$F_2(q^n, q^k) = (Q_k - 1)\frac{q^k - q^n}{q^k + q^n + 1} + \frac{1}{q^{3k} - q^n}.$$

In this example, $S(q^n, q^k) = 0$ for F_1 and $S(q^n, q^k) = (q^k - q^n)/(q^k + q^n + 1)$ for F_2. The denominators of the rational function $T(q^n, q^k)$ are the same for F_1 and F_2, and can be written in the form

$$q^{3k} - q^n = \left(q^k - q^{\frac{n}{3}}\right)\left(q^k + \left(\frac{1}{2} - \frac{\sqrt{3}i}{2}\right) q^{\frac{n}{3}}\right)\left(q^k + \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) q^{\frac{n}{3}}\right).$$

Therefore $q\mathcal{Z}$ is applicable to both. In terms of time and space requirements, it takes `qsumrecursion` 0.340 seconds and 1345764 bytes to compute a qZ-pair for $F_1(q^n, q^k)$ as opposed to 1295.260 seconds and 4266058620 bytes for the case of $F_2(q^n, q^k)$. [1] This leads to a possibly good improvement in the implementation of $q\mathcal{Z}$: one first applies the decomposition problem to $F(q^n, q^k)$, and then computes a qZ-pair (L, G) for $T(q^n, q^k)$. Since $(1, S(q^n, q^k))$ is a qZ-pair for $(Q_k - 1)S(q^n, q^k)$, a qZ-pair for $F(q^n, q^k)$ is $(L, LS(q^n, q^k) + G)$.

5 How to use the criterion

First we consider the question of how to recognize if a given polynomial can be written in some desired forms.

Lemma 5 *A monic irreducible polynomial $p(q^n, q^k) \in \overline{\mathbb{C}(q, q^n)}[q^k]$ has the form (12) iff*

$$p(q^{n+1}, q^{k+c}) = q^c p(q^n, q^k). \tag{17}$$

Proof : If $p(q^n, q^k)$ has the form (12) then (17) evidently holds. Conversely, if (17) holds, then

$$p(q^{n+m}, q^{k+mc}) = q^{mc} p(q^n, q^k), \ \ m \in \mathbb{N}. \tag{18}$$

Let $p(q^n, q^k) = q^k - \varphi(q^n), \ \ \varphi(q^n) \in \overline{\mathbb{C}(q, q^n)}$. Equality (18) gives

$$\varphi(q^{n+m}) = q^{mc}\varphi(q^n), \ \ m \in \mathbb{N}. \tag{19}$$

By using the similar argument as that used in the proof of Lemma 4, Section 4, we obtain the relation

$$\varphi(q^n) = \lambda q^{cn}, \ \ \lambda \in \overline{\mathbb{C}(q)}.$$

Consequently,

$$p(q^n; q^k) = q^k - \lambda q^{cn}.$$

By setting $\gamma = -\lambda$, we get what was claimed. □

Lemma 6 *A monic polynomial $f(q^n; q^k) \in \overline{\mathbb{C}(q, q^n)}[q^k]$ can be written in the form*

$$(q^k + \gamma_1 q^{cn})(q^k + \gamma_2 q^{cn}) \dots (q^k + \gamma_d q^{cn}), \ \ c \in \mathbb{Q}, \gamma_1, \dots, \gamma_d \in \overline{\mathbb{C}(q)} \tag{20}$$

iff

$$f(q^{n+1}, q^{k+c}) = \alpha f(q^n, q^k), \ \ \alpha \in \overline{\mathbb{C}(q)}. \tag{21}$$

[1] All the reported timings were obtained on 400Mhz, 1Gb RAM, SUN SPARC SOLARIS.

