QUICKLY EXCLUDING $K_{2,r}$ FROM PLANAR GRAPHS*

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§The way the name of the author appears is a possible phonetic transcription, in the latin alphabet, of the original name: Δημήτριος Μ. Θηλυκός.

Abstract

We prove that any planar graph that does not contain $K_{2,r}$ as a minor has treewidth $\leq r+2$.

1 introduction

In this paper we consider finite graphs without loops or multiple edges. We will denote the vertex (edge) set of a graph G as $V(G)$ $(E(G))$.

A tree-decomposition of a graph G is a pair $D = (X, T)$ with $T = (I, F)$ a tree and $X = \{X_i \mid i \in I\}$ a family of subsets of $V(G)$, one for each vertex of T, such that

- $\bigcup_{i \in I} X_i = V(G)$.
- for all edges $\{v, w\} \in E(G)$, there exists an $i \in I$ with $v \in X_i$ and $w \in X_i$.
- for all $i, j, k \in I$: if j is on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.

The width of a tree-decomposition $(\{X_i \mid i \in I\}, T = (I, F))$ is $\max_{i \in I} |X_i| - 1$. The treewidth of a graph G is the minimum width over all possible tree-decompositions of G.

Given a graph H, we say that a graph G is *H-minor free* if G does not contain H as a minor (for the definition of the minor containment, see Section 2). The following has been proven in [13].

Theorem 1 *For any planar graph H there exist a constant c_H such that any H-minor free graph G has treewidth $\leq c_H$.*

The above result has been a basic step for the proof of the Graph Minors Theorem (formerly known as Wagner's Conjecture), developed by Robertson and Seymour, in their graph minors series (for a survey, see [12]). A simpler proof of Theorem 1, with a much better bound for c_H, was given by Robertson, Seymour, and Thomas in [14], where $c_H \leq 20^{2(2|V(H)|+4|E(H)|)^5}$ (additional results on general bounds for c_H can be found in [11]).

Much research has been done towards proving tighter bounds for c_H when H is restricted to certain families of planar graphs. Such kind of bounds have been found in [1] (trees), [9] (cycles and subgraphs of cycles), [5] (disjoint copies of K_3), and [3] (graphs that are minors of a circus graph and a $(2 \times k)$-grid). Lastly, a bound for c_H, in the case H is a $K_{2,r}$, has been found in [6] (we denote as $K_{2,r}$ the complete bipartite graph that have r vertices in the one part and 2 vertices in the other).

Theorem 2 *Let r be a positive integer. Then any $K_{2,r}$-minor free graph has treewidth $\leq 2r - 2$.*

The above result has certain applications in distributing computing as it provides a partial characterization for the class of graphs that allow k-label Interval Routing Schemes under dynamic cost edges (in short, this class is denoted as k-$\mathcal{LIRS}$). In particular, in [6] it is proved that k-$\mathcal{LIRS}$ is closed under taking of minors. Combining this with the fact that no graph in k-$\mathcal{LIRS}$ contains K_{2k+1} as a subgraph (see [10]), it follows that graphs that allow k-label Interval Routing Schemes under dynamic cost edges have treewidth at most $4k$ (for a survey on Interval Routing Schemes and other Compact routing methods see [8]).

In this paper we prove a tighter upper bound for c_H when $K_{2,r}$ is excluded as a minor from planar graphs.

Theorem 3 *Let $r \geq 1$ and G be a planar $K_{2,r}$-minor free graph. Then,* $\text{treewidth}(G) \leq r + 2$.

Consequently, our result implies that the planar graphs that allow k-label Interval Routing Schemes under dynamic cost edges, have treewidth $\leq 2k + 3$.

2 Definitions and Preliminary Results

We will assume that all the graphs we deal with are connected, as this does not influence the generality of our results. Let G be a graph. Given a vertex $v \in V(G)$, we denote as $N_G(v)$ the vertices of G that are adjacent to v and we set $d_G(v) = |N_G(v)|$. Moreover, if $S \subseteq V(G)$ we set $N_G(S) = \{N_G(v) \mid v \in S\} - S$. If $v, u \in V(G)$, then we denote as $D_G(v, u)$ the length of the shortest path connecting v and u in G. We also define $G - v = G[V(G) - \{v\}]$. We call *clique* of a graph G any complete subgraph of G (if it contains 3 vertices we call it *triangle*). If $S \subseteq V(G)$, we call the graph $(S, \{\{v,u\} \in E(G) \mid v,u \in S\})$ the *subgraph of G induced by S* and we denote it as $G[S]$. We say that a graph G is a *minor* of a graph H if H can be obtained from G by a series of vertex/edge deletions or/and edge contractions. (a contraction of an edge $\{u,v\}$ in G is the operation that replaces u and v by a new vertex whose neighbors are the vertices that where adjacent to u and/or v). We say that $G \leq H$ ($G \subseteq H$) if G is a minor (subgraph) of H. Notice that $G \subseteq H$ implies that $G \leq H$. Clearly, a graph G is *H-minor free* if $H \not\leq G$. The following is easy (for a formal proof see [2]).

Lemma 1 *Let G, H be graphs where $G \leq H$. Then* $\text{treewidth}(G) \leq \text{treewidth}(H)$.

A proof of the following can be found in [4].

Lemma 2 *Let $(\{X_i \mid i \in I\}, T)$ be a tree-decomposition of graph G. For any clique K of G, there exists an $i \in I$ with $V(K) \subseteq X_i$.*

Let G be a graph and $\mathcal{S}$ be a collection of non-empty subsets of $V(G)$. We define as $G^{(\mathcal{S})} = (V(G), E(G) \cup (\bigcup_{S \in \mathcal{S}} E^{(S)}))$ where $E^{(S)} = \{\{v,u\} \mid v,u \in S \text{ and } v \neq u\}$. Moreover, if $\mathcal{S} = \{S_1, \ldots, S_q\}$ we define $G^{<\mathcal{S}>} = (V(G) \cup \{v_1^{\text{new}}, \ldots, v_q^{\text{new}}\}, E(G) \cup (\bigcup_{1 \leq i \leq q} E^{<S_i>}))$ where $\{v_1^{\text{new}}, \ldots, v_q^{\text{new}}\} \cap V(G) = \emptyset$ and $E^{<S_i>} = \{\{v_i^{\text{new}}, u\} \mid u \in S_i\}, 1 \leq i \leq q$.

An easy consequence of Lemma 2 is the following (see also [6]).

Lemma 3 *Let G be a graph and $\mathcal{S}$ be a collection of non-empty subsets of $V(G)$ such that $\forall_{S \in \mathcal{S}}$ $G[S]$ is a clique of G. Then* $\text{treewidth}(G) = \text{treewidth}(G^{<\mathcal{S}>})$.

Given two graphs G_1, G_2 we set $G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. The following lemma describes a widely known way of merging tree-decompositions (see e.g. [6, 7]). We present it in a form suitable for the objectives of our paper.

