J. Symbolic Computation (1998) **11**, 1–??

Integrating Factors and ODE Patterns

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(Received 25 November 1997)

A systematic algorithm for building integrating factors of the form $\mu(x, y')$ or $\mu(y, y')$ for non-linear second order ODEs is presented. When such an integrating factor exists, the scheme returns the integrating factor itself, without solving any differential equations. The scheme was implemented in Maple, in the framework of the *ODEtools* package. A comparison between this implementation and other computer algebra ODE-solvers in solving non-linear examples from Kamke's book is shown.

1. Introduction

From a practical point of view, when developing solving methods for ODEs, what we actually do is to attempt to determine families of ODEs which can be transformed into algebraic problems or simple ODEs such as[†] $y' = F(x)$ or $y' = F(y)$ by changes of variables or equivalent processes. For high order ODEs, one hopes that such a simplification of the problem will be possible after successive reductions of order. Some more powerful schemes are also able to exploit other information, as for instance the ODE's symmetries, and so to try a multiple reduction of order at once (see for instance MacCallum's review and Stephani's book).

In the specific case of high order ODEs, the methods based on determining integrating factors play a relevant role in the general solving scheme (Anco and Bluman 1997), leading to ODEs of reduced order, which solvability is usually viewed as a separate problem. Now, although in principle it can always be determined whether a given ODE is exact (a total derivative), there is no known universal scheme for making ODEs exact. Actually, for n^{th}order ODEs, integrating factors arise as solutions of an n^{th}order linear PDE in $n+1$ variables, and to solve this PDE is a major problem in itself.

Bearing this in mind, this paper presents a method for obtaining integrating factors of the form $\mu(x, y')$ and $\mu(y, y')$ for non-linear second order ODEs, using a different approach, based only on a computerized analysis of the pattern of the given ODE. That

[†] Throughout this article, we use the notation $y = y(x), y' = \frac{dy}{dx}, y^{(n)} = \frac{d^n y}{dx^n}$.

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is, for a given ODE, if an integrating factor with such a functional dependence exists, the scheme returns the integrating factor itself, without solving any differential equations.

The exposition is organized as follows. In sec. 2, the use of integrating factors for solving ODEs is briefly reviewed. In sec. 3, the scheme for obtaining the aforementioned integrating factors $\mu(x, y')$ or $\mu(y, y')$ is presented and some examples are given. Sec. 4 contains some statistics concerning the new solving method and the second order non-linear ODEs found in Kamke's book, as well as a comparison of performances of computer algebra packages in solving a related subset of these ODEs. In sec. 5 the computer algebra implementation of the scheme in the framework of the ODEtools package (Cheb-Terrab et al. 1997) is outlined and a description for the package's new command, **redode**, is presented. Finally, the conclusions contain some general remarks about this work and its possible extensions.

Aside from this, in the Appendix, a table containing extra information concerning integrating factors for some of Kamke's ODEs is presented.

2. Integrating factors and reductions of order

2.1. First order ODEs

The idea of looking for an integrating factor (μ) is usually presented in the framework of solving a given first order ODE, say,

$$y' = \Phi(x, y) \tag{2.1}$$

If by multiplying Eq.(2.1) by a factor $\mu(x, y)$, the ODE becomes a total derivative†,

$$\mu(x, y)\,(y' - \Phi(x, y)) = \frac{d}{dx}\,R(x, y) \tag{2.2}$$

for some function R, then one can look for μ as a solution to the first order PDE:

$$\frac{\partial \mu}{\partial x} + \frac{\partial}{\partial y}\left(\mu\,\Phi\right) = 0 \tag{2.3}$$

which arises as the exactness condition for the problem (see Eq.(2.7)). Once μ has been obtained, $R(x, y)$ - an implicit form solution - can be calculated as a line integral.

Although to solve Eq.(2.3) for μ is as difficult as the original problem, it turns out that for a given $N(x, y)$, when a solution of the form $\mu(x, y) = \widetilde{\mu}(q)\,N(x, y)$ exists - q is either x or y - μ can be determined by solving an auxiliary linear first order ODE. For example, introducing $\mu(x, y) = \widetilde{\mu}(x)\,N(x, y)$ and $M(x, y) = N(x, y)\,\Phi(x, y)$, one obtains:

$$\widetilde{\mu}(x) = C_1\, e^{-\int \frac{1}{N}\left(\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x}\right) dx} \tag{2.4}$$

and a solution $\widetilde{\mu}(x)$ exists only when the integrand in above does not depend on y. This

† In this paper we use the term "integrating factor" in connection with the explicit form of the ODE, i.e., the ODE, turned exact by taking the product $\mu\, ODE$, is assumed to be of the form $y'' = \Phi(x, y, y')$ or $y'' - \Phi(x, y, y') = 0$.

gives both an existence condition and an explicit solution to the problem; however, the advantages of the scheme are only apparent since there is no way to determine in advance what would be the appropriate $N(x, y)$.

2.2. High order ODEs

Integrating factors for high order ODEs are defined as in the first order case. Here, we consider $\mu(x, y, y', ..., y^{(n-1)})$ to be an integrating factor for an n^{th}order ODE, say

$$y^{(n)} = \Phi(x, y, y', ..., y^{(n-1)}) \tag{2.5}$$

if after multiplying the explicit ODE by μ we obtain a total derivative:

$$\mu\left(y^{(n)} - \Phi\right) = \frac{dR}{dx} \tag{2.6}$$

for some function $R(x, y, y', ..., y^{(n-1)})$. To determine μ, one can try to solve for it in the exactness condition (see Murphy's book, p. 221):

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial H}{\partial y''}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial H}{\partial y^{(n)}}\right) = 0 \tag{2.7}$$

where $H \equiv \mu\left(y^{(n)} - \Phi\right)$. Now, Eq.(2.7) is always of the form

$$A(x, y, y', ..., y^{(2n-3)}) + y^{(2n-2)} B(x, y, y', ..., y^{(n-1)}) \tag{2.8}$$

where A is of degree n-1 in $y^{(n)}$ and linear in $y^{(k)}$ for $n < k \leq (2n-3)$, so that Eq.(2.8) can be split into an overdetermined system of PDEs for μ. For example, for second order ODEs Eq.(2.7) is of the form

$$A(x, y, y') + y''\ B(x, y, y') \tag{2.9}$$

Hence, by taking $A(x, y, y') = 0$ and $B(x, y, y') = 0$ we have a system of two PDEs for μ:

$$\begin{aligned} A(x, y, y') \equiv & \\ & \left(\frac{\partial^2 \mu}{\partial y' \partial x} + \left(\frac{\partial^2 \mu}{\partial y' \partial y}\right) y' - \frac{\partial \mu}{\partial y}\right)\Phi + \left(\frac{\partial^2 \Phi}{\partial y' \partial x} - \frac{\partial \Phi}{\partial y} + \left(\frac{\partial^2 \Phi}{\partial y' \partial y}\right) y'\right)\mu \\ & + \left(\frac{\partial^2 \mu}{\partial y^2}\right) y'^2 + \left(\left(\frac{\partial \mu}{\partial y'}\right)\frac{\partial \Phi}{\partial y} + \left(\frac{\partial \mu}{\partial y}\right)\frac{\partial \Phi}{\partial y'} + 2\left(\frac{\partial^2 \mu}{\partial x \partial y}\right)\right) y' \\ & + \left(\frac{\partial \mu}{\partial y'}\right)\frac{\partial \Phi}{\partial x} + \left(\frac{\partial \mu}{\partial x}\right)\frac{\partial \Phi}{\partial y'} + \frac{\partial^2 \mu}{\partial x^2} = 0 \end{aligned} \tag{2.10}$$

$$\begin{aligned} B(x, y, y') \equiv & \\ & 2\frac{\partial \mu}{\partial y} + \left(\frac{\partial^2 \mu}{\partial y'^2}\right)\Phi + 2\left(\frac{\partial \mu}{\partial y'}\right)\frac{\partial \Phi}{\partial y'} + \mu\frac{\partial^2 \Phi}{\partial y'^2} + \frac{\partial^2 \mu}{\partial y' \partial x} + \left(\frac{\partial^2 \mu}{\partial y' \partial y}\right) y' = 0 \end{aligned} \tag{2.11}$$

Nonetheless, there are no general rules which might help in solving these PDEs†.

