PENALTY METHODS FOR AMERICAN OPTIONS WITH STOCHASTIC VOLATILITY

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Abstract. The American early exercise constraint can be viewed as transforming the two dimensional stochastic volatility option pricing PDE into a differential algebraic equation (DAE). Several methods are described for forcing the algebraic constraint by using a penalty source term in the discrete equations. The resulting nonlinear algebraic equations are solved using an approximate Newton iteration. The solution of the Jacobian is obtained using an incomplete LU (ILU) preconditioned PCG method. Some example computations are presented for option pricing problems based on a stochastic volatility model, including an exotic American chooser option written on a put and call with discrete double knockout barriers and discrete dividends.

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1. Introduction. A derivative is a security whose value depends on one or more underlying factors. Derivative markets are rapidly growing. For example, the total notional value of outstanding derivatives was $5 trillion in 1990 and was over $20 trillion in 1994.

Options are a form of derivative security that give the holder the right, but not the obligation, to buy or sell an asset for a specified exercise price at some future time. It is of great interest to financial institutions to be able to determine the value of an option as a function of the underlying factors and time.

Utilizing models of asset prices based on stochastic differential equations, a non-stochastic partial differential equation (PDE) for the price of the option can be derived [28]. This PDE has the familiar form of the multi-dimensional convection-diffusion equation.

Many options have an early exercise feature. This allows the holder of the option to exercise the option at any time during its life [28]. An option with this early exercise feature is known as an American option. An option which cannot be exercised early is termed a European option. Assuming that investors act optimally, the value of an American option cannot fall below the value that would be obtained if it was exercised early. Effectively, this means that the option pricing partial differential equation becomes a differential algebraic equation (DAE).

If an implicit method is used to solve the basic option pricing PDE, then the algebraic constraint (due to the early exercise feature) should, in general, also be handled implicitly. One method for incorporating the constraint is to view the problem as a linear complementarity problem [28] and then use projected SOR [6] to solve the discrete algebraic equations. However, in regions where it is not optimal to exercise the option early, this method simply reduces to unaccelerated SOR for solving the sparse linear system. Unaccelerated SOR iterative methods have, of course, been supplanted by the more robust PCG-like techniques [20, 24, 26, 23]. Projected SOR can be accelerated using a multigrid method [5]. While multigrid methods can sometimes be spectacularly successful, they must often be tuned to the problem at hand. Care must be taken with the choice of smoother, and the prolongation and restriction operators. For example, in [5], the smoother must be adjusted to fit early exercise and non-early exercise parts of the computational domain. It is therefore a daunting task, at the present time, to produce black box option pricing software based on multigrid techniques which can be used in day-to-day financial applications. An alternative method based on linear programming [9] has recently been proposed. However, if the underlying PDE is more than one dimensional, then the linear programming method used in [9] may become computationally infeasible.

The objective of this article is to develop a general method for handling the American early exercise feature. We simply view the problem as a nonlinear differential algebraic system (DAE), where the (in general) nonlinear constraint can be imposed using a penalty method. The resulting system of nonlinear algebraic equations is then solved using Newton iteration, where the nonsymmetric Jacobian at each nonlinear iteration is solved using PCG-like methods [23]. The advantages of this approach are

- Software can be developed based on black box *off-the-shelf* components. The sparse Jacobian is solved using a standard method. The Jacobian itself can be constructed using a variety of techniques.
- Since we regard the system as nonlinear right from the start, there is no difficulty incorporating more sophisticated discretization methods such as nonlinear flux limiters [25, 19]. In many cases, such as for an option pricing PDE based on a stochastic volatility model [15], the PDE has large regions which are convection dominated, and hence standard central or upstream weighting methods are inappropriate.
- Incorporation of other types of constraints (e.g. time dependent barriers [30]) can be done in a straightforward fashion, since the algorithm does not depend on the form of the constraint or the form of the PDE.

In this paper, we give examples of the use of this technique for pricing options based on a stochastic volatility model. To illustrate the flexibility of this approach, we include an example of an exotic American chooser option [16] written on a barrier put and call, which has a complex early exercise constraint.

2. Stochastic Volatility. Recently, there has been some interest in models where the volatility of the underlying asset is a random variable [17, 21, 15, 27]. Consider an option which is a function of the asset price s and the variance v, where s and v evolve according to [15]:

$$\begin{aligned} ds &= \mu s dt^* + \sqrt{v} s dz_1 \\ dv &= \kappa(\theta - v) dt^* + \sigma \sqrt{v} dz_2 \end{aligned} \tag{1}$$

where z_1, z_2 are Wiener processes [28]. Note that $\sqrt{v}$ is the instantaneous volatility of the asset price s. Stochastic volatility models are considered to be a more realistic specification of stock price movement than models with constant volatility such as the classic Black-Scholes [3] analysis. Following the usual steps [16], the PDE for the value of an option $U = U(s, v, t^*)$ is:

$$\frac{vs^2}{2} U_{ss} + \rho\sigma v s U_{sv} + \frac{\sigma^2 v}{2} U_{vv} + rsU_s + (\kappa(\theta - v) - \lambda v) U_v - rU + U_{t^*} = 0 \tag{2}$$

where:

$$\begin{aligned} \rho &= \text{correlation between } dz_1, dz_2 \\ \sigma &= \text{volatility of volatility} \\ \kappa &= \text{mean reversion time constant} \\ \theta &= \text{mean reverting value of } v \\ \lambda &= \text{market price of volatility risk} \\ r &= \text{interest rate} \end{aligned} \tag{3}$$