Lemma 4 *Let $G_i, 0 \leq i \leq q$ be graphs and $\mathcal{S} = \{S_i \mid , 1 \leq i \leq q\}$ a collection of vertex sets such that $\forall_{1 \leq i \leq q} S_i = V(G_0) \cap V(G_i)$. Suppose also that $\forall_{0 \leq i \leq q}$*treewidth$(G_i) \leq k$ *and $\forall_{1 \leq i \leq q}$ S_i induces a clique in G_i and G_0. Then,* treewidth$(G) \leq k$ *where $G = \bigcup_{0 \leq i \leq r} G_i$.*

Proof. We first choose a set of indices I and a partition $\{I_0, I_1, \ldots, I_q, I_{q+1}\}$ of I such that $\forall_{0 \leq i \leq q}$ $D^i = (\{X_j \mid j \in I^i\}, (I^i, F^i))$ is a tree decomposition of G_i with width $\leq k$ and $I_{q+1} = \{h_1, \ldots, h_q\}$. From Lemma 2 we have that, for $i = 1, \ldots, q$, S_i will be a subset of some node, say X_{l_i}, of D^0 and of some node, say X_{j_i}, of D^i. We set $X_{h_i} = S_i, 1 \leq i \leq q$ and $F = \{\{l_1, h_1\}, \{h_1, j_1\}, \ldots, \{l_q, h_q\}, \{h_q, j_q\}\} \cup (\bigcup_{0 \leq i \leq q} F^i)$. It is now easy to see that $(\{X_m \mid m \in I\}, (I, F))$ is a tree decomposition of G with width$\leq k$. □

Let G be a graph and let $v \in V(G)$. We set $r = \max\{D_G(v,u) \mid u \in V(G)\}$ and, for $i = 0, \ldots, r$, we define $G_i = G[V_i \cup \cdots \cup V_r]$ where $V_i = \{u \in V(G) \mid D_G(v,u) = i\}$. For $i = 0, \ldots, r$, let $\{C_i^1, \ldots, C_i^{q_i}\}$ be the connected components of G_i. For $i = 0, \ldots, r$ and $j = 1, \ldots, q_i$, we define $V_i^j = V_i \cap V(C_i^j)$. Clearly, $\{V_0^1, V_1^1, \ldots, V_1^{q_1}, \ \ldots \ , V_r^1, \ldots, V_r^{q_r}\}$ is a partition of $V(G)$ and we denote it as $\mathcal{W}(G,v)$. Let now U be a set containing $|\mathcal{W}(G,v)|$ vertices and let $f : U \to \mathcal{W}(G,v)$ be a bijection, mapping each vertex of U to a set of $\mathcal{W}(G,v)$. We define the directed graph (U, F) where $(x,y) \in F$ iff $\phi(x) \cap N_G(\phi(y)) \neq \emptyset$ and $D_G(v, \phi(x)) + 1 = D_G(v, \phi(y))$. It is easy to see that $T_G = (U, F)$ is a directed tree rooted on $\phi^{-1}(V_0^1) = \phi^{-1}(\{v\})$. We call the triple $(\mathcal{W}(G,v), \phi, T_G)$ *v-representation* of G (for an example of a v-representation of a graph G, see Figure 1).

Clearly, for any $i, 1 \leq i \leq r$, any vertex in V_i is adjacent with a vertex in V_{i-1}. A direct consequence of this fact is the following.

Lemma 5 *Let G be a graph and $(\mathcal{W}(G,v), \phi, T_G)$ a v-representation of G, $v \in V(G)$. Then, for any vertex $x \in V(T_G)$ and any vertex $u \in \phi(x)$, there exist a path in G connecting v and u that has no internal vertex that belongs to $\phi(x)$ or to the image of a descendant of y in T_G.*

Let G be a graph and $(\mathcal{W}(G,v),\phi,T_G)$ a v-representation of G for some $v \in V(G)$. We will denote the root of T_G as x_r. Let $y \in V(T_G)$ and let $\{y_1,\ldots,y_q\}$ be the set of the children of y in T_G (if y is a leaf of T_G then this set is empty). We set $\mathcal{T}(y) = \{\phi(y) \cap N_G(\phi(y_i)) \mid 1 \leq i \leq q\}$. and if $y \neq x_r$ we set $\tau(y) = N_G(\phi(y)) \cap \phi(x)$ where x is the unique parent of y. We also set $\tau(x_r) = \emptyset$. It is now easy to see that

$$\forall_{y \in V(T_G)} \mathcal{T}(y) \quad = \quad \{\tau(y_1),\ldots,\tau(y_q)\}. \tag{1}$$

We define $G_y = G[\phi(y) \cup \tau(y)]$, $\mathcal{S}_y = \{\tau(y)\} \cup \mathcal{T}(y)$, $\overline{G}_y = \bigcup_{z \in T_G^y} G_z$, and $\overline{\mathcal{S}}_y = \bigcup_{z \in V(T_G^y)} \mathcal{S}_z$ where T_G^y is the subtree of T_G induced by y and its descendants. Let G be a graph and $V_1, V_2 \subseteq V(G)$. We write $V_1 \sim V_2$ if $V_1 \cap V_2 = \emptyset$ and $V_i \subseteq N_G(V_{3-i}), i = 1,2$.

Lemma 6 *Let G be a $K_{2,r}$-minor free planar graph and $(\mathcal{W}(G,v),\phi,T_G)$ a v-representation of G for some $v \in V(G)$. Then, for any vertex y of T_G, the following hold.*

a. $\tau(y) \sim \phi(y)$

b. $G[\phi(y)]^{<\mathcal{T}(y)>}$ *is connected.*

c. $G_y^{<\mathcal{S}_y>}$ *is planar.*

d. $|\tau(y)| < r$.

Proof. (a) follows directly from the definitions of $\tau(y)$ and $\phi(y)$. We will prove (b) by showing that $G[\phi(y)]^{<\mathcal{T}(y)>}$ is isomorphic to a graph that can be obtained from a connected subgraph of G only through edge contractions. Recall that $G_i = G[\{u \in V(G) \mid D(v,u) \geq i\}]$ and let $\{y_1,\ldots,y_q\}$ be the set of children of y in T_G (if y is a leaf then (b) follows trivially as, in this case, $G[\phi(y)]$ is connected, and $\mathcal{T}(y) = \emptyset$). Let $d = D_G(v_y, v)$ where v_y is any vertex in $\phi(y)$. For any vertex $z \in \{y, y_1,\ldots,y_q\}$, we set $\overline{G}_z^\circ = G[V(\overline{G}_z) - \tau(z)]$. Notice that $\overline{G}_y^\circ$ is one of the connected

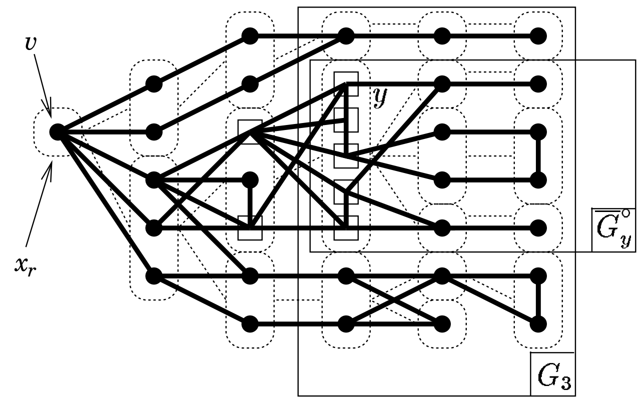

Figure 1: A v-representation of a graph G (the vertices in G_y are depicted as squares).

