Symmetries and First Order ODE patterns

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Abstract

A scheme for determining symmetries for certain families of first order ODEs, without solving any differential equations, and based mainly in matching an ODE to patterns of invariant ODE families, is presented. The scheme was implemented in Maple, in the framework of the *ODEtools* package and its ODE-solver. A statistics of the performance of this approach in solving the first order ODE examples of Kamke's book [1] is shown.

Key words:
First order ordinary differential equations; symmetry methods; symbolic computation.

PROGRAM SUMMARY

Title of the software package: Extension to the Maple ODEtools package

Catalogue number: (supplied by Elsevier)

Software obtainable from: CPC Program Library, Queen's University of Belfast, N. Ireland (see application form in this issue)

Licensing provisions: none

Operating systems under which the program has been tested: UNIX systems, Macintosh, Windows (AT 386, 486 and Pentium based) systems, DEC VMS, IBM CMS

Programming language used: Maple V Release 4 and 5

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Memory required to execute with typical data: 16 Megabytes.

Keywords: First order ordinary differential equations, symmetry methods, invariant ODE patterns, symbolic computation

Nature of mathematical problem
Analytical solving of first order ordinary differential equations using symmetry methods.

Methods of solution
Matching ODEs to the patterns of a pre-determined set of invariant ODE families.

Restrictions concerning the complexity of the problem
The computational scheme presented works when the input ODE has a symmetry of one of the forms considered in this work. This set of symmetry patterns can be extended, but there are symmetry patterns for which the ideas of this work cannot be applied.

Typical running time
The methods being presented were implemented in the framework of the ODEtools Maple package. On the average, over Kamke's [1] first order non-trivial examples (see sec. 4), the ODE-solver of ODEtools is now spending $\approx$ 5 sec. per ODE when *successful*, and $\approx$ 20 sec. when *unsuccessful*. When considering only ODEs of first degree in y', these timings drop by 50 %. The timings of this paper were obtained using Maple R5 on a Pentium 200 - 128 Mb. of RAM - running Windows95.

Unusual features of the program
The computational scheme being presented is able to find symmetries by matching a given ODE to the patterns of a varied set of invariant ODE families. When such a symmetry exists, the routines can explicitly determine it, without solving any differential equations, and use it to return a closed form solution without requiring further participation from the user. The invariant ODE families that are covered include, as particular cases, more than 70% of Kamke's first order examples, as well as many subfamilies for which there is no standard classification. Many of these subfamilies cannot be solved by using other methods nor by ODE-solvers of other computer algebra systems. The combination of this *symmetry & pattern matching* approach with the standard classification methods of the ODEtools package succeeds in solving 93 % of Kamke's first order examples.

LONG WRITE-UP

1 Introduction

One of the most remarkable aspects of Lie's method of symmetries for differential equations (DEs) is its generality: roughly speaking, all solving schemes for DE's can be correlated to particular forms of the corresponding symmetry generators [2,3]. However, from a practical point of view, the problem of determining symmetry generators for a given n^{th}order ODE is not trivial: one needs to solve a related *determining* n^{th} *order PDE* in $n+1$ variables for the components of the infinitesimal symmetry generator (infinitesimals) [4].

The usual strategy for tackling this determining PDE consists of restricting the cases handled to the universe of ODEs having *point* symmetries, so that the infinitesimals depend on just two variables, and then the determining PDE splits into a system of PDEs. Although the solving of such a PDE system may be a major problem in itself [5], the hope is that one will be able to solve it by taking advantage of the fact that it is overdetermined. For first order ODEs, however, this strategy of splitting the determining PDE is inapplicable, so that in this case most of the computer algebra implementations of Lie methods just do not work.

A computational alternative to this situation was presented in two recent works [6], where the authors explore different ideas for finding particular solutions for an *unsplit* determining PDE, and use these ideas to tackle first and second order ODEs. Although the performance obtained with that approach is good indeed [1], those works did not explore the fact that, by imposing certain restrictions on the form of the infinitesimals, these can be determined directly in closed form. In such cases, the search for symmetries can be reduced to matching an ODE to (pre-determined) invariant ODE patterns.

Bearing all this in mind, this paper presents an approach for finding symmetries based on the matching of a first order ODE to the patterns of a selected set of invariant ODE families. Nine different symmetry patterns were considered. Concretely, the scheme consists of analyzing an input ODE in order to determine if it belongs to one of these nine ODE families, and if so determine the symmetry, all without solving any auxiliary differential equations. Due to the simplicity and generality of the scheme, it appears to us appropriate for a computer algebra implementation in the framework of ODE solvers and we have implemented it in Maple R5. Other relevant features of the scheme are:

- the determination of the symmetries of the types considered, when they exist, is systematic;
- the scheme works at a remarkably high speed (see sec. 4);
- a good variety of first order ODEs is covered, including families not solved by other methods or ODE-solvers (see sec. 3 and 4) and 73% of the examples in Kamke's book.

[1] Symmetries for more than 85% of Kamke's first order ODE examples were found by looking for particular solutions to Eq.(2) in [6].

The exposition is organized as follows. In sec. 2, symmetry methods for first order ODEs are briefly described. In sec. 3, the body of this work, the computational schemes for obtaining the aforementioned symmetries of various different types are presented and some examples are given. Sec. 4 contains statistics concerning the performance of the new scheme in determining symmetries for the first order ODE examples of Kamke's book, as well as a brief discussion about the symmetries of these examples and their use as a testing arena for ODE-solvers. Finally, the conclusions contain general remarks about this work and its possible extensions.

2 Symmetry methods for 1st order ODEs

Generally speaking, given a first order ODE,

$$y' = \Phi(x, y), \tag{1}$$

the key point of Lie's solving method is that the knowledge of a (Lie) group of transformations leaving Eq.(1) invariant reduces the problem of finding the equation's solution to solving a line integral or a quadrature [2,3]. Despite the subtleties which arise when considering different cases, we can summarize the computational task of finding a Lie symmetry of Eq.(1), as the finding of a pair of functions $\{\xi(x,y),\ \eta(x,y)\}$ satisfying

$$\eta_x + (\eta_y - \xi_x)\,\Phi - \xi_y \Phi^2 - \xi\,\Phi_x - \eta\,\Phi_y = 0 \tag{2}$$

Once a solution to Eq.(2) is found, one can either look for the canonical coordinates, say $\{r, s(r)\}$, of the associated Lie group, to be used to reduce Eq.(1) to a quadrature; or, alternatively, one can work directly with the original variables and use the expressions found for $\{\xi(y,x),\ \eta(y,x)\}$ to build an implicit form solution in terms of a line integral [2]:

$$\int \frac{dy - \Phi\, dx}{\eta - \xi\, \Phi} = \text{constant} \tag{3}$$

Despite the generality and apparent simplicity of this scheme, the problem is that there are no general rules which might help in solving Eq.(2), and hence Lie methods are generally viewed as not useful for tackling first order ODEs [2].

[2] The same happens with high order linear ODEs, for which particular solutions are also symmetries, so that their determination is usually considered as difficult as solving the original problem.

3 Symmetries and first order ODE patterns

The starting point of this work is the observation that, in some cases, the knowledge of the form of the invariant ODE family associated to a certain symmetry pattern allows one to determine symmetries for all members of the family by just performing algebraic operations. For example, the general Bernoulli ODE

$$y' = f(x)\, y + h(x)\, y^{\alpha} \tag{4}$$

where f and h are arbitrary functions of x, is the invariant ODE family associated to the symmetry pattern:

$$[\xi = 0, \;\; \eta = y^{\alpha}\, e^{(1-\alpha)\int f(x) dx}] \tag{5}$$

and in this case the symmetry parameters, that is, $f(x)$ and α, can be obtained from a given Bernoulli ODE just by inspection.

