Continuity Adjustments to Triangular Bézier Patches That Retain Polynomial Precision

Computer Science Department
University of Waterloo
Research Report CS-2000-01

Stephen Mann

Abstract

In this paper, I discuss a method for increasing the continuity between two polynomial patches by adjusting their control points. The method described in this paper leaves the control points unchanged if the patches already meet with the desired level of continuity. Next I give two C^0 degree n polynomial interpolation schemes that reproduce degree n polynomials, and show how to apply my continuity increasing scheme to these interpolants without decreasing their polynomial precision. The second of these interpolants is interesting in its own right, as it requires less data than other methods. Finally, I apply my continuity method to Clough-Tocher methods, and create split domain schemes with top-level polynomial precision.

1 Introduction and Background

There are many solutions to the following problem: Given a triangulated set of points in the plane, with height values and derivatives at the points, find a smooth function that interpolates the values and derivatives at the data points. Here smooth means for the surface to be C^k, with $k \geq 0$. See one of several survey papers for an overview of schemes to solve this problem [6, 7]. In this paper, I will investigate piecewise polynomial schemes for interpolating such data. The standard method for interpolating this data with polynomials is to split the data triangles into three or more pieces [2, 13], which allows for a C^1 construction with compatible twist vectors at the corners of the patches.

I am concerned with constructions that achieve polynomial precision. Similar constructions exists for (and this work is based on) split domain schemes [10] and for non-polynomial schemes [5].

Through out I will assume that we have a non-degenerate triangulation of the data with no overlapping triangles. Further, I will not worry about boundary conditions.

I will use the multi-variate Bernstein-Bézier representation for polynomials. The description of the Bernstein-Bézier representation that I give here just touches on the topics that I need for this paper. For a more complete discussion on triangular Bézier patches, see any introductory text on CAGD, such as Farin's [4].

I will index the control points using standard multiindex notation. Figure 1 is a schematic illustrating this labeling for quintic patches. Many of the figures in this paper will be of this schematic form; although the control points are regularly placed in a plane in the diagram, they represent points in three-space. I

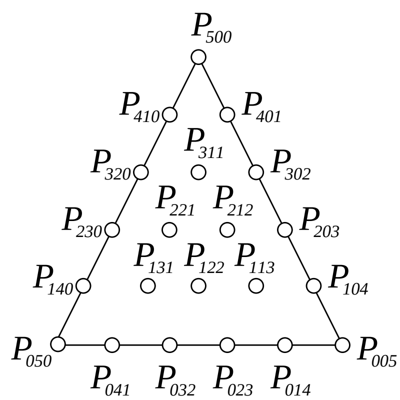

Figure 1: Quintic Bézier polynomial with vertices labeled.

will also use the barycentric form of the polynomials, where every point in a domain triangle is expressed as an affine combination of the triangle corners, $\triangle V_0V_1V_2$:

$$t = t_0V_0 + t_1V_1 + t_2V_2,$$

with $t_0 + t_1 + t_2 = 1$. In this formulation, triangular Bézier patches have the following form:

$$B(t) = \sum_{\vec{i},|\vec{i}|=n} P_{\vec{i}} B^n_{\vec{i}}(t),$$

where $\vec{i} = (i_0, i_1, i_2)$ is a multiindex, the $P_{\vec{i}}$ are the coefficients (or control points) for the patch, and the $B^n_{\vec{i}}$ are the degree n Bernstein polynomials:

$$B^n_{\vec{i}}(t) = \frac{n!}{i_0!i_1!i_2!} t_0^{i_0} t_1^{i_1} t_2^{i_2}.$$

The derivative and continuity analysis used in this paper is simplified by using the *polar form* or *blossom* of the polynomial [14]. For a degree n polynomial B, the polar form of B (denoted ϖB) is an n-variate function satisfying the following:

- ϖB is symmetric;
- ϖB is multi-affine in each argument;
- $\varpi B(u^{<n>}) = B(u)$,

where $\varpi B(u^{<n>})$ is ϖB evaluated with all n of its arguments equal to u. The polar form has a nice relation to the Bézier control points of a triangular patch. In particular, over a domain triangle $\triangle V_0V_1V_2$, $P_{i,j,k} = \varpi B(V_0^{<i>}, V_1^{<j>}, V_2^{<k>})$.

The coefficients of a Bézier patch give us information about the derivatives of the patch. In particular, the derivatives in the direction of the triangle edges are proportional to simple differences of the control points:

$$\begin{aligned}
B(1,0,0) &= P_{n00}, \\
d_{e_{01}} B(1,0,0) &= n(P_{n-1,1,0} - P_{n,0,0})/|V_1 - V_0|, \\
d_{e_{02}} B(1,0,0) &= n(P_{n-1,0,1} - P_{n,0,0})/|V_1 - V_0|.
\end{aligned}$$

Here, e_{01} is the unit directional derivative from V_0 to V_1; e_{02} is similar. Derivatives at the other corners and higher order derivatives are computed in a similar fashion. Thus, if we are given position and derivative information at the corners of the patch, it is easy to find settings of the control points to interpolate this information. In the following discussion, I will merely state that we set control points to match the derivatives and not give the formulas.

In my diagrams, when a group of control points are set using the derivative information at one of the V_i, I will circle those points with a dashed circle, as in Figure 10. Conversely, these dashed regions also indicate the number of derivatives needed at the corresponding V_i (i.e., at each data point, we need to have the position and the appropriate derivatives for setting the circled control points). Control points covered by more than one dashed circle will be set by averaging the values computed for each set of derivatives.

If we want two neighboring patches to meet with C^k continuity, then there are simple settings of the control points to achieve this. To achieve a C^0 join, the boundary control points of two patches have to be identical. To achieve C^1 continuity between patches F and G over $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$ respectively, blossoming tells us that

$$\begin{aligned} \varpi G(D_0, V_1^{<i>}, V_2^{<j>}) &= \varpi F(D_0, V_1^{<i>}, V_2^{<j>}) \\ &= a_0 \varpi F(V_0, V_1^{<i>}, V_2^{<j>}) + a_1 \varpi F(V_1, V_1^{<i>}, V_2^{<j>}) + \\ &\quad\ a_2 \varpi F(V_2, V_1^{<i>}, V_2^{<j>}), \end{aligned}$$

where (a_0, a_1, a_2) are the barycentric coordinates of D_0 relative to $\triangle V_0V_1V_2$ and $i + j = n - 1$, $i, j \geq 0$. Geometrically, the condition is that the four control points in each of the the neighboring triangular panels of the two patches must be coplanar (for example, in Figure 3 each shaded and hashed group of four points must be coplanar). Higher order conditions exist; later in this paper I will show the C^2 and C^3 conditions.

The rest of this paper will proceed as follows: First, I will describe an averaging technique for adjusting the control points of two neighboring patches to obtain C^k continuity between these two adjacent patches. The important feature of the averaging scheme is that if the patches already meet with C^k continuity, then the control points remain unchanged. As examples of how this continuity scheme is useful, I will present two methods for constructing C^0 interpolants that set all the degrees of freedom using derivative data at the data points of the triangulation. Both of these interpolants will reproduce maximal degree polynomials; i.e., the degree n interpolant will reproduce degree n polynomials. By using my averaging scheme to increase the continuity between the patches we obtain higher order continuity without losing any of the polynomial precision. As further examples, I will show how to apply my scheme to Clough-Tocher schemes.