components of G_d and that, for $1 \leq i \leq q$, $\overline{G}^\circ_{y_i}$ is one of the connected components of G_{d+1}. Using Relation 1 and the fact that $\forall_{1\leq i\leq q}\ \tau(y_i) \sim \phi(y_i)$, one can easily see that $G[\phi(y)]^{<\{\mathcal{T}(y)\}>}$ is isomorphic to the graph occurring if we contract in $\overline{G}^\circ_y$ all the edges of $\overline{G}^\circ_{y_i}, 1 \leq i \leq q$. This completes the proof of (b). As (c) and (d) are trivial for the case where y is the root of T_G, we assume that x is the unique parent of y in T_G. We will prove (c) by showing that $G_y^{<\mathcal{S}_y>}$ is isomorphic to a minor of G. Using Lemma 5, we can choose $|\tau(y)|$ paths of G, each connecting one of the vertices of $\tau(y)$ with v and without internal vertices in $V(\overline{G}_y)$. Let V^* (E^*) be the vertices (edges) of these paths. Clearly $G^* = (V^*, E^*) \cup \overline{G}_y$ is a subgraph of G. Let G' be the graph obtained from G^* if we contract all edges that do not have both endpoints in $\{v\} \cup V(\overline{G}_y)$, until this it not possible any more. Notice that G' is isomorphic to $\overline{G}_y^{<\{\tau(y)\}>}$. Similarly now to the proof of (b), we contract, in G', all the edges of $\overline{G}^\circ_{y_i}, 1 \leq i \leq q$. Using the same arguments as in the proof of (b), we can see that the occurring graph is isomorphic to $G_y^{<\mathcal{S}_y>}$ and this proves (c). Suppose, towards a contradiction to (d), that $|\tau(y)| \geq r$. Notice that $\overline{G}^\circ_y$ is a connected subgraph of G' and that $\phi(y) \sim \tau(y)$. Therefore, if we contract in G' all the edges of $\overline{G}^\circ_y$ we have a graph that is isomorphic to $K_{2,|\tau(y)|}$, a contradiction. $\square$

Lemma 7 *Let G be a $K_{2,k}$-minor free planar graph and $(\mathcal{W}(G,v),\phi,T_G)$ a v-representation of G for some $v \in V(G)$. Suppose also that for any $y \in V(T_G)$* $\text{treewidth}(G_y^{(\mathcal{S}_y)}) \leq k$*. Then,* $\text{treewidth}(G) \leq k$*.*

Proof. Let x_r be the root of T_G. As G is a subgraph of $\overline{G}_{x_r}^{(\overline{\mathcal{S}}_{x_r})}$, it is enough to prove that $\forall_{x \in V(T_G)}$ the following relation holds.

$$\text{treewidth}(\overline{G}_x^{(\overline{\mathcal{S}}_x)}) \quad \leq \quad k. \tag{2}$$

If x is a leaf of T_G then $\overline{\mathcal{S}}_x = \mathcal{S}_x$, $\overline{G}_y = G_y$, and Relation (2) follows because $\overline{G}_x^{(\overline{\mathcal{S}}_x)} = G_x^{(\mathcal{S}_x)}$. Assume now that Relation (2) holds if we replace x by any of the children $\{y_1,\ldots,y_q\}$ of a non-leaf vertex $y \in V(T_G)$. We will prove that Relation (2) holds if we replace x by y as well. From Relation 1 we have that $\mathcal{T}(y) = \{\tau(y_1),\ldots,\tau(y_q)\} = \mathcal{S}_y \cap (\bigcup_{1\leq i\leq q}\overline{\mathcal{S}}_{y_i}) = \{V(G_y^{(\mathcal{S}_y)}) \cap V(\overline{G}_{y_i}^{(\overline{\mathcal{S}}_{y_i})}) \mid 1 \leq i \leq q\}$. As, for $i = 1,\ldots,q$, $\tau(y_i)$ induces a clique in $G_y^{(\mathcal{S}_y)}$ and $\overline{G}_{y_i}^{(\overline{\mathcal{S}}_{y_i})}$, the claim now follows if we apply Lemma 4 for $G_y^{(\mathcal{S}_y)}, \overline{G}_{y_1}^{(\overline{\mathcal{S}}_{y_1})},\ldots,\overline{G}_{y_q}^{(\overline{\mathcal{S}}_{y_q})}$, and $\mathcal{T}(y)$. □

We define the graph class $\mathcal{D}_r$ containing all the graphs $G = (V_0 \cup V_1 \cup V_2 \cup V_3, E)$ that satisfy the following conditions.

(1) V_0, V_1, V_2, V_3 are disjoint sets.

(2) $V_0 = \{v_0\}$, $N_G(V_0) = V_1$, and $N_G(V_3) \subseteq V_2$.

(3) $G[V_2 \cup V_3]$ is connected.

(4) $V_1 \sim V_2$.

(5) G is planar.

(6) $|V_1| < r$ and all vertices in V_3 have degree less than r.

From now on, given a graph $G = (V_\rho \cup \ldots \cup V_{\rho+h}, E), \rho \geq 0, h \geq 1, \forall_{\rho \leq i < j \leq \rho+h} V_i \cap V_j = \emptyset$, we will use the notation $V_i(G) = V_i, i = \rho, \ldots, \rho + h$ (we call $V_\rho, \ldots, V_{\rho+h}$ *parts* of G).

Finally, if $G \in \mathcal{D}_r$, we set $\mathcal{S}(G) = \{V_1\} \cup \{N_G(v) \mid v \in V_3\}$ and we define $G\big|_\emptyset = G^{(\mathcal{S}(G))}$. We also define $\tilde{\mathcal{D}}_r = \{G\big|_\emptyset \mid G \in \mathcal{D}_r\}$.

Lemma 8 *Let G be a $K_{2,r}$-minor free planar graph and $(\mathcal{W}(G, v), \phi, T_G)$ a v-representation of G for some $v \in V(G)$. Then $\forall_{y \in V(T_G)}$ $G_y^{(\mathcal{S}_y)}$ is a subgraph of a graph in $\tilde{\mathcal{D}}_r$.*

Proof. Notice that $G_y^{<\mathcal{S}_y>} = (V_0 \cup V_1 \cup V_2 \cup V_3, E)$ where $V_0 = N_{G_y^{<\mathcal{S}_y>}}(\tau(y)) - \phi(y)$, $V_1 = \tau(y)$, $V_2 = \phi(y)$ and $V_3 = N_{G_y^{<\mathcal{S}_y>}}(\phi(y)) - \tau(y)$. We first claim that $G_y^{<\mathcal{S}_y>} \in \mathcal{D}_r$. Indeed, Conditions (1) and (2) follow directly from the definition of G_y. Condition (3)–(5) follow from Lemma 6.(a)–(c). Finally, Condition (6) follows combining Relation (1) and Lemma 6.(d). Notice now that $\mathcal{S}_y = \{V_1\} \cup \{N_G(v) \mid v \in V_3\}$ and thus $(G_y^{<\mathcal{S}_y>})^{(\mathcal{S}_y)} = G_y^{<\mathcal{S}_y>}\big|_\emptyset$. As now $G_y^{(\mathcal{S}_y)} \subseteq (G_y^{(\mathcal{S}_y)})^{<\mathcal{S}_y>} = (G_y^{<\mathcal{S}_y>})^{(\mathcal{S}_y)}$, the result follows. □

It is now clear that Theorem 3 follows directly from Lemmata 1, 7, 8, and the following.

Lemma 9 *Let $r \geq 1$. If $G \in \mathcal{D}_r$ then $G\big|_\emptyset$ has treewidth $\leq r + 2$.*

It is easy to see that, for any graph $G \in \mathcal{D}_r$, one can construct a graph $H \in \mathcal{D}_r$ where $\forall_{v \in V_3(G)} d_G(v) \geq 3$ and treewidth$(G\big|_\emptyset) \leq$ treewidth$(H\big|_\emptyset)$ (use Lemma 3, setting $\mathcal{S} = \{N_G(v) \mid v \in V_3(G) \text{ and } d_G(x) \leq 2\}$). Therefore, we may add the following condition in the definition of $\mathcal{D}_r$, without harming the generality of our results.