The idea then was to consider the possible determination of symmetries by just performing algebraic operations, but for more general invariant ODE families. For example, Eq.(5) is a particular case of the symmetry pattern

$$[\xi = 0, \;\; \eta = \mathcal{F}(x)\, \mathcal{G}(y)] \tag{6}$$

to which corresponds the invariant ODE family

$$y' = \mathcal{H}(x)\, \mathcal{G}(y) + \frac{\mathcal{F}_x\, \mathcal{G}(y)}{\mathcal{F}(x)} \int^{y} \frac{da}{\mathcal{G}(a)} \tag{7}$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are arbitrary functions of their arguments. The ODE family Eq.(7), for instance, includes as particular cases ODEs of type separable, linear, Bernoulli, some Riccati and Abel subfamilies, etc., but mainly many other ODEs for which there is no classification nor specific solving methods available. As we show below, determining if a given ODE is of the form of Eq.(7), including in that calculating $\mathcal{F}$ and $\mathcal{G}$, is as systematic (though not so straightforward) as in the Bernoulli ODE example above.

We then started considering whether it would be possible to set up a computational scheme for first order ODEs, based on Lie's theory, but organized, as in the simplest cases, just around some matching pattern routines. At this point, two questions arose:

1. The knowledge that the ODE matches the required pattern may or may not allow one to get the symmetries performing only algebraic operations. For example, the general Riccati

ODE

$$y' = f_2(x)\, y^2 + f_1(x)\, y + f_0(x) \tag{8}$$

is the invariant ODE family associated to the symmetry pattern

$$[\xi = 0,\ \eta = \mathcal{F}(x)(y + \mathcal{H}(x))^2] \tag{9}$$

but to determine $\mathcal{F}$ and $\mathcal{H}$ in terms of f_2, f_1 and f_0 for a given Riccati ODE one needs to solve an auxiliary second order linear ODE with variable coefficients, equivalent and as difficult to solve as the original ODE;

2. From a practical point of view, the efficacy of a first implementation of the ideas being presented would depend on a convenient selection of a set of symmetry patterns and related invariant ODE families.

After analyzing the symmetries of Kamke's 576 first order ODE examples and taking into account practical issues (see discussions in sec. 3), our selection for the set of symmetry patterns to consider at first was:

$$\begin{array}{ll}
[\xi = \mathcal{F}(x)\, \mathcal{G}(y),\ \eta = 0], & [\xi = 0,\ \eta = \mathcal{F}(x)\, \mathcal{G}(y)] \\
[\xi = \mathcal{F}(x) + \mathcal{G}(y),\ \eta = 0], & [\xi = 0,\ \eta = \mathcal{F}(x) + \mathcal{G}(y)] \\
[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(x)], & [\xi = \mathcal{F}(y),\ \eta = \mathcal{G}(y)] \\
[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)], & [\xi = \mathcal{F}(y),\ \eta = \mathcal{G}(x)] \\
[\xi = a\, x + b\, y + c,\ \eta = f\, x + g\, y + h] &
\end{array} \tag{10}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions and $\{a, b, c, f, g, h\}$ are arbitrary constants. For the first eight symmetry patterns, it was possible to set up a matching pattern routine, mainly determining whether or not a given ODE matches one of the related invariant ODE families, and if so directly return the symmetry by performing only algebraic operations. For the pattern $[\xi = a\, x + b\, y + c,\ \eta = f\, x + g\, y + h]$, we took advantage of the natural splitting of the determining PDE into an overdetermined linear system of *algebraic* equations. Concerning the patterns $[\xi = \mathcal{F}(x) + \mathcal{G}(y),\ \eta = 0]$ and $[\xi = 0,\ \eta = \mathcal{F}(x) + \mathcal{G}(y)]$, they do not contribute to solving Kamke's examples, but we added them for generality; we noted that the related invariant ODE families have practically no intersection with those of the other patterns.

As an additional remark, for ODEs of types separable, linear and inverse-linear [3], their pattern and corresponding symmetry are easy to determine so that a little subroutine for them was prepared separately, and the computer routines for the patterns of Eq.(10) work assuming that the given ODE is not of one of these types.

[3] We classify here an ODE as *inverse linear* when by changing variables $x \rightarrow y^*(x^*)$, $y(x) \rightarrow x^*$, the resulting ODE in y^* is linear.

3.1 The symmetry patterns $[\xi = \mathcal{F}(x)\,\mathcal{G}(y),\ \eta = 0]$ and $[\xi = 0,\ \eta = \mathcal{F}(x)\,\mathcal{G}(y)]$

Starting with the symmetry pattern $[\xi = \mathcal{F}(x)\,\mathcal{G}(y),\ \eta = 0]$, the purpose is to determine whether a given ODE has a symmetry of this form, and if so, to determine $\mathcal{F}(x)$ and $\mathcal{G}(y)$.

In order to avoid integrals in the pattern for the invariant ODE family, we rewrite the symmetry in terms of new arbitrary functions f and g, without loss of generality, using $\mathcal{F}(x) = f_x^{-1}$ and $\mathcal{G}(y) = \exp(\int g(y)dy)$:

$$[\xi = \frac{\mathrm{e}^{\int g(y)dy}}{f_x}, \eta = 0] \tag{11}$$

The related invariant ODE family is then given by [4]:

$$y' = \Phi(x,y) \equiv \frac{f_x}{g(y)f(x) + \mathcal{J}(y)} \tag{12}$$

where $\mathcal{J}$ is an arbitrary functions of y. Now, any member of the family satisfies:

$$\frac{1}{\Phi^2}\frac{\partial^2\left(\ln(\Phi)\right)}{\partial x \partial y} = \frac{g(y)\mathcal{J}_y - g_y \mathcal{J}(y)}{f_x} \tag{13}$$

that is, the combination of operations in the left-hand-side of the above will lead to an expression in which the variables separate by product [5]. For such an ODE, the factors in Eq.(13) containing x give $\mathcal{F}$, and g can be expressed in terms of Φ and $\mathcal{F}$ by taking the appropriate derivative:

$$\mathcal{F}\frac{\partial}{\partial x}\left(\frac{1}{\mathcal{F}\,\Phi}\right) = g(y) \tag{14}$$

The separability of Eq.(13), together with Eq.(14) being an expression only depending on y, are the *existence conditions* for a symmetry of the type under consideration, or in other words, they *classify* an ODE as member of family Eq.(12). These conditions are necessary, and also sufficient since the only solution Φ to Eq.(14) is the right-hand-side of Eq.(12).

Once the problem of classifying an ODE as member of this family and finding a symmetry for it is solved, the ODE's solution, in terms of $\mathcal{F}$ and $\mathcal{G}$, is given by

$$C_1 - \frac{1}{\mathcal{G}(y)}\int \frac{dx}{\mathcal{F}(x)} + \int^y \frac{\mathcal{J}(a)}{\mathcal{G}(a)}da = 0 \tag{15}$$

[4] These invariant families associated to given symmetries can be obtained in various ways, all of them equivalent to solving Eq.(2) for Φ - the right-hand-side of the ODE; see [3].
[5] When Eq.(13) is zero, the ODE as a whole is already separable.

where $\mathcal{J}$ can be expressed in terms of $\mathcal{F}$, $\mathcal{G}$ and Φ from Eq.(12).