2 Increasing Continuity Between Two Patches

Given two polynomial patches over adjacent triangles of our domain (i.e., where the two domain triangles share exactly one edge) that meet with some level of continuity, we would like to increase the continuity with which our patches meet. This increased continuity will require adjusting the control points of one or both patches. However, I will further require my adjustment scheme to leave unchanged any control points that already meet the continuity conditions. As a starting point, I use a variation of the method of Foley-Opitz [5], who devised such a construction for cubics.

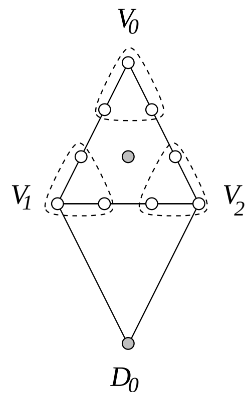

Figure 2: Foley-Opitz cross-boundary scheme.

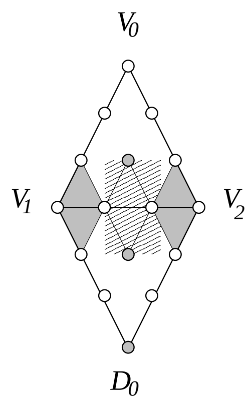

Figure 3: Two cubics meeting along a common boundary.

Foley and Opitz were working with triangulated data having first derivatives at the data points. They were fitting *hybrid-cubics* to the data (see their paper for details on hybrid-patches), and as part of their construction they found a cubic precision construction for two adjacent patches. Their construction (illustrated in Figure 2) constructs a patch for each data triangle. The patch for $\triangle V_0V_1V_2$ is constructed by using the data at the V_i to set the white control points, and the positional data at D_0 is used to set the shaded control point (i.e., the center point of the patch is set so that the patch interpolates the z-value at D_0 when evaluated at D_0). The patch for $\triangle V_2V_1D_0$ is built in a symmetric fashion.

As illustrated schematically in Figure 3, these two patches will share boundary control points (since both patches compute the boundary points from the data at V_1 and V_2). To meet with C^1 continuity, each of the three panels of four control points must be coplanar. The gray panels will be coplanar because both patches compute these control points consistent with the derivative information at V_1 and V_2. However, the hashed panel will not, in general, be coplanar; these four points will be coplanar if and only if the data at the V_i and at D_0 come from a common cubic.

To achieve C^1 continuity in the general case (i.e., non-coplanar hashed panels panels, as illustrated on the left in Figure 4), Foley and Opitz in effect extend both panels to the neighboring triangle as shown in the middle of Figure 4. They then average the two points on either side, which results in coplanar panels (the right in Figure 4). Note that if the data at V_1 and V_2 of Figure 2 come from a common cubic,

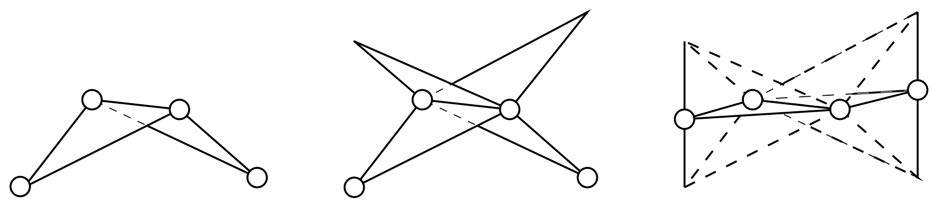

Figure 4: Adjusting the panels to meet C^1.

then panels on the left of Figure 4 will be coplanar, and the averaging will have no effect. The result is that their construction builds two cubic patches that meet with C^1 continuity and reproduce cubic polynomials if the data at all four vertices comes from a common cubic.

2.1 C^0 Continuity

Most constructions build common boundary curves before setting the interior control points, so the patches meet with C^0 continuity as a first step. However, if we knew our construction might build patches that did not meet with C^0 continuity, we could use the average of each pair of control points along the boundary (one from each patch) as the boundary points. The patches would then have a common boundary, and if the initial boundary points were already identical, this averaging would not change them.

2.2 C^2 Continuity

We can extend the Foley-Opitz averaging scheme to more interior vertices to achieve higher order continuity. For example, if we apply it to the next layer of control points, we can achieve C^2 continuity.

To have C^2 continuity, we first must have C^1 continuity. Given that we have C^1 continuity between patches F and G defined over $\triangle V_0V_1V_2$ and $\triangle V_2V_1D_0$ respectively, the following additional condition must hold for C^2 continuity:

$$\varpi G(D_0, V_0, V_1^{<i>}, V_2^{<j>}) \quad = \quad \varpi F(D_0, V_0, V_1^{<i>}, V_2^{<j>}),$$

for $i + j = n - 2$, $i, j \geq 0$. We can construct these two points from the control points of the polynomials as

$$\begin{aligned}
\varpi F(D_0, V_0, V_1^{<i>}, V_2^{<j>}) &= a_0 \varpi F(V_0, V_0, V_1^{<i>}, V_2^{<j>}) + a_1 \varpi F(V_1, V_0, V_1^{<i>}, V_2^{<j>}) + \\
&\quad a_2 \varpi F(V_2, V_0, V_1^{<i>}, V_2^{<j>}), \\
\varpi G(D_0, V_0, V_1^{<i>}, V_2^{<j>}) &= b_0 \varpi F(D_0, D_0, V_1^{<i>}, V_2^{<j>}) + b_1 \varpi F(V_1, D_0, V_1^{<i>}, V_2^{<j>}) + \\
&\quad b_2 \varpi F(V_2, D_0, V_1^{<i>}, V_2^{<j>}),
\end{aligned}$$

where (a_0, a_1, a_2) are the barycentric coordinates of D_0 relative to $\triangle V_0V_1V_2$ and (b_2, b_1, b_0) are the barycentric coordinates of V_0 relative to the triangle $\triangle D_0V_2V_1$.

Geometrically, these condition require certain groups of nine control points along the boundary of one patch to be in a special relationship to the corresponding control points of the neighboring patch as described by Farin [3] and later by Lai [8]. The vertices adjacent to the shaded triangles in each of the three diagrams in Figure 5 show the groups of vertices affecting C^2 continuity in the quartic case. Figure 6 illustrates these constraints. First, the dark shaded panels must be coplanar (this is the C^1

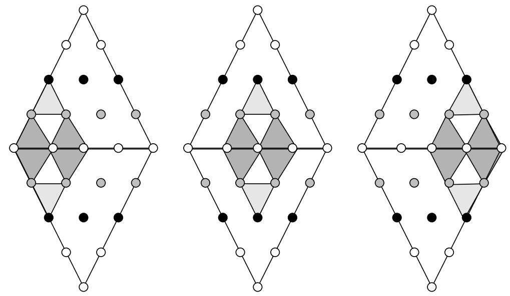

Figure 5: Quintic control points affecting C^2 continuity.