Proof: If $f(q^n;q^k)$ has the form (20) then $f(q^{n+1},q^{k+c}) = q^{cd} f(q^n,q^k)$. Therefore, (21) holds. Conversely, if (21) holds, then

$$f(q^{n+m}, q^{k+mc}) = \alpha^m f(q^n, q^k), \quad m \in \mathbb{N}. \tag{22}$$

Let

$$f(q^n;q^k) = (q^k - \varphi_1(q^n)) \dots (q^k - \varphi_d(q^n)), \quad \varphi_1(q^n), \dots, \varphi_d(q^n) \in \overline{\mathbb{C}(q, q^n)}.$$

Consider any irreducible factor $p_i(q^n;q^k) = q^k - \varphi_i(q^n)$ of $f(q^n;q^k)$. From (22) there exists an integer $j, 1 \le j \le d$, such that

$$(q^k - \varphi_i(q^n)) = q^k - q^{-mc}\varphi_j(q^{n+m}),$$

or equivalently,

$$\varphi_i(q^n) = q^{-mc}\varphi_j(q^{n+m}).$$

Again by using the similar argument as that used in the proof of Lemma 4, one obtains the relation

$$\varphi_i(q^n) = \alpha_i q^{cn}, \quad \alpha_i \in \overline{\mathbb{C}(q)}.$$

This means

$$p_i(q^n, q^k) = q^k - \alpha_i q^{cn}, \quad 1 \le i \le d.$$

By setting $\gamma_i = -\alpha_i$, we get what was claimed. □

Let $w(q^n,q^k) \in \mathbb{C}(q)[q^n,q^k]$, and $c \in \mathbb{Q}$. Denote by A_c the transformation

$$q^n \to q^{n+1}, \ q^k \to q^{k+c}.$$

Let $t_0(q^n,q^k) = \gcd(A_c w, w)$. Define the sequence of computation

$$t_i(q^n,q^k) = \gcd(A_c t_{i-1}, t_{i-1}), \quad i = 1, 2, \dots \tag{23}$$

where the termination condition in (23) takes place when $\deg_{q^k} t_i(q^n,q^k) = \deg_{q^k} t_{i-1}(q^n,q^k)$, i.e., the degree w.r.t. q^k stops decreasing. (Note that the number of gcd computation in (23) is guaranteed to be finite.) Set $w_c(q^n,q^k) = t_{i-1}(q^n,q^k)$. The following Lemma gives us a possibility to find $\deg_{q^k} w_c(q^n,q^k)$ for all $c \in \mathbb{Q}$ such that $w_c(q^n,q^k) \ne 1$.

Lemma 7 *Let $w(q^n,q^k) \in \mathbb{C}[q^n,q^k]$. Let $t_0(q^n,q^k) = \gcd(A_c w, w)$. Then $w_c(q^n,q^k)$ is the maximal factor of $t_0(q^n,q^k)$ of the form*

$$\prod_{i=1}^{s}(q^k - \gamma_i q^{cn}), \quad \gamma_i \in \overline{\mathbb{C}(q)}. \tag{24}$$

Proof: Lemma 5 allows one to remove all reducible factors which are not of the form (12) from $t_0(q^n,q^k)$. Lemma 6 guarantees that $w_c(q^n,q^k)$ can be written in the form (20). □

The following theorem shows how to use the criterion for an arbitrary rational function.

Theorem 2 *Let $g(q^n,q^k) \in \mathbb{C}[q^n,q^k], \deg_{q^k} g(q^n,q^k) > 0$. Extract from $g(q^n,q^k)$ the maxmimal factors $v_1(q^n) \in \mathbb{C}(q)[q^n]$, $v_2(q^k) \in \mathbb{C}(q)[q^k]$. Set $w(q^n,q^k) = g(q^n,q^k)/(v_1(q^n)v_2(q^k))$. Let $c_0, \dots, c_m$ be all rational values of c such that $w_c(q^n,q^k) \ne 1$. Set $\delta_i = \deg_{q^k} w_{c_i}(q^n,q^k)$. Then $w(q^n,q^k)$ can be represented as the product of polynomials of the form*

$$k + \gamma q^{cn}, \quad c \in \mathbb{Q} \setminus \{0\}, \gamma \in \overline{\mathbb{C}(q)}, \tag{25}$$

iff

$$\delta_0 + \dots + \delta_m = \deg_{q^k} w(q^n,q^k). \tag{26}$$

Proof: If $w(q^n, q^k)$ can be represented as the product of polynomials of the form (25) then (26) holds since the $w_{c_0}(q^n, q^k), \ldots, w_{c_m}(q^n, q^k)$ are pairwise relatively prime. If (26) holds, then any irreducible factor $p(q^n, q^k)$ of $w(q^n, q^k)$ divides one of the $w_{c_0}(q^n, q^k), \ldots, w_{c_m}(q^n, q^k)$. This implies that $p(q^n, q^k)$ is of the form (25). □