In the specific case of *linear* high order ODEs, one can also look for an integrating factor by trying to solve the adjoint ODE (see for instance Murphy's book), which however is of equal order and may be no easier to solve.

Alternatively, a possible strategy for directly obtaining R instead of looking for μ can be formulated as follows. Consider the first order linear operator associated to Eq.(2.5)

$$\mathrm{A} : f \rightarrow \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + ... + \Phi \frac{\partial f}{\partial y^{(n-1)}} \tag{2.12}$$

where x, y and its derivatives are all treated as independent variables on the same footing. Now

$$A(R) = 0 \tag{2.13}$$

and there are n functionally independent solutions (first integrals) to the problem. In some cases, a first integral R such that $dR/dy^{(n-1)} \neq 0$ can be obtained as the solution to a subset of the characteristic strip of $A(R) = 0$ or by other means.

3. Integrating Factors and ODE patterns

Since the classical way for determining integrating factors leads to problems similar in difficulty to solving the ODEs themselves, we consider here a different approach, based on careful matching of an ODE pattern.

The starting point is the observation that it is trivial to solve the inverse problem; i.e., to find the most general ODE having a given μ. In fact, from Eq.(2.6), we have

$$\mu(x, y, y', ..., y^{(n-1)}) = \frac{\partial R}{\partial y^{(n-1)}} \tag{3.1}$$

and hence the reduced ODE R is of the form

$$R = G(x, y, ..., y^{(n-2)}) + \int \mu \, dy^{(n-1)} \tag{3.2}$$

for some function G. Inserting Eq.(3.2) into Eq.(2.13) and solving for $y^{(n)}$ leads to the general form of an ODE having μ as integrating factor:

$$y^{(n)} = \frac{-1}{\mu} \left[\frac{\partial}{\partial x} \left(\int \mu \, dy^{(n-1)} + G \right) + ... + y^{(n-1)} \frac{\partial}{\partial y^{(n-2)}} \left(\int \mu \, dy^{(n-1)} + G \right) \right] \tag{3.3}$$

The expression above then becomes an *ODE pattern* which, for some families of integrating factors, one can try to match against an input ODE. The generation of such *matching pattern routines* is difficult, even for restricted subfamilies of integrating factor, but once built, they are a powerful and computationally efficient way to reduce the order of the corresponding ODEs (see sec. 4).

† In a recent work by Anco and Bluman (1997), the authors arrive at Eq.(2.9) and Eq.(2.11) - numbered there as (3.5) and (3.8) - departing from the adjoint linearised system corresponding to a given ODE; the possible splitting of Eq.(2.8) into an overdetermined system for μ is also mentioned. However, in formula (3.5) of that work, y'' of Eq.(2.9) above appears replaced by $\Phi(x, y, y')$, and the authors discuss possible alternatives to tackle Eqs.(2.9) and (2.11) instead of Eqs.(2.10) and (2.11).

3.1. Second order ODEs and the integrating factor family $\mu(x, y')$

In the case of second order ODEs, if instead of considering the general case $\mu(x, y, y')$ we restrict the family of integrating factors under consideration to $\mu(x, y')$, Eq.(3.2) - the reduced ODE - becomes

$$R(x, y, y') = F(x, y') + G(x, y) \tag{3.4}$$

for some functions G and F, where

$$\mu(x, y') = F_{y'}(x, y') \tag{3.5}$$

(we denote $F_{y'} = \frac{\partial F}{\partial y'}$). Eq.(3.3) can then be written in terms of F and G as

$$y'' = -\frac{F_x(x, y') + G_x(x, y) + G_y(x, y)\, y'}{F_{y'}(x, y')} \tag{3.6}$$

The idea is now to build a routine to determine if a given ODE can be written in the form Eq.(3.6), in which case it will have an integrating factor of the form $\mu(x, y')$, and if so, determine also F and G, directly leading to the reduced ODE Eq.(3.4). The feasibility of such a computational routine is based on the following theorem.

Theorem 3.1. *Given a nonlinear second order ODE*

$$y'' = \Phi(x, y, y') \tag{3.7}$$

($\frac{\partial \Phi}{\partial y} \neq 0$) † *having an integrating factor of the form* $\mu(x, y')$*, then a pair of functions* $F(x, y')$ *and* $G(x, y)$ *satisfying*

$$\Phi(x, y, y') = -\frac{F_x(x, y') + G_x(x, y) + G_y(x, y)\, y'}{F_{y'}(x, y')} \tag{3.8}$$

and leading to the reduced ODE Eq.(3.4) can be systematically determined.

Proof. We then divide the proof in two steps. In the first step we *assume* that, given Eq.(3.7), it is always possible to determine $\mu(x, y')$ up to a factor depending on x; that is, to find some $\mathcal{F}(x, y')$ satisfying

$$\mathcal{F}(x, y') = \frac{\mu(x, y')}{\widetilde{\mu}(x)} \tag{3.9}$$

for some unknown function $\widetilde{\mu}(x)$. We then prove that the knowledge of $\mathcal{F}(x, y')$ is enough to determine $\widetilde{\mu}(x)$, leading to the desired integrating factor $\mu(x, y')$.

In a second step, we prove our assumption, that is, we show how to find $\mathcal{F}(x, y')$ satisfying Eq.(3.9), concluding the proof of the theorem.

† ODEs *missing* y may also have integrating factors of the form $\mu(x, y')$, which cannot be determined using the scheme here presented. However, such integrating factors are not relevant here since these ODEs can always be reduced to first order by a simple change of variables.

3.1.1. Determination of $\widetilde{\mu}(x)$ when $\mathcal{F}(x,y')$ is known

Starting with the first aforementioned step, we assume that we can determine $\mathcal{F}(x,y')$. It follows from Eqs.(3.5), (3.8) and (3.9) that

$$\frac{\partial}{\partial y}\left(\Phi(x,y,y')\ \mathcal{F}(x,y')\right) = \frac{G_{y\,x}(x,y) + G_{y\,y}(x,y)\,y'}{\widetilde{\mu}(x)} \tag{3.10}$$

so that by taking coefficients of y' in $\frac{\partial\Phi}{\partial y}\mathcal{F}$ we obtain

$$\begin{aligned} \varphi_1 &\equiv \Phi_y(x,y,y')\,\mathcal{F}(x,y') - y'\,\frac{\partial}{\partial y'}\left(\Phi_y(x,y,y')\,\mathcal{F}(x,y')\right) = \frac{G_{y\,x}(x,y)}{\widetilde{\mu}(x)} \\ \varphi_2 &\equiv \frac{\partial}{\partial y'}\left(\Phi_y(x,y,y')\,\mathcal{F}(x,y')\right) = \frac{G_{y\,y}(x,y)}{\widetilde{\mu}(x)} \end{aligned} \tag{3.11}$$

Similarly, we obtain

$$\begin{aligned} \varphi_3 &\equiv -\frac{\partial}{\partial y'}\left(\Phi(x,y,y')\ \mathcal{F}(x,y')\right) = \frac{F_{y'\,x}(x,y') + G_y(x,y)}{\widetilde{\mu}(x)} \\ \varphi_4 &\equiv \frac{\partial}{\partial y'}\mathcal{F}(x,y') = \frac{F_{y'\,y'}(x,y')}{\widetilde{\mu}(x)} \end{aligned} \tag{3.12}$$

Now, since Eq.(3.7) is nonlinear by hypothesis, either φ_2 or φ_4 is different from zero, so that at least one of the pairs of ratios $\{\varphi_1,\ \varphi_2\}$ or $\{\varphi_3,\ \varphi_4\}$ can be used to determine $\widetilde{\mu}(x)$ as the solution of an auxiliary first order linear ODE. For example, if $\varphi_2 \neq 0$,

$$\frac{\partial}{\partial y}\left(\varphi_1(x,y)\ \widetilde{\mu}(x)\right) = \frac{\partial}{\partial x}\left(\varphi_2(x,y)\ \widetilde{\mu}(x)\right) \tag{3.13}$$

and we obtain

$$\widetilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_2}\left(\frac{\partial\varphi_1}{\partial y} - \frac{\partial\varphi_2}{\partial x}\right)dx} \tag{3.14}$$