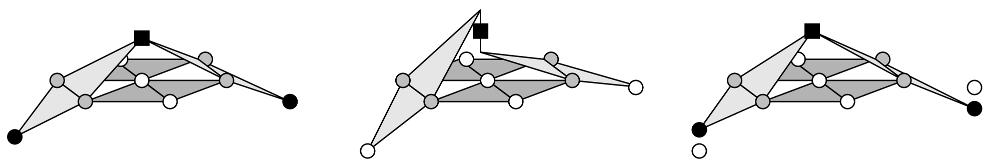

Figure 6: The C^2 constraints (left) and construction (middle and right).

condition). Next, if we take the three vertices connected by the light shaded panel, and extend them in the ratio given by the two domain triangles, we get a point. If we do the same extension using the six corresponding points of the neighboring patch, we get another point. For the patches to meet with C^2 continuity, these two points must be the same (the black square in the left diagram of Figure 6). Such a condition must hold at all three groups of points illustrated in Figure 5.

In general the groups of nine control points will not have this property, and if F and G meet with C^1 continuity along their common boundary then we have a situation more like the middle diagram of Figure 6. In this case, my scheme is to average the two extrapolated points (giving the black square in the middle figure), and then extrapolate in the other direction to get the black points of the right diagram (the white points indicate the initial positions of these control points).

Note that if a set of nine control points is already in an acceptable C^2 configuration, then this averaging will leave the control points unchanged.

2.3 C^3 and Beyond

The C^1 and C^2 conditions illustrate the two types of conditions that will occur: one is a coplanarity requirement and the other a constructed point that must be common to both patches. When we go to higher order continuity, the C^k condition first require C^{k-1} continuity and then impose either the coplanarity or common point condition. The last condition may require some construction to set up the points in the condition, but if the condition is not met, we can use one of the averaging schemes discussed earlier and find new settings of the kth layer of control points to satisfy this condition.

For example, the C^3 condition is that

$$\begin{aligned}
\varpi G(D_0, D_0, V_0, V_1^{<i>}, V_2^{<j>}) &= \varpi F(D_0, D_0, V_0, V_1^{<i>}, V_2^{<j>}) \\
&= a_0 \varpi F(D_0, V_0, V_0, V_1^{<i>}, V_2^{<j>}) + a_1 \varpi F(D_0, V_0, V_1, V_1^{<i>}, V_2^{<j>}) + \\
&\quad a_2 \varpi F(D_0, V_0, V_2, V_1^{<i>}, V_2^{<j>}), \qquad (1)
\end{aligned}$$

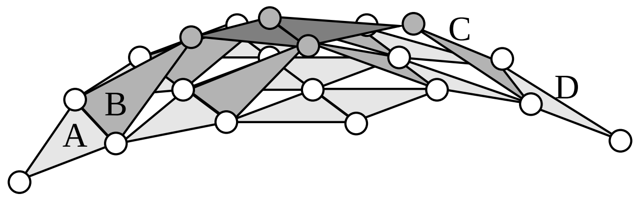

Figure 7: C^3 conditions

where $i + j = n - 3$, $i, j \geq 0$.

Figure 7 illustrates these conditions. In this figure, the white points (connected with the light gray panels) are the control points of our two patches. The medium gray panels extend from the control points to construct the gray points, and the dark gray panel illustrates the constraint given in Equation 1.

To test if two patches meet with C^3 continuity, consider each group of 10 control points along the boundary in each patch and first test the C^2 conditions. If those conditions are met, then extend panel A of Figure 7 to get the third point of panel B, and extend panel D to get the third point of panel D. For the patches to meet with C^3 continuity, the dark gray panels must be coplanar.

For my construction, first adjust the control points to make the patches meet with C^2 continuity. Then extend panels A and D to build the third points of panels B and C. We can now use the adjustment schemed used for C^1 continuity (Figure 4) to make the gray panels coplanar. This may cause panels A and B to no longer be coplanar; in this case, extend panel B to get a new position for the third point of panel A. A similar adjustment is made for panels C and D.

Again we see that if the two patches originally meet with C^3 continuity, then none of the control points of the two patches will be changed by this construction.

These constructions can be applied to achieve C^4 and higher continuity. For odd continuity, we will apply an adjustment similar to that of Figure 4, possibly causing us to push the adjustment back down the construction to adjust the control points of the patch as we did for C^3 continuity. And for even continuity, we will have a construction similar to that of Figure 6, again possibly needing us to push the adjustment back down the construction to adjust the control points of the patch.

In all cases, if the patches already meet with C^k continuity, the adjustments required to achieve C^{k+j} continuity will leave unchanged the control points in the first $k + 1$ layers along the boundary.

2.4 Continuity Conditions as Averaging of Polynomials

Mike Floater has pointed out that the continuity conditions can be viewed as averaging the two polynomials. If p_1 is the initial polynomial patch constructed for one triangle, and p_2 is the initial polynomial patch constructed for the adjacent triangle, then $P = (p_1+p_2)/2$ is the average of these two polynomials. What the above averaging schemes do is select layers of control points from P until the desired continuity is met. Since both sides are setting their control points from the same P, the two patches meet with the desired continuity.

Note that with Floater's view of the continuity control point adjustments, we still need a process similar to the ones described above to construct the control points of P, since at least one of p_1 or p_2 will need to be reparameterized over the other's domain.

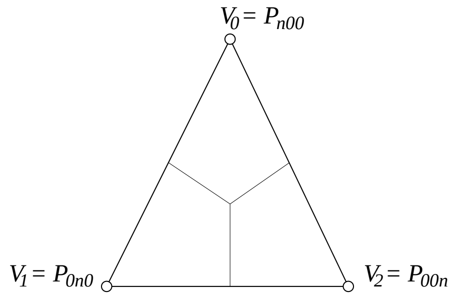

Figure 8: Control points in each region are set by data at the closest V_i. Control points between regions have averaged values.

3 Fitting a Single Polynomial Per Triangle

In this section, I will present two simple degree n interpolants that have degree n polynomial precision, but only produce C^0 surfaces. For each, I will show how to use my scheme for increasing continuity to increase the continuity of these each interpolant while maintaining degree n polynomial precision.

3.1 The First Interpolant

This interpolant requires $\lfloor 2n/3 \rfloor$ complete derivatives at each data points, although I will not use all of the higher order derivatives. I will use this data to set all of the control points. Some control points will be influenced by the derivatives at more than one corner. In these cases, I will use an averaging scheme to set these values in a symmetric manner. The result will be a C^0 interpolant that reproduces degree n polynomials.

More precisely, for P_{ijk}, with $i > j$, $i > k$, I use the data at V_0 to set the value of P_{ijk}. If $i < j = k$, then I compute two values using the data at V_0 and V_1 and average them. If $i = j = k$, then I compute three values, using the data at V_0, V_1, and V_2 and average the result. The remaining cases are handled in a similar fashion. This divides the data triangle into three regions as illustrated in Figure 8.

With this scheme, when constructing a degree n patch, if the data at all three corners comes from a single degree n polynomial, then this scheme reproduces that polynomial. In cases where a control vertex is the average of two or three values computed from the corner data, these values will be identical, since the corner data come from a single polynomial.

The piecewise interpolant filling a triangular network will create a C^0 piecewise polynomial surface. The C^0 continuity conditions are met because the boundary control points between two adjacent patches are computed using the same data. However, in general the patches will not meet with C^1 continuity, since in the case when n is even, the middle boundary point is an average of two values (Figure 9), and the panels adjoining this boundary point will not be coplanar, and when n is odd, a similar problem occurs for the cross-boundary points.

