DISCRETE PARISIAN AND DELAYED BARRIER OPTIONS: A GENERAL NUMERICAL APPROACH

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Abstract

In this paper we present a numerical method for the valuation of derivative securities which have a payoff dependent upon the amount of time during the life of the contract that some underlying variable lies within a specified range. We concentrate in particular on the examples of Parisian options and delayed barrier options, but our approach is easily adapted to other cases such as switch options and step options. Available analytic pricing formula are based on the assumption that the underlying variable is monitored continuously. In contrast, we consider discrete (e.g. daily or weekly) sampling. Given that path-dependent option values are known to be generally very sensitive to sampling frequency, this is an important advantage of our numerical approach.

1 Introduction

Recently there has been some interest in the valuation of derivative securities for which the payoff depends on the amount of time during the life of the contract that some underlying variable such as an asset price or interest rate lies within a specified range. Switch options and corridor options have been examined by Pechtl (1995). Switch options have a payoff of some predetermined amount multiplied by the fraction of the contract life for which the underlying asset lies above or below a given level. Corridor options are similar, the distinction being that the payoff depends on the time spent by the underlying asset between an upper and a lower bound. As Pechtl observes, corridor options can be seen as the difference between two switch options. Interest rate range notes (see Turnbull (1995)) are floating rate contracts for which the interest paid depends on the amount of time that a particular market interest rate lies within specified limits. Linetsky (1996) provides an excellent general overview of developments in the area and a detailed analysis of step options. These are an extension of barrier options. Rather than having an immediate knock-out as soon as an asset price reaches a barrier, these contracts provide for a more gradual reduction in the option payoff, the amount of the decrease depending on the amount of time that the underlying asset spends beyond the barrier. Parisian options (Chesney, Jeanblanc-Picqué, and Yor (1997); Chesney, Cornwall, Jeanblanc-Picqué, Kentwell, and Yor (1997)) are another variation of barrier options. In these contracts, the knock-out (or knock-in) provision applies if the underlying asset remains continuously beyond a barrier for a pre-specified amount of time. Cumulative Parisian options (referred to here and in Linetsky (1996) as delayed barrier options) are similar except that the barrier provision is defined in terms of the total amount of time spent beyond the barrier.

As Linetsky (1996) notes, there are at least a several reasons why these two modifications of standard barrier options have been introduced. First, it is well known that it can be quite difficult to apply traditional delta hedging techniques in the case of standard barrier options because both the option payoff and the option delta are discontinuous at the barrier at observation times.[1] Second, it has been suggested that market participants have attempted to temporarily manipulate prices of underlying assets so as to trigger barrier events. Because the time required to do so is longer in the case of Parisian and step options, such contracts are clearly less vulnerable to this sort of manipulation.

Although analytic or quasi-analytic solutions for pricing these various types of options (switches, range notes, steps, Parisians, delayed barriers) have been provided in the references cited above, it is also desirable to have a numerical method available. Existing analytic solutions are all based on the assumption that the underlying asset price follows geometric Brownian motion, i.e. it is

[1]This has led authors such as Bowie and Carr (1994) and Derman, Ergener, and Kani (1995) to propose alternative static strategies to hedge at least approximately (and exactly in some special cases).

lognormally distributed.[2] The existence of the "volatility smile" for vanilla options indicates that it is worth investigating alternative distributional assumptions. For instance, Cox (1996) argues that the constant elasticity of variance (CEV) process is more consistent with the smile than geometric Brownian motion due to the embedded negative correlation between volatility and stock price changes. It is also worth noting in this regard that Boyle and Tian (1997) report significant pricing differences for barrier options between CEV and lognormal models, even when implied volatilities (in terms of vanilla options) are the same. As departures from lognormality usually preclude analytic techniques, numerical methods must be developed to handle such cases. Another limitation of analytic approaches is that they typically are available only for cases where the underlying variable is continuously monitored, but in practice most contracts are discretely monitored. Moreover, there are often very large differences in path-dependent option values between the two cases (see, e.g. Cheuk and Vorst (1996)). The ability to incorporate discrete monitoring is another potential advantage of a numerical approach. Yet another possible benefit is the ability to handle cases where barriers are time-varying.[3]

The main objective of this paper is to provide a general numerical approach for pricing options with a payoff which depends on the amount of time during the contract life for which the underlying asset value lies within a specified range. For expositional simplicity, we deal only with the cases of discretely monitored Parisian and delayed barrier options under the usual Black-Scholes assumptions. Extensions to other types of contracts such as switch options and step options are straightforward, as is an examination of the CEV process case. The outline of the paper is as follows. Section 2 formulates the option pricing problems to be examined. Section 3 describes the numerical algorithm. Section 4 presents some illustrative computations. Section 5 concludes.

2 Formulation

We begin by considering the discrete Parisian option pricing problem. It can be formulated as a two dimensional partial differential equation (PDE) using a similar approach to that described in Wilmott, Dewynne, and Howison (1993) for valuing Asian and lookback options. We consider the option to be a function of a new deterministic state variable J, which is the current running sum of the number of successive observations where the underlying asset price S is over (under) the barrier for an up (down) option. We define a Parisian up option as an option where the counter J is incremented on an observation date if S is above the barrier, and reset to zero if it is below the

[2] To derive a closed form solution for the value of interest rate range notes, Turnbull (1995) assumes that interest rates are normally distributed, implying that bond prices are lognormal.