Example:

$$y' = \Phi(x,y) \equiv -\frac{(y+b)^2}{(x+a)\left(1+(y+b)^2(x+a)\sin(y)\right)} \tag{16}$$

For this ODE Eq.(13) gives an expression in which the variables separate by product

$$\frac{1}{\Phi^2}\frac{\partial^2(\ln(\Phi))}{\partial x \partial y} = -\frac{((y+b)\cos(y)+2\sin(y))(x+a)^2}{(y+b)^3} \tag{17}$$

So $\mathcal{F} = (x+a)^2$, and from Eqs. (14) and (11) $\mathcal{G} = \mathrm{e}^{(-1/(y+b))}$, that is, an expression only depending on y. Hence, this ODE has the symmetry $[\xi = \mathcal{F}(x)\,\mathcal{G}(y),\ \eta = 0]$, and from Eq.(15) its solution is given by:

$$C_1 - \int^y \sin(z)\mathrm{e}^{\left(\frac{1}{z+b}\right)}dz + \frac{\mathrm{e}^{\left(\frac{1}{y+b}\right)}}{x+a} = 0 \tag{18}$$

Concerning the symmetry pattern $[\xi = 0,\ \eta = \mathcal{F}(x)\,\mathcal{G}(y)]$, and its related invariant ODE family,

$$y' = \frac{\mathcal{G}(y)}{\mathcal{F}(x)}\left(\mathcal{F}_x \int^y \frac{1}{\mathcal{G}(a)}da + \mathcal{H}(x)\mathcal{F}(x)\right) \tag{19}$$

where $\mathcal{H}$ is an arbitrary function of x, we note that if a given ODE has a symmetry of the form $[\xi(x,y),\,\eta(x,y)]$, then, by changing variables as in $x \to y^*(x^*),\ \ y(x) \to x^*$, the resulting ODE will have a symmetry of the form $[\eta(y^*,x^*),\,\xi(y^*,x^*)]$. Hence, by transforming Eq.(19) using this change of variables, the resulting ODE will have a symmetry of the form $[\xi = \mathcal{G}(x^*)\mathcal{F}(y^*),\ \ \eta = 0]$, which can thus be determined using the scheme just outlined. Changing the variables back, we obtain the symmetries for Eq.(19).

3.2 The symmetry patterns $[\xi = \mathcal{F}(x) + \mathcal{G}(y),\ \eta = 0]$ *and* $[\xi = 0,\ \eta = \mathcal{F}(x) + \mathcal{G}(y)]$

The ODE family invariant under $[\xi = \mathcal{F}(x) + \mathcal{G}(y),\ \eta = 0]$ is given by

$$y' = \Phi(x,y) \equiv \left[\left(\mathcal{G}_y \int \frac{dx}{(\mathcal{G}(y)+\mathcal{F}(x))^2} - \mathcal{J}(y)\right)(\mathcal{G}(y)+\mathcal{F}(x))\right]^{-1} \tag{20}$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{J}$ are arbitrary functions of their arguments. To determine if a given ODE matches the pattern above, we first note that Eq.(20) satisfies [6] :

$$\Phi \frac{\partial^2}{\partial x^2}\left(\Phi^{-1}\right) = \frac{\mathcal{F}_{xx}}{\mathcal{G}(y) + \mathcal{F}(x)} \tag{21}$$

Hence, by differentiating the reciprocal of the right-hand-side of Eq.(21) w.r.t y, we arrive at an expression where the variables separate by product:

$$\frac{\partial}{\partial y}\left[\left(\Phi \frac{\partial^2}{\partial x^2}\left(\Phi^{-1}\right)\right)^{-1}\right] = \frac{\mathcal{G}_y}{\mathcal{F}_{xx}} \tag{22}$$

Thus, for a given ODE, the separability of Eq.(22) is a *necessary condition* for the existence of a symmetry of form $[\xi = \mathcal{F}(x) + \mathcal{G}(y),\ \eta = 0]$. A sufficient condition can be formulated by noting that the determining PDE for the problem,

$$\mathcal{F}_x \Phi + \mathcal{G}_y \Phi^2 + \left(\mathcal{F}(x) + \mathcal{G}(y)\right) \Phi_x = 0 \tag{23}$$

involving the unknowns $\mathcal{F}$ and $\mathcal{G}$, can be rewritten in terms of $\mathcal{F}(x) + \mathcal{G}(y)$

$$\frac{\partial}{\partial x}\left(\left(\mathcal{F}(x) + \mathcal{G}(y)\right)\Phi\right) + \Phi^2 \frac{\partial}{\partial y}\left(\mathcal{F}(x) + \mathcal{G}(y)\right) = 0 \tag{24}$$

Now in Eq.(22), the factors depending on x give the reciprocal of $\mathcal{F}_{xx}$, from where we obtain $\mathcal{F}(x) + \mathcal{G}(y)$ just dividing Eq.(21) by the expression found for $\mathcal{F}_{xx}$. The solution for any member of the family Eq.(20), in terms of $\mathcal{F} + \mathcal{G}$, is then given by

$$C_1 + \int^x \frac{da}{\mathcal{F}(a) + \mathcal{G}(y)} + \int^y \mathcal{J}(a) da = 0 \tag{25}$$

where $\mathcal{J}$ can be expressed in terms of $\mathcal{F} + \mathcal{G}$, $\mathcal{G}_y$ and Φ from Eq.(20).

Concerning the symmetry pattern $[\xi = 0,\ \eta = \mathcal{F}(x) + \mathcal{G}(y)]$, and its related invariant ODE family,

$$y' = \left(\mathcal{F}_x \int \frac{dy}{\left(\mathcal{G}(y) + \mathcal{F}(x)\right)^2} + \mathcal{H}(x)\right)\left(\mathcal{G}(y) + \mathcal{F}(x)\right) \tag{26}$$

[6] In Eq.(21), when the left-hand-side is zero, the ODE is already of type "inverse-linear".

where $\mathcal{H}$ is an arbitrary function, if a given ODE has a symmetry of this form, by changing variables $x \rightarrow y^*(x^*), \quad y(x) \rightarrow x^*$, the resulting ODE will have a symmetry of the form $[\xi = \mathcal{G}(x^*) + \mathcal{F}(y^*), \; \eta = 0]$, which can then be determined using the scheme just outlined.

Example:

$$y' = 3\left(1 + \frac{x^2}{y^2}\right)\arctan\left(\frac{y}{x}\right) + \frac{1 - 2\,y}{x} + \frac{(1 - 3\,y)\,x}{y^2} \tag{27}$$

Changing variables $x \rightarrow y^*(x^*), \quad y(x) \rightarrow x^*$, we obtain

$$y^{*\,\prime} = \Phi(x^*, y^*) \equiv \frac{y^* x^{*2}}{3\,y^*\left(x^{*2} + y^{*2}\right)\arctan\left(\frac{x^*}{y^*}\right) + y^{*2}\,(1 - 3\,x^*) + x^{*2}\,(1 - 2\,x^*)} \tag{28}$$

For this ODE, Eq.(22) gives an expression in which the variables separate by product

$$\frac{\partial}{\partial y^*}\left[\left(\Phi\,\frac{\partial^2}{\partial x^{*2}}\left(\Phi^{-1}\right)\right)^{-1}\right] = -\frac{x^{*4}}{3\,y^{*3}} \tag{29}$$

Hence, from Eqs.(21) and (22),

$$\mathcal{F}(x^*) + \mathcal{G}(y^*) = -\frac{x^{*2} + y^{*2}}{2\,y^{*2}x^{*2}}, \quad \mathcal{G}_{y^*} = \frac{1}{y^{*3}} \tag{30}$$

