Ununfoldable Polyhedra

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Abstract

A well-studied problem is that of unfolding a convex polyhedron into a simple planar polygon. In this paper, we study the limits of unfoldability. We give an example of a polyhedron with convex faces that cannot be unfolded by cutting along its edges. We further show that such a polyhedron can indeed be unfolded if cuts are allowed to cross faces. Finally, we prove that "open" polyhedra with convex faces may not be unfoldable no matter how they are cut.

1 Introduction

A classic open question in geometry [4, 6, 14, 17] is whether every convex polyhedron can be cut along its edges and flattened into the plane without any overlap. Such a collection of cuts is called an *edge unfolding* of the polyhedron, and the resulting simple polygon is called a *net*. While the first explicit description of this problem is by Shephard in 1975 [17], Grünbaum believes that it has been implicit since at least the time of Albrecht Dürer, circa 1500 [14]. It is widely conjectured that the answer to this question is yes.

While unfoldings were originally used to make paper models of polyhedra [5, 20], unfoldings have important industrial applications. For example, sheet metal bending is an efficient process for manufacturing [8, 19]. In this process, the desired object is approximated by a polyhedron, which is unfolded into a collection of polygons. Then these polygons are cut out of a sheet of material, and each piece is folded into a portion of the object's surface using a bending machine. Multipiece unfoldings are used partly for practical reasons such as efficient area use, but mainly because little theory on unfolding nonconvex polyhedra is available, and thus heuristics must be used [14].

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There are two freely available heuristic programs for constructing edge unfoldings of polyhedra: the Mathematica package UnfoldPolytope [13], and the Macintosh program HyperGami [9]. Neither program has ever failed, and HyperGami even works for some nonconvex polyhedra. There are also several commercial heuristic programs; an example is Touch-3D [11], which supports nonconvex polyhedra by using multiple pieces when needed.

It is known that if we allow cuts across the faces as well as along the edges, then every *convex* polyhedron has an unfolding. Two such unfoldings are known. The simplest to describe is the *star unfolding* [1, 2], which cuts from a generic point on the polyhedron along shortest paths to each of the vertices. The second is the *source unfolding* [12, 16], which cuts along points with more than one shortest path to a generic source point.

There has been little theoretical work on unfolding nonconvex polyhedra. In what may be the only paper on this subject, Biedl et al. [3] show the positive result that certain classes of orthogonal polyhedra can be unfolded. They show the negative result that not all nonconvex polyhedra have edge unfoldings. Two of their examples are given in Figure 1. The first example is rather trivial: the top box must unfold to fit inside the hole of the top face of the bottom box, but there is insufficient area to do so. The second example is closer to a convex polyhedron in the sense that every face is homeomorphic to a disk.

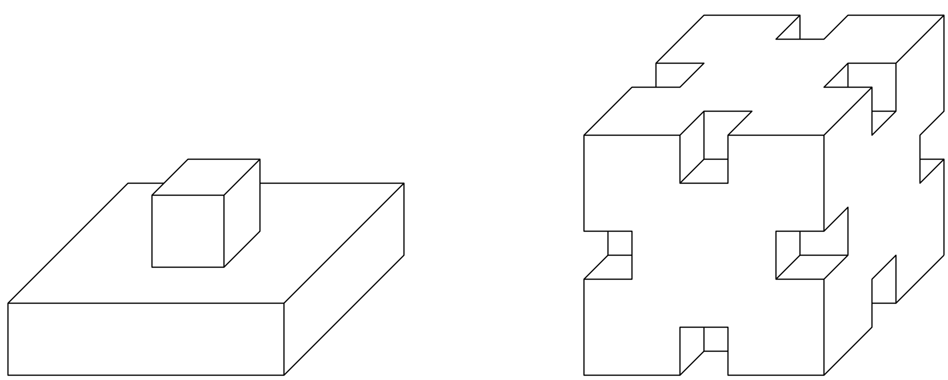


Figure 1: Orthogonal polyhedra with no edge unfoldings from [3].

Neither of these examples is satisfying because they are not "topologically convex." A polyhedron is *topologically convex* if its graph is the graph of some convex polyhedron. The first example of Figure 1 is ruled out because faces of a convex polyhedron are always homeomorphic to disks. The second example is ruled out because there are pairs of faces that share more than one edge, which is impossible for a convex polyhedron[1]. In general, Steinitz's theorem [7, 10, 18] tells us that a polyhedron is topologically convex precisely if its graph is 3-connected and planar.