(7) $\forall_{v \in V_3} d_G(v) \geq 3$.

We will devote the rest of this paper to the proof of Lemma 9. In the next section we will develop the main tools required for the proof of Lemma 9. The main proof will be given in Section 4.

3 The classes $\mathcal{Z}_r$ and $\mathcal{Q}_r$

A graph $G = (V_1 \cup V_2, E)$ is called an *r-fence*, if it can be written in the following form: $V = V_1 \cup V_2$, with $V_i = \{v_1^i, \ldots, v_r^i\}, i = 1, 2$ and $E = \{(v_j^i, v_{j'}^{i'}) \mid v_j^i \neq v_{j'}^{i'},\ |j - j'| \leq 1,\ i, i' \in \{1, 2\}\}$. An example of a 12-fence is given in Figure 2.

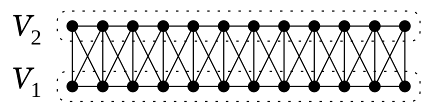

Figure 2: A 12-fence.

Lemma 10 *If* $G = (V_1 \cup V_2, E)$ *is an r-fence then* $\text{treewidth}(G^{(\{V_1, V_2\})}) \leq r + 1$.

Proof. Take the tree-decomposition $(\{X_i \mid i \in I\}, T)$ where T is a path with r vertices and $X_1 = \{v_1^1, \ldots, v_r^1, v_1^2, v_2^2\}, X_i = X_{i-1} \cup \{v_{i+1}^2\} - \{v_{i-1}^1\}, i = 2, \ldots, r - 1$. It is easy to see that this is a tree-decomposition of G with treewidth $\leq r + 1$. □

Let $\mathcal{Z}_r$ be the collection of graphs $G = (V_1 \cup V_2, E)$ that can be constructed as follows:

1. Take two disjoint sets of vertices $V_1 = \{v_1^1, \ldots, v_{k_1}^1\}, V_2 = \{v_1^2, \ldots, v_{k_2}^2\}$ with $k_1, k_2 < r$ and add edges $\{v_1^i, v_2^i\}, \ldots, \{v_{k_i - 1}^i, v_{k_i}^i\}, i = 1, 2$ and edges $\{v_1^1, v_1^2\}, \{v_{k_1}^1, v_{k_2}^2\}$. ($V_i, i = 1, 2$, will be the *parts* of the graph under construction.)

2. Add a maximal set of edges such that

   a. the graph stays planar,

   b. any vertex in V_1 (resp. V_2) is adjacent to at least one vertex in V_2 (resp. V_1),

   c. the resulting planar graph can be embedded such that the outer face is formed by the cycle $(v_1^1, \ldots, v_{k_1}^1, v_{k_2}^2, v_{k_2 - 1}^2, \ldots, v_1^2, v_1^1)$.

Notice that the graph constructed so far is outerplanar.

3. The construction is completed by setting $E^j = E(G[V_j]), j = 1, 2$ and applying the following operation for an arbitrary number of times:

   For some edge $\{v_i^2, v_{i+1}^2\} \in E^2$ and a set of vertices $V_{l,r}^1 = \{v_l^1, \ldots, v_{l+r}^1\} \subseteq V_1, l, r \geq 1$ such that $E(G[V_{l,r}^1]) \subseteq E^1$ and $\{v_i^2, v_l^1\}, \{v_{i+1}^2, v_{l+r}^1\} \in E(G)$ we set

   (i) $E^1 \leftarrow E^1 - E(G[V_{l,r}^1])$

   (ii) $E^2 \leftarrow E^2 - \{\{v_i^2, v_{i+1}^2\}\}$

   (iii) $E(G) \leftarrow E(G) \cup \{\{v_i^2, v_l^1\}, \ldots, \{v_i^2, v_{l+r}^1\}\} \cup \{\{v_{i+1}^2, v_l^1\}, \ldots, \{v_{i+1}^2, v_{l+r}^1\}\}$.

For an example of the construction of a graph in $\mathcal{Z}_{14}$ see Figure 3.

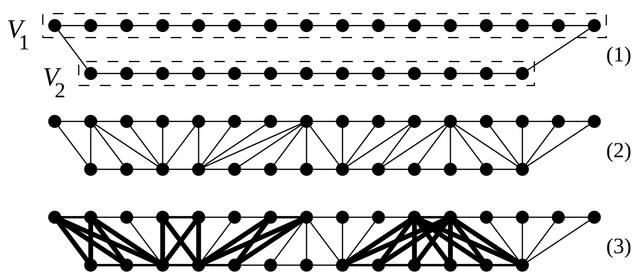

Figure 3: The construction of a graph in $\mathcal{Z}_{14}$.

If $G \in \mathcal{Z}_r$, then define $G\big|_\emptyset = G^{(\{V_1(G), V_2(G)\})}$. Also, we define $\tilde{\mathcal{Z}}_r = \{G\big|_\emptyset \mid G \in \mathcal{Z}_r\}$.

Lemma 11 *If $G \in \mathcal{Z}_r$, then* treewidth$(G\big|_\emptyset) \leq r$.

Proof. We will use induction on r. If $r \leq 3$, then the Lemma is trivial. We assume that lemma holds for any $r \leq k$. We will prove that if $G = (V_1 \cup V_2, E) \in \mathcal{Z}_{k+1}$, then treewidth$(G\big|_\emptyset) \leq k+1$.

It is easy to see that any graph $H \in \mathcal{Z}_{k+1}$, with at least one part of cardinality $< k$, is a subgraph of a graph $G \in \mathcal{Z}_{k+1}$ where $|V_1(G)| = |V_2(G)| = k$. Therefore, from Lemma 1, we may assume that both parts of G have cardinality k.

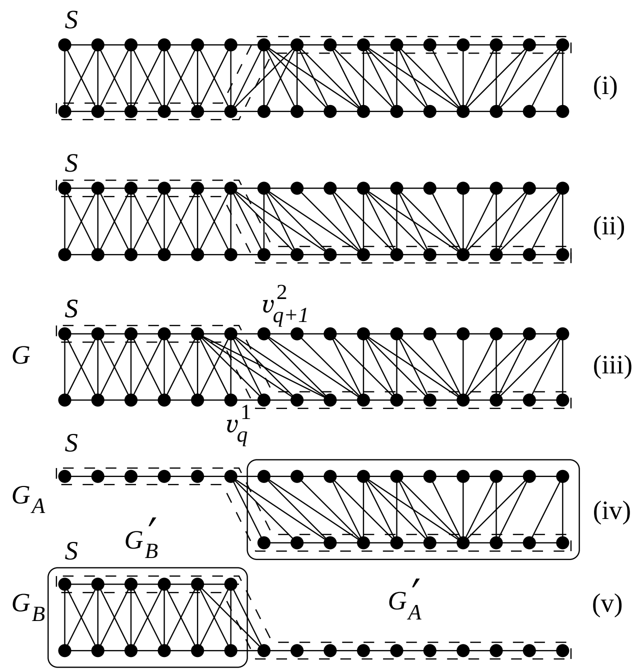

Figure 4: An example of the proof of Lemma 11.