If $\varphi_4 \neq 0$,

$$\widetilde{\mu}(x) = e^{\displaystyle\int \frac{1}{\varphi_4}\left(\frac{\partial\varphi_3}{\partial y'} - \frac{\partial\varphi_4}{\partial x}\right)dx} \tag{3.15}$$

Eqs.(3.14) and (3.15) alternatively give both an explicit solution to the problem and an existence condition, since a solution $\widetilde{\mu}(x)$ - and hence an integrating factor of the form $\mu(x,y')$ - can exist if the integrand in Eq.(3.14) or Eq.(3.15) only depends on x. $\triangle$

Example: Kamke' ODE 37.

$$y'' = -2\,y\,y' - f(x)\left(y' + y^2\right) + g(x) \tag{3.16}$$

For this ODE, $\mathcal{F}(x, y')$ was determined (see sec. 3.1.2) as:

$$\mathcal{F}(x, y') = 1 \tag{3.17}$$

from which (Eq.(3.10))

$$\frac{G_{y\,x}(x,y) + G_{y\,y}(x,y)\,y'}{\widetilde{\mu}\,(x)} = -2y\,f(x) - 2y' \tag{3.18}$$

and then as in Eq.(3.11) we obtain

$$\begin{aligned} \varphi_1 &= -2y\,f(x) \\ \varphi_2 &= -2 \end{aligned} \tag{3.19}$$

Using this in Eq.(3.14), we get

$$\widetilde{\mu}\,(x) = e^{\displaystyle\int f(x)\,dx} \tag{3.20}$$

and so, from Eq.(3.9), since $\mathcal{F}(x, y') = 1$, $\mu(x, y') = \widetilde{\mu}\,(x)$.

3.1.2. Determination of $\mathcal{F}(x, y')$

We now prove our assumption, that is, we show how to obtain a function $\mathcal{F}(x, y')$ satisfying Eq.(3.9) from the knowledge of $\Phi(x, y, y')$.

Since we have already assumed that the given ODE has an integrating factor of the form $\mu(x, y')$, then there exist some functions $F(x, y')$ and $G(x, y)$ such that it is possible to rewrite $\Phi(x, y, y')$ - the right-hand-side (RHS) of the given ODE - as in Eq.(3.6). We then start by considering the expression

$$\Upsilon \equiv \frac{\partial \Phi}{\partial y} = -\frac{G_{x\,y}(x,y) + G_{y\,y}(x,y)\;y'}{F_{y'}(x, y')} \tag{3.21}$$

and the possible cases.

<u>Case A</u>

The first case happens when the ratio $G_{x\,y}(x,y)/G_{y\,y}(x,y)$ *depends* on y; i.e., $G_{x\,y}(x,y)$ and $G_{y\,y}(x,y)$ are linearly independent w.r.t y. To determine whether this is the case, note that we cannot just analyze the mentioned ratio itself since it is unknown. However, we can always select the factors of Υ containing y, and check if this expression *also* contains y'. If so, we just determine $F_{y'}(x, y')$ up to a factor depending on x, that is, the required $\mathcal{F}(x, y')$, as the reciprocal of the factors of Υ which depend on y' but not y. $\triangle$

Example: Kamke' ODE 226.

This ODE is presented in Kamke's book already in exact form, so we start by rewriting it in explicit form as

$$y'' = \frac{x^2 y y' + x y^2}{y'} \tag{3.22}$$

We determine Υ (Eq.(3.21)) as

$$\Upsilon = \frac{x(xy' + 2y)}{y'} \tag{3.23}$$

The only factor of Υ containing y is:

$$xy' + 2y \tag{3.24}$$

and since this also depends on y', $\mathcal{F}(x, y')$ is immediately given by

$$\mathcal{F}(x, y') = y' \tag{3.25}$$

<u>Case B</u>

When the expression formed by all the factors of Υ containing y does not contain y', it is impossible to determine whether one of the functions $\{G_{x\,y}(x,y),\ G_{y\,y}(x,y)\}$ is zero, or alternatively their ratio does not depend on y. We then proceed by assuming the former, build an expression for $\mathcal{F}(x, y')$ as in Case A, and determine $\widetilde{\mu}\,(x)$ as explained in the previous subsection. If this doesn't lead to the required integrating factor, we then proceed as follows.

<u>Case C</u>

In this case, we assume that neither $G_{x\,y}(x,y)$ nor $G_{y\,y}(x,y)$ are zero and their ratio is a function of just x, so that we have

$$\begin{aligned} G_{x\,y}(x,y) &= v_1(x)\ w(x,y) \\ G_{y\,y}(x,y) &= v_2(x)\ w(x,y) \end{aligned} \tag{3.26}$$

for some unknown functions $v_1(x)$ and $v_2(x)$, such that Eq.(3.21) can be factored as

$$\Upsilon = w(x,y)\ \frac{(v_1(x) + v_2(x)\ y')}{F_{y'}(x, y')} \tag{3.27}$$

for some function $w(x,y)$ which *can always be determined* as the factors of Υ depending on y. To determine $F_{y'}(x,y')$ up to a factor depending on x, we then need to determine the ratio $v_1(x)/v_2(x)$. For this purpose, from Eq.(3.26) we build an auxiliary PDE for $G_y(x,y)$,

$$G_{x\,y}(x,y) = \frac{v_1(x)}{v_2(x)}\, G_{y,y}(x,y) \tag{3.28}$$

The general solution of Eq.(3.28) is given by

$$G_y(x,y) = \mathcal{G}\,(y + p(x)) \tag{3.29}$$

where $\mathcal{G}$ is an arbitrary function of its argument and for convenience we introduced

$$p'(x) \equiv v_1(x)/v_2(x) \tag{3.30}$$

We can now determine $p'(x)$, that is, the ratio v_1/v_2 we were looking for, as follows. Taking into account Eq.(3.26), we arrive at

$$v_2(x)\, w(x,y) = \mathcal{G}'(y + p(x)) \tag{3.31}$$

By taking the ratio between this expression and its derivative w.r.t y we obtain

$$\mathcal{H}(y + p(x)) \equiv \frac{\partial w}{\partial y}/w = \frac{\mathcal{G}''(y + p(x))}{\mathcal{G}'(y + p(x))} \tag{3.32}$$

that is, a function of $y + p(x)$ only, which we can determine since we know $w(x, y)$. If $\mathcal{H}' \neq 0$, we obtain $p'(x)$ as

$$p'(x) = \frac{\partial \mathcal{H}}{\partial x}/\frac{\partial \mathcal{H}}{\partial y} = \frac{\left(\frac{\partial^2 w}{\partial y^2 \partial x}\right)\ w - \left(\frac{\partial w}{\partial y}\right) \frac{\partial w}{\partial x}}{\left(\frac{\partial^2 w}{\partial y^2}\right)\ w - \left(\frac{\partial w}{\partial y}\right)^2} \tag{3.33}$$

Once we determined $p'(x)$, from Eq.(3.27) we determine $\mathcal{F}(x, y')$ as

$$\mathcal{F}(x, y') = \frac{(p' + y')\ w}{\Upsilon} \tag{3.34}$$

where Υ, $w(x, y)$ and $p'(x)$ are now all known. $\triangle$

Example: Kamke' ODE 136.