The class of topologically convex polyhedra includes all *convex-faced* polyhedra (i.e., polyhedra whose faces are all convex) that are homeomorphic to spheres. (This will be proved formally later.) Schevon and other researchers [3, 15] have asked whether all such polyhedra can be unfolded by cutting along edges. In other words, can the conjecture that every convex polyhedron is edge unfoldable be extended to topologically convex polyhedra?

[1] We disallow polyhedra with "floating vertices," that is, vertices with angle π on both incident faces, which can just be removed.

In this paper we prove that the answer to this question is no; that is, we construct a family of convex-faced polyhedra that are homeomorphic to spheres and have no edge unfoldings. We go on to show that cuts across faces can unfold some convex-faced polyhedra that cannot be unfolded with cuts only along edges. This is the first demonstration that general cuts are more powerful than edge cuts.

We also consider the problem of constructing a convex-faced polyhedron that cannot be unfolded even using general cuts. If such a polyhedron exists, the theorem that every convex polyhedron is generally unfoldable (using, for example, the star or source unfolding) cannot be extended to topologically convex polyhedra. As a step towards this goal, we present an "open" convex-faced polyhedron that cannot be unfolded. Finding a "closed" ununfoldable polyhedron is an intriguing open problem.

2 Basics

We begin with formal definitions and some basic results about polyhedra and unfoldings.

We define a *polyhedron* to be a connected set of closed planar polygons in 3-space such that (1) any intersection between two polygons in the set is a collection of vertices and edges common to both polygons, and (2) each edge is shared by at most two polygons in the set. A *closed* polyhedron is one in which each edge is shared by exactly two polygons. If a polyhedron is not closed, we call it an *open* polyhedron; the *boundary* of an open polyhedron is the set of edges covered by only one polygon.

In general, we may allow the polygonal faces to be multiply connected (i.e., have holes). However, in this paper we concentrate on convex-faced polyhedra. A polyhedron is *convex-faced* if every face is strictly convex, that is, every interior angle is strictly less than π. Thus, in particular, every face of a convex-faced polyhedron is simply connected.

Next we prove that these polyhedra are a subclass of *topologically convex* polyhedra, that is, polyhedra whose graphs are the graphs of convex polyhedra. A *convex polyhedron* is a closed polyhedron whose interior is a convex set—equivalently, the open line segment connecting any pair of points on the polyhedron's surface is interior to the polyhedron.

Theorem 1 (Steinitz's Theorem [7, 10, 18]) *A graph is the graph of a convex polyhedron precisely if it is 3-connected and planar.*

Corollary 1 *Every convex-faced closed polyhedron that is homeomorphic to a sphere is topologically convex.*

Proof: Because the polyhedron is homeomorphic to a sphere, its graph G must be planar. It remains to show that G is 3-connected. If there is a vertex v whose removal disconnects G into at least two components G_1 and G_2, then there must be a "belt" wrapping around the polyhedron that separates G_1 and G_2. This belt has only one vertex (v) connecting G_1 and G_2, so it must consist of a single face. This face touches itself at v, which is impossible for a convex polygon. Similarly, if there is a pair (v_1, v_2) of vertices whose removal disconnects G into at least two components G_1 and G_2, then again there must be a "belt" separating the two components, but this time the belt connects G_1 and G_2 at two vertices (v_1 and v_2).

Thus, the belt can consist of up to two faces, and these faces share two vertices v_1 and v_2. This is impossible for strictly convex polygons. $\square$

An *unfolding* of a polyhedron P is a union C of a finite number of polygonal chains on P, each with a finite number of vertices, such that cutting along C results in a connected surface $P - C$ that can be flattened into the plane (that is, isometrically embedded) without overlap. An *edge unfolding* of P is an unfolding of P that is just a union of P's edges. We sometimes call unfoldings *general unfoldings* to distinguish them from edge unfoldings. We call a polyhedron *unfoldable* if it has a general unfolding, and *edge unfoldable* if it has an edge unfolding. Similarly, we call a polyhedron [*edge*] *ununfoldable* if it is not [edge] unfoldable.

If P is an open polyhedron, we limit attention to unfoldings that contain only a finite number of boundary points of P. Every neighborhood of a boundary point of P in C must contain an interior point of P in C; otherwise, the boundary point could be removed from C without changing $P - C$. Hence, the collection of boundary points of P in C can be reduced down to a finite set without any effect.