3.1.1 Increasing Continuity

Two neighboring patches built with this construction will only meet with C^0 continuity. However, note that only one or two panels along the boundary are out of C^1 alignment, as illustrated in Figure 9. Thus, if we apply the C^1 adjustment to these panels (a single panel when the degree is odd, two panels when

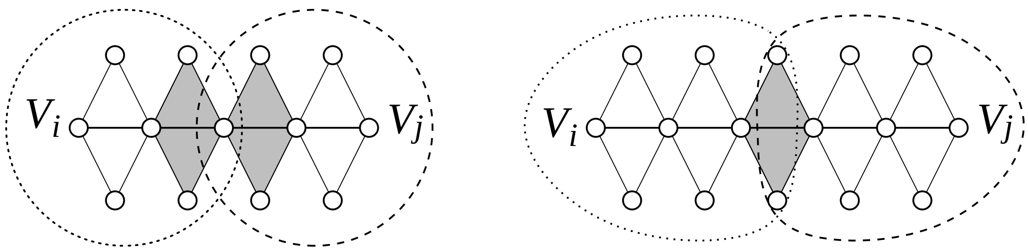

Figure 9: Join is not C^1 because shaded panels not coplanar.

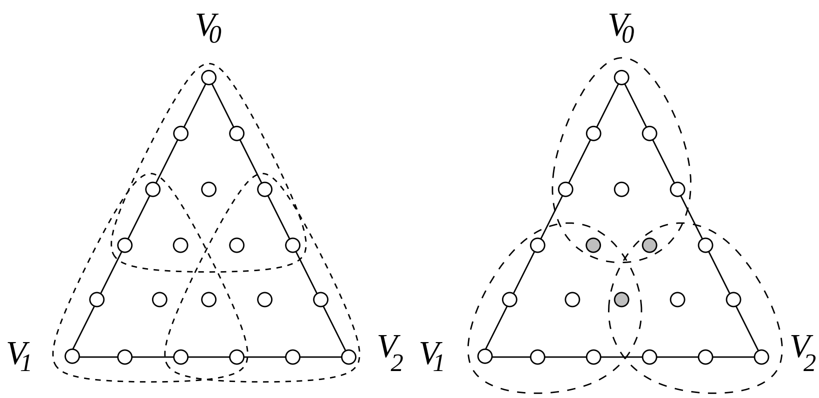

Figure 10: Derivatives needed for the quintic version of first scheme.

the degree is even), then the two patches will meet with C^1 continuity.

We must be careful, however, that the vertices adjusted for one boundary do not affect the C^1 connection along another boundary. In particular, degree 5 is the lowest degree for which we can use this C^1 adjustment on a patch network, since as illustrated in Figure 10, the shaded vertices are the ones we need to adjust, and each shaded vertex only affects the C^1 continuity across a single boundary.

In general, while we can apply the averaging construction given in this paper to pairs of patches, to apply it to a network of patches the degree of patch required by my constructions is $4k+1$, where k is the level of continuity desired. This requirement is needed so that the vertices adjusted to achieve C^k continuity along one boundary are not involved in the C^k conditions along another boundary. This $4k+1$ condition agrees with the result of Ženíšek [16].

3.1.2 Example

As an example, Figure 10 indicates the derivatives needed for the quintic case. On the left I have an illustration of the full derivatives at the data points. However, I do not need nor use all of these derivatives. The figure on the right indicates the needed derivatives. In this figure, we set the white vertices using the derivative information at data point within the same region. The shaded vertices are set to the average of the two values computed from the derivative information at the data vertices in the two regions in which they lie. Note that we only have an averaged center value for even degree patches.

Having set the control points of the patch using the corner data results in a C^0 patch network. To construct C^1, we take this patch network, and adjust the shaded points of Figure 10 using the Foley-Optiz

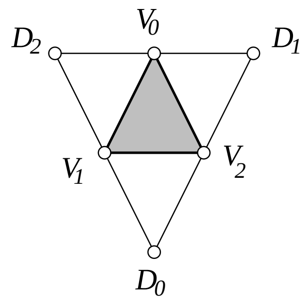

Figure 11: Interpolant to be constructed over shaded triangle.

scheme.

3.2 The Second Interpolant

With any interpolant that achieves polynomial precision, you need to set the degrees of freedom in a patch to match enough data to specify a polynomial of the desired degree. In many constructions, some of that data (position and derivatives) comes from the corner points, and then to achieve the desired continuity, the remaining data is specified as cross-boundary derivatives along the edges of the data triangle. The data at the corner points is difficult enough to obtain, but the cross-boundary data along the edges of the data triangle is even less convenient to get.

The interesting feature of the Foley-Optiz interpolant (and the Clough-Tocher interpolant based on their ideas) is that while it needs position and first derivatives at the corners, it needs no cross-boundary derivatives. Instead, it uses data stored at other vertices in the triangulation to completely specify the degrees of freedom in the patch and achieve polynomial precision. This is the basis for my second interpolant; some of the degrees of freedom in the patch will be set using the data at the triangle corners, and the remaining degrees of freedom will be set using data at other vertices in the triangulation. The advantages of this method are (a) it requires less data overall and (b) it does not need the cross-boundary derivative data. The disadvantage is a loss of locality in the construction.

This construction requires fewer derivatives at the data points, and instead uses data from neighboring triangles to achieve maximal polynomial precision. The construction is slightly different for even and odd degree. I will begin with odd degree $n = 2d+1$, for which I will construct a degree n patch over the triangle $V_0V_1V_2$ and I require the position and d complete derivatives at each of the six vertices shown in Figure 11

The construction works as follows (note that it only uses $d-1$ derivatives at the D_i):

1. For P_{ijk} where $i \geq d$, set P_{ijk} to match the relevant derivative data at V_0 (similarly, we will match the derivative data at V_1 and V_2 when $j \geq d$ and $k \geq d$).

2. Construct three degree n patches. Each patch will use the settings from the previous step and set the remaining control points so that the patch interpolates the data at one of the D_i (only position and $d-1$ derivatives are needed at D_i).

   This step requires setting up and solving a linear system involving the unknown control points.

3. Average the interior control points from the three patches as described below.

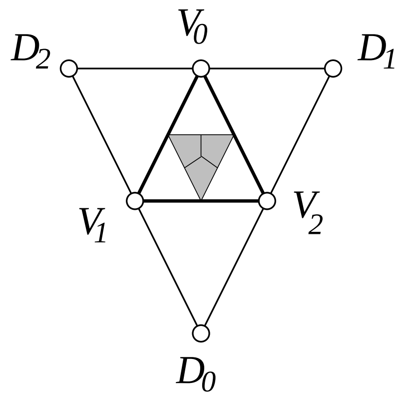

Figure 12: Averaging scheme for second scheme. I have multiple choices for control points in shaded region.

3.2.1 Averaging Scheme

The goal of the averaging scheme is to blend in a symmetric manner the vertices for which I have computed multiple values while maintaining polynomial precision. In my construction, I will have a set of vertices corresponding to the shaded vertices of Figure 14 for which I have computed multiple values.

The control points are divided into three regions; within a region, the values computed for one of the D_i's is used. On the boundaries between regions, an average of values is used. Figure 12 illustrates the scheme I used. The following states how the values are computed for one region in this diagram (note the differences between this averaging scheme and the one illustrated in Figure 8).

- For $i < n/3$ (the vertices in the shaded regions), I will use those values computed for the closest D_i.
- For $i > j = k$ (the vertices along the boundary between two regions), I will average the two values computed for the D_i on either side (and not use the values computed for the opposite D_i).
- For $i = j = k = n/3$ (the center point), I will average all three values computed for the D_i

The values in the other two regions are computed in a similar fashion.