[3] Kunitomo and Ikeda (1992) examine the valuation of barrier options where the barriers are exponential functions of time. More general cases require numerical methods. Neither Chesney, Jeanblanc-Picqué, and Yor (1997) (for Parisian options) nor Linetsky (1996) (for step options) consider cases of time-varying barriers.

barrier. A Parisian down option is similarly defined, except that the counter is incremented if S is below the barrier, and reset to zero if it is above the barrier.

A Parisian knock-in option is an option which pays off a positive amount only if S is above (below) the barrier for a specified number of observation dates for an up (down) option. A Parisian knock-out option ceases to have value if S is above (below) the barrier for the required number of observations for an up (down) option. In the following, we let J^* be the critical number of observations before the option is either knocked-in or knocked-out. For an up option, the time evolution of J is given by

$$\frac{dJ}{dt} = \lim_{\lambda \to \infty} \sum_i H(S - S^*)\delta(t - t_i) - J\lambda H^-(S^* - S)\delta(t - t_i) \tag{1}$$

where S^* is the barrier value, t denotes time, t_i indicates an observation date, $\delta(\cdot)$ is the delta function, and

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

$$H^-(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases} \quad .$$

Similarly, the time evolution of J can be defined for a down option as

$$\frac{dJ}{dt} = \lim_{\lambda \to \infty} \sum_i H(S^* - S)\delta(t - t_i) - J\lambda H^-(S - S^*)\delta(t - t_i) \tag{2}$$

Equation (1) can be integrated over a time interval spanning an observation date t_i to give (for an up option)

$$J(S, t^+) = \begin{cases} J(S, t^-) + 1 & S \geq S^* \\ 0 & S < S^* \end{cases} \tag{3}$$

where $t^+ = t + \epsilon$ and $t^- = t - \epsilon$ ($\epsilon > 0;\ \epsilon \to 0$). For a down option, we obtain (from equation (2))

$$J(S, t^+) = \begin{cases} J(S, t^-) + 1 & S \leq S^* \\ 0 & S > S^* \end{cases} \tag{4}$$

Consequently, equations (1-2) specify that $J = J(S, t)$ is a counter which tracks the number of successive observations for which the underlying asset price lies above (below) the barrier. The counter J is reset to zero if the asset price falls below (above) the barrier at an observation date. This is illustrated in Figure 1 for an up option. We can now consider a Parisian option under the usual Black-Scholes assumptions that markets are frictionless, that there is a constant interest rate

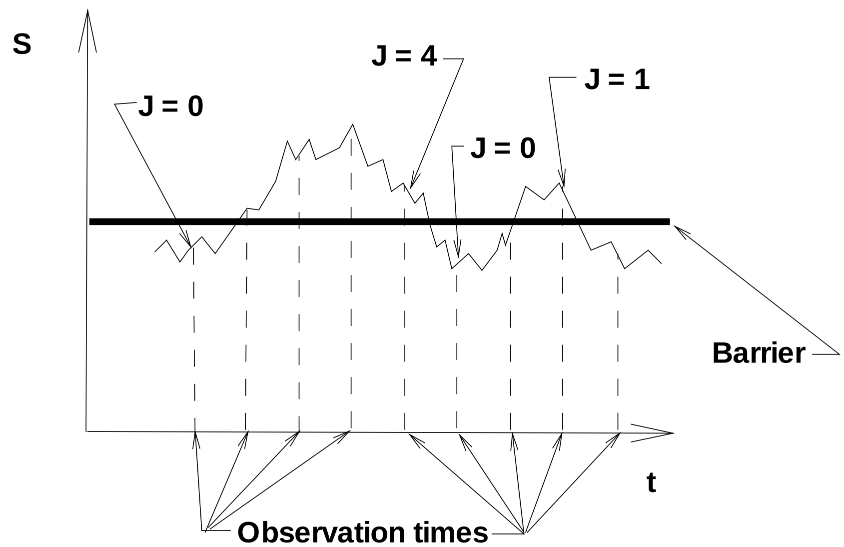

FIGURE 1: *Illustration of the value of counter J at various observation times for a Parisian up option. The counter is incremented when the underlying asset price is above the barrier, and reset to zero if it falls below the barrier.*

r, and that S evolves according to:

$$dS = \mu S dt + \sigma S dz \tag{5}$$

where μ is the expected rate of return on S, σ is the volatility, and z is a Wiener process. The evolution of J is given by:

$$dJ = \frac{dJ}{dt} dt \quad . \tag{6}$$

Let $U = U(S, J, t)$ denote the price of an option. Then, following standard arbitrage arguments (see, e.g. Wilmott, Dewynne, and Howison (1993)), the following PDE is obtained for the option value:

$$U_t + \frac{dJ}{dt} U_J + \frac{\sigma^2 S^2}{2} U_{SS} + rSU_S - rU = 0 \quad . \tag{7}$$