If G is a k-fence, then the result follows from Lemma 10. Suppose that G is not an k-fence. We set $q = \max\{i \mid G[\{v_1^1, \dots, v_i^1, v_1^2, \dots, v_i^2\}]$ is an i-fence$\}$ (clearly, $q < k$). It is easy to see that $N_G(v_q^h) \cap \{v_{q+1}^{3-h}, \dots, v_k^{3-h}\} = \emptyset$, for some h that is either 1 or 2. According to the value of h, we set $S = \{v_1^{3-h}, \dots, v_q^{3-h}, v_{q+1}^h, \dots, v_k^h\}$ (an example of a graph where $h = 2$ is depicted in Figure 4.(i) – examples of graphs where $h = 1$ are depicted in Figures 4.(ii) and 4.(iii)). We also set $G_A = G[\{v_{q+1}^h, \dots, v_k^h\} \cup V_{3-h}(G)]$, $G_B = G[V_h(G) \cup \{v_1^{3-h}, \dots, v_q^{3-h}\}]$ (for the case of the graph in Figure 4.(iii), graphs G_A and G_B are depicted in Figures 4.(iv) and 4.(v) respectively). As $G\big|_\emptyset$ is a subgraph of $G\big|_\emptyset^{(\{S\})}$, from Lemma 1, it is enough to prove that $G\big|_\emptyset^{(\{S\})}$ has treewidth $\leq k+1$. Towards this, we will show that it is possible to apply Lemma 4 for $G_A^{(\{S,V_2\})}$, $G_B^{(\{S,V_1\})}$, and $\{S\}$. Indeed, it is easy to see that $G\big|_\emptyset^{(\{S\})} = G_A^{(\{S,V_2\})} \cup G_B^{(\{S,V_1\})}$, $V(G_A^{(\{S,V_2\})}) \cap V(G_B^{(\{S,V_1\})}) = S$, and that S induces a clique in both $G_A^{(\{S,V_2\})}$ and $G_B^{(\{S,V_1\})}$. Clearly, what remains is to prove that both $G_A^{(\{S,V_2\})}$ and $G_B^{(\{S,V_1\})}$ have treewidth $\leq k+1$.

Let $V_i' = \{v_{q+1}^i, \dots, v_k^i\}, i = 1, 2$. It is not hard to see that $G_A' = G_A[V_1' \cup V_2']$ is a subgraph of

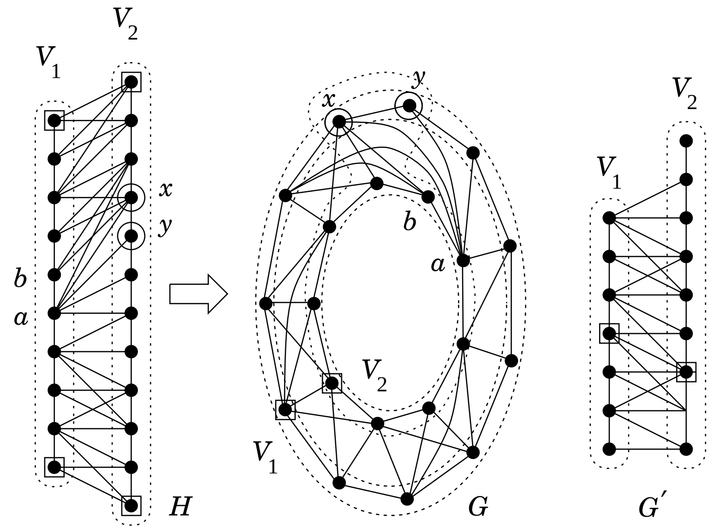

Figure 5: The construction of a graph in $\mathcal{Q}_{12}$ and an example of the proof of Lemma 12.

a graph in Z_{k-q+1}. Therefore, ${G'_A}^{(\{V'_1,V'_2\})}$ is a subgraph of a graph in $\tilde{Z}_{k-q+1}$ and, as $k-q+1 \leq k$, by the induction hypothesis, treewidth$({G'_A}^{(\{V'_1,V'_2\})}) \leq k-q+1$. As $G_A^{(\{S,V_2\})}$ contain q vertices more than ${G'_A}^{(\{V'_1,V'_2\})}$, one can easily see that it has treewidth $\leq k+1$.

We now define $V'_i = \{v^i_1, \ldots, v^i_q\}, i = 1,2$. Clearly, $G'_B = G_B[V'_1 \cup V'_2]$ is a q-fence and, from Lemma 10, treewidth$({G'_B}^{(\{V'_1,V'_2\})}) \leq q+1$. As $G_B^{(\{S,V_2\})}$ contains $k-q$ vertices more than ${G'_B}^{(\{V'_1,V'_2\})}$, one can easily see that it has treewidth $\leq k+1$. □

Let $\mathcal{Q}_r$ be the collection of all the graphs $G = (V_1 \cup V_2, E)$ that are the result of the identification of vertices v^i_1 and $v^i_{|V_i(H)|}, i = 1,2$ and edges $\{v^1_1, v^2_1\}$ and $\{v^1_{|V_i(H)|}, v^2_{|V_i(H)|}\}$ of a graph in $H \in \mathcal{Z}_{r+1}$ (we use the notation $V_i(H) = \{v^i_1, \ldots, v^i_{|V_i(H)|}\}, i = 1,2$). We may assume that $v^i_{|V_i(H)|}, i = 1,2$ are not any more vertices of G and that $\{v^1_{|V_1(H)|}, v^2_{|V_2(H)|}\}$ is not any more an edge in G (for an example of a graph in $\mathcal{G}_{12}$, see Figure 5).

If $G \in \mathcal{Q}_r$ and $e = \{x,y\}$ is an edge of $G[V_2(G)]$, we define $G\big|_e = G^{(\{V_1(G) \cup \{x,y\}, V_2(G)\})}$. Also, we define $\tilde{\mathcal{Q}}_r = \{G\big|_e \mid G \in \mathcal{Q}_r$ and e is an edge of $G[V_2(G)]\}$. For an example of the construction of a graph in $\mathcal{Q}_{12}$ see Figure 5.

Lemma 12 *If $G \in \mathcal{Q}_r$ and $e \in E(G[V_2])$ then* $\text{treewidth}(G\big|_e) \leq r+2$.

Proof. Let $e = \{x, y\}$. Notice that there will exist a vertex $a \in V_1(G)$ that is adjacent to both x and y in G. Let now $b \in V_1(H)$ be any neighbor of a in G. It is easy to see that $G' = G[V(G) - \{x, y, a, b\}]$ is a subgraph of a graph in $\mathcal{Z}_{r-2}$ (see Figure 5). From Lemmata 1 and 11 we have that $H = G'^{(\{V_1(G)-\{a,b\}, V_2(G)-\{x,y\}\})}$ has treewidth $\leq r-2$ and, as H is a subgraph of $G\big|_e$ containing four vertices less, we can easily see that $\text{treewidth}(G\big|_e) \leq r+2$. □

4 The class $\mathcal{P}_r$

We are now ready to prove Lemma 9. Note that if G is a graph in $\mathcal{D}_r$, then $G[V_2]$ is outerplanar. Recall that outerplanar graphs have treewidth ≤ 2 (see e.g. [2]). Using this fact we can easily see that, if $G \in \mathcal{D}_r$ for $r \leq 3$, then $G\big|_\emptyset$ has treewidth $\leq r+1$. Therefore, Lemma 9 holds for $r \leq 3$. In what follows we will prove that it also holds when $r \geq 4$.