Now for a given $F(q^n, q^k) \in \mathbb{C}(q)(q^n, q^k)$, rewrite F in the form (14) where

$$T(q^n, q^k) = \frac{f(q^n, q^k)}{g(q^n, q^k)}$$

is in reduced form. Extract from $g(q^n, q^k)$ the maximal factor $v_1(q^n) \in \mathbb{C}(q)[q^n], v_2(q^k) \in \mathbb{C}(q)[q^k]$. We know that $v_2(q^k)$ can be written as the product of factors of the form

$$q^k - \gamma, \gamma \in \overline{\mathbb{C}(q)}.$$

As for $v_1(q^n)$, if it cannot be written in the form $q^{an}, a \in \mathbb{Z}$, we can conclude that $q\mathcal{Z}$ is not applicable to $F(q^n, q^k)$. Set

$$w(q^n; q^k) = \frac{g(q^n, q^k)}{v_1(q^n) v_2(q^k)}.$$

It remains to investigate whether $w(q^n; q^k)$ can be decomposed into factors of the form (25). Let w' be an element from $\mathbb{C}(q)[q^x, q^n, q^k]$ obtained from w by substituting k by $k + x$ and n by $n + 1$. Let

$$S(q^x; q^n) = Resultant_{q^k}(w, w'),\ S(q^x; q^n) \in \mathbb{C}(q)[q^x, q^n].$$

Find all rational values of x such that $S = 0$, i.e., w and w' have a non-trivial common factor. To attain this goal, consider S as a polynomial in q^n with coefficients which are polynomials of q^x. Let $G(q^x)$ be the gcd of all these coefficients. Now we need to find all values of $x \in \mathbb{Q} \setminus \{0\}$ such that $G(q^x) = 0$. (An algorithm for this is given in the following paragraph.) Let $x_0, \ldots, x_m$ be the set of all non-zero rational numbers such that $G(q^{x_i}) = 0, 0 \leq i \leq m$. We now apply Lemma 7 to find $c_0, \ldots, c_d$ from the set $\{x_0, \ldots, x_m\}$ such that $deg_{q^k} w_c(q^n, q^k) \neq 0$. Set $\delta_i = \deg_{q^k} w_{c_i}(q^n, q^k)$. To check whether the criterion holds, it is sufficient to check if relation (26) is satisfied.

Now on the search for $x_0, \ldots, x_m$, suppose we have an equation of the form

$$a_m(q) X^m + \cdots + a_1(q) X + a_0(q) = 0 \tag{27}$$

where $a_i(q) \in \mathbb{C}[q], 0 \leq i \leq m$. We would like to find all values of X that satisfy (27) where X is of the form $X = q^x, x \in \mathbb{Q} \setminus \{0\}$. One solution is to generate a finite set of candidates for x. Then substitute each element of the set into (27) and check for equality. One way is to consider any monomial q^j of $a_m(q)$. There must exist a monomial q^l in one of the polynomials $a_{m-1}(q), \ldots, a_0(q)$, say $a_d(q)$, such that $q^j q^{xm} = q^l q^{xd}$, i.e., $j + mx = l + xd$. (Otherwise $q^j q^{mx}$ does not vanish.) By examining all monomials in $a_{m-1}(q), \ldots, a_0(q)$, we generate a set of candidates for x. Since each monomial of $a_i(q), 0 \leq i \leq m$ generates a set of candidates for x, one can choose a set of monomials as generators of different candidate sets, and then take the intersection of these sets. Note that any solution $X = q^x$ of (27) is an algebraic function of q whose Puiseux series (a fraction power series) has only one term. Using equation (27) for this series we can find the value of the exponent x by Newton's polygon [10]. This describes another way to generate a set of candidates for x.