Alternative averaging scheme exist; the important property that any scheme should have is that it is symmetric.

3.2.2 Even degree

For degree $n = 2d$, if we use $d - 1$ derivatives, then do not have enough constraints at a single D_i to set the interior control points. However, if we use d derivatives, then the data at the corners will give two settings for the middle control point along each boundary. For our purposes, it is sufficient to use the latter approach, and take the average of the two values for each of these three boundary vertices, and then continue the construction as outlined for the odd degree. Note that for degrees 2 and 4, the derivatives constraints fully specify the control points of the patch. It is not until degree 6 that we have control points left unset by the derivatives (Figure 13).

Each patch of the resulting interpolant reproduces degree n polynomials, uses less data at the D_i than the interpolant for odd degrees (but uses more data at the V_i), and the patches constructed for a triangulation meet with C^0 continuity.

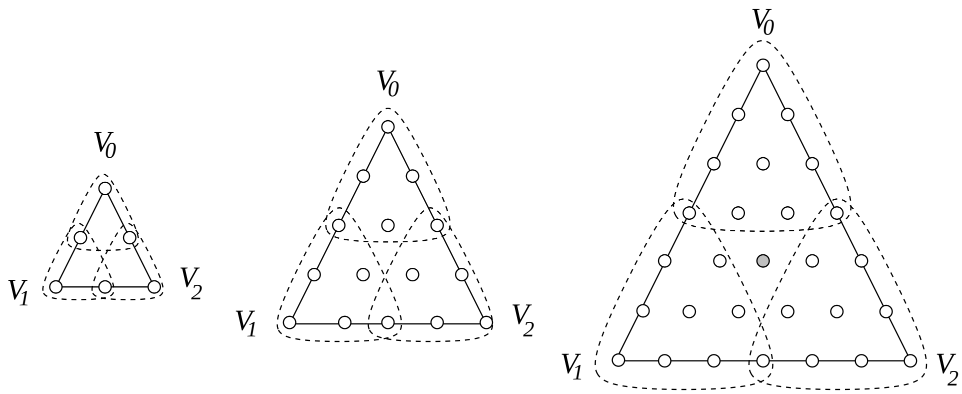

Figure 13: Even degrees 2, 4, and 6.

3.2.3 Continuity

This second construction also reproduces degree n polynomials when the data at all six vertices of Figure 11 come from a polynomial. This is because the data at the three V_i and any one of the D_i uniquely specify a polynomial. Thus, if the data at all six vertices come from a single polynomial, then all versions of the shaded vertices will be identical, and the averaging scheme will leave them unchanged.

This second construction is also only C^0. It is clear that it will be C^0, since the boundary vertices of a patch are computed using only the data stored at the two data vertices adjacent to that patch edge. Further, in general two neighboring patches will not meet with C^1 continuity because for the center panel of control points will not be coplanar.

Again, we can increase the continuity to C^k if our patch is of degree $4k + 1$ using the averaging schemes described in this paper.

3.2.4 Examples

Figure 14 illustrates steps one and two of the this scheme for quintics. In this figure, the white circles represent control points set using the data at the corresponding V_i. The remaining shaded points are constructed by fitting a patch to the white points and the position and derivative information at D_0. This process is repeated for D_1 and D_2, giving three values for each of the shaded control points.

Having set the control points of the patch using the corner data results in a C^0 patch network. To construct C^1, we take this patch network, and adjust the shaded points of Figure 10 using the Foley-Optiz scheme.

Remark: This quintic element is similar to Bell's triangle [1], with the primary difference being in the setting of the degrees of freedom for achieving C^1 continuity across the triangle boundaries (the shaded points of Figure 10).

As a second example, I will build a C^2 interpolant. Degree 9 is the minimum degree for which the C^2 adjustment can be applied without affecting the cross-boundary derivatives along the other two boundaries. As illustrated in Figure 15, the vertices circled with the dashed lines have been set to match derivatives at the boundaries. Looking at the vertices in the dotted region, the white ones and the corresponding groups of six vertices along the bottom edge will meet the C^2 conditions (without adjustment) with the patch across this boundary since all vertices in each group of six have been set to

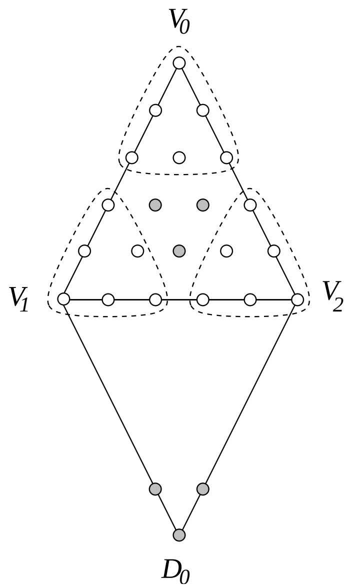

Figure 14: Construction for second scheme illustrated for $n = 5$.

agree with common derivative information; only the black vertices need to be adjusted to achieve a C^2 join along this boundary (once the one gray vertex has been adjusted to achieve C^1 continuity across this boundary). Further note that these two black vertices do not affect the C^2 continuity across the other two boundaries.

The following scheme will build a C^2 degree 9 interpolant:

1. Set the circled control points in Figure 15 to interpolate the derivatives at the corners.
2. Construct three patches, each of which sets the remaining interior control points to interpolate the derivative data at one of the three neighboring vertices (the D_i of Figure 11).
3. Apply the averaging scheme of Section 3.2.1 to combine the three settings of the interior vertices. Note that the averaging scheme will average the three values for the center point, but will select one of the other values to use for the remaining points. For example, when fitting to a configuration like the one in Figure 11, the two black vertices and the corresponding gray vertex of Figure 15 will be set to the values given by the patch that interpolates the data at D_0.

   At this point, we have constructed a C^0 interpolant with ninth degree precision.
4. Apply the Foley-Opitz averaging scheme (Section 2) to the gray vertices.
5. Apply the C^2 averaging scheme (Section 2.2) to the black vertices and the similar white vertices.

3.2.5 Solve-ability of Linear System

This second construction requires solving linear systems to find the interior control points. For example, for degree 5, we have three unknowns (P_{122}, P_{212}, and P_{221}). To solve for the unknowns, we have to solve the following system of equations:

$$\sum_{\vec{i},|\vec{i}|=5} P_{\vec{i}} B^5_{\vec{i}}(t) \quad = \quad D_0$$

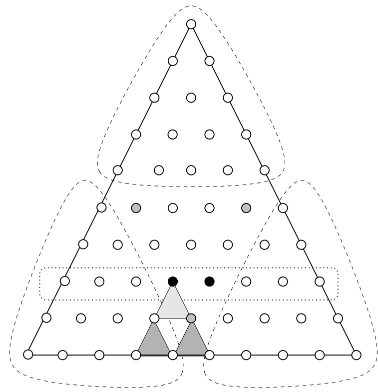

Figure 15: Adjusting a degree 9 to meet its neighbor C^2.

$$
\begin{aligned}
5\sum_{\vec{i},|\vec{i}|=4}(P_{\vec{i}+e_0}-P_{\vec{i}+e_2})B^4_{\vec{i}}(t) &= N^1_0\\
5\sum_{\vec{i},|\vec{i}|=4}(P_{\vec{i}+e_0}-P_{\vec{i}+e_1})B^4_{\vec{i}}(t) &= N^2_0
\end{aligned}
$$

where N^1_0 (N^2_0) signifies the directional derivative stored at D_0 in the direction from D_0 to V_2 (V_1) and $e_0=(1,0,0)$, $e_1=(0,1,0)$, $e_2=(0,0,1)$, If the data triangles are symmetric across their common edge so that we evaluate the above equations at $t=(-1,1,1)$ then the equations simplify to $AX=B$, where (after rearranging terms)

$$
\begin{aligned}
A &= \begin{bmatrix} -1 & 1 & 1\\ 3 & -3 & -4\\ 3 & -4 & -3 \end{bmatrix}\\
X &= [P_{122}P_{212}P_{221}]'
\end{aligned}
$$

with B being the known values in our equations. The matrix A has a condition number of about 70. If our triangles are not symmetric, the condition number of this matrix will be greater. However, unless there is a degeneracy in the triangulation, the system of equations should always have a solution.