We define the *curvature* of an interior vertex v to be a discrete analog of Gaussian curvature, namely 2π minus the sum of the face angles at v. (For vertices on the boundary of the polyhedron, we count the incident face angles on both sides.) Hence, the neighborhood of a zero-curvature vertex can be flattened into the plane, the neighborhood of a positive-curvature vertex (for example, a cone) requires a cut in order to be flattened, and the neighborhood of a negative-curvature vertex (for example, a saddle) requires two or more cuts to avoid self-overlap.

The following lemmas state some basic facts about unfoldings.

Lemma 1 *Any unfolding of a polyhedron P is acyclic, and spans every nonboundary vertex of P with nonzero curvature.*

Proof: If an unfolding C contained a cycle, its removal would disconnect P, a contradiction. (Above we excluded the possibility of C containing a cycle on the boundary of an open polyhedron P.) If C did not contain a particular (nonboundary) point with nonzero curvature, neighborhoods of that point could not be flattened. $\square$

Lemma 2 *If v is a vertex of a polyhedron P with negative curvature, then any unfolding of P must include more than one edge incident to v.*

Proof: Suppose some cutting C includes only a single cutting edge incident to v. Let N be $P \cap B$, where B is a small ball around v. Neighborhood N unfolds to a small disk that self-overlaps by precisely the absolute value of the curvature of v. $\square$

3 Witch's Hat

This section describes properties of the open polyhedron shown in Figure 2, called a *witch's hat* or just *hat* for short. Hats are parameterized by two acute angles α and β, and a length $\ell > 1$; we denote the corresponding hat by $H(\alpha, \beta, \ell)$ or simply H when the parameters are obvious. H consists of two parts. The top part is an open tetrahedron consisting of three identical isosceles triangles, where the lower angles are α and the bottom sides have unit length. The bottom part is an open truncated tetrahedron consisting of three identical symmetric trapezoids, where the lower angles are β, the top length is 1, and the bottom length is ℓ.

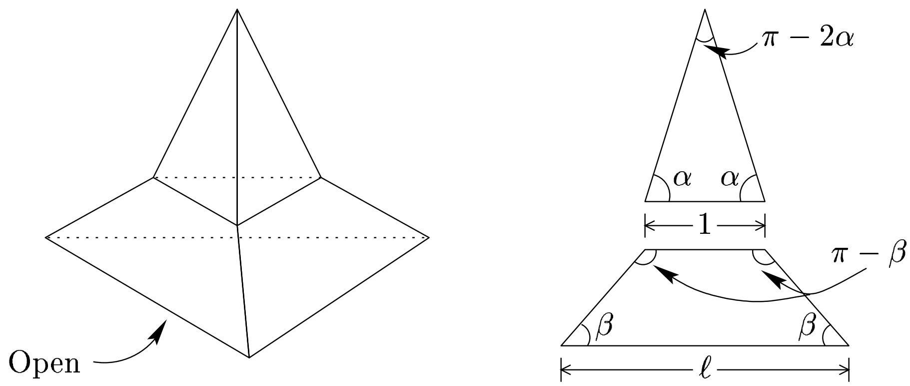


Figure 2: A witch's hat and its constituent faces.

The key to our constructions of ununfoldable polyhedra is that almost all of the vertices have negative curvature. In particular, hats are designed to have the property that the vertices in between the top and bottom parts (called *middle vertices*) have negative curvature, that is, the sum of the faces angles at a middle vertex is greater than 2π. Thus, we choose the angles α and β so that

$$2\alpha + 2(\pi - \beta) > 2\pi, \quad \text{i.e.,} \quad \alpha > \beta.$$

Lemma 3 *If $\alpha > \beta$, $H(\alpha, \beta, \ell)$ has at most one edge unfolding up to symmetry.*

Proof: Suppose we had an edge unfolding C of H. By Lemma 1, C is a forest of H's edges that spans the top vertex and the middle vertices. By Lemma 2, none of the middle vertices can be leaves of C. Hence, C must include one of the bottom vertices; otherwise, C would be a four-vertex forest with at most one leaf. None of the boundary edges can be in C, so each bottom vertex must have degree zero or one in C. If C includes more than one bottom vertex (as a leaf), there must be a path in C through the interior of P, which disconnects P, a contradiction.

Hence, C is a five-vertex forest with at most two leaves, so it must be a path from the top vertex to a bottom vertex that passes through all the middle vertices and no other bottom vertices. There are three choices for the bottom vertex, and two such paths for each choice; but up to symmetry, there is precisely one such path, shown in bold in the left of Figure 3. □

To refer to the faces in this edge unfolding, we introduce the concept of the index of a face. Define the index of a trapezoid to be the shortest distance in the dual graph to a triangle. Similarly, define the index of a triangle to be the shortest distance in the dual graph to a trapezoid. For example, the first trapezoid (the trapezoid with index one) is incident to the first triangle and the second trapezoid.