Using these values, Eq.(24) is identically satisfied, so that Eq.(28) has the symmetry

$$[\xi = \frac{1}{y^{*2}} + \frac{1}{x^{*2}}, \; \eta = 0] \tag{31}$$

and hence Eq.(27) has the symmetry $[\xi = 0, \eta = y^{-2} + x^{-2}]$. Finally, using Eq.(25) and changing variables back, the solution to the original ODE Eq.(27) is given by:

$$C_1 + x^2\left(-2\,y + 2\,x\arctan\left(\frac{y}{x}\right) + 1\right) = 0 \tag{32}$$

3.3 The symmetry patterns $[\xi = \mathcal{F}(x), \; \eta = \mathcal{H}(x)]$ and $[\xi = \mathcal{G}(y), \; \eta = \mathcal{J}(y)]$

The invariant ODE family associated to $[\xi = \mathcal{F}(x), \; \eta = \mathcal{H}(x)]$ is given by:

$$y' = \Phi(x, y) \equiv \frac{1}{\mathcal{F}(x)}\left(\mathcal{H}(x) + K\left(y - \int\frac{\mathcal{H}(x)}{\mathcal{F}(x)}dx\right)\right) \tag{33}$$

where $\mathcal{F}$, $\mathcal{H}$ and K are arbitrary functions of their arguments. To determine $\mathcal{F}(x)$ and $\mathcal{H}(x)$ we first build an expression depending on x and y only through K

$$Q\left(y - \int \frac{\mathcal{H}(x)}{\mathcal{F}(x)} dx\right) \equiv \frac{\Phi_y}{\Phi_{yy}} = \frac{K_y}{K_{yy}} \tag{34}$$

At this point, the cases $Q_y \neq 0$ and $Q_y = 0$ must be considered separately. When $Q_y \neq 0$, we can obtain the ratio $\mathcal{H}(x)/\mathcal{F}(x)$ just by taking

$$\Upsilon \equiv \frac{Q_x}{Q_y} = -\frac{\mathcal{H}(x)}{\mathcal{F}(x)} \tag{35}$$

From the knowledge of Υ we can obtain an explicit expression for $\mathcal{F}$ as follows. In the determining PDE Eq.(2) for the problem,

$$\mathcal{H}_x - \mathcal{F}_x\Phi - \mathcal{F}(x)\Phi_x - \mathcal{H}(x)\Phi_y = 0 \tag{36}$$

we remove $\mathcal{H}$ and introduce Υ using Eq.(35)

$$\left(\Upsilon_x + \Phi_x - \Upsilon(x)\,\Phi_y\right)\mathcal{F}(x) + \left(\Upsilon(x) + \Phi\right)\mathcal{F}_x = 0 \tag{37}$$

from where $\mathcal{F}$ is given by

$$\mathcal{F}(x) = C_1\,\mathrm{e}^{\displaystyle\int \left(\frac{\Upsilon(x)\,\Phi_y - \Upsilon_x - \Phi_x}{\Phi + \Upsilon(x)}\right) dx} \tag{38}$$

Setting the integration constant $C_1 = 1$ without loss of generality and using this result in conjunction with Eq.(35), we obtain $\mathcal{H}(x)$. The *necessary* and *sufficient* conditions for the existence of the symmetry here are summarized as:

$$\frac{\partial}{\partial y}\left(\frac{Q_x}{Q_y}\right) = 0, \qquad \frac{\partial}{\partial y}\left(\frac{\Upsilon(x)\,\Phi_y - \Upsilon_x - \Phi_x}{\Phi + \Upsilon(x)}\right) = 0 \tag{39}$$

As for the other case, when $Q_y = 0$, we note that Q satisfies

$$Q_y = -\frac{\mathcal{F}(x)}{\mathcal{H}(x)} Q_x \tag{40}$$

which implies that $Q_x = 0$ too, so Q is a constant C_1. The right-hand-side of the invariant ODE family Eq.(33) is then restricted to the solution Φ of (see Eq.(34))

$$\frac{\Phi_y}{\Phi_{yy}} = C_1 \tag{41}$$

hence Eq.(33) - the invariant ODE family - becomes

$$y' = \Phi \equiv A(x) + B(x)\, \mathrm{e}^{\frac{y}{C_1}} \tag{42}$$

where A and B are arbitrary functions. For a given ODE of this type, A and B can be determined by inspection, and the determining PDE for the infinitesimals $[\mathcal{F}(x), \mathcal{H}(x)]$ naturally splits up into a system of two equations for $\mathcal{F}$ and $\mathcal{H}$:

$$-C_1\, \mathcal{F}_x - \frac{C_1\, \mathcal{F}(x) B_x}{B} - \mathcal{H}(x) = 0 \tag{43}$$

$$\mathcal{H}_x - \mathcal{F}_x\, A - \mathcal{F}(x) A_x = 0 \tag{44}$$

This system is easily solved for $\mathcal{F}(x)$ as

$$\mathcal{F}(x) = \frac{\mathrm{e}^{-\int \frac{A}{C_1} dx}}{B} \left(C_2 \int B\, \mathrm{e}^{\int \frac{A}{C_1} dx} dx + C_3 \right) \tag{45}$$

where C_2 and C_3 are arbitrary constants. Plugging back this result for $\mathcal{F}$ into Eq.(43) we obtain $\mathcal{H}$. We note here that the family of infinitesimals above parameterized by C_2 and C_3 is wider than what is necessary to integrate Eq.(42). So, without loss of generality, we choose $C_2 = 0$ and $C_3 = 1$, to finally obtain

$$\mathcal{F}(x) = \frac{\mathrm{e}^{-\int \frac{A}{C_1} dx}}{B}, \quad \mathcal{H}(x) = A\, \mathcal{F}(x) \tag{46}$$

Once the problems of classifying an ODE as member of this family and determining a symmetry for it are solved, the ODE solution in terms of $\mathcal{F}$ and $\mathcal{H}$ (A and B) is given by:

Case $Q_y = 0$

$$y = \int A(x) dx - C_1 \ln \left(\left(C_2 - \int \mathrm{e}^{\int \frac{A(x)}{C_1} dx} B(x) dx \right) {C_1}^{-1} \right) \tag{47}$$

Case $Q_y \neq 0$

$$C_1 + \int^y \left(K \left(a - \int \frac{\mathcal{H}(x)}{\mathcal{F}(x)} dx \right) \right)^{-1} da - \int \frac{dx}{\mathcal{F}(x)} = 0 \tag{48}$$

where from Eq.(33) $K = \Phi \mathcal{F}(x) - \mathcal{H}(x)$.

Concerning $[\xi = \mathcal{G}(y),\ \eta = \mathcal{J}(y)]$, and its related invariant ODE family

$$y' = \frac{\mathcal{J}(y)}{\mathcal{G}(y) + \tilde{K} \left(x - \int \frac{\mathcal{G}(y)}{\mathcal{J}(y)} dy \right)}, \tag{49}$$

as with the other symmetry patterns, we can determine if a given ODE belongs to this family and then find a symmetry for it by changing variables $x \to y^*(x^*),\quad y(x) \to x^*$, and using the procedure explained above.