4 Clough-Tocher Interpolants

When fitting a single polynomial patch per triangle, we either have to construct a patch of high degree relative to the continuity (as was done in the previous section) or we need to solve the vertex consistency problem [12]. An alternative to these two choices is to fit more than one patch per face, as done by Clough-Tocher [2], Powell-Sabin [13], and Morgan-Scott [11]. In this section, I will use my continuity adjustment ideas to build Clough-Tocher interpolants, which fits three polynomial patches per triangle.

The first step of my Clough-Tocher interpolants is to fit a single polynomial patch to the data. The write-up below proceeds by using the second interpolant of Section 3 (the one described in Section 3.2), but you could instead use for the first step the interpolant described in Section 3.1; however, note that use of the first interpolant would require more derivatives at the data points.

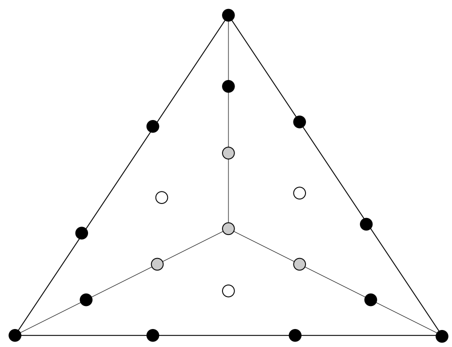

Figure 16: Clough-Tocher control points.

Given a triangle $\triangle P_0P_1P_2$ (whose projection into the x-y plane forms the domain triangle $\triangle D_0D_1D_2$) with normals N_0, N_1, and N_2, and using the control point shading from Figure 16, the Clough-Tocher construction fits three cubic Bézier patches to the data, setting the z-values of the control points with the following steps:

1. The black control points are set to interpolate the position and normals at the data points.
2. The white control points are set to get a C^1 join across the *macro-triangle* boundaries (the boundaries of the data triangle).
3. The gray control points are set to get a C^1 join across the *mini-triangle* boundaries (the boundaries between the subtriangles).

The only degrees of freedom in this construction are in the second step. Each of the white control points has a linear degree of freedom.

The standard Clough-Tocher scheme sets each white control point by creating a cross-boundary tangent vector field that is linearly varying in one domain direction, and quadratically varying in the remaining directions. This choice of direction is not unique: we can choose any domain direction (other than the one parallel to the boundary) in which to have linear variation; we just have to ensure that the same direction is chosen for both patches.

4.1 C^2 Split Point

Although the original Clough-Tocher interpolant constructs a C^1 join along the interior and exterior boundaries, the three patches meeting at the split point meet with C^2 continuity at this point [15]. This is seen in the construction of the previous section since once we set the three interior black points and the three white points, the last step sets the three gray points by averaging the surrounding black and white points, or (for the center point) three gray points. I.e., this is the de Casteljau evaluation of a quadratic patch, where we use the subpatches as the corners of the three Clough-Tocher patches.

Such a construction is said to have *super-continuity* at the split point, and results in good stability properties [9]. In this paper, I will use the C^2 split point construction to set the center control points of some of my Clough-Tocher interpolants.

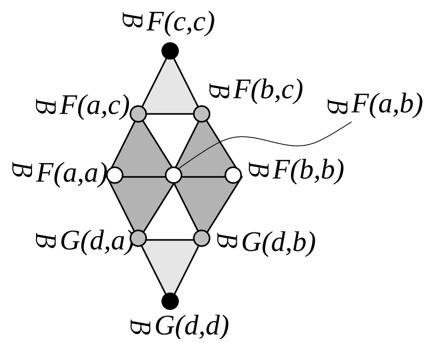

Figure 17: Alternate C^2 adjustment scheme.

4.2 Alternative C^2 Adjustment

In some of the constructions below, we will have a configuration of nine control points where seven of the points are already set, and we want to set the remaining two control points so that the group satisfies the C^2 condition. Consider the control points in Figure 17, where we have the control points and blossom values of a quadratic defined over two domain triangles $\triangle abc$ and $\triangle bad$. We have already seen how to adjust $\varpi F(c,c)$ and $\varpi G(d,d)$ to achieve C^2 continuity without losing polynomial precision when the nine points are in a C^1 configuration. In this section, I show how to set $\varpi F(b,c)$ and $\varpi G(d,b)$ to get C^2 continuity when the nine points are in a C^1 configuration.

Using the blossom of the polynomial, if we have C^1 continuity and we want to adjust $\varpi F(b,c)$ and $\varpi G(d,b)$ to get C^2 continuity, then the following two equations must hold:

$$\begin{aligned} \varpi F(b,c) &= \alpha\varpi F(b,b) + \beta\varpi F(a,b) + \gamma\varpi G(b,d), \\ \alpha\varpi G(d,b) + \beta\varpi G(d,a) + \gamma\varpi G(d,d) &= \alpha'\varpi F(b,c) + \beta'\varpi F(c,a) + \gamma'\varpi F(c,c), \end{aligned}$$

where $c = \alpha b + \beta a + \gamma d$ and $d = \alpha' b + \beta' a + \gamma' c$. Note that $\alpha' = -\alpha/\gamma$.

Substituting the first equation into the second, we get

$$\alpha\varpi G(d,b) + \beta\varpi G(d,a) + \gamma\varpi G(d,d) = \alpha'\big(\alpha\varpi F(b,b) + \beta\varpi F(a,b) + \gamma\varpi G(b,d)\big) + \beta'\varpi F(c,a) + \gamma'\varpi F(c,c)$$

and solving for $\varpi G(b,d)$ gives us

$$\begin{aligned} (\alpha - \alpha'\gamma)\varpi G(b,d) &= \\ 2\alpha\,\varpi G(b,d) &= \alpha'\big(\alpha\varpi F(b,b) + \beta\varpi F(a,b)\big) - \beta\varpi G(d,a) - \gamma\varpi G(d,d) + \beta'\varpi F(c,a) + \gamma'\varpi F(c,c). \end{aligned}$$

Thus, for a non-degenerate triangulation, we have a unique solution. Note that uniqueness implies that if the nine control points come from a single polynomial (of degree two or higher), then this method for setting $\varpi F(b,c)$ and $\varpi G(d,b)$ will leave them unchanged and polynomial precision is retained.