It is convenient to convert equation (7) into an equation forward in time by substituting $\tau = T - t$

where T is the expiry date of the option to give:

$$U_\tau = \frac{dJ}{dt} U_J + \frac{\sigma^2 S^2}{2} U_{SS} + rSU_S - rU \tag{8}$$

where $\frac{dJ}{dt}$ can be written in terms of $\tau = T - t$:

$$\frac{dJ}{dt} = \begin{cases} \lim_{\lambda\to\infty} \sum_i H(S - S^*)\delta(\tau - \tau_i) - J\lambda H^-(S^* - S)\delta(\tau - \tau_i) & \text{for an up option} \\ \lim_{\lambda\to\infty} \sum_i H(S^* - S)\delta(\tau - \tau_i) - J\lambda H^-(S - S^*)\delta(\tau - \tau_i) & \text{for a down option} \end{cases} \quad . \tag{9}$$

The term $\frac{dJ}{dt} U_J$ in equation (8) is zero except at observation times τ_i. As $\tau \to \tau_i$ equation (8) reduces to

$$U_\tau - \frac{dJ}{dt} U_J \simeq 0 \quad . \tag{10}$$

If $\tau^+ = \tau_i + \epsilon$ and $\tau^- = \tau_i - \epsilon$ then equation (10) specifies mathematically that the solution is constant along characteristics. The financial implication of this is that the option value is continuous across the sampling date.[4] This can be written as

$$U(S, J^+, \tau^+) = U(S, J^-, \tau^-) \tag{11}$$

where

$$J^- = \begin{cases} J^+ + 1 & S \geq S^* \\ 0 & S < S^* \end{cases} \tag{12}$$

for an up option and

$$J^- = \begin{cases} J^+ + 1 & S \leq S^* \\ 0 & S > S^* \end{cases} \tag{13}$$

for a down option.

A delayed barrier option is triggered when the underlying asset price stays above (below) the barrier for a total number of observations $J = J^*$, in contrast to a Parisian option where the barrier is triggered only if the observations are consecutive. In the delayed barrier case, equation (9) becomes

$$\frac{dJ}{dt} = \begin{cases} \sum_i H(S - S^*)\delta(\tau - \tau_i) & \text{for an up option} \\ \sum_i H(S^* - S)\delta(\tau - \tau_i) & \text{for a down option} \end{cases} \quad . \tag{14}$$

[4]See Section 8.3 of Wilmott, Dewynne, and Howison (1993) for more detailed arguments along these lines in the context of vanilla options where discrete dividends are paid by the underlying asset.

Consequently, the value of J^- in equation (11) for a delayed barrier up option is

$$J^- = \begin{cases} J^+ + 1 & S \geq S^* \\ J^+ & S < S^* \end{cases} \quad , \tag{15}$$

whereas for a delayed barrier down option

$$J^- = \begin{cases} J^+ + 1 & S \leq S^* \\ J^+ & S > S^* \end{cases} \quad . \tag{16}$$

We now describe the payoff and boundary conditions. Denoting the exercise price by K, the terminal payoff conditions are

$$\begin{aligned} U(S, J = J^*, \tau = 0) &= \begin{cases} \max(S-K,0) & \text{for a call} \\ \max(K-S,0) & \text{for a put} \end{cases} \\ U(S, J \neq J^*, \tau = 0) &= 0 \end{aligned} \tag{17}$$

for an in option and

$$\begin{aligned} U(S, J = J^*, \tau = 0) &= 0 \\ U(S, J \neq J^*, \tau = 0) &= \begin{cases} \max(S-K,0) & \text{for a call} \\ \max(K-S,0) & \text{for a put} \end{cases} \end{aligned} \tag{18}$$

$$\tag{19}$$

for an out option. The intuition here is straightforward. We simply determine whether or not the option has been knocked-in or knocked-out according to the value of J relative to J^* and then apply the usual call and put option payoffs for states where the option is potentially exercisable.

We now consider the boundary condition at $J = J^*$, $S < \infty$, when $\tau > 0$. For an in option, we solve equation (8) ignoring the U_J term:

$$U_\tau = \frac{\sigma^2 S^2}{2} U_{SS} + rSU_S - rU \ ; \ J = J^* \quad . \tag{20}$$

This, of course, just reflects the fact that this option has been knocked-in and so we really have a standard put or call here. For an out option, we apply the Dirichlet condition

$$U(S, J = J^*, \tau) = 0 \ ; \ 0 \leq S < \infty \tag{21}$$

to incorporate the knocked-out status of the option. Similarly, when $J = J^*$, $S \to \infty$ we have

$$U(S \to \infty, J = J^*, \tau) = \begin{cases} S & \text{for a call} \\ 0 & \text{for a put} \end{cases} . \tag{22}$$

for an in option and

$$U(S \to \infty, J = J^*, \tau) = 0 \tag{23}$$

for an out option. Finally, we consider the boundary condition where $J \neq J^*$, $S \to \infty$. Initially ($\tau = 0$), we have

$$U(S \to \infty, J \neq J^*, \tau = 0) = 0 \tag{24}$$

for an in option and

$$U(S \to \infty, J \neq J^*, \tau = 0) = \begin{cases} S & \text{for a call} \\ 0 & \text{for a put} \end{cases} . \tag{25}$$

for an out option. These conditions are also used when $\tau > 0$ except at observation times τ_i, when the boundary conditions at $S \to \infty$, $J \neq J^*$ are reset according to conditions (11). Note that no reset takes place at $J = J^*$.