We begin by writing the ODE in explicit form as

$$y'' = \frac{h(y')}{x - y} \tag{3.35}$$

Here Υ (Eq.(3.21)) is determined as

$$\Upsilon = -\frac{h(y')}{(x - y)^2} \tag{3.36}$$

and $w(x, y)$ as

$$w(x, y) = \frac{1}{(x - y)^2} \tag{3.37}$$

Then $\mathcal{H}(y + p(x))$ (Eq.(3.32)) becomes

$$\mathcal{H} = \frac{2}{x - y} \tag{3.38}$$

and hence, from Eq.(3.33), $p'(x)$ is

$$p'(x) = -1 \tag{3.39}$$

so from Eq.(3.34):

$$\mathcal{F}(x,y') = \frac{1-y'}{h(y')} \tag{3.40}$$

Case D

We now discuss how to obtain $p'(x)$ when $\mathcal{H}'(y+p(x)) = 0$. We consider at first the case in which $\mathcal{H} = 0$, hence $\mathcal{G}' = 0$, and so, recalling Eq.(3.29), we see that

$$G(x,y) = B_1\ (y+p(x))^2 + B_2\ (y+p(x)) + g(x) \tag{3.41}$$

for some function $g(x)$ and some constants B_1, B_2. Recalling Eq.(3.8), $\Phi(x,y,y')$ takes the form

$$\Phi(x,y,y') = -\frac{F_x(x,y') + g'(x) + (2B_1\ (y+p(x)) + B_2)(y'+p'(x))}{F_{y'}(x,y')} \tag{3.42}$$

We can now obtain explicit equations where the only unknown is $p(x)$ as follows. First, from the knowledge of Υ and Φ we build the two explicit expressions:

$$\Lambda \equiv \frac{1}{\Upsilon} = -\frac{F_{y'}}{2B_1\ (y'+p'(x))} \tag{3.43}$$

and

$$\Psi \equiv \frac{\Phi(x,y,y')}{\Upsilon} - y = \frac{F_x + g'(x)}{2B_1\ (y'+p'(x))} + p(x) + \frac{B_2}{2B_1} \tag{3.44}$$

It is now clear from Eq.(3.43) and Eq.(3.44) that Λ and Ψ are related by the following equation:

$$\frac{\partial}{\partial x}\left((y'+p'(x))\ \Lambda\right) + \frac{\partial}{\partial y'}\left((y'+p'(x))\ \Psi\right) = p(x) + \frac{B_2}{2B_1} \tag{3.45}$$

where the only unknowns are $p(x)$, B_1, and B_2. By differentiating the equation above w.r.t y' and x we obtain two equations where the only unknown is $p'(x)$:

$$\Lambda_{y'}p''(x) + (\Lambda_{xy'} + \Psi_{y'y'})(y'+p'(x)) + \Lambda_x + 2\Psi_{y'} = 0 \tag{3.46}$$

$$\Lambda\ p'''(x) + (\Lambda_{xx} + \Psi_{y'x})(y'+p'(x)) + (\Lambda_x + \Psi_{y'})p''(x) + \Psi_x = p'(x) \tag{3.47}$$

As a shortcut, if $(\Lambda_{xy'} + \Psi_{y'y'})/\Lambda_{y'}$ depends on y', then we can build a linear algebraic equation for $p'(x)$ by solving for $p''(x)$ in Eq.(3.46) and differentiating w.r.t. y'. Otherwise, in general we obtain $p'(x)$ by solving a linear algebraic equation built by eliminating $p''(x)$ between Eq.(3.46) and Eq.(3.47)†.

† From Eq.(3.43), $\Lambda \neq 0$, so that Eq.(3.47) always depends on $p'''(x)$, and solving Eq.(3.46) for $p''(x)$ and substituting twice into Eq.(3.47) will lead to the desired equation for $p'(x)$. If Eq.(3.46) depends on $p'(x)$ but not on $p''(x)$, then Eq.(3.46) itself is already a linear algebraic equation for $p'(x)$.

If Eq.(3.46) depends neither on $p'(x)$ nor on $p''(x)$ this scheme will not succeed. However, it is possible to prove that in that case the original ODE is already linear, and easy to solve. To see this, we set to zero the coefficients of $p'(x)$ and $p''(x)$ in Eq.(3.46), obtaining:

$$\Lambda_{y'} = \Lambda_{x y'} + \Psi_{y' y'} = \Lambda_x + 2\Psi_{y'} = 0 \tag{3.48}$$

from which Λ is a function of x only, and then

$$\Psi_{y' y'} = 0 \tag{3.49}$$

If we now rewrite $F(x, y')$ as in

$$F(x, y') = Z(x, y') - g(x) - \Lambda (y' + p')^2 \, B_1 \tag{3.50}$$

and introduce this expression in Eq.(3.43), we obtain $Z_{y'} = 0$; similarly, using this result, Eq.(3.44), Eq.(3.49) and Eq.(3.50) we obtain $Z_x = 0$. Hence, Z *is a constant.* Finally, taking into account that Z is constant, Eq.(3.50) and Eq.(3.42), we see that the ODE Eq.(3.7) which led us to this case is just a non-homogeneous linear ODE of the form

$$(y+p)'' + (\Lambda'(y+p)' - 2(y+p) - B_2/B_1)/2\Lambda = 0 \tag{3.51}$$

which homogeneous part does not depend on $p(x)$:

$$y'' + \frac{\Lambda'(x)}{2\Lambda(x)}\, y' - \frac{y}{\Lambda(x)} = 0 \tag{3.52}$$

and which solution is in any case straightforward. $\triangle$

Example: Kamke' ODE 66.

This ODE is given by

$$y'' = a\,(c + b x + y)\left(y'^2 + 1\right)^{3/2} \tag{3.53}$$

Proceeding as in Case A, we determine Υ, $w(x, y)$, and $\mathcal{H}(y + p(x))$ as

$$\Upsilon = a\left(y'^2 + 1\right)^{3/2}; \qquad w(x, y) = 1; \qquad \mathcal{H} = 0 \tag{3.54}$$

From the last equation we realize that we are in Case D. We determine Λ and Ψ (Eqs. 3.43, 3.44) as:

$$\begin{aligned} \Lambda &= \frac{1}{\left(y'^2 + 1\right)^{3/2} a} \\ \Psi &= c + b\,x \end{aligned} \tag{3.55}$$

We then build Eq.(3.45) for this ODE:

$$\frac{p''(x)}{\left(y'^2+1\right)^{3/2}\,a}+c+b\,x=p(x)+\frac{B_2}{2B_1} \tag{3.56}$$

Differentiating w.r.t. y' leads to Eq.(3.46):

$$-3\frac{p''(x)\,y'}{\left(y'^2+1\right)^{5/2}\,a}=0 \tag{3.57}$$

from which it follows that $p''(x)=0$. Using this in Eq.(3.47) we obtain:

$$p'(x)=b \tag{3.58}$$

after which Eq.(3.34) becomes

$$\mathcal{F}(x,y')=\frac{y'+b}{a\left(y'^2+1\right)^{3/2}} \tag{3.59}$$

Case E

We now show how to obtain $p'(x)$ when $\mathcal{H}'(y+p(x))=0$ and $\mathcal{H}=\mathcal{G}''/\mathcal{G}'$ is a constant, B_1, which is different from zero; so $\mathcal{G}'$ is an exponential function of its argument $(y+p(x))$ and hence from Eq.(3.29)

$$G(x,y)=B_2e^{(y+p(x))B_1}+(y+p(x))B_3+g(x) \tag{3.60}$$

for some constants B_2, B_3 and some function $g(x)$. In this case, it is always possible to arrive at an algebraic equation for $p'(x)$, though the case entails some subtleties. First of all, $\Phi(x,y,y')$ will be of the form

$$\Phi(x,y,y')=-\frac{F_x(x,y')+g'(x)+\left(B_2B_1e^{(y+p(x))B_1}+B_3\right)(y'+p'(x))}{F_{y'}(x,y')} \tag{3.61}$$

Now, taking advantage of the fact that we explicitly know B_1, we build our first explicit expression by dividing $B_1e^{yB_1}$ by Υ:

$$\Lambda\equiv-\frac{F_{y'}}{B_2e^{p(x)B_1}\,(y'+p'(x))} \tag{3.62}$$

We now multiply Φ by Λ and subtract $B_1e^{B_1y}$ to obtain our second explicit expression:

$$\Psi\equiv\frac{1}{B_2e^{p(x)B_1}}\left(\frac{F_x+g'(x)}{y'+p'(x)}+B_3\right) \tag{3.63}$$