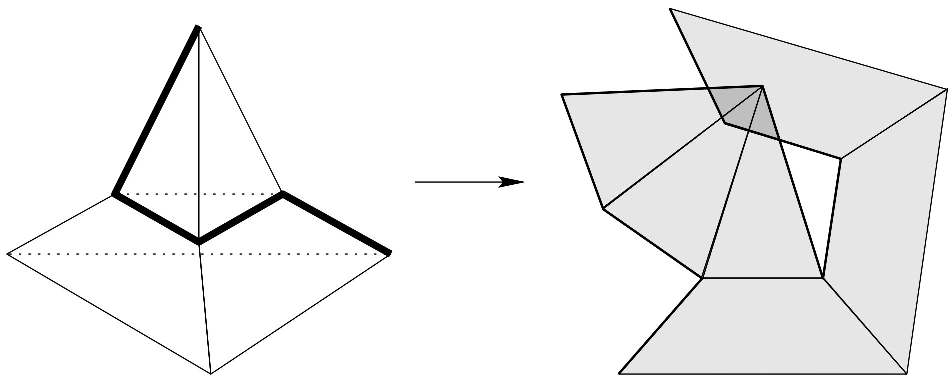


Figure 3: The only possible edge unfolding of a hat when $\alpha > \beta$. In this case, overlap occurs.

Lemma 4 *For any $\beta < \pi/3$, there is an $\hat{\alpha}$ such that the convex-faced open polyhedron $H(\alpha, \beta, \ell)$ has no edge unfolding, for all $\alpha > \hat{\alpha}$ and for all ℓ. If $\beta \geq \pi/3$, then $H(\alpha, \beta, \ell)$ is edge unfoldable for all α and ℓ.*

Proof: First enforce that $\hat{\alpha} \geq \beta$, so that the only possible edge unfolding is the one from Lemma 3. Consider the line segment e connecting the top-right vertex of the first trapezoid and the top-right vertex of the third trapezoid. If $\beta = \pi/3$ (see Figure 4), the top edges of the trapezoids lie on a regular hexagon, so e is perpendicular to the top edge of the first trapezoid. As we let α approach $\pi/2$, the first triangle approaches occupying the half-infinite rectangle R outlined by dotted lines in Figure 4. This implies that the first triangle (and hence all of the triangles) will not intersect any trapezoids. Changing ℓ does not affect this. If $\beta > \pi/3$, the trapezoids are pushed further away from R, so the hat is still edge unfoldable.

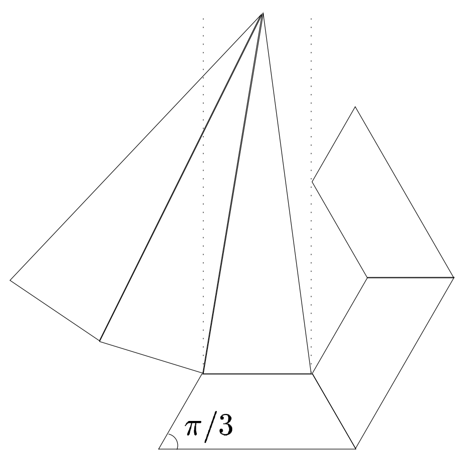


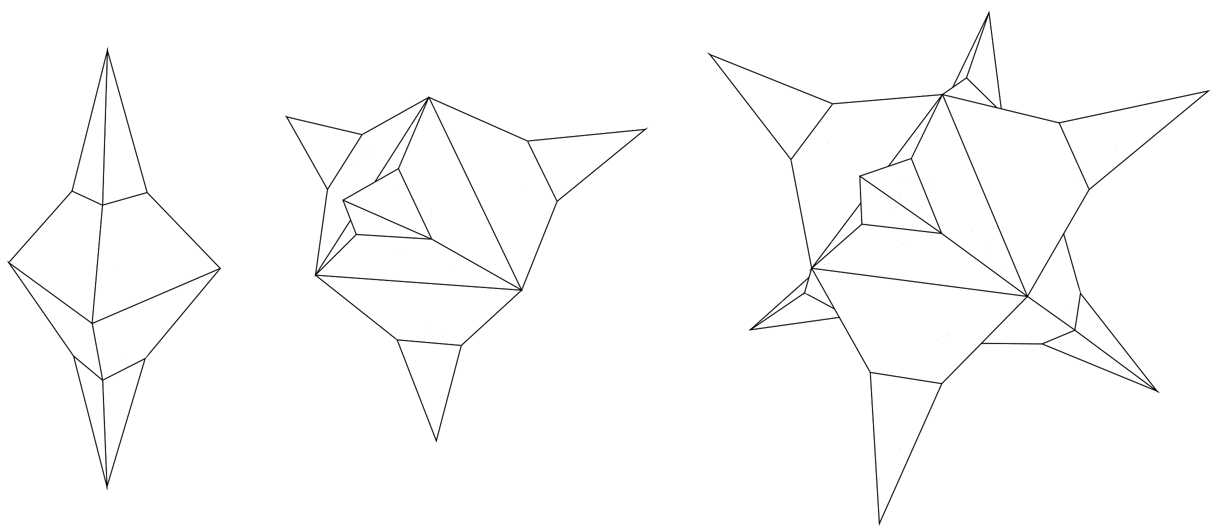
Figure 4: Hats are edge-unfoldable for $\beta = \pi/3$, and indeed for any $\beta \geq \pi/3$.