Our first step will be the definition, for $r \geq 4$, of a subclass of $\mathcal{D}_r$ which we will denote as $\mathcal{P}_r$. The main property of $\mathcal{P}_r$ is that any graph in $\mathcal{D}_r$ is a minor of some graph in $\mathcal{P}_r$. Using this fact, it will be enough to prove Lemma 9 for $\mathcal{P}_r$ instead of $\mathcal{D}_r$ which is much easier.

We define $\mathcal{P}_r, r \geq 4$ as the set of graphs that can be constructed from a graph $G \in \mathcal{D}_r, r \geq 4$, by applying the following five steps:

(a) Consider a planar embedding of G. Let $v \in V_0[G] \cup V_3[G]$. From Conditions (2) and (3) we have that all the vertices in $G[N_G(v)]$ have degree ≤ 2 (otherwise, $K_{3,3} \leq G$). Using Condition (7), we have that, for any vertex $v \in V_3[G]$, there exist a set of edges E_v, with endpoints in $V_1(G)$, such that $G_v = (V(G), E(G) \cup E_v)$ remains planar and all the vertices in $G_v[N_{G_v}(v)]$ have degree 2. Moreover, such a set E_{v_0} exist also for the unique vertex in V_0 in case $|V_1| \geq 3$. In case $V_1 = \{v_1^1, v_2^1\}$, we set $E_{v_0} = \{v_1^1, v_2^1\}$ where $V_0(G) = \{v_0\}$. In case

$|V_1| = 1$ we set $E_{v_0} = \emptyset$. We define $G_a = (V(G), E(G) \cup (\bigcup_{v \in V_0[G] \cup V_3[G]} E_v))$. Notice that the fact that Condition (3) holds for G, implies that $G_a[V_2(G_a)]$ is a connected outerplanar graph. Clearly, $G_a \in \mathcal{D}_r$.

(b) If there exist an edge $\{a, b\} \notin E(G_a)$ such that $a \in V_1(G_a) \cup V_2(G_a)$, $b \in V_2(G_a)$ and $(V(G_a), E(G_a) \cup \{\{a, b\}\})$ is a planar graph, then add this edge to G_a. We repeat this step until no such edge can be added in G_a. We denote the resulting graph as G_b. Clearly, $G_b \in \mathcal{D}_r$. Notice that if $|V_1(G_b)| \geq 3$, all faces in the planar embedding of G_b correspond to triangles (if $|V_1(G_b)| \leq 2$ then the same holds for the planar embedding of $G_b - v_0$).

(c) If there is a biconnected component in $G_b[V_2]$ that contains only two vertices, say a, b, then it is easy to see that there exist at least one vertex $d \in V_1(G_b)$ such that $\{a, d\}, \{b, d\} \in E(G_b)$ (in the graph G_b of Figure 6, d can be a_1^1 or a_2^1). In this case, we add a new vertex c to $V_2(G_b)$, and add edges $\{\{a, c\}, \{b, c\}, \{c, d\}\}$. We repeat this step until all the biconnected components of $G_b[V_2]$ contain at least 3 vertices. We denote the resulting graph as G_c and observe that $G_c \in \mathcal{D}_r$.

(d) If there is a triangle of $G_c[V_2(G_c)]$ with vertices a, b, and c such that no vertex of $V_3(G_c)$ is adjacent to all its vertices, then add a new vertex d in $V_3(G_c)$, and add edges $\{a, d\}, \{b, d\}, \{c, d\}$. Repeat this step until no such a triangle exist any more. We denote the resulting graph as G_d. Notice also that $G_d \in \mathcal{D}_r$.

(e) Let A be the articulation vertices of $G_d[V_2(G_d)]$. Let $x \in A$. If $|V_1(G_d)| \geq 3$, we observe that the set $N_G(x) \cap V_1(G_d)$ can be partitioned into two vertex sets V_x, V_x', each containing consecutive vertices of the cycle formed by the vertices of $V_1[G_d]$ (in Figure 6, $V_x = \{v_3^1\}, V_x' = \{v_6^1, v_1^1\}$). Let $V_x = \{a_i, \dots, a_{(i+\sigma-1 \bmod |V_1(G_d)|)+1}\}$ (in Figure 6, $i = 3$, $\sigma = 0$). If $|V_1(G_d)| \leq 2$ we set $V_x = \{v_1^1\}$ (notice that, in this case, $v_1^1 \in V_1(G_d) = N_{G_d}(x) \cap V_1(G_d)$).

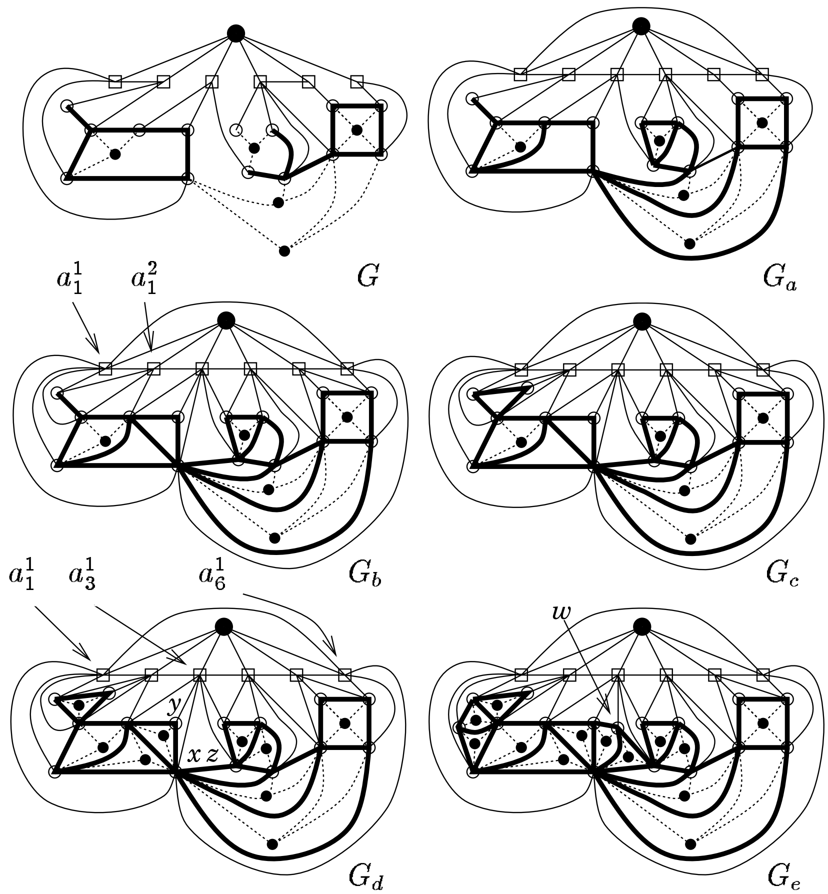

Figure 6: The construction of a graph in $\mathcal{P}_7$

Since all the faces of G_d (or $G_d - v_0$ in case $|V_1(G_d)| \leq 2$) are triangles, there will exist two vertices $y, z \in V_2(G_d)$ such that $\{x, y\}, \{x, z\}, \{y, a_i\}, \{z, a_{i+\sigma}\} \in E(G_d)$ (Notice that y, z belong to different connected components of $G_d[V_2(G_d) - \{x\}]$). We now construct graph G_e by applying, for any $x \in A$, the following operations: (i) we remove from G the edges in $\{\{x, t\} \mid t \in V_x\}$, (ii) we add a new vertex w in $V_2(G_d)$ and two new vertices u, v in $V_3(G_d)$, (iii) we add in $E(G_d)$ edges $\{w, x\}, \{w, y\}, \{w, z\}, \{u, x\}, \{u, y\}, \{u, w\}, \{v, x\}, \{v, z\}$, and $\{v, w\}$, and (iv) we add in $E(G_d)$ the edge set $\{\{w, t\} \mid t \in V_x\}$. The resulting graph G_e becomes a member of $\mathcal{P}_r$. Notice that $G_e[V_2(G_e)]$ is a biconnected outerplanar graph and that $G_e \in \mathcal{D}_r$.