Now, as in Case D, Λ and Ψ are related by

$$\frac{\partial}{\partial x}\Big((y'+p'(x))\,\Lambda\Big)+(y'+p'(x))\;p'(x)\Lambda B_1+\frac{\partial}{\partial y'}\Big((y'+p'(x))\,\Psi\Big)$$

$$= \frac{B_3}{B_2 e^{p(x) B_1}} \tag{3.64}$$

where the only unknowns are B_2, B_3 and $p(x)$. We build a first equation for $p'(x)$ by differentiating Eq.(3.64) with respect to y'

$$\left(p''(x) + p'(x)^2 B_1\right) \Lambda_{y'} + p'(x) \left(y' \Lambda_{y'} B_1 + \Lambda B_1 + \Lambda_{xy'} + \Psi_{y'y'}\right) \tag{3.65}$$
$$+2\Psi_{y'} + \Lambda_x + y' \Lambda_{xy'} + y' \Psi_{y'y'} = 0$$

The problem now is that, due to the exponential on the RHS of Eq.(3.64), differently from Case D, we are not able to obtain a second expression for $p'(x)$ by differentiating w.r.t x. The alternative we have found to determine $p'(x)$ can be summarized as follows. We first note that if $\Lambda_{y'} = 0$, Eq.(3.65) is already a linear algebraic equation for p'†, so that we are only worried with the case $\Lambda_{y'} \neq 0$. With this in mind, we divide Eq.(3.65) by $\Lambda_{y'}$ and, *if* the resulting expression depends on y', we directly obtain a linear algebraic equation in $p'(x)$ just differentiating w.r.t y'.$\triangle$

Example:

$$y'' = \frac{y' \; (x y' + 1) \left(-2 + e^y\right)}{y' x^2 + y' - 1} \tag{3.66}$$

Proceeding as in Case A, we determine Υ, $w(x, y)$, and $\mathcal{H}(y + p(x))$ as

$$\Upsilon = \frac{y'(xy' + 1)e^y}{y' x^2 + y' - 1}; \qquad w(x, y) = e^y; \qquad \mathcal{H} = 1 \tag{3.67}$$

From the last equation we know that we are in Case E. We then determine Λ and Ψ as in Eqs. (3.62) and (3.63):

$$\begin{aligned} \Lambda &= \frac{y' x^2 + y' - 1}{y'(xy' + 1)} \\ \Psi &= -2 \end{aligned} \tag{3.68}$$

Now, we build Eq.(3.64):

† We can see this by assuming $\Lambda_{y'} = 0$ and that the Eq.(3.65) does not contain p', and then arriving at a contradiction as follows. We first set the coefficients of p' in Eq.(3.65) to zero, arriving at

$$0 = B_1 \; \Lambda + \Psi_{y'y'} = 2\Psi_{y'} + \Lambda_x + \Psi_{y'y'} y' \tag{A}$$

Eliminating $\Psi_{y'y'}$ gives

$$2\Psi_{y'} = B_1 \; \Lambda \; y' - \Lambda_x$$

Differentiating the expression above w.r.t y' and since $\Lambda_{y'} = 0$ we have,

$$2\Psi_{y'y'} = B_1 \; \Lambda$$

Finally using Eq.(A), $0 = \Lambda$, contradicting $F_{y'} \neq 0$.

$$\frac{1}{xy'+1}\left(\left(p''+{p'}^2+{y'}^2\frac{(xp'-1)}{xy'+1}\right)\left(x^2+1-\frac{1}{y'}\right)+2\,xp'-2\right)=\frac{B_3}{B_2\,e^p} \tag{3.69}$$

and, differentiating w.r.t. y', we obtain (Eq.(3.65)),

$$\frac{2\,xy'+1-(x^3+x){y'}^2}{{y'}^2(xy'+1)^2}\left(p''+{p'}^2\right)+\frac{2\,y'-1-2\,x+xy'}{(xy'+1)^3}(xp'-1)=0 \tag{3.70}$$

Proceeding as explained, dividing by $\Lambda_{y'}$ and differentiating w.r.t. y' gives

$$\frac{\partial}{\partial y'}\left({y'}^2\frac{2\,y'-1-2\,x+xy'}{(xy'+1)\,(2\,xy'+1-(x^3+x){y'}^2)}\right)(xp'-1)=0 \tag{3.71}$$

Solving for $p'(x)$ gives $p'(x)=1/x$, from which (Eq.(3.34)):

$$\mathcal{F}(x,y')=\left(y'-\frac{1}{x}\right)\frac{y'x^2+y'-1}{y'(xy'+1)} \tag{3.72}$$

Case F

The final branch occurs when Eq.(3.65) divided by $\Lambda_{y'}$ does not depend on y' (so that we will not be able to differentiate w.r.t y'). In this case we can build a linear algebraic equation for $p'(x)$ as follows. Let us introduce the label $\beta(x,p',p'')$ for Eq.(3.65) divided by $\Lambda_{y'}$, so that Eq.(3.65) becomes:

$$\Lambda_{y'}(x,y')\;\beta(x,p',p'')=0 \tag{3.73}$$

Again, since we obtained Eq.(3.65) by differentiating Eq.(3.64) with respect to y', we see that Eq.(3.64) can be written in terms of β by integrating Eq.(3.73) with respect to y':

$$\Lambda(x,y')\beta(x,p',p'')+\gamma(x,p',p'')=\frac{B_3}{B_2 e^{p(x)B_1}} \tag{3.74}$$

where $\gamma(x,p',p'')$ is the constant of integration, and can be determined explicitly in terms of x, p' and p'' by comparing Eq.(3.74) with Eq.(3.64). Taking into account that $\beta(x,p',p'')=0$, we see that Eq.(3.74) reduces to:

$$\gamma(x,p',p'')=\frac{B_3}{B_2 e^{p(x)B_1}} \tag{3.75}$$

We can remove the unknowns B_2 and B_3 after multiplying Eq.(3.75) by $e^{p(x)B_1}$, differentiating with respect to x, and then dividing once again by $e^{p(x)B_1}$. We now have our second equation for p', which we can build explicitly in terms of p', since we know $\gamma(x,p',p'')$ and B_1:

$$\frac{d\gamma}{dx}+B_1\;p'\gamma=0 \tag{3.76}$$

Eliminating the derivatives of p' between Eq.(3.73) and Eq.(3.75) leads to a linear algebraic equation in p'. Once we have p', the determination of $\mathcal{F}(x, y')$ follows directly from Eq.(3.34). □

3.2. Integrating factors of the form $\mu(y, y')$

Just as in the previous section, the following first integral is associated with the integrating factor family $\mu(y, y')$:

$$R(x, y, y') = F(y, y') + G(x, y) \tag{3.77}$$

where $\mu = F_{y'}$, and to this integrating factor corresponds the ODE pattern:

$$y'' = -\frac{G_x(x, y) + (F_y(y, y') + G_y(x, y))y'}{F_{y'}(y, y')} \tag{3.78}$$

For this ODE family, it would be possible to build a matching pattern routine as done in the previous section for the case $\mu(x, y')$. However, it is straightforward to notice that under the transformation $y(x) \to x,\ x \to y(x)$, we obtain Eq.(3.6) from Eq.(3.78), so that the transformed ODE *will* have an integrating factor of the form $\mu(x, y')$. This means that the above pattern can be matched by merely changing variables in the given ODE and trying to match Eq.(3.6). It follows that any explicit 2nd order ODE having an integrating factor of the form $\mu(y, y')$ can be reduced to a first order ODE by first changing variables, and then using the scheme outlined in the previous section, unless the resulting ODE is linear.