Example: a symmetry of the form $[\xi = \mathcal{F}(x),\, \eta = \mathcal{H}(x)]$ *for Kamke's first order ODE 84*

$$y' = f(ax + by) \tag{50}$$

For this ODE, Eq.(35) leads to a constant expression:

$$\frac{Q_x}{Q_y} = -\frac{\mathcal{H}(x)}{\mathcal{F}(x)} = \frac{a}{b} \tag{51}$$

Since the conditions Eq.(39) are satisfied, Eq.(38) gives the first infinitesimal $\xi = \mathcal{F}(x) = 1$; hence $\eta = \mathcal{H}(x) = -a/b$, and using Eq.(48) the solution to the ODE follows as

$$C_1 + \int^y \frac{b}{a + b\, f(ax + b\, z)} dz - x = 0 \tag{52}$$

Example: an ODE with symmetry of the form $[\xi = \mathcal{G}(y),\, \eta = \mathcal{J}(y)]$

$$y' = \frac{y^2}{\sin(y - x) - x^2 + 2\,xy} \tag{53}$$

Changing variables $x \to y^*(x^*),\quad y(x) \to x^*$, we obtain

$$y^{*\,\prime} = \Phi(x^*, y^*) \equiv \frac{-y^{*2} + 2\,x^* y^* - \sin(y^* - x^*)}{x^{*2}} \tag{54}$$

For this ODE, Eq.(35) gives

$$\frac{Q_{x^*}}{Q_{y^*}} = -\frac{\mathcal{H}(x^*)}{\mathcal{F}(x^*)} = -1 \tag{55}$$

So, from Eq.(38), $\mathcal{F} = x^{*2}$ and $\mathcal{H} = x^{*2}$. Since Eqs.(39) are satisfied, Eq.(54) has the symmetry $[\xi = x^{*2},\ \eta = x^{*2}]$ and hence Eq.(53) has the symmetry $[\xi = y^2,\ \eta = y^2]$. Finally, using Eq.(48) and changing the variables back, the solution to Eq.(53) is given by:

$$\frac{1}{y} + \int^{y-x} \frac{1}{a^2 - \sin(a)} da + C_1 = 0 \tag{56}$$

3.4 The symmetry patterns $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$ *and* $[\xi = \mathcal{G}(y),\ \eta = \mathcal{F}(x)]$

Differently from the cases treated in the previous subsections, by changing variables $x \rightarrow y^*(x^*),\ y(x) \rightarrow x^*$, we have that

$$[\xi = \mathcal{G}(y),\ \eta = \mathcal{F}(x)] \rightarrow [\xi = \mathcal{F}(y^*),\ \eta = \mathcal{G}(x^*)]$$

so, this change of variables will not transform the problem of finding $[\xi = \mathcal{G}(y),\ \eta = \mathcal{F}(x)]$ into that of finding $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$ since ξ remains depending on the *dependent* variable and η on the *independent* one; to handle both problems in general, two different schemes are required. There is however one situation in which a single scheme can determine a symmetry of any of these two types. Since this scheme is simple and suitable for extensions, we concentrate the discussion on it at first and only discuss the general cases in the next subsection.

3.4.1 A scheme for ODEs with functions or non-integer powers involving both x and y

To start with, by rewriting the symmetry $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$ in terms of new functions $f(x)$ and $g(y)$ as $[\xi = 1/f_x,\ \eta = -1/g_y]$, we obtain the invariant ODE family as:

$$y' = \frac{f_x}{g_y} \left(K\left(f(x) + g(y)\right) - 1 \right) \tag{57}$$

where f, g and K are arbitrary functions of their arguments. A departure point for matching a given ODE to the pattern above is based on the following observation: if the ODE *contains functions or non-integer powers involving both* x *and* y, then, from Eq.(57), a necessary condition for the existence of the symmetry is that at least one of these functions - say M - is a function of $f(x) + g(y)$, and then the ratio

$$\frac{M_x}{M_y} = \frac{f_x}{g_y} \tag{58}$$

is an expression where the variables x and y separate by product. This separability is thus a necessary condition for the existence of the symmetry. Now, recalling that $[\xi = 1/f_x,\ \eta = -1/g_y]$, the factors in Eq.(58) containing x give ξ^{-1}, and the factors containing y give $-\eta$. Finally, a sufficient condition for $[\xi,\ \eta]$ to be a symmetry is that it satisfies Eq.(2).

Example: Kamke's first order ODE 85

$$y' - x^{a-1}\ y^{1-b}\ \mathcal{H}\left(\frac{x^a}{a} + \frac{y^b}{b}\right) = 0 \tag{59}$$

This ODE contains the arbitrary function $\mathcal{H}\left(\frac{x^a}{a} + \frac{y^b}{b}\right)$ and the ratio of its partial derivatives separates by product as in Eq.(58):

$$\frac{(\mathcal{H}\left(\frac{x^a}{a} + \frac{y^b}{b}\right))_x}{(\mathcal{H}\left(\frac{x^a}{a} + \frac{y^b}{b}\right))_y} = \frac{y x^a}{y^b x} \tag{60}$$

from where a pair of infinitesimals satisfying Eq.(2) is given by

$$[\xi = \frac{x}{x^a}, \eta = -\frac{y}{y^b}] \tag{61}$$

To understand what is behind this way of orienting the search for the symmetry, we note that the invariant ODE family associated to a given pair of infinitesimals can be written as

$$I_1(\xi, \eta, \eta_1) = K(I_0(\xi, \eta)) \tag{62}$$

where η_1 is the first extension of η, I_0 and I_1 are the differential invariants of orders zero and one associated to the symmetry group, and K is an arbitrary function of I_0. For example, in the case under consideration, $\xi = 1/f_x$, $\eta = -1/g_y$ and

$$I_0 = f(x) + g(y),\ \ I_1 = -\frac{f_x}{g_y\, y' + f_x} \tag{63}$$

Since there are no functions in I_1 simultaneously containing x and y, if a given ODE contains such functions, at least one of them must come from I_0 and thus be a function of $(f + g)$.

Equivalent situations happen with other symmetry patterns too. For instance, when looking for symmetries of the form $[\xi = \mathcal{G}(y),\, \eta = \mathcal{F}(x)]$, rewriting the symmetry pattern as $[\xi = g_y,\, \eta = -f_x]$, where $g(y)$ and $f(x)$ are arbitrary functions, I_0 is given by

$$I_0 = f(x) + g(y) \tag{64}$$

Hence, if a given ODE has this type of symmetry and contains functions involving both x and y, one of them - say M - will satisfy $M_x/M_y = f_x/g_y$ and lead to the symmetry by separating the factors depending on x and on y.

Example:

$$y' = -\tan\left(\arctan\left(\frac{x}{y}\right) + \mathcal{H}\left(x^2 + y^2\right)\right) \tag{65}$$

Here $\mathcal{H}$ depends on both x and y and the ratio of its partial derivatives separates by product

$$\frac{(\mathcal{H}(x^2+y^2))_x}{(\mathcal{H}(x^2+y^2))_y} = \frac{f_x}{g_y} = \frac{x}{y} \tag{66}$$

from where we get a pair of infinitesimals of the form $[\xi = \mathcal{G}(y), \eta = \mathcal{F}(x)]$ satisfying Eq.(2):

$$[\xi = y,\ \eta = -x] \tag{67}$$

Now, the two invariant ODE families, respectively associated to $[\xi = \mathcal{F}(x), \eta = \mathcal{G}(y)]$ and $[\xi = \mathcal{G}(y), \eta = \mathcal{F}(x)]$, can be written in terms of an I_0 of the form $f(x) + g(y)$. Therefore, when there are functions or non-integer powers present in the given ODE, a scheme for finding symmetries of both types can be summarized as follows:

(1) Select, in the given ODE, all functions and non-integer powers containing x *and* y;
(2) Loop over each of these mappings (M) by performing the following operations:
  (a) calculate the ratio $R = M_y/M_x$;
  (b) when R separates by product, select the factors - say X - depending on x;
    - if $[X, -X/R]$ satisfies Eq.(2), return a symmetry of type $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$;
    - if $[-R/X, 1/X]$ satisfies Eq.(2), return a symmetry of type $[\xi = \mathcal{G}(y),\ \eta = \mathcal{F}(x)]$.