4.3 Cubic Precision Clough-Tocher

In an earlier paper, I showed how to use the Foley-Opitz cross-boundary construction to achieve cubic precision [10]. Basically, you set the center control point of each subpatch so that the subpatch interpolates the vertex on the neighboring triangle. If the data at the triangle corners and the neighboring

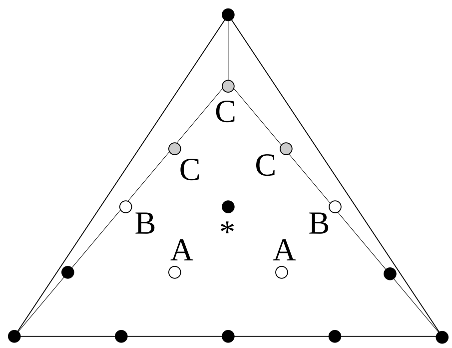

Figure 18: Quartic Clough-Tocher control points.

corners come from a single polynomial, then each of the three subpatches will be a piece of that polynomial. In the remainder of this section, I will apply the continuity ideas to higher degree patches, allowing me to achieve top-level polynomial precision with higher continuity than with a single polynomial per data triangle.

4.4 Quartic Precision Clough-Tocher

Rather than fit cubic patches to the data, we can fit higher degree patches. The problem becomes how to set the degrees of freedom available in the higher degree patch. My approach is to initially set these degrees of freedom to achieve top-order polynomial precision, and then adjust to get the highest order continuity between patches. As a first example, I show how to construct a quartic Clough-Tocher interpolant, which proceeds as follows:

1. Fit a single quartic patch to interpolate position, first, and second derivatives (although the second derivatives are interpolated only if adjacent vertices come from a common quartic) as described in Section 3.2.
2. Perform a 3-to-1 split on each patch.
3. Adjust the interior control points (labeled in Figure 18) as follows:
   (a) (A) Set using the Foley-Opitz C^1 construction (Section 2).
   (b) (B) Set to be coplanar with neighboring black point and two neighboring points labeled (A). This satisfies one of the C^1 continuity conditions across the mini-triangle boundaries.
   (c) (C) Construct with the C^2 split point construction (Section 4.1).

Note that all the black vertices and in particular the (*) vertex have values determined at step 2 of this algorithm.

The result is a construction for which we need position, first, and second derivatives at the data points, with the resulting surface have the following properties:

- Quartic precision if the appropriate data comes from a common quartic.
- C^1 across all boundaries.
- C^2 at split point.

The only gain of this construction over the cubic form is the higher precision.

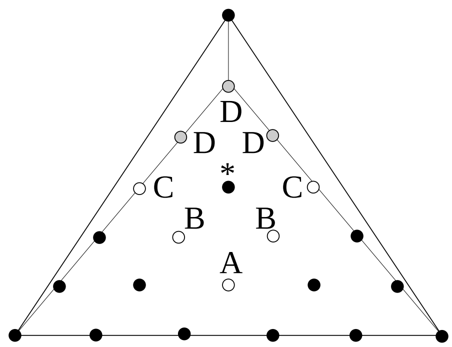

Figure 19: Quintic Clough-Tocher control points.

4.5 Quintic Precision Clough-Tocher

Using a quartic Clough-Tocher interpolant gave no improvement in continuity. Going to fifth degree, we find some improvement, as illustrated by the following scheme:

1. Fit a single quintic patch to interpolate position, first, and second derivatives as described in Section 3.2.
2. Perform a 3-to-1 split on each patch.
3. Adjust the interior control points (labeled in Figure 19) as follows:
   (a) (A) Set using the Foley-Opitz C^1 construction. (Section 2).
   (b) (B) Set using the C^2 adjustment across macro boundary (Section 2.2).
   (c) (C) Set to be coplanar with neighboring black point and two neighboring points labeled (B). This satisfies one of the C^1 continuity conditions across the mini-triangle boundaries.
   (d) (D) Construct with the C^2 split point construction (Section 4.1).

Note that all the black vertices and in particular the (*) vertex have values determined at step 2 of this algorithm.

The result is a construction for which we need position, first, and second derivatives at the data points, with the resulting surface have the following properties:

- Quintic precision if the appropriate data comes from a common quintic.
- C^2 across macro boundaries
- C^2 at split point
- C^1 across mini-boundaries

It is unclear if this patch is an improvement over the single polynomial patch that can be fit to the same data; while this Clough-Tocher patch is C^2 over the macro-boundaries (the single quintic patch is only C^1 across those boundaries), it is only C^1 across the mini-boundaries whereas the single polynomial is C^∞ on the interior. However, despite being only C^1 on the interior boundaries, this Clough-Tocher patch is C^2 at the ends of each boundary, which should result in a lower C^2 discontinuity across the mini-triangle boundaries than otherwise expected.

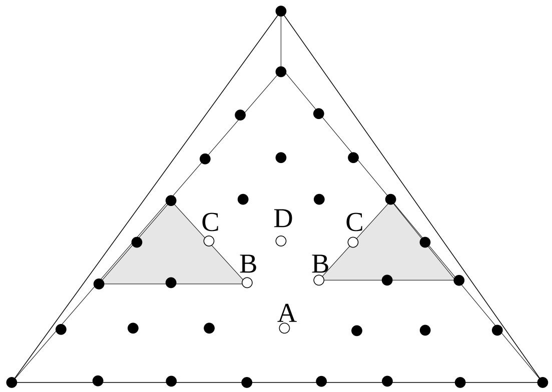

Figure 20: Septic Clough-Tocher control points.

4.6 Septic Precision Clough-Tocher

Using a sixth degree Clough-Tocher interpolant gives no significant improvements over the fifth degree interpolant. The first truly interesting interpolant is the seventh degree interpolant, which allows us to achieve C^2 continuity. The following are the steps in this construction:

1. Fit a single septic patch to interpolate position, first, second, and third derivatives as described in Section 3.2.
2. Perform a 3-to-1 split on each patch.
3. Adjust the interior control points (labeled in Figure 20) as follows:
   (a) (A) Set using the Foley-Opitz C^1 construction (Section 2).
   (b) (B) Set using the C^2 adjustment across macro boundary (Section 4.1).
   (c) (C) Set using the alternate C^2 across mini-triangle boundary (Section 4.2); these conditions involve the control points in one of the shaded regions and the corresponding points across the mini-triangle boundary.
   (d) (D) Solve for D to satisfy C^2 conditions (see below).

Note that all the black vertices and in particular the (*) vertex have values determined at step 2 of this algorithm.