Now consider the case when $\beta < \pi/3$. Then e is contained in R, and hence the top edge of the third trapezoid hangs into R. As we let α grow, the first triangle will occupy closer and closer to the entire rectangle R. Hence, if we make α sufficiently large, the first triangle will intersect the third trapezoid. Figure 3 is an example of this case. □

4 No Edge Unfolding

This section presents a convex-faced closed polyhedron that has no edge unfolding. The basic idea is to take a closed polyhedron made out of equilateral triangles, and replace each face with a hat; see Figure 5. Again, the key is to make almost all of the vertices have negative curvature. In particular, we need to make the vertices of the original polyhedron have negative curvature.

The simplest polyhedron one might imagine is a doubly covered triangle. In other words, consider gluing two identical hats to each other at their bases, as shown in Figure 5(a). This implies that four trapezoids come together at a vertex of the original polyhedron (the shared base of the hats). However, because β is acute, 4β must be less than 2π, so negative curvature is unachievable. Indeed, this polyhedron is edge unfoldable.



(a) (b) (c)

Figure 5: Gluing hats to the faces of (a) a doubly covered triangle, (b) a regular tetrahedron, and (c) a regular octahedron. Theorem 2 says that (c) cannot be edge unfolded.

The next simplest polyhedron one could imagine is the tetrahedron; see Figure 5(b). Now there are 3 hats and hence 6 trapezoids that come together at each vertex of the tetrahedron. To achieve negative curvature at these vertices, we must have $\beta > \pi/3$. But by Theorem 4, it is impossible to make edge-ununfoldable hats with $\beta \geq \pi/3$.

Hence, we need all the vertices on the base polyhedron to have degree at least four. The simplest of these polyhedra is the octahedron; see Figure 5(c).

Theorem 2 *Replacing each face of the regular octahedron with the hat $H(\alpha, \beta, \ell)$ results in a convex-faced closed polyhedron that has no edge unfolding, provided $\pi/4 < \beta < \pi/3$, $\ell > 2$, and α is sufficiently large.*

Proof: Because $\alpha > \beta$, all middle vertices have negative curvature. Furthermore, there are four hats and hence eight trapezoids incident to each octahedron vertex, so the sum of face angles at each octahedron vertex is 8β, which is more than 2π because $\beta > \pi/4$. In other words, the only vertices with non-negative curvature are the tops of the hats.

By Lemmas 1 and 2, any unfolding C must be a spanning forest of the polyhedron, whose leaves are only at the tops of the hats. Let n be the number of leaves of C, so that $n \leq 8$. Then C has at most $n - 2$ vertices with degree more than two. This implies that at least two of the hats whose tops are leaves of C have no middle vertices of degree more than two. Let H be such a hat.

We claim there are two possible edge unfoldings of H up to symmetry. Let t denote the top vertex of H; let m_1, m_2, m_3 denote the middle vertices of H; and let b_1, b_2, b_3 denote the corresponding bottom vertices of H (see Figure 6). Precisely one of the middle vertices of H is incident to t in C; by symmetry, let it be m_1. Because m_1 cannot be a leaf of C, and we assume it has degree at most two, it must be incident to exactly one of m_2, m_3, and b_1. If m_1 is incident to m_2, then m_2, m_3, b_3 must be a path in C. In other words, we would have the situation in Figure 6(a). The case where m_1 is incident to m_3 is symmetric. If m_1 is incident to b_1, then b_2, m_2, m_3, b_3 must be a path in C, the situation in Figure 6(b).