It is worth concluding this section by noting how this setup can be easily generalized to other contexts. Different types of securities can be valued by simply changing the payoff and boundary conditions suitably. For example, discretely monitored switch or step options may be priced by making the terminal payoff an appropriate function of J. Parisian options or delayed barrier options with double barriers can be handled by changing the way the counter variable J is incremented (through redefining the H and H^- functions) and the payoff functions and/or boundary conditions. Changing the H and H^- functions over time would allow us to examine situations where the specified range in the contract is time-varying. The CEV process case can be dealt with simply by changing $\sigma^2 S^2$ to $\sigma^2 S^\alpha$ (where α is a parameter) in equation (8). It is easy to handle cases where the underlying asset pays either a continuous dividend yield or a discrete dollar dividend. There is no requirement that observation dates be equally-spaced, so weekend effects with daily sampling pose no difficulty. Finally, the same basic methodology can readily be adapted to higher dimensional settings (though obviously at a cost of much greater computational complexity). Examples might include stochastic volatility models and options on more than one asset.

3 The Numerical Algorithm

Away from observation dates, equation (8) has no J dependence. From equations (11-16), we can see that information is required only at specific values of J, in particular at $J = 0, 1, 2, \ldots, J^*$. This is shown in Figure 2. Consequently, at most $J^* + 1$ lines of constant J are required. Typically,

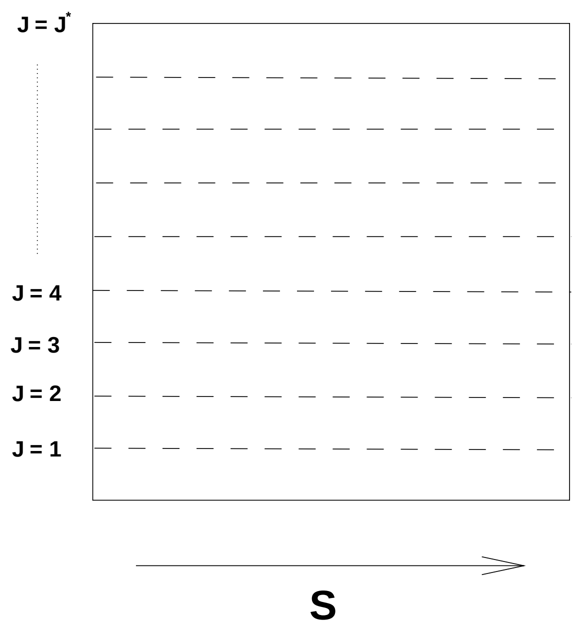

FIGURE 2: *Numerical domain for Parisian and delayed barrier options. At most, $J^* + 1$ lines of constant J are required.*

since J^* is quite small, (e.g. < 20), this means that the pricing problems at hand consist of a small number of one dimensional problems which exchange information only at observation dates.

Several different solution methods are possible. For instance, it may be feasible in some situations to use an analytic solution for the one dimensional PDE between sampling dates and then to generate a new initial condition at each sampling date. However, to maintain as much generality as possible, we do not explore this type of approach here, relying on a fully numerical technique. Within the set of numerical alternatives one possible approach would be to use a standard finite element or finite difference algorithm to solve each of the independent one dimensional PDEs, with initial and boundary conditions given in equations (17-25). At each observation date, the jump conditions (11) would be applied. This is shown schematically for a Parisian option in Figure 3. Yet another possibility is to employ a two dimensional finite element or finite difference technique. Due to superior grid spacing flexibility, the finite element approach is the preferred choice here. The illustrative computations below were in fact calculated using a two dimensional finite ele-

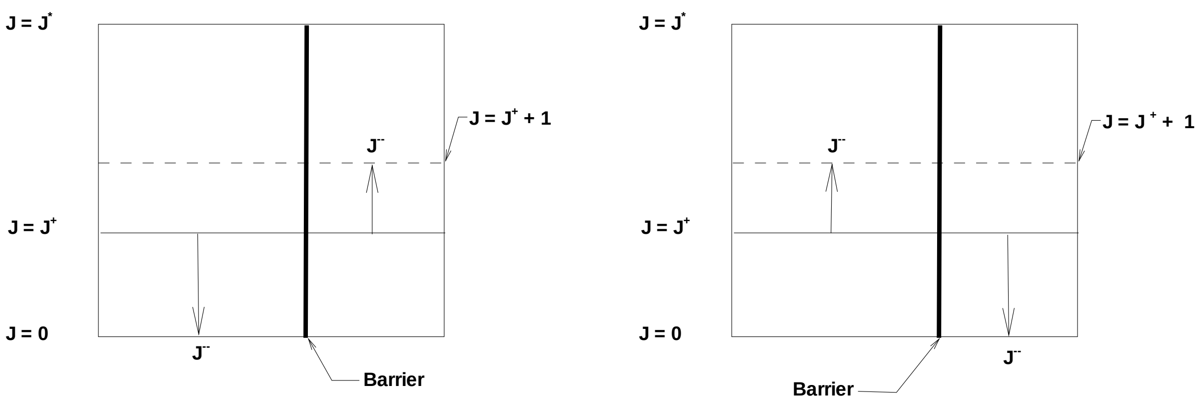

FIGURE 3: *Jump conditions for a Parisian option at observation times. Left: up option, right: down option.*