3.3. The Connection to PDEs

Let $R(x, y, y')$ be a first integral of Eq.(3.7). We rewrite Eq.(2.13) by renaming $y' \equiv z$

$$\frac{\partial R}{\partial x} + z\frac{\partial R}{\partial y} + \Phi(x, y, z)\frac{\partial R}{\partial z} = 0 \tag{3.79}$$

From Theorem 3.1, if a given PDE of the form Eq.(3.79) has a particular solution of the form $R(x, y, z) = F(x, z) + G(x, y)$, such that $R(x, y, z)$ is nonlinear in y or z; or $R(x, y, z) = F(y, z) + G(x, y)$, such that $R(x, y, z)$ is nonlinear in y or z^{-1}, then F and G can be determined in a systematic manner.

Although this is a natural consequence of the previous sections, it is worth mentioning that the determination of R using the scheme here presented *does not require* solving the characteristic strip of Eq.(3.79), thus being a genuine alternative.

4. Tests

After plugging the reducible-ODE scheme here presented into ODEtools, we tested the scheme and routines using Kamke's non-linear 246 second order ODE examples†. The purpose was to confirm the correctness of the returned results and to determine which of these ODEs have integrating factors of the form $\mu(x, y')$ or $\mu(y, y')$. The test consisted of determining μ and testing the exactness condition Eq.(2.7) of the product μ times ODE.

† Kamke's ODEs 6.247 to 6.249 cannot be made explicit and are then excluded from the tests.

We then ran a comparison of performances in solving a related subset of Kamke's examples using different computer algebra ODE-solvers (Maple, Mathematica, MuPAD and the Reduce package Convode). The idea was to situate the new scheme in the framework of a sample of relevant packages presently available. As a secondary goal, we were also interested in comparing the solving performance of the new scheme with the one of the symmetry scheme implemented in ODEtools.

Finally we considered the table of integrating factors for second order non-linear ODEs found in Murphy's book and the answers for them returned by all these ODE-solvers.

4.1. The *reducible-ODE* solving scheme and Kamke's ODEs

To run the test with Kamke's ODEs, the first step was to classify these ODEs into: *missing x*, *missing y*, *exact* and *reducible*, where the latter refers to the new scheme. The reason for such a classification is that ODEs missing variables are straightforwardly reducible, so they are not the relevant target of the new scheme. Also, ODEs already in exact form can be easily reduced after performing a simple check for exactness; before running the tests all these ODEs were rewritten in explicit form by isolating y''. For classifying the ODEs we used the **odeadvisor** command from ODEtools. All the integrating factors found satisfied the exactness condition Eq.(2.7). The classification we obtained for these 246 ODEs is as follows

Classification	ODE numbers as in Kamke's book
99 ODEs are missing x or missing y	1, 2, 4, 7, 10, 12, 14, 17, 21, 22, 23, 24, 25, 26, 28, 30, 31, 32, 40, 42, 43, 45, 46, 47, 48, 49, 50, 54, 56, 60, 61, 62, 63, 64, 65, 67, 71, 72, 81, 89, 104, 107, 109, 110, 111, 113, 117, 118, 119, 120, 124, 125, 126, 127, 128, 130, 132, 137, 138, 140, 141, 143, 146, 150, 151, 153, 154, 155, 157, 158, 159, 160, 162, 163, 164, 165, 168, 188, 191, 192, 197, 200, 201, 202, 209, 210, 213, 214, 218, 220, 222, 223, 224, 232, 234, 236, 237, 243, 246
13 are in exact form	36, 42, 78, 107, 108, 109, 133, 169, 170, 178, 226, 231, 235
40 ODEs are *reducible* with integrating factor $\mu(x, y')$ or $\mu(y, y')$ *and* missing x or y	1, 2, 4, 7, 10, 12, 14, 17, 40, 42, 50, 56, 64, 65, 81, 89, 104, 107, 109, 110, 111, 125, 126, 137, 138, 150, 154, 155, 157, 164, 168, 188, 191, 192, 209, 210, 214, 218, 220, 222, 236
28 ODEs are *reducible* and not missing x or y	36, 37, 51, 66, 78, 97, 108, 123, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215, 226, 235

Table 1. Missing variables, exact and *reducible* Kamke's 246 second order non-linear ODEs.

From the table above, $\approx 30\%$ of these 246 Kamke's ODEs are reducible to first order using the scheme here presented. Also, although the symmetry scheme is more general, and finds symmetries for 191 of these 246 ODEs, its implementation in ODEtools is not reducing five ODEs which are being reduced by the new scheme. These are the ODEs numbered 36, 37, 123, 215, and 235. For ODE 215, which we write in explicit form as

$$\text{ode} := \quad y'' = \frac{\left(6\,y^2 - \frac{a}{2}\right) {y'}^2}{4\,y^3 - a\,y - b} - f(x)\,y' \tag{4.1}$$

the integrating factor found using the computer algebra implementation of the new scheme (see sec. 5) was†:

```
> mu = intfactor(ode,y(x));
```

$$\mu = \frac{1}{y'} \tag{4.2}$$

This integrating factor leads to a reduced ODE which can be solved as well, resulting in the following implicit solution in terms of an elliptic integral†:

```
> odsolve(ode);
```

$$\int e^{-\int f(x)dx} dx - C_1 \int^y \frac{1}{\sqrt{-4\,\_a^3 + \_a\,a + b}} d\_a + C_2 = 0 \tag{4.3}$$

The integrating factor found for ODE 37 (see Eqs.(3.16) and (3.20)) lead to a reduction of order resulting in the most general first order Riccati type ODE; in this example **odsolve** just returns the reduction of order obtained using the new scheme. For ODE 123, the integrating factor found was $1/y$, and the reduced ODE is also a generic Riccati ODE. Finally, ODE 235 appears in Kamke's book written in exact form, but the ODE is interesting because it contains three arbitrary functions, of the first derivative, the dependent and the independent variables, respectively. Such an arbitrary dependence makes this ODE almost intractable for most computer algebra ODE-solvers and related packages. We then first isolated the highest derivative as to make the ODE *non-exact*

$$\text{ode} := \quad y'' = \frac{-\left(G(y)\,y' + F(x)\right)}{H(y')} \tag{4.4}$$

The integrating factor here found is

```
> mu = intfactor(ode,y(x));
```

$$\mu = H(y') \tag{4.5}$$

It is interesting to note that none of the other computer algebra ODE-solvers tested during this work succeeded in solving or reducing the order of any of these five ODEs, even though the corresponding integrating factors depend on only one variable.

Concerning timings, it is worth mentioning that in the specific subset of 28 Kamke's examples which are not missing variables, the average *time consumed* by **odsolve** in solving each ODE using the new scheme was 2.5 sec, while using symmetries this time jumps to 21 sec. These tests were peformed using a Pentium 200, 64 Mb RAM, running Windows 95. In summary: for ODEs having an integrating factor of the form $\mu(x, y')$ or $\mu(y, y')$, the new scheme seems to be, on the average, $\approx$ 10 times faster than the symmetry scheme.

† In what follows, the *input* can be recognized by the Maple prompt `>`.

† For ODE 6.215, there is a typographical mistake in Kamke's book concerning the reduced ODE: instead of ...$\sqrt{4y^3 - g_2 y - g_3}$..., one should read ...$\sqrt{4\,y^3 - g_2 y - g_3}$....

4.2. Comparison of performances

With the classification presented in Table 1. in hands, we used different computer algebra systems to run a comparison of performances in solving these ODEs having integrating factors of the form $\mu(x, y')$ or $\mu(y, y')$. For our purposes, the interesting subset is the one comprised of the 28 ODEs not already missing variables (see Table 1.). The results we obtained are summarized in the following table‡:

	Kamke's ODE numbers			
	Convode	Mathematica 3.0	MuPAD 1.3	ODEtools
Solved:	51, 166, 173, 174, 175, 176, 179.	78, 97, 108, 166, 169, 173, 174, 175, 176, 178, 179, 206.	78, 97, 108, 133, 166, 169, 173, 174, 175, 176, 179,	51, 78, 97, 108, 133, 134, 135, 136, 166, 169, 173, 174, 175, 176, 178, 179, 193, 196, 203, 204, 206, 215.
Totals:	7	12	11	22
Reduced:				36, 37, 66, 123, 226, 235.
Totals:	0	0	0	6

Table 2. *Performances in solving 28 Kamke's ODEs having an integrating factor* $\mu(x, y')$ *or* $\mu(y, y')$

As shown above, while the scheme here presented is finding first integrals in all the 28 ODE examples, opening the way to solve 22 of them to the end, the next scores are only 12 and 11 ODEs, respectively solved by Mathematica 3.0 and MuPAD 1.3.