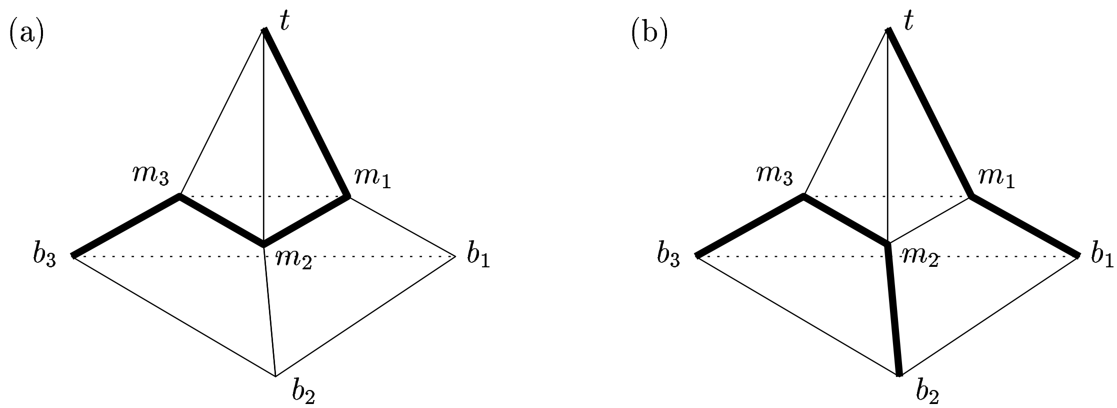


Figure 6: The possible edge unfoldings of a hat H on the octahedron whose top has degree one and whose middle vertices all have degree two.

Case (a) is an unfolding of the hat itself, so by Theorem 4 cannot exist because $\beta < \pi/3$ and α is sufficiently large. In Case (b), one of the pieces of the hat unfolds into three triangular faces off of which hang two trapezoids, as shown in Figure 7. Letting α approach $\pi/2$, the bottoms of the triangular faces approach being collinear. Because $\ell > 2$, the two trapezoids will intersect for α sufficiently large. □

5 General Unfoldings

In this section we demonstrate that a polyhedron without an edge unfolding may have a general unfolding. Hence cuts across faces can indeed be more powerful than edge cuts. Figure 8 shows an example of a hat that admits general unfoldings but not edge unfoldings.

Theorem 3 *For all $3\pi/10 < \beta < \pi/3$, $H(\alpha, \beta, \ell)$ is a convex-faced open polyhedron with an unfolding but no edge unfolding, provided α is sufficiently large.*

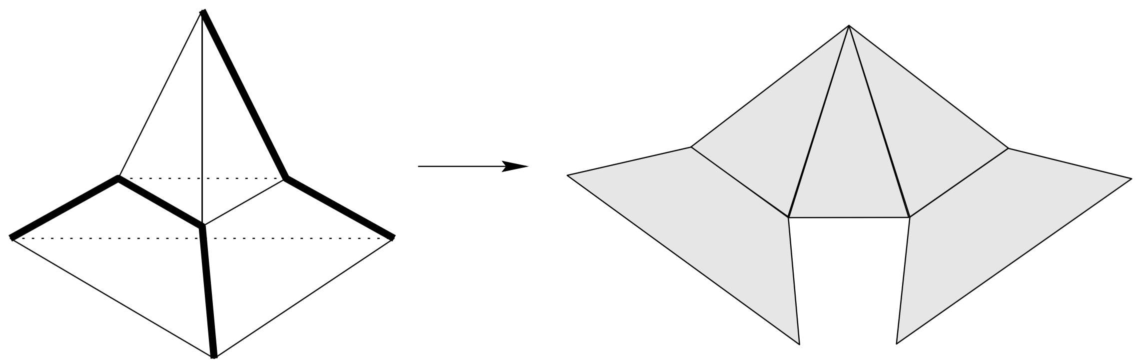


Figure 7: One piece of the flattened polyhedron from the unfolding in Figure 6(b).