ment method, described in detail in Forsyth, Zvan, and Vetzal (1997) and Forsyth, Vetzal, and Zvan (1997). This approach was selected because it was very simple to modify the general two dimensional finite element method for the types of problems considered here, though if one was particularly concerned with computational speed it would probably be better to solve a set of one dimensional PDEs between the sampling dates.

Note that discontinuities may be introduced at observation times. This means that it is more efficient to use a fully implicit method than either an explicit method or a Crank-Nicolson method, since the time step size required to preclude the formation of spurious oscillations in the numerical solution becomes very small for these latter two approaches. See Zvan, Vetzal, and Forsyth (1997, 1998) for a much more detailed discussion of this point. Our algorithm also incorporated an automatic time step size selector, as described in Zvan, Vetzal, and Forsyth (1997). In practice, we also use a normalized computational domain, by defining a new variable $y = J/J^*$, so that the maximum value for $y = 1$, independent of J^*. The jump conditions at observation times are then altered. For example, equation (12) becomes

$$y^- = \begin{cases} y^+ + \Delta y & S \geq S^* \\ 0 & S < S^* \end{cases} \tag{26}$$

with $\Delta y = 1/J^*$. The other jump conditions are modified in a similar fashion.

4 Numerical Examples

We consider both Parisian and delayed barrier options, using the common data given in Table 1. Since only a fixed number of nodes are required in the J direction, we carried out convergence studies by increasing the number of nodes in the S direction. A non-uniform grid was employed with very fine spacing near the barrier $S = S^*$.

TABLE 1: *Common data for the numerical examples.*

σ	.25
r	0.05
Time to expiry	0.5 years
Barrier observation times	daily (1/250 of a year)
Exercise price	\$100
Barrier location	\$120

Table 2 demonstrates the convergence of the solution for the price, delta $= U_S$ and gamma $= U_{SS}$ for a Parisian option with $J^* = 10$. The automatic time step size selector was adjusted so that

TABLE 2: *Convergence of Parisian call option, $J^* = 10$ days. The number of nodes is fixed in the J direction (11 nodes for $J^* = 10$ days). Variable grid spacing is used in the S direction. The fine grid run also uses smaller time steps.*

Nodes (S direction)	Quantity	$S = 100$	$S = 110$	$S = 120$
221	price	3.1547	2.9904	1.3855
	delta	0.061958	−0.097592	−0.20120
	gamma	−0.013847	−0.015508	0.0016005
Smaller time steps				
441	price	3.1585	2.9942	1.3840
	delta	0.062142	−0.097818	−0.20252
	gamma	−0.013870	−0.015553	0.0014075

approximately twice as many time steps were taken for the fine grid case. As a result, the differences between the find and coarse grid numbers reflect the effects of both time and space discretization errors. The solution appears to be adequately resolved with 441 nodes in the S direction. This grid spacing and these time step parameters are used in all subsequent calculations. As an additional check on the accuracy of the numerical results, we considered a Parisian option with $J^* = 1$. In this case, a Parisian option becomes simply a discrete barrier option, with daily sampling. Figure 4 shows the results for gamma $= U_{SS}$ for both the grid sizes (and time step parameters) given in