Concerning the six reductions of order returned by **odsolve**, it must be said that neither MuPAD nor Mathematica provides a way to convey them, so that perhaps their ODE-solvers are obtaining first integrals for these cases but the routines are giving up when they cannot solve the problem to the end.

Maple R4 is not present in the table since it is not solving any of these 28 ODEs. This is understandable since in R4 the only methods implemented for high order non-linear ODEs are those for ODEs which are missing variables. This situation is being resolved in the upcoming Maple R5, where the ODEtools routines are included in the Maple library, and the previous ODE-solver has been replaced by **odsolve**†.

Although the primary goal of this work is just to obtain first integrals for second order ODEs, it is also interesting to comment on the six ODEs shown in Table 2. for which the new scheme succeeds in determining integrating factors but the reduced ODEs remain unsolved. First of all, for ODEs 36, 37 and 123 the reduction of order lead to general

‡ When building the statistics for ODEtools, we passed to **odsolve** the optional argument `[reducible]`, meaning: try the reducible scheme, and if it does not solve the problem just give up. To solve the reduced ODE all of **odsolve**'s methods, including symmetries, were used. The input and output in the respective format for all the packages tested are available in http://dft.if.uerj.br/odetools/mu_odes.zip.

† However, the scheme here presented was not ready when the development library was closed; the *reducible* scheme implemented in Maple R5 is able to determine, when they exist, integrating factors only of the form $\mu(y')$.

Riccati type ODEs, so that in these cases no more than a reduction of order should be expected. Concerning ODE 235 (Eq.(4.4)), the reduced ODE is:

$$\int^{y'} H(\_a)d\_a - \int^{y} -G(\_a)d\_a - \int -F(x)dx + C_1 = 0 \tag{4.6}$$

Methods for solving such a first order ODE are known only for very special explicit combinations of H, G and F. Concerning ODEs 66 and 226, the obtained reduced ODEs are the same as those shown in Kamke's book, and are out of the scope of **odsolve**.

4.3. The *reducible-ODE* scheme and Murphy's table of integrating factors

There is an explicit paragraph in Murphy's book concerning integrating factors of the form $\mu(y')$, where it is shown a table with four second order non-linear ODE families for which $\mu(y')$ is already known. The first two families are trivial in the sense that they are already missing variables. The third of these ODE families is:

$$\text{ode} := \quad y'' = P(x)\, y' + Q(y)\, {y'}^2 \tag{4.7}$$

where P and Q are arbitrary functions of its arguments; this is actually Liouville's ODE. The integrating factor mentioned in the book is the same found by the scheme here presented: y'; and the corresponding reduced ODE can be solved in implicit form:

```
> odsolve(ode);
```

$$\int e^{\int P(x)dx} dx - \int^{y} e^{-\int Q(\_a)d\_a + C_1} d\_a + C_2 = 0 \tag{4.8}$$

The fourth ODE family is the most general second order ODE having $1/y'$ as integrating factor (see sec. 5):

$$\text{ode} := \quad y'' = \frac{\partial R(x,y)}{\partial x} y' + \frac{\partial R(x,y)}{\partial y}\, {y'}^2 \tag{4.9}$$

for some function $R(x, y)$. Here the new scheme finds the integrating factor $1/y'$ and returns the reduced ODE

$$\ln(y') - R(x,y) + C_1 = 0 \tag{4.10}$$

actually a generic first order ODE†.

5. Computer algebra implementation

We implemented the scheme for finding integrating factors described in sec. 3 in the framework of the ODEtools package (Cheb-Terrab et al. 1997), taking advantage of its set of programming tool routines specifically designed to work with ODEs. The implementation consists of:

† For the third ODE family, Mathematica 3.0 returns a wrong answer and MuPAD 1.3 gives up, while for the fourth family, Mathematica gives up and MuPAD returns an ERROR message.

- The plugging of the *reducible-ODE* solving scheme here presented in the block of methods for nonlinear second order ODEs of the ODEtools command **odsolve**;
- The extension of the capabilities of the ODEtools **intfactor** command to determine integrating factors for non-linear second order ODEs using the scheme here presented;
- A new user-level routine, **redode**, returning the most general explicit ODE having a given integrating factor (Eq.(3.3));

The computational implementation follows straightforwardly the explanations of sec. 3 and includes three main routines, for determining $\mathcal{F}(x, y')$, $\widetilde{\mu}\,(x)$ and the reduced ODE $R(x, y, y')$, respectively. Callings to these routines were in turn added to both the **intfactor** and **odsolve** commands, so that the scheme becomes available at user-level.

A test of this implementation in **odsolve** and some related examples are found in sec. 4. Since detailed descriptions of the ODEtools commands are found in the On-Line help, we have restricted this section to a description of the new command **redode** followed by two examples.

Description of **redode**

Command name: **redode**
Feature: returns the n^{th}order ODE having a given integrating factor
Calling sequence:

```
> redode(mu, n, y(x));
> redode(mu, n, y(x), R);
```

Parameters:

`n`	- indicates the order of the requested ODE.
`mu`	- an integrating factor depending on x, $y, ..., y^{(n-1)}$.
`y(x)`	- the dependent variable.
`R`	- optional, the expected reduced ODE depending on x, $y, ..., y^{(n-1)}$.

Synopsis:

• Given an integrating factor $\mu(x, y, ...y^{(n)})$, **redode**'s main goal is to return the ODE of order n having μ as integrating factor

$$y^{(n)} = \frac{-1}{\mu}\left[\frac{\partial}{\partial x}\left(\int \mu\, dy^{(n-1)} + G\right) + ... + y^{(n-1)}\frac{\partial}{\partial y^{(n-2)}}\left(\int \mu\, dy^{(n-1)} + G\right)\right]$$

where $G \equiv G(x, y, ...y^{(n-2)})$ is an arbitrary function of its arguments (see sec. 3). This command is useful to identify the general ODE problem related to a given μ, as well as to understand the possible links between the integrating factor scheme for reducing the order and other reduction schemes (e.g., symmetries).

• When the expected *reduced ODE* (differential order $n-1$), here called R, is also given as argument, the routine proceeds as follows. First, a test to see if the requested ODE exists is performed:

$$\mu(x, y, ...y^{(n-1)}) = \nu(x, y, ...y^{(n-2)})\ \frac{\partial}{\partial y^{(n-1)}} R(x, y, ..., y^{(n-1)}) \tag{5.1}$$

for some function $\nu(x, y, ...y^{(n-1)})$. If the problem is solvable, **redode** will then return an n^{th}order $ODE^{(n)} = y^{(n)} - \Phi(x, y, ..., y^{(n-1)})$ satisfying

$$\mu(x, y, ..., y^{(n-1)})\ ODE^{(n)} = \frac{d}{dx} R(x, y, ..., y^{(n-1)}) \tag{5.2}$$

that is, an ODE having as first integral $R +$ *constant.*

• When the given μ does not depend on $y^{(n-1)}$ and R is non-linear in $y^{(n-1)}$, the requested n^{th}order ODE nevertheless exists if R can be solved for $y^{(n-1)}$.

Examples:

The **redode** command is interesting mainly as a tool for generating solving schemes for given ODE families; we illustrate with two examples.

1. Consider the family of second order ODEs having as integrating factor $\mu = F(x)$ - an arbitrary function - such that the reduced ODE has the same integrating factor. We want to set up an algorithm such that, given a second order linear ODE,

$$y'' = \psi_1(x)\, y' + \psi_2(x)\, y + \psi_3(x) \tag{5.3}$$

where there are no restrictions on $\psi_1(x)$, $\psi_2(x)$ or $\psi_3(x)$, the scheme determines if the ODE belongs to the family just described, and if so it also determines $F(x)$. The knowledge of $F(x)$ will be enough to build a closed form solution for the ODE.