More symmetries from the differential invariant of order zero

In the specific case in which I_0 is of the form $A(x) + B(y)$, steps (2.2a) and (2.2b) of the scheme just outlined directly lead to symmetries of the form $[\mathcal{F}(x),\, \mathcal{G}(y)]$ and $[\mathcal{G}(y),\, \mathcal{F}(x)]$. Now,

1) there are other symmetry patterns for which I_0 has the form $A(x) + B(y)$ and the knowledge of R and X (steps (2.2a) and (2.2b)) leads to the appropriate symmetry;

2) there are various ODE families for which I_0 depends on x and y but not in the form of a sum of functions. In these cases R will not separate by product but its determination will lead to symmetries as well.

As an example of 1), consider the symmetry pattern

$$[\xi = \mathcal{F}(x)\,\mathcal{G}(y),\ \ \eta = 1] \tag{68}$$

If we rewrite this symmetry in terms of new functions $f(x)$ and $g(y)$ as $[-g_y/f_x, 1]$, I_0 and the invariant ODE family are given by

$$\begin{aligned} I_0 &= f(x) + g(y), \\ y' &= -\frac{f_x}{K(I_0) + g_y} \end{aligned} \tag{69}$$

where K is an arbitrary function. Here R of step (2.2b) gives

$$R = \frac{(K(I_0))_y}{(K(I_0))_x} = \frac{g_y}{f_x} = -\xi \tag{70}$$

from where the symmetry is $[-R, 1]$.

As an example of 2), consider the symmetry pattern and its equivalent format

$$[\xi = 1,\ \ \eta = \mathcal{F}(x) + y\,\mathcal{H}(x)]\ \equiv\ [\xi = 1,\ \ \eta = \frac{f(x) + y\,h_x}{h(x)}] \tag{71}$$

I_0 and the related invariant ODE family are given by

$$\begin{aligned} I_0 &= \frac{y}{h(x)} - \int \frac{f(x)}{(h(x))^2} dx \\ y' &= h(x)\,K(I_0) + \frac{f(x) + y\,h_x}{h(x)} \end{aligned} \tag{72}$$

where K is an arbitrary function. Here, step (2.2b) of the scheme summarized in the previous subsection gives a ratio which is not separable by product

$$R = \frac{(K(I_0))_y}{(K(I_0))_x} = -\frac{h(x)}{f(x) + y\,h_x} = -\frac{1}{\eta} \tag{73}$$

but its determination leads to a symmetry of the form $[1, -1/R]$. Moreover, in this case the factors - say X - depending on x give the ratio itself, so that this symmetry is also of the form $[-R/X, 1/X]$. Hence, by removing the test for separability in step (2.2b), the scheme will automatically find symmetries for the invariant family Eq.(72) too.

Example: Kamke's first order ODE 433

$$(xy' + y + 2\,x)^2 - 4\,xy - 4\,x^2 - 4\,a = 0 \tag{74}$$

After isolating y', the routine detects the mapping $(x, y) \rightarrow \sqrt{xy + x^2 + a}$ involving both x and y, whose partial derivatives give the ratio R (not separable by product)

$$R = \frac{(\sqrt{xy + x^2 + a})_y}{(\sqrt{xy + x^2 + a})_x} = \frac{x}{y + 2x} \tag{75}$$

so that, after checking the cancellation of Eq.(2), according to step (2.2b) the scheme returns

$$[\xi = 1,\ \eta = -\frac{x}{2\,x + y}] \tag{76}$$

3.4.2 The symmetry pattern $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$ *(the general case)*

We now turn to the formal problem of determining symmetries of the form $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$, that is, symmetries for ODEs belonging to the invariant family Eq.(57) in the general case. One of the first things we noticed is that our trials to exploit the knowledge of the form of the invariant ODE were unsuccessful. We then used the *diffalg* [8] and *standard form* [9] packages - two Maple packages for reducing PDE systems to a *canonical form* by basically adding integrability conditions and removing redundancies. After processing the problem using these packages, it was possible to solve the resulting system of PDEs in closed form, but the resulting intermediate and final expressions were huge in size. So, we worked on formulating the problem and a computational sequence of calculations so as to have almost reasonable-sized expressions in all steps. To start with, we considered a general first order ODE written as

$$y' = \mathrm{e}^{\phi(x,y)} \tag{77}$$

where $\phi(x, y)$ is an arbitrary function. The system of PDEs representing the problem of finding the infinitesimals consists of Eq.(2), together with two equations indicating the restrictions to the functional dependence of the infinitesimals ξ and η,

$$\eta_y - \xi_x - \xi\,\phi_x - \eta\,\phi_y = 0 \qquad \xi_y = 0 \qquad \eta_x = 0 \tag{78}$$

We also set up a set of inequations entering our problem, given by

$$\begin{array}{lll} A(x,y) \equiv \phi_{x,y} \neq 0, & B(x,y) \equiv \phi_{y,y} + {\phi_y}^2 \neq 0, & C(x,y) \equiv \phi_{x,x} - {\phi_x}^2, \neq 0, \\ \xi_x \neq 0, & \eta_y \neq 0, & \end{array} \tag{79}$$

respectively meaning that we assume Eq.(57) is not separable, nor linear, nor inverse-linear, and that we are only interested in the cases in which ξ effectively depends on x and η on y. The calculations are rather long and the intermediate steps present no particular interest. The results we obtained and the computational sequence of calculations we worked on can be summarized as follows.

Case I

This case happens when

$$D \equiv [2A_{xy} + \phi_x A_y - A_x \phi_y + (\phi_x \phi_y + 2A)A]A - 3A_x A_y = 0 \tag{80}$$

Here, the necessary and sufficient conditions for the existence of a symmetry of the form $[\xi = \mathcal{F}(x),\ \eta = \mathcal{G}(y)]$ are

$$\begin{aligned} &E_1 \equiv 3A_x^2 + [(\phi_x^2 + 2C)A - 2A_{xx}]A \neq 0 \\ &E_2 \equiv [2A_{yy} + (2B - \phi_y^2)A]A - 3A_y^2 = 0 \\ &E_3 \equiv [(28A_x + 4\phi_x A)A^3 - (\phi_y A + A_y)E_1]E_1 - 8A^4 E_{1,x} = 0 \end{aligned} \tag{81}$$

When these conditions are satisfied, the symmetry is given by

$$\begin{aligned} &\eta = \exp\left(\int \frac{4(A_x - \phi_x A)A^3 + (\phi_y A - A_y)E_1}{2AE_1} dy\right) \\ &\xi = -\frac{4A^3\eta}{E_1} \end{aligned} \tag{82}$$

Case II

This case happens when D, defined in Eq.(80), is different from zero, and

$$
\begin{aligned}
&(6A_xA_{yy}D + [6A_xDB - 3\phi_y^2A_xD - 2A_{yy}D_x + (\phi_y^2D_x - 2D_xB)A]A)A \\
&+[(3AD_x - 9A_xD)A_y - 3D^2]A_y + AD_yD = 0
\end{aligned}
\tag{83}
$$

Here, the necessary and sufficient conditions for the existence of a symmetry are

$$
\begin{aligned}
&E_4 \equiv [2A_{yy} + (2B - \phi_y^2)A]A - 3A_y^2 \neq 0 \\
&E_5 \equiv 4A^3D - D^2 + ([2A_{xx} - (\phi_x^2 + 2C)A]A - 3A_x^2)E_4 = 0 \\
&E_6 \equiv -AE_{4,y}D + [(E_{4,x} - \phi_yD)A + 3A_yD + (A\phi_x - 3A_x)E_4]E_4 = 0
\end{aligned}
\tag{84}
$$

When these conditions are satisfied, the symmetry is given by

$$
\begin{aligned}
&\eta = \exp\left(\int \frac{(A\phi_x - A_x)E_4 - (A_y + A\phi_y)D}{2AD} dy\right) \\
&\xi = -\frac{E_4\eta}{D}
\end{aligned}
\tag{85}
$$

Once the classifying rule for members of the invariant family Eq.(57) and for their symmetries are obtained, the corresponding ODE's solution in terms of ξ and η is given by Eq.(3).