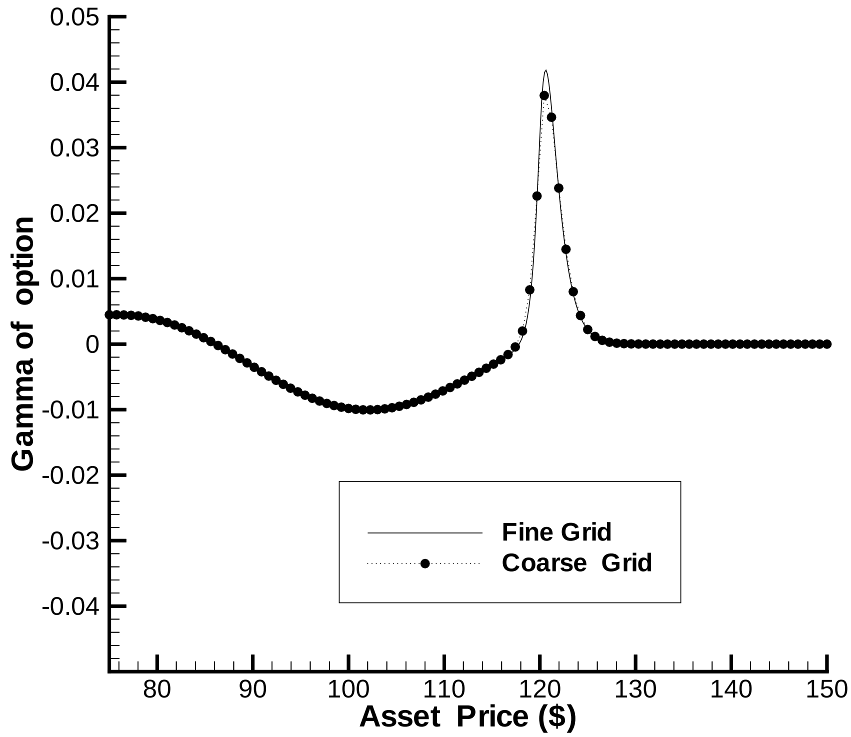

FIGURE 4: *Comparison of fine and coarse grid results for Gamma (U_{SS}) for a knock-out Parisian call option, which is knocked-out if there is a single observation over the barrier of \$120. This is actually a standard discrete barrier option with daily observation. It is a worst case for the discretization errors. The plots of the option value U and delta (U_S) for the two different grid sizes are indistinguishable. The fine grid used 441 nodes and the coarse grid used 221 nodes (as in Table 2). The fine grid run also used smaller time steps, so this figure indicates the effect of both time and space discretization errors.*

Table 2. Plots of the price U and delta $= U_S$ are indistinguishable for the coarse and fine grid runs, and hence are not shown. Figure 4 indicates that the errors for U_{SS} are very small, with the only observable error being a slight underestimation of the peak value of U_{SS} on the coarse grid. It is worth emphasizing that the represents a worst case scenario for the errors. The gamma is harder to estimate than either the delta or the price, since it has such a sharp peak. Moreover, the peak is sharpest with more frequent sampling. Fine grid results are shown in all the following figures.

The two panels of Figure 5 plot the price of a knock-out call option of the Parisian and delayed barrier types, for various values of J^*. The solution profiles exhibit broadly similar patterns.

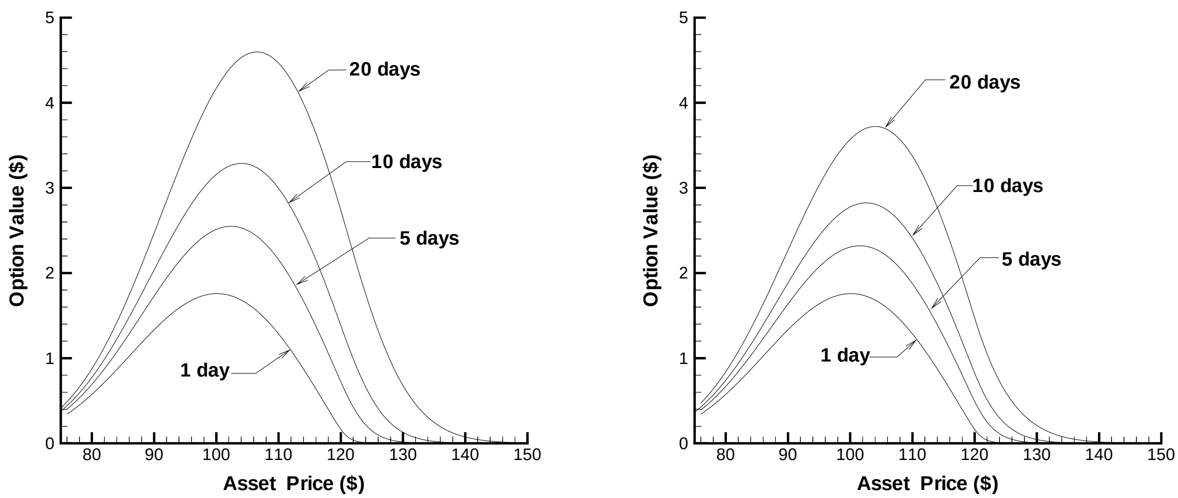

FIGURE 5: *Knock-out call option value for various values of J^*. Left: Parisian option; right: delayed barrier option.*

Observe that when $J^* = 1$, these two options are exactly the same contract, and as noted above, they are equivalent to a standard discrete barrier option with daily monitoring. Note that the option payoff with $J^* = 1$ is not discontinuous at the barrier — it has positive value when S exceeds the barrier value of \$120 because we assume that the next observation time is one day from the current time, implying that there is still some probability of S dropping below the barrier before knock-out. For $J^* > 1$, the delayed barrier option is worth less than the corresponding Parisian option, the difference increasing with J^*. This is because it is less probable that the underlying asset will remain above the barrier for a number of consecutive days than for a total number of days over the contract life. In other words, there is a lower knock-out probability for the Parisian option, so it is more valuable.