If a graph $H \in \mathcal{P}_r$ is constructed by a graph $G \in \mathcal{D}_r$ after applying steps (a)–(e), we call H *triangular extension* of G. Notice that if H is a triangular extension of G, then G is a minor of H. For an example of the construction of a graph in $\mathcal{P}_r$, see Figure 6. Clearly, $\mathcal{D}_r \subseteq \mathcal{P}_r, r \geq 4$.

Let $G = (V_0 \cup V_1 \cup V_2 \cup V_3, E) \in \mathcal{P}_r$. Notice that, as $G[V_2]$ is outerplanar, G has a planar embedding where each vertex $v \in V_0 \cup V_3$ (or just V_3, in case $|V_1| \leq 2$) is at the inside of the cycle C_v of G formed by the vertices of $N_G(v)$ (recall that any cycle of G defines two areas in any planar embedding of G: one finite and one infinite; we say that a vertex is *inside* a cycle when the embedding maps it to a point of its finite area).

We call this planar embedding of G *outerplanar embedding.* We also call the set $\mathcal{R}(G) = \{C_v \mid v \in V_3\}$ *set of regions* of G and, for any region R of $\mathcal{R}(G)$, we denote as v_R the vertex of V_3 that is inside it. We will denote as v_0 the unique vertex in $V_1(G)$ and will use the notation $V_1 = \{v_1^1, \ldots, v_{|V_1(G)|}^1\}$ where, if $|V_1| \geq 3$, $C_0 = (v_1^1, \ldots, v_{|V_1(G)|}^1, v_1^1)$ is the cycle of G formed by the vertices of $N_G(v_0)$. We say that an edge $\{x, y\}$ *belongs to* a region R if x and y are consecutive vertices of R. Notice that an edge of $G[V_2]$ can belong to either one or two regions in $\mathcal{R}(G)$. We denote as $E^{\text{ext}}(G)$ ($E^{\text{int}}(G)$) the edges that belong to one (two) region(s) of $\mathcal{R}(G)$. Given a region $R \in \mathcal{R}(G)$, we define as $E_R^{\text{int}}(G)$ the set of edges in $E^{\text{int}}(G)$ that belong to R. Given an edge $e \in E^{\text{ext}}(G)$ we denote as $R(e)$ the unique region to which e belongs.

Let $G = (V_0 \cup V_1 \cup V_2 \cup V_3, E) \in \mathcal{P}_r$ and consider an outerplanar embedding of G. Notice that $E^{\text{ext}}(G) \neq \emptyset$. If $e = \{x, y\} \in E^{\text{ext}}(G)$, we set $G\big|_e = G[V_1 \cup V_2]^{(\mathcal{S}(G,e))}$ where $\mathcal{S}(G, e) = \{N_G(v) \mid v \in V_3\} \cup \{V_1 \cup \{x, y\}\}$ (we point out that, in contrary to the definition of $G\big|_\emptyset$, the vertices in $V_0 \cup V_3$ are not considered to be vertices of $V(G\big|_e)$). If $R \in \mathcal{R}(G)$ and $e = \{a, b\} \in E_R^{\text{int}}(G)$, we define $S_{R,e}$ as the vertex set of a cycle $C_{R,e} = (a, v_i^1, \ldots, v_{i'}^1, b, a)$ of G where both v_0 and v_R are vertices of the same area of $C_{R,e}$ in the outerplanar embedding of G (vertices $v_i^1, \ldots, v_{i'}^1$ are consecutive in the cyclic order of C_0). We also define $\mathcal{S}_R = \{S_{R,e} \mid e \in E_R^{\text{int}}\}$.

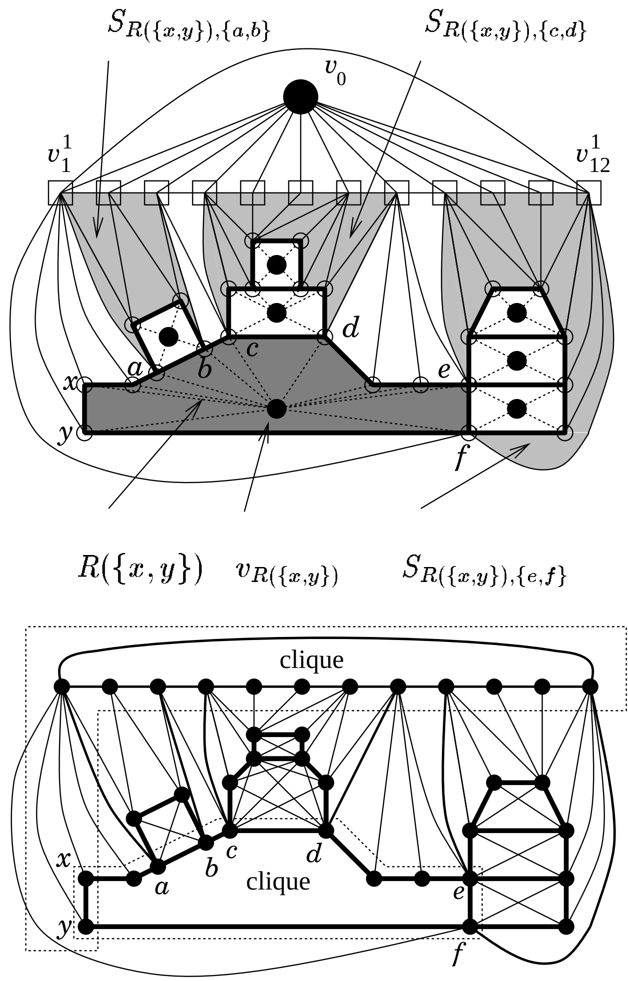

Figure 7: An example of a graph $G \in \mathcal{P}_7$ and of the graph $G\big|_e$.

Finally, for each member $S_{R,e}$ of $\mathcal{S}_R$, we define $V_{R,e}^{\text{in}}$ as the set of the vertices of $V_2(G)$ that are inside the cycle $C_{R,e}$. A general example of the given definitions is depicted in Figure 7.

In the next (and last) lemma we exploit the fact that the graphs in $\mathcal{P}_r$ contain subgraphs of graphs that also belong in $\mathcal{P}_r$ but with smaller number of regions.

Lemma 13 *Let* $G \in \mathcal{P}_r, r \geq 4$ *and* $e = \{x, y\} \in E^{\text{ext}}(G)$. *Then* $\text{treewidth}(G\big|_e^{(\mathcal{S}_{R(e)})}) \leq r + 2$.

Proof. Let $G = (V_0 \cup V_1 \cup V_2 \cup V_3, E) \in \mathcal{P}_r$. Clearly, $G\big|_e = G[V_1 \cup V_2]^{(\mathcal{S}(G,e))}$. We set $H = G[V_1 \cup V_2]^{(\mathcal{S}(G,e) \cup \mathcal{S}_{R(e)})}$ and we have to prove that $\text{treewidth}(H) \leq r + 2$. We will use induction on the number of regions of G.