3.5 The symmetry pattern $[\xi = a\,x + b\,y + c,\ \eta = f\,x + g\,y + h]$

We finally consider the symmetry pattern $[\xi = a\,x + b\,y + c,\ \eta = f\,x + g\,y + h]$. First of all, to obtain the related invariant ODE family $y' = \Phi(x, y)$, we need to solve for Φ the related determining PDE

$$
f + (g - a)\,\Phi - b\,\Phi^2 - (ax + by + c)\,\Phi_x - (fx + gy + h)\,\Phi_y = 0 \tag{86}
$$

The characteristic strip associated to this PDE is given by

$$
\frac{dy}{-fx - gy - h} = \frac{dx}{-ax - by - c} = \frac{d\Phi}{-f - \Phi\,g + \Phi\,a + b\Phi^2} \tag{87}
$$

The system above, which should be tackled by first solving $dy/dx = F(x, y)$, to obtain y as a function of x, and then solving $d\Phi/dx = G(x, y(x), \Phi)$, does not yield a solution since we were not able to obtain an *explicit* solution $y(x)$ for the first ODE. As an alternative, it is possible to solve Eq.(86) for the unknown *constants* (a, b, c, f, g and h), arriving at a set of existence conditions for a solution plus expressions for a, b, c, f, g and h in terms of Φ. We note however that this formal solving of the problem involves fifth order derivatives of Φ and expressions of huge size, turning the approach very inefficient, if not just impracticable. We then followed the approach used in [6], which mainly consists of considering the natural splitting of Eq.(86) into a linear system of algebraic equations[7] for a, b, c, f, g and h.

Example: Kamke's first order ODE number 189

$$x^{n(m+1)-m}\, y' - a y^n - b\, x^{n(m+1)} = 0 \tag{88}$$

For this ODE, Eq.(86) splits into an overdetermined algebraic system of twelve equations

$$\begin{array}{lll} 0 = anf, & 0 = cbm, & 0 = anh, \\ 0 = b^2 m, & 0 = ba^2, & 0 = 2\, b^2 a, \\ 0 = b^3, & 0 = b(g - am - a), & 0 = a(cmn - cm + cn), \\ 0 = f, & 0 = ab(n - m + mn), & 0 = a(g - a - ng - am + amn + an) \end{array} \tag{89}$$

which has as solution

$$b = 0, \quad c = 0, \quad f = 0, \quad h = 0, \quad a = a, \quad g = am + a \tag{90}$$

from where a symmetry for Eq.(88) is given by

$$[\xi = x, \ \ \eta = y\,(m+1)] \tag{91}$$

4 Tests and performance

The set of schemes here presented was implemented in Maple R5, in the framework of the ODEtools package [6]. The implementation consists of two separate routines for each of the symmetry patterns discussed in sec. 3, respectively accomplishing the following tasks:

(1) determine whether a given ODE has such a symmetry, and if so calculate it;
(2) use the symmetry to integrate the given ODE, taking advantage of the form of the invariant ODE family.

[7] This splitting of Eq.(86) is obtained by taking the coefficients of x and y.

We have tested this Maple implementation extensively, to confirm the correctness of the returned results and to produce statistics concerning the classification of Kamke's first order ODE examples according to their symmetries. Such statistics can be used as a guideline for adding more symmetry patterns to the solving scheme, as well as for extending Kamke's test suite with ODEs belonging to more varied invariant ODE families.

For the purpose of the test, instead of working with the whole set of Kamke's 576 examples, we previously discarded ODEs of type separable, linear, inverse-linear and Bernoulli: their recognition and solution are straightforward and they are not the main target of this work. Also, all the examples which are *unsolvable* in the sense that the class is too general and Kamke provides only a discussion, e.g. the general Abel ODE number 50, were excluded as well [8]. To perform the tests, we used the six files containing the Maple input for Kamke's examples available in the web at http://dft.if.uerj.br/odetools.html (see [6]).

4.1 Classification of Kamke's first order ODE examples

The classification we obtained for Kamke's 576 first order ODEs is shown below divided into three tables. In each table, we followed the division by Kamke into ODEs of: first, second, third, and higher degree on y'; respectively organized as the examples: 1 to 367, 368 to 517, 518 to 544 and 545 to 576. The first table contains the total number of ODEs of easy and of unsolvable types aforementioned and excluded from the test:

	Degree in y'				
Class	1	2	3	higher	Totals
separable	44	32	11	8	95
linear	29	1	0	0	30
inverse linear	7	0	0	0	7
Bernoulli	29	0	0	1	30
unsolvable	12	3	0	1	16
Total of excluded ODEs	121	36	11	10	178

Table 1. Kamke's ODEs excluded from the tests: 178 of 576.

In the above, all ODEs missing x or y are classified as separable. Also, the high degree ODEs 442 and 557, respectively classified above as "linear" and "Bernoulli", appear as such after isolating y'. The second table shows the standard classification for the remaining 398 Kamke examples which conformed our test suite. There are ODEs matching more than one

[8] The ODEs we classified as *unsolvable* are those numbered in Kamke's book as 50, 55, 56, 74, 79, 82, 111, 202, 219, 250, 269, 331, 370, 461, 503 and 576.

classification (we called *rational* ODEs all those for which the right-hand-side can be written as a rational function of x and y, no matter what other type they are):

Class	Degree in y' 1	2	3	higher	Totals
Abel	61	0	0	0	61
Clairaut	0	10	3	3	16
Riccati	61	0	0	0	61
dAlembert	36	48	4	7	95
exact	28	0	0	0	28
homogeneous	99	53	3	4	159
rational	139	56	0	2	197
None	27	11	0	6	44
Total of ODEs	246	114	16	22	398

Table 2. Classification of Kamke's 398 ODEs used in the test.

The third table is concerned with these 398 ODEs and shows the total number of them having linear symmetries or matching any of the invariant ODE patterns discussed in sec. 3:

Class	Degree in y' 1	2	3	higher	Totals
$[\mathcal{F}(x)\,\mathcal{G}(y), 0]$ or $[0, \mathcal{F}(x)\,\mathcal{G}(y)]$	25	0	0	0	25
$[\mathcal{F}(x), \mathcal{H}(x)]$ or $[\mathcal{G}(y), \mathcal{J}(y)]$	30	16	1	3	50
$[\mathcal{F}(x), \mathcal{G}(y)]$ or $[\mathcal{G}(y), \mathcal{F}(x)]$	54	48	3	9	114
linear symmetries	102	78	12	11	203
Total of ODEs	132	88	13	14	247

Table 3. Classification of the 398 ODEs according to the schemes of sec. 3

From the results above, it can be seen that the set of schemes here presented finds symmetries for 247 ODEs, thus providing solutions for 62% of these 398 Kamke examples. These 247 solved examples include 30 ODEs of Abel type, 18 of Riccati type and 9 of *"no"* type. If we add the ODEs of easy type of Table 1., we arrive at a symmetry-based scheme for classifying, finding symmetries and returning answers for 73% of the whole set of Kamke's examples.

We note that more than one half of these 398 Kamke examples have linear symmetries, and most of them are of the form $[\xi = \alpha\, x,\ \eta = y]$ (α is a non-zero constant). In Table 3., these examples appear also classified as having symmetries of the form $[\mathcal{F}(x),\ \mathcal{G}(y)]$.