Figure 6 plots the deltas for the Parisian and delayed barrier options. These change quite rapidly near the barrier value of \$120, especially for the $J^* = 1$ case. Of course, this merely demonstrates again the well-known difficulties in using delta hedging strategies for barrier options as small changes in the underlying asset price can imply potentially large hedging errors. It is worth noting, however, that these effects are mitigated somewhat when $J^* > 1$: the larger is the value of J^*, the less steep are the slopes of the graphs near the barrier. This is particularly true for the Parisian option case. Another measure of possible hedging problems is provided in Figure 7, which plots the option gammas. The sharp peaks near the barrier are again indicative of such problems. The differences between the Parisian and delayed barrier option cases when $J^* > 1$ are perhaps most apparent here, as the peak gets noticeably smoother (and shifts lower and to the right) for

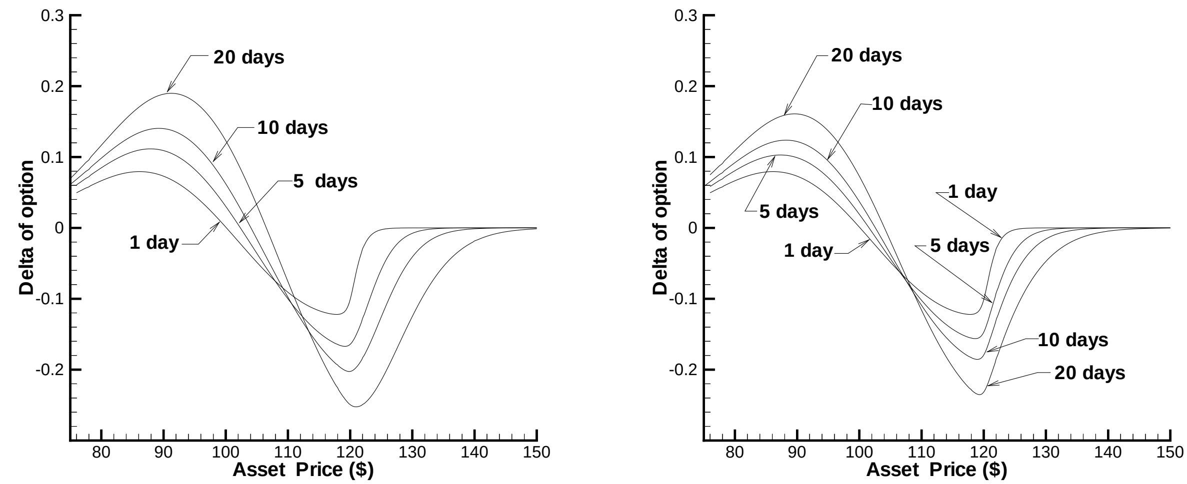

FIGURE 6: *Delta (U_S) of knock-out call option for various values of J^*. Left: Parisian option; right: delayed barrier option.*

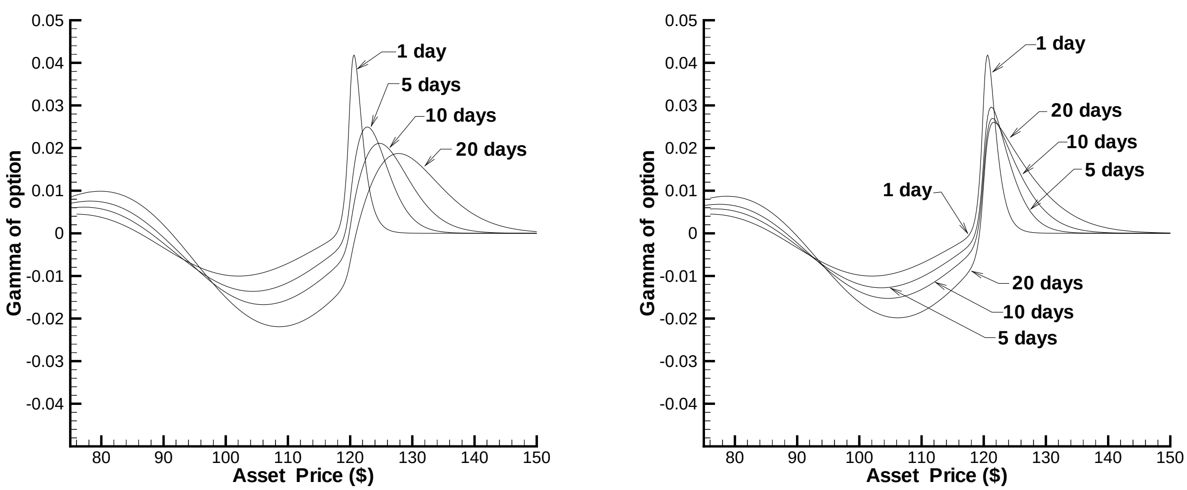

FIGURE 7: *Gamma (U_{SS}) of knock-out call option for various values of J^*. Left: Parisian option; right: delayed barrier option.*

the Parisian option in particular. Note that even though both of these types of options (especially the Parisian) are evidently easier to hedge near the barrier than standard discrete barrier options, this is not necessarily true for all values of the underlying asset price. For instance, the relatively large negative value of gamma near $S = 110$ when J^* is, say, 20, indicates possible hedging losses were S to move rapidly, even for a delta-neutral portfolio.

Finally, the three panels of Figure 8 plot the value, delta, and gamma of a knock-in Parisian call option for different values of J^*. The solution profiles for the option value are broadly similar, declining with J^* in an obvious reflection of the lower knock-in probability associated with higher values of J^*. Again, the standard discrete barrier option case of $J^* = 1$ poses the biggest hedging problem near the barrier of \$120. This is most evident from the plot of the gamma of the option.