The last step requires solving a cycle of constraints on the D vertices. As illustrated in Figure 21, the D vertices of Figure 20 are linked by the C^2 continuity conditions of Section 2.2. Each shaded region in this figure represents one set of C^2 continuity conditions. The continuity equations give us the following system of equations that must hold:

$$\begin{aligned}
a_0 D_2 + a_1 d_{20,2} + a_2 d_{20,1} &= b_0 D_0 + b_1 d_{02,2} + b_2 d_{02,1} \\
a_0' D_0 + a_1' d_{01,2} + a_2' d_{01,1} &= b_0' D_1 + b_1' d_{10,2} + b_2' d_{10,1} \\
a_0'' D_1 + a_1'' d_{12,2} + a_2'' d_{12,1} &= b_0'' D_2 + b_1'' d_{21,2} + b_2'' d_{21,1},
\end{aligned}$$

where (a_0, a_1, a_2) are the barycentric coordinates of V_2 relative to $\triangle V_0 S V_1$, (b_0, b_1, b_2) are the coordinates of V_0 relative to $\triangle V_2 V_1 S$, etc. Only the D_i of this figure remain unset at the last step of the construction. The result is a set of three linear equations in three unknowns (the D_i). Further, since S is the split

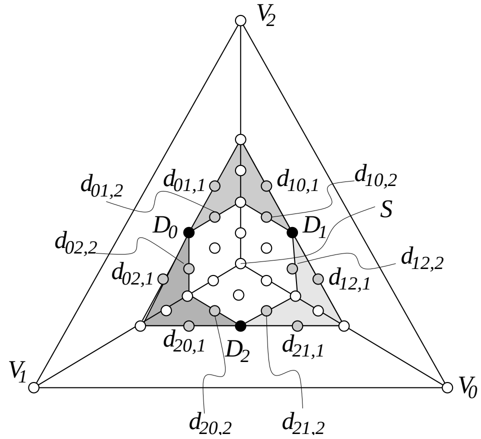

Figure 21: Cycle of C^2 conditions.

point of $\triangle V_0V_1V_2$, the (a_0, a_1, a_2) barycentric coordinates of Section 2.2 are all $(-1, 3, 1)$. Expressing this linear system in matrix form $AD = C$, with $D = [D_0\ D_1\ D_2]^t$, we find the D by computing

$$A^{-1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 \end{bmatrix}.$$

Thus, the last step of the construction always has a solution.

The result is a construction for which we need position, first, second, and third derivatives at the data points, with the resulting surface have the following properties:

- Septic precision if the appropriate data comes from a common septic.
- C^2 across all boundaries.
- C^3 at corners and split point.

However, see the conclusions for a comment on the stability of this scheme.

4.7 Octic Precision Clough-Tocher

The seventh degree interpolant has an extra cycle of constraints to solve. When we increase the degree to eight, this cycle disappears and we get a simpler C^2 construction:

1. Fit a single octic patch to interpolate position, first, second, third, and fourth derivatives (although the fourth derivatives are interpolated only if adjacent vertices come from a common octic surface) as described in Section 3.2.
2. Perform a 3-to-1 split on each patch.
3. Adjust the interior control points (labeled in Figure 22) as follows:
   (a) (A) Set using the Foley-Opitz C^1 construction (Section 2).
   (b) (B) Set using the C^2 adjustment across macro boundary (Section 2.2).
   (c) (C) Set using the alternate C^2 across mini-triangle boundary (Section 4.2); these conditions involve the control points in one of the shaded regions and the corresponding points across the mini-triangle boundary.

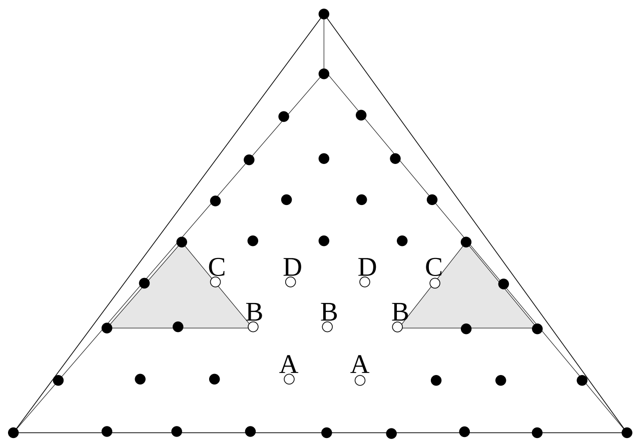

Figure 22: Octic Clough-Tocher control points.

(d) (D) Set using the C^2 adjustment across mini boundary (Section 2.2).

The result is a construction for which we need position, first, second, third, and fourth derivatives at the data points, with the resulting surface have the following properties:

- Octic precision if the appropriate data comes from a common octic.
- C^2 across all boundaries.
- C^4 at split point.

5 Summary and Future Work

In this paper, I have discussed two constructions that use derivative information to construct degree n triangular patches that interpolate data at the vertices of the triangulation and that reproduce degree n polynomials. In the initial constructions, the patches meet with only C^0 continuity. I then showed how to adjust the control points of these patches to achieve higher order continuity without losing polynomial precision. The second of these interpolants is interesting in its own right, as it requires less data than other methods. I then showed how to apply this continuity adjustment method to Clough-Tocher schemes.

One issue I have not addressed is the stability of my Clough-Tocher constructions. Lai and Schumaker [9] discuss this issue in detail, and it is one that should be investigated for my schemes. In particular, note that my degree seven Clough-Tocher interpolant is probably not a stable local basis, since it violates the stability formula $d \geq 3k + 2$. A related issue is that as yet, I have not implemented any of the new schemes discussed in this paper. Such implementations and testing would be a further indication of the properties of these interpolants.

Another idea is to apply these ideas to Powell-Sabin and Morgan-Scott interpolants.

6 Acknowledgments

Many thanks to Larry Schumaker, Tom Lyche, and Mike Floater for their discussions of the topics in this paper.

References

[1] K. Bell. A refined triangular plate bending element. *Internation Journal of Numerical Methods in Engineering*, 1:101–122, 1969.

[2] R. Clough and J. Tocher. Finite element stiffness matrices for analysis of plates in bending. In *Proceedings of Conference on Matrix Methods in Structural Analysis*, 1965.

[3] Gerald Farin. Triangular Bernstein-Bézier patches. *CAGD*, 3(2):83–127, August 1986.

[4] Gerald E. Farin. *Curves and Surfaces for Computer Aided Geometric Design*. Academic Press, Inc., fourth edition, 1997.

[5] Thomas A. Foley and Karsten Opitz. Hybrid cubic bézier triangle patches. In T Lyche and L Schumaker, editors, *Mathematical Methods for Computer Aided Geometric Design II*, pages pp 275–286. Academic Press, 1992.

[6] Richard Franke. Scattered data interpolation: tests of some methods. *Mathematics of Computation*, 38(157):181–200, January 1982.

[7] Richard Franke and Gregory Nielson. Scattered data interpolation: A tutorial and survey. In Hans Hagen, editor, *Geometric Modeling: Methods and Applications*. Springer-Verlag, 1991.

[8] Ming-Jun Lai. Geometric interpretation of smoothness conditions of triangular polynomial patches. *CAGD*, 14(2):191–199, 1997.

[9] Ming-Jun Lai and Larry L. Schumaker. Macro-elements and stable local bases for splines on clough-tocher triangulations. Manuscript.

[10] Stephen Mann. Cubic precision clough-tocher interpolation. *CAGD*, 16(2):85–88, February 1999.

[11] John Morgan and Ridgway Scott. A nodal basis for c^1 piecewise polynomials of degree $n \geq 5$. *Mathematics of Computing*, 131(29):736–740, 1975.

[12] Jorg Peters. Joining smooth patches at a vertex to form a C^k surface. *CAGD*, 9:387–411, 1992.

[13] Michael J. D. Powell and Malcolm A. Sabin. Piecewise quadratic approximations on triangles. *ACM Transactions of Mathematical Software*, 3(4):316–325, December 1977.

[14] Lyle Ramshaw. Blossoming: a connect-the-dots approach to splines. Technical report, Digital Systems Research Center, 1987.

[15] Larry Schumaker. Personal communication.

[16] A. Ženíšek. Interpolation polynomials on the triangle. *Numerische Mathematik*, 15:283–296, 1970.