Concerning the symmetry patterns $[\mathcal{F}(x) + \mathcal{G}(y), 0]$ and $[0, \mathcal{F}(x) + \mathcal{G}(y)]$, it is curious that there are no Kamke examples having these types of symmetries. Also, we detected only one example (Kamke's ODE 488) with a non-linear symmetry of the form $[\mathcal{G}(y),\ \mathcal{J}(y)]$. A similar situation happens with the symmetry pattern $[\mathcal{G}(y),\ \mathcal{F}(x)]$: in the 560 solvable ODEs of Kamke's suite, we detected only ten (numbers: 86, 339, 365,446, 451, 511, 512, 515, 555 and 561) having symmetries of this type, and all these symmetries are also linear of the form $[\xi = \alpha\, y,\ \eta = x]$. In the same line, disregarding ODEs already having linear symmetries and those of Table 1., in Kamke's suite we detected only seven ODEs (numbers: 16, 33, 201, 212, 366, 394 and 574) having symmetries involving arbitrary functions, 37 ODEs (9% of 398) with symmetries involving radicals and 61 ODEs (15% of 398) - most of them of Riccati type - with symmetries involving exponentials, logarithms, trigonometric or special functions.

All this makes us think that, as a testing arena for computer algebra implementations of ODE-solvers, Kamke's set of examples is good but in some sense incomplete. Actually, if on one hand we have a test suite with symmetries associated to invariance under scaling, rotations, etc. which are usual in nature, on the other hand it would be interesting to have more examples with polynomial non-linear symmetries, symmetries involving functions (standard and arbitrary) and more varied symmetry patterns in general. The origin of these limitations may perhaps be found in that Kamke's examples are from a time when computers were not available. Then, disregarding the ODEs of Table 1., apart from some examples involving tricky changes of variables, most of the remaining 398 examples are not so difficult to solve for nowadays computer algebra ODE-solvers.

4.3 Test of the ODE-solver of ODEtools with the 1st order Kamke examples

Although the main purpose of this paper is to present a computational scheme for determining symmetries for some first order ODE families, it is interesting to see how **odsolve** - the ODE-solver of ODEtools - perform when the new routines are merged with the old ones. The resulting solving strategy is basically as follows [9]

(1) If the ODE is of type separable, linear, inverse-linear or Bernoulli, return an answer in terms of integrals;
(2) if the ODE is in exact form, or belongs to a solvable subfamily of type Abel or Riccati then return an answer in terms of integrals;

[9] For ODEs of high degree this strategy is slightly different.

(3) if the ODE has linear symmetries or matches any of the invariant ODE family patterns discussed in sec. 3, then return an answer built using these symmetries;
(4) otherwise, try looking for a pair of infinitesimals as a particular solution for the determining PDE Eq.(2) as explained in [6].

The new routines enter in step (3), and the performance of the strategy above with all of Kamke's 560 solvable examples jumps from the aforementioned 73 %, using only the routines presented in this paper, to 93 % of examples solved. This performance is summarized as follows

Degree in y'	ODEs	Solved	Average time	
			solved	*fail*
1	355	323	2.2 sec.	9.9 sec.
2	147	140	8.8 sec.	61.1 sec.
3	27	26	7.2 sec.	17 sec.
higher	31	30	13.4 sec.	25.2 sec.
Total:	560	519	≈ 5 sec.	≈ 20 sec.

Table 4. *Kamke's first order ODEs, solved by* **odsolve***:* 93%

It is interesting to mention that, at least with Kamke's examples, the symmetry approach is complementary with the standard methods for classifiable ODEs, thus resulting in such a high performance. Actually, the total number of ODEs solved using symmetry methods but before inserting the new routines in the ODE-solver was also high: 90%. The number and classification of Kamke's 1^{st} order ODEs still not solved by **odsolve** is now:

Class	Kamke's numbering
rational	452, 480, 485
Riccati	22, 25
Abel	37, 42, 43, 45, 47, 48, 49, 145, 146, 147, 151, 169, 185, 205, 206, 234, 237, 253, 257, 265
NONE	80, 81, 83, 87, 121, 128, 340, 350, 367, 395, 460, 506, 510, 543, 572

Table 5. *Kamke's 1^{st} order solvable ODEs for which* **odsolve** *fails:* 7%

One of the relevant differences between the new and the old symmetry schemes of **odsolve** is that the new schemes are systematic. Another difference is in the timings. For ODEs of the types discussed in sec. 3, the new routines are working more than 10 times faster (average over Kamke's examples) than the Maple R5 routines for finding symmetries as particular solutions to Eq.(2). This fact resulted in the replacement, in the Maple version under development, of the corresponding R5 routines by the new ones presented in sec. 3.

5 Conclusions

This paper presented a set of schemes for finding symmetries for various first order ODE families, as well as its computer algebra implementation in Maple R5, in the framework of the ODEtools package. This implementation proved to be a valuable tool for tackling first order ODEs, as shown in sec. 4, resulting in a concrete step towards *classifying* and correspondingly *solving* first order ODEs according to their symmetries. Moreover, this symmetry scheme enlarges the ODE solving power of computer algebra systems. Actually, none of the examples presented in this paper - half of them from Kamke's book - can be solved by any of Macsyma 2.3, Mathematica 3.0, MuPAD 1.4, or the Reduce package CONVODE.

One of the interesting aspects of classifying ODEs by their symmetries is that the classification has a concrete geometrical meaning. Also relevant, in the symmetry approach presented, solutions to rather general ODE families are obtained just using matching-pattern schemes, as in the easier linear, separable, etc. cases. It is also not difficult to imagine the addition of new routines related to other symmetry patterns, in order to extend further the ODE-solving power. All this leads us to believe that symmetry approaches based on matching pattern routines will play a relevant role in future computer algebra ODE-solvers.

It is curious that, apparently, none of the ODE-solvers of Mathematica, Macsyma, Reduce or MuPAD are tackling first order ODEs by systematically looking for their symmetries. Perhaps that is due to the usual belief that symmetry methods are of no help in solving first order ODEs. In contrast, we recall that the symmetry classification scheme here presented leads to the solution to 73% of the first order Kamke's examples, and when combined with standard classification schemes this performance jumps to 93%. Also, the fact that the symmetries are found using matching pattern routines resulted in a very fast ODE-solver, taking on the average $\approx$ 5 sec. in each solved example, and $\approx$ 20 sec. in each unsolved one. We are actually not aware of such a high performance by any ODE-solver not using symmetry methods with first order ODEs. Concerning Maple R5, the situation is a bit different: this system is already looking for symmetries for first order ODEs as particular solutions to the determining PDE Eq.(2). We note however that the routines here presented work remarkably faster, and the search for the symmetries is systematic. These features motivated the replacement, in the Maple version under development, of corresponding routines for first order ODEs by the ones presented in this work.

We also note that the ideas here discussed cannot be used to solve all types of ODEs one can imagine. For example, they cannot be used to systematically find symmetries for Riccati or Abel ODEs but for some subfamilies, so that a combination of symmetry schemes with standard schemes seems to be the way to achieve the best performance. Concerning what would be the best combination of methods, a testing arena should be chosen first and the result will depend on this choice. A collection of testing examples related to real problems, as found in Kamke's book, appears to be an appropriate choice. On the other hand, as commented in sec. 4, this book appears to us an incomplete testing arena, and to have the required testing suite, extensions of the examples of Kamke's book are necessary.

Finally, some of the ideas presented here for first order ODEs can be considered in the framework of high order ODEs and first order PDEs as well. We are presently working in some prototypes in these directions [10] and expect to succeed in obtaining reportable results in the near future.

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