5 Conclusion

This paper has described a general numerical method for pricing derivative securities with a payoff dependent on how much time an underlying variable lies within a certain range before the contract matures. The particular examples considered were discretely monitored Parisian and delayed barrier options. It was shown that such options (particular the Parisian type) were easier to hedge than standard discrete barrier options when the underlying asset price is near the barrier. It was also demonstrated that this is not necessarily the case for other values of the underlying asset price.

The method described here is simple to adapt to handle other types of contracts such as step options and switch options, as well as more complex contractual provisions such as time-varying barriers, irregularly-spaced monitoring dates, and an additional asset. It is also straightforward to incorporate dividends (either in the form of a continuous dividend yield or as a discrete dollar payment). Finally, we note that since our methods are based on the general framework for path-dependent options described in Wilmott, Dewynne, and Howison (1993), it is also possible to use techniques similar to those described in this paper to value other types of derivative securities such as Asian options and lookbacks. We leave these applications to future research.

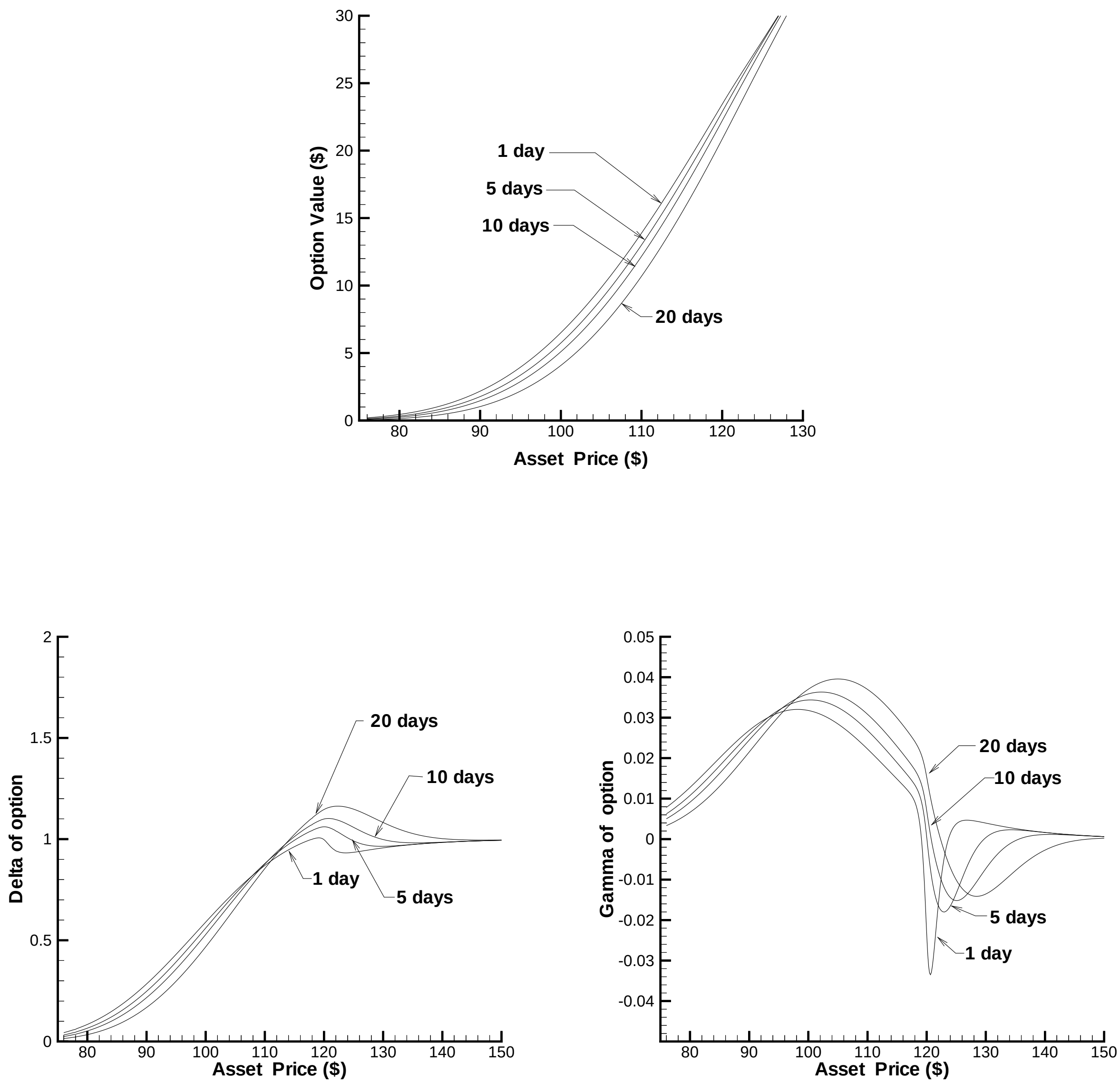

FIGURE 8: *Knock-in Parisian call option, various values of J^*. Top: option value; lower left: delta (U_S); lower right: gamma (U_{SS}).*

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