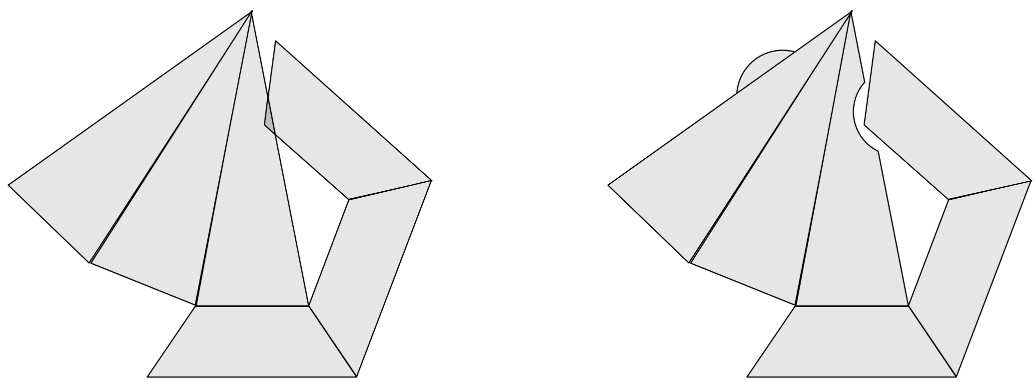


Figure 8: An unfoldable hat with no edge unfolding.

Proof: Consider the only possible edge unfolding of the polyhedron from Lemma 3; refer to Figure 8. At $\beta = \pi/3$, the top-right vertex of the third trapezoid is on a vertical line through the top-right vertex of the first trapezoid, and the bottom-right vertex of the third trapezoid is strictly right of this vertical line. Here "vertical" is defined to be perpendicular to the first trapezoid's top edge. At $\beta = 3\pi/10$, the top edges of the trapezoids are on a regular pentagon; hence, the top-right and bottom-right vertices of the third trapezoid are on a vertical line through the center of the first trapezoid's top edge.

Thus, for $3\pi/10 < \beta < \pi/3$, the top-right vertex of the third trapezoid is on a vertical line through a point on the open right half of the first trapezoid's top edge, and the entire third trapezoid lies to the right of this vertical line. This implies that for sufficiently large α, the first triangle and no other triangles intersect the third trapezoid. Therefore, the polyhedron has no edge unfolding, but if we cut out a portion of the first triangle around the intersection and glue it to the third triangle, we obtain an unfolding. $\square$

Using this result, we can show that there exist *closed* polyhedra that are generally unfoldable but not edge unfoldable:

Theorem 4 *For all $3\pi/10 < \beta < \pi/3$, replacing each face of the regular octahedron with $H(\alpha, \beta, \ell)$ results in a convex-faced closed polyhedron with an unfolding but no edge unfolding, provided α is sufficiently large, and ℓ is sufficiently large relative to α.*