6 Implementation

We now show a Maple implementation of the criterion on $T(q^n, q^k)$ in (14). Let $N = q^n, K = q^k$. The main procedure, `is_qZ_applicable`, has the following calling sequence is_qZ_applicable(T_NK,N,K,q);

where T_NK is a rational function in N and K. If each irreducible factor over $\overline{\mathbb{C}(q)(N)}$ of the denominator of T_NK can be written in the form $K-\gamma N^c$ or, equivalently, $q^k-\gamma q^{cn}$, $c\in\mathbb{Q},\gamma\in\overline{\mathbb{C}(q)}$, the procedure returns `true`; otherwise it returns `false`.

Note that `infolevel` is used to illustrate the main steps of the program.

Example 3 Consider the rational function
$T(N,K)=$

$$\frac{q}{K^2(q^2K^5-q^2K^3N-q^3NK^2+q^3N^2-3K^5+3K^3N+3qNK^2-3qN^2)}.$$

$T(N,K)$ satisfies the criterion since its denominator can be written as
$(q^2-3)K^2(K-N^{\frac{1}{2}})(K+N^{\frac{1}{2}})(K-q^{\frac{1}{3}}N^{\frac{1}{3}})(K+\frac{1}{2}(1-3^{\frac{1}{2}}i)q^{\frac{1}{3}}N^{\frac{1}{3}})(K+\frac{1}{2}(1+3^{\frac{1}{2}}i)q^{\frac{1}{3}}N^{\frac{1}{3}})$.

```
>infolevel[is_qZ_applicable] := 3:
>T := q/K^2/(q^2*K^5-q^2*K^3*N-q^3*N*K^2+q^3*N^2-3*K^5+3*K^3*N+3*q*N*K^2-3*q*N^2);
>is_qZ_applicable(T,N,K,q);
"extract from g(N,K) the maximal factor v(K)"
"result: v(K) = K^2*(q^2-3)"
"set w(N,K) = g(N,K)/v(K)"
"result: w(N,K) = q*N*K^2-q*N^2-K^5+K^3*N"
"set w'(N,K,x) = w(N*q,K*x)"
"result: w'(N,K,x) = q^2*N*K^2*x^2-q^3*N^2-K^5*x^5+K^3*x^3*q*N"
"Find all values of x of the form q^b, b rational "
"such that w and w' have a non-trivial common factor"
"The set of candidates for b is: {1/2, 1/3, 2/5, 3/7, 3/8, 4/11, 5/13}"
"Substitution done. The solution is {1/2, 1/3}"
"degree checking: 5 = 5?"
"applicable!"
```

$$true$$

Example 4 Consider the rational function

$$T(N,K)=\frac{1}{K^2N+K^3+K^2-N^2-NK-N}.$$

$T(N,K)$ does not satisfy the criterion since its denominator can be written as

$$(K-N^{\frac{1}{2}})(K+N^{\frac{1}{2}})(N+K+1).$$

```
>T := 1/(K^2*N+K^3+K^2-N^2-N*K-N);
```

$$T:=\frac{1}{K^2N+K^3+K^2-N^2-NK-N}$$

```
>is_qZ_applicable(T,N,K,q);
"extract from g(N,K) the maximal factor v(K)"
"result: v(K) = 1"
"set w(N,K) = g(N,K)/v(K)"
"result: w(N,K) = K^2*N+K^3+K^2-N^2-N*K-N"
"set w'(N,K,x) = w(N*q,K*x)"
```

```
"result: w'(N,K,x) = K^2*x^2*N*q+K^3*x^3+K^2*x^2-N^2*q^2-N*q*K*x-N*q"
"Find all values of x of the form q^b, b rational "
"such that w and w' have a non-trivial common factor"
"The set of candidates for b is: {1/2}""
"Substitution done. The solution is {1/2}"
"degree checking: 2 = 3?"
"not applicable!"
```

$$false$$

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