To start with we obtain the first order ODE having $F(x)$ as integrating factor via

```
> ode_1 := redode(F(x), y(x));
```

$$\mathrm{ode}_1 := y' = -\frac{1}{F(x)} \left(y\ \frac{dF(x)}{dx} + \_F1(x) \right) \tag{5.4}$$

where $\_F1(x)$ is an arbitrary function. To obtain the second order ODE aforementioned we pass `ode_1` as argument (playing the role of the *reduced* ODE) together with the integrating factor $F(x)$ to obtain

```
> ode_2 := redode(F(x), y(x), ode_1);
```

$$\mathrm{ode}_2 := y'' = -\frac{1}{F(x)} \left(2\, y'\ \frac{dF(x)}{dx} + y \frac{d^2 F(x)}{dx^2} + \frac{d\_F1(x)}{dx} \right) \tag{5.5}$$

Taking this general ODE pattern as departure point, we setup the required solving scheme by comparing coefficients in Eq.(5.3) and Eq.(5.5), obtaining

$$\frac{-2}{F(x)} \frac{dF(x)}{dx} = \psi_1(x), \qquad \frac{-1}{F(x)} \frac{d^2 F(x)}{dx^2} = \psi_2(x) \tag{5.6}$$

By solving the first equation, we get $F(x)$ as

$$F(x) = C_1\, e^{-\int \frac{\psi_1(x)}{2} dx} \tag{5.7}$$

and by substituting this result into the second one we get the pattern identifying the ODE family

$$\frac{1}{2}\frac{d\psi_1(x)}{dx} - \frac{1}{4}\left(\psi_1(x)\right)^2 - \psi_2(x) = 0 \tag{5.8}$$

Among the ODE-solvers of Maple R4, Mathematica 3.0, MuPAD 1.3 or Convode (Reduce), only those of MuPAD and Maple succeed in solving this ODE family.

2. Consider the second order ODE family having as integrating factor $\mu = F(x)$ - an arbitrary function - *also having* the symmetry[†] $[\xi = 0, \eta = F(x)]$, and such that the reduced ODE is the most general first order linear ODE

$$\mathrm{ode}_1 := y' = A(x)\, y + B(x) \tag{5.9}$$

where $A(x)$ and $B(x)$ are arbitrary functions. To start with, we obtain the aforementioned second order ODE having the integrating factor $F(x)$ as in Example 1.

```
> ode_2 := redode(F(x), y(x), ode_1);
```

$$y'' = \left(\frac{dA(x)}{dx} + \frac{\left(\frac{dF(x)}{dx}\right) A(x)}{F(x)}\right) y + \left(A(x) - \frac{\frac{dF(x)}{dx}}{F(x)}\right) y' + \frac{dB(x)}{dx} + \frac{\left(\frac{dF(x)}{dx}\right) B(x)}{F(x)} \tag{5.10}$$

In this step, `ode_2` is in fact the most general second order linear ODE. If we now impose the symmetry condition[‡] X(`ode_2`)=0, where $X = [0,\, F(x)]$ we arrive at the following restriction on $A(x)$

$$-F(x)\frac{dA(x)}{dx} - 2\left(\frac{dF(x)}{dx}\right) A(x) + \frac{\left(\frac{dF(x)}{dx}\right)^2}{F(x)} + \frac{d^2F(x)}{dx^2} = 0 \tag{5.11}$$

Solving this ODE for $A(x)$, introducing the result into Eq.(5.10) and disregarding the non-homogeneous term (irrelevant in the solving scheme) we obtain the homogeneous ODE family pattern:

$$y'' = \left(\frac{1}{2}\frac{dH(x)}{dx} + \frac{3}{4}\frac{\left(\frac{dH(x)}{dx}\right)^2}{\left(H(x)\right)^2} - \frac{1}{2}\frac{\frac{d^2H(x)}{dx^2}}{H(x)}\right) y + H(x)\, y' \tag{5.12}$$

where we introduced $H(x) = (F(x))^{-2}$. Although this ODE family appears more general than the one treated in Example 1., the setting up of a solving scheme here is easier: one just need to check if the coefficient of y in a given ODE is related to the coefficient of y' as in equation Eq.(5.12), in which case the integrating factor is just $\frac{1}{\sqrt{H(x)}}$.

[†] Here we denote the infinitesimal symmetry generator by $[\xi, \eta] \equiv \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}$

[‡] For linear ODEs, symmetries of the form $[0, F(x)]$ are also symmetries of the homogeneous part.

6. Conclusions

This paper presented a systematic method for obtaining integrating factors of the form $\mu(x, y')$ and $\mu(y, y')$ - when they exist - for second order non-linear ODEs, as well as its computer algebra implementation in the framework of the ODEtools package. The scheme is new, as far as we know, and the implementation has proven to be a valuable tool since it leads to reductions of order for varied ODE examples, as shown in sec. 4. Actually, the implementation of the scheme is solving ODEs which cannot be solved by using standard methods or other computer algebra ODE-solvers; furthermore, it involves simple operations, resulting in answers about 10 times faster than the symmetry scheme.

It is also worth mentioning that restricting the dependence of μ in Eq.(2.7) to $\mu(x, y')$, *does not lead* to a straightforwardly solvable problem except for few or simple cases. Moreover, when the determination of μ from Eq.(2.7) is frustrated, there is no manner to determine whether such a solution $\mu(x, y')$ exists. It is then a pleasant surprise to see such integrating factors - provided they exist - being determined in all cases and without solving any differential equations, convincing us of the value of the new scheme.

On the other hand, the method presented is not as general as for instance the symmetry scheme. We are restricting the problem to the universe of second order ODEs having integrating factors depending only on two variables - the general case is $\mu(x, y, y')$ - and even so, the method is as yet unable to find integrating factors of the form $\mu(x, y)$.

Some natural extensions of this work then would be to develop a scheme for building integrating factors of the forms considered in this work but for higher order ODEs, at least for restricted ODE families yet to be determined. We are presently working on some prototypes in these directions[†] and expect to succeed in obtaining reportable results in the near future.

Acknowledgments

This work was supported by the State University of Rio de Janeiro (UERJ), Brazil, and by the Symbolic Computation Group, Faculty of Mathematics, University of Waterloo, Ontario, Canada. The authors would like to thank T. Kolokolnikov[‡] for fruitful discussions.

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† See http://dft.if.uerj.br/odetools.html

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Appendix A

This appendix contains some additional information which may be useful as a reference for developing computer algebra implementations of this work, or for improving the one here presented.

As explained in sec. 3.1.2, the scheme presented can be subdivided into six different cases: A, B, C, D, E and F. Actually, there are just five cases since case B is always either A or C. From the point of view of a computer implementation of the scheme it is interesting to know what one would expect from such an implementation concerning Kamke's ODEs and the cases aforementioned. We then display here both the integrating factors obtained for the 28 Kamke's ODEs used in the tests (see sec. 4) and the "case" corresponding to each ODE.

Integrating factor	Kamke's book ODE-number	Case
1	36	D
$e^{\int f(x)dx}$	37	A
y'^{-1}	51, 166, 169, 173, 175, 176, 179, 196, 203, 204, 206, 215	C
$\frac{b+y'}{\left(1+y'^2\right)^{3/2}}$	66	D
x	78	D
x^{-1}	97	A
y	108	D
y^{-1}	123	A
$\frac{1+y'}{\left(y'-1\right)y'}$	133	C
$\frac{y'-1}{\left(1+y'\right)y'}$	134	C
$\frac{y'-1}{\left(1+y'\right)\left(1+y'^2\right)}$	135	C
$\frac{y'-1}{h(y')}$	136	C
$\frac{x}{2\,xy'-1}$	174	C
$(1+y')^{-1}$	178	C
$\frac{1}{y'\left(1+2\,yy'\right)}$	193	C
y'	226	A
$h(y')$	235	C

Table A.1 Integrating factors for Kamke's *reducible* and ODEs not missing variables.