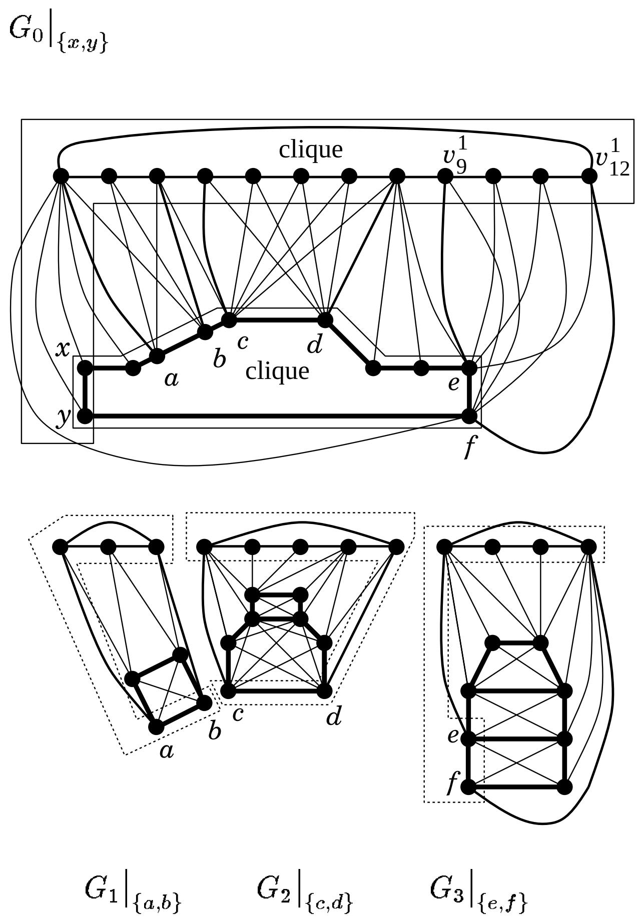

Figure 8: An example of the proof of Lemma 13.

If $|\mathcal{R}(G)| = 1$, then $\mathcal{S}(G, e) = \{V_2, V_1 \cup \{x, y\}\}$, $\mathcal{S}_{R(e)} = \{V_2\}$ and thus $H = G[V_1 \cup V_2]^{(\{V_2, V_1 \cup \{x,y\}\})} \in \tilde{\mathcal{Q}}_r$. The result now follows from Lemma 12.

Suppose now that lemma holds for any graph $G \in \mathcal{P}_r$ where $|\mathcal{R}(G)| < l, l \geq 2$. We will prove that it also holds when $|\mathcal{R}(G)| = l$.

Let $E^{\rm int}_{R(e)}(G) = \{e_1, \ldots, e_t\}$ and $\mathcal{S}_{R(e)} = \{S_{R(e),e_1}, \ldots, S_{R(e),e_t}\}$. For any $i, \leq i \leq t$, we set $e_i = \{u_i, u_i'\}$, $G_i = G[\{v_0\} \cup S_{R(e),e_i} \cup V^{\rm in}_{R(e),e_i}]$, and $V_j(G_i) = V(G_i) \cap V_j, 0 \leq j \leq 3$.

Notice that, if $|V_1(G_i)| = 1$ then $G_i \in \mathcal{P}_r$. In case $|V_1(G_i)| > 1$, we observe first that $V_1(G_i)$ will contain exactly two vertices w_i, w_i' with degree 1 in $G_i[V_1(G_i)]$ (in Figure 8, if

$e = \{x,y\}$ and $e_i = \{e,f\}$, then $w_i = v_9^1$ and $w_i' = v_{12}^1$). Notice that exactly two edges, in $\{\{w_i,u_i\},\{w_i',u_i\},\{w_i,u_i'\},\{w_i',u_i'\}\}$ are not edges of $E(G_i)$. Moreover, these two edges cannot have common endpoints. W.l.o.g. we assume that they are $\{w_i,u_i\}$ and $\{w_i',u_i'\}$. We now set $G_i \leftarrow (V(G_i), E(G_i) \cup \{\{w_i,w_i'\},\{w_i,u_i\}\})$ and observe that G_i is now a member of $\mathcal{P}_r$.

Notice now that $e_i \in E^{\text{ext}}(G_i)$, and, as $|\mathcal{R}(G_i)| < l$, from the induction hypothesis, we have that treewidth$(G_i\big|_{e_i}^{(\mathcal{S}_{R(e_i)})}) \leq r+2$. Clearly, $G_i\big|_{e_i} \subseteq G_i\big|_{e_i}^{(\mathcal{S}_{R(e_i)})}$ and from Lemma 1, treewidth$(G_i\big|_{e_i}) \leq r+2$, $1 \leq i \leq t$. Notice also that $S_{R(e),e_i}$ induces a clique in $G_i\big|_{e_i}, 1 \leq i \leq t$.

We now set $G_0 = G[V(G) - (\bigcup_{1\leq i\leq t} V^{\text{in}}_{R(e),e_i}) - \{v_0, v_{R(e)}\}]^{(\mathcal{S}_{R(e)})}$ where $V_1(G_0) = V_1$ and $V_2(G_0)$ is the vertex set of $R(e)$. Observe that $G_0 \in \mathcal{Q}_r$ and therefore, $G_0\big|_e \in \tilde{\mathcal{Q}}_r$. From Lemma 12, we have that treewidth$(G_0\big|_e) \leq r+2$. Notice also that $S_{R(e),e_i}$ induces a clique in $G_0\big|_e, 1 \leq i \leq t$.

It is easy to verify that $\forall_{1\leq i\leq t} V(G_0\big|_e) \cap V(G_i\big|_{e_i}) = S_{R(e),e_i}$ and $H = G_0\big|_e \cup (\bigcup_{1\leq i\leq r} G_i\big|_{e_i})$. Applying now Lemma 4 for $G_0\big|_e, G_1\big|_{e_1}, \ldots, G_t\big|_{e_t}$ and $\mathcal{S}_{R(e)}$ we have the required. □

For an example of the proof of Lemma 13 see Figure 8.

Proof of Lemma 9. As we mentioned in the beginning of Section 4, Lemma 9 follows easily when $r \leq 3$. Let $G \in \mathcal{D}_r, r \geq 4$ and let H be the triangular extension of G. Clearly, $H \in \mathcal{P}_r$, and $G \leq H$. Considering H as a member of $\mathcal{P}_r$, we observe that $S(G) \subseteq S(H)$ and therefore, $G\big|_\emptyset \leq H\big|_\emptyset$. Let $e = \{x,y\}$ be any edge in $E^{\text{ext}}(H)$. Using the fact that $S(H,e) - \{V_1(H) \cup \{x,y\}\} = S(H) - \{V_1(H)\}$, it is easy to see that $H\big|_\emptyset \subseteq H\big|_e^{<\mathcal{S}(H)>}$. Clearly, all the members of $\mathcal{S}(H)$ induce a clique in $H\big|_e$ and, from Lemma 3, we have that treewidth$(H\big|_e^{<\mathcal{S}(H)>}) =$ treewidth$(H\big|_e)$. Since $H\big|_e \subseteq H\big|_e^{(\mathcal{S}_{R(e)})}$, the result follows from Lemmata 1 and 13. □

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