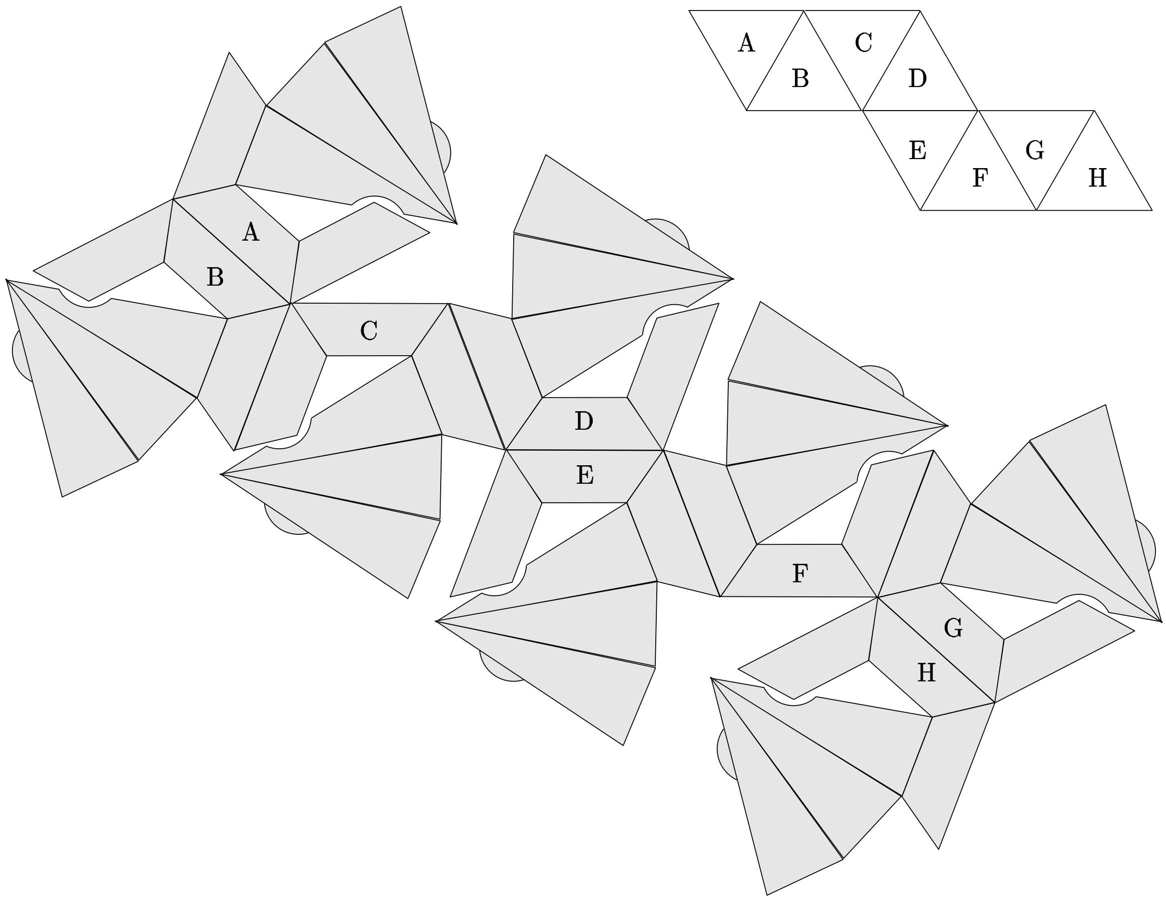


Figure 9: An unfolding of the edge-unfoldable polyhedron from Figure 5(c). We show the correspondence to an unfolding of the octahedron.

Proof: By Theorem 2, the polyhedron is edge ununfoldable. Figure 9 shows that it is generally unfoldable. For a given value of α, this unfolding is valid for sufficiently large values of ℓ. Note that Theorems 2 and 3, which we are using here, do not constrain the value of ℓ, except that ℓ must be more than 2. $\square$

6 No General Unfolding

This section makes a step towards finding a closed polyhedron with no general unfolding, by presenting an open polyhedron with this property. More precisely, we give an open polyhedron with triangular faces that cannot be unfolded no matter how its surface is cut.

The basic idea is to connect several triangles in a cycle, all sharing a common vertex v, as shown in Figure 10. By connecting enough triangles and/or adjusting the triangles to have large enough angle incident to v, we can arrange for vertex v to have negative curvature.

Theorem 5 *This polyhedron is ununfoldable if v has negative curvature.*

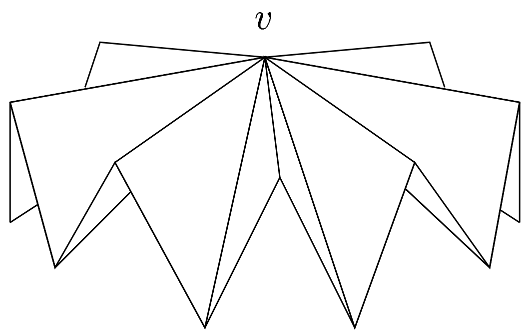


Figure 10: An open polyhedron with no unfolding.

Proof: Suppose for contradiction that the polyhedron has an unfolding. By Lemma 1, it is a (nonempty) forest. If the forest has a leaf anywhere other than at v or on the polyhedron's boundary, then the incident cut can be glued (uncut) without affecting the unfolding. Thus, we can assume that all leaves are at v or on the boundary. But by Lemma 2, v cannot be a leaf. Now any tree in the (nonempty) forest has at least two leaves, and thus in particular there is a path that cuts from one point on the boundary to another (possibly identical) point on the boundary. But this cut must disconnect the polyhedron into two pieces, a contradiction to the definition of an unfolding. □

7 Conclusion

We have presented new examples of ununfoldable polyhedra. In particular, we have demonstrated that the classic conjecture about edge unfoldability of convex polyhedra cannot be generalized to convex-faced polyhedra (homeomorphic to spheres). We have further shown that general unfoldings can be nontrivially more powerful than edge unfoldings. Finally, we have proved that the result about general unfoldability of convex polyhedra cannot be extended to convex-faced *open* polyhedra.

Our work raises several interesting open questions. The major open question is whether there is a convex-faced *closed* polyhedron with no general unfolding. We conjecture that a witch's hat with suitably chosen parameters cannot be generally unfolded. But even if this is true, it seems difficult to join witch's hats as the faces of a triangulated polyhedron, and maintain that there is no general unfolding, in contrast to what we did for edge unfoldings. Another open question is whether our example of a closed polyhedron with no edge unfolding can be modified to have triangular faces only, instead of the current mix of triangular and trapezoidal faces.

Acknowledgments

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