Equation (2) is solved backward in time from the expiry date of the option $t^* = T$ to the current time $t^* = 0$. Equation (2) can be converted to the familiar form of an equation forward in time by substituting $t = T - t^*$ to give:

$$U_t = \frac{vs^2}{2} U_{ss} + \rho \sigma v s U_{sv} + \frac{\sigma^2 v}{2} U_{vv} + rsU_s + \left(\kappa(\theta - v) - \lambda v\right) U_v - rU. \tag{4}$$

Following some algebraic manipulations, equation (4) can be put into the following form:

$$U_t + \mathbf{V} \cdot \nabla U = \nabla \cdot \mathbf{D} \cdot \nabla U - rU \tag{5}$$

where:

$$\mathbf{D} = \frac{1}{2} \begin{pmatrix} vs^2 & \rho \sigma s v \\ \rho \sigma s v & \sigma^2 v \end{pmatrix} \tag{6}$$

and

$$\mathbf{V} = - \begin{pmatrix} rs - vs - \rho \sigma s / 2 \\ \kappa(\theta - v) - \lambda v - \sigma^2/2 - \rho \sigma v / 2 \end{pmatrix}. \tag{7}$$

Equation (5) has the form of the convection-diffusion equation. The initial conditions depend on the contractually agreed payoff function. For a vanilla (standard) put or call, the initial conditions (at $t = 0$ or equivalently at $t^* = T$) are:

$$\begin{aligned} U(s, v, 0) &= \max(s - E, 0) \; ; \quad \text{Call} \\ &= \max(E - s, 0) \; ; \quad \text{Put} \\ & \quad E = \text{ exercise price.} \end{aligned} \tag{8}$$

Other boundary conditions for this equation can be determined by examining the original equation (2). Letting $v, s \to 0$ we obtain:

$$\begin{aligned} U_t &= rsU_s + \kappa \theta U_v - rU; \qquad v \to 0 \\ U_t &= \frac{\sigma^2 v}{2} U_{vv} + (\kappa(\theta - v) - \lambda v) U_v - rU; \qquad s \to 0 \end{aligned} \tag{9}$$

For $s \to \infty$ we have:

$$\begin{aligned} U &= s \qquad \text{Call} \\ U &= 0 \qquad \text{Put.} \end{aligned}$$

Finally, noting that as $v \to \infty$ then $U_v \to 0$, so

$$U_t = \frac{vs^2}{2} U_{ss} + rsU_s - rU; \quad v \to \infty. \tag{10}$$

3. Discretization. We will now discretize equation (5) using a standard Galerkin finite element method for the diffusion terms. For the convective terms, we will use a finite volume approach. Formally, a finite volume discretization can be considered to be a Galerkin method with a special quadrature rule [13], so that in a mathematical sense, a Galerkin finite element method is being used for all terms in the equation. However, it is more intuitively appealing to use a geometric finite volume approach for discretizing the convective term.

Consider a discrete two dimensional computational domain R which is tiled by triangles. Let N_i be the usual C^0 Lagrange basis functions defined on triangles. Then,

$$
\begin{aligned}
N_i &= 1 \text{ at node } i \\
&= 0 \text{ at all other nodes} \\
\sum_j N_j &= 1 \text{ everywhere in the solution domain.}
\end{aligned}
\tag{11}
$$

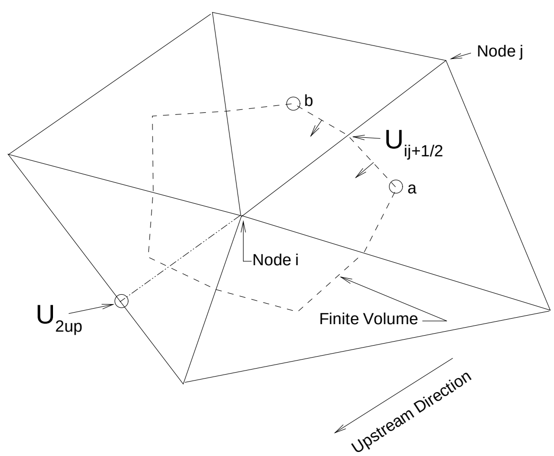
If $U^n = \sum_j U_j^n$ where $U_j^n = U(s_j, v_j, t^n)$ is the value of U at (s_j, v_j, t^n), then the discretization of equation (5) is given by:

$$
\begin{aligned}
A_i \left(\frac{U_i^{n+1} - U_i^n}{\Delta t} \right) &= (1-\beta) \left(\sum_{j \in \eta_i} \gamma_{ij} (U_j^{n+1} - U_i^{n+1}) + \sum_{j \in \eta_i} \vec{L_{ij}} \cdot \mathbf{V}_i U_{ij+1/2}^{n+1} - A_i r U_i^{n+1} \right) \\
&\quad + \beta \left(\sum_{j \in \eta_i} \gamma_{ij} (U_j^n - U_i^n) + \sum_{j \in \eta_i} \vec{L_{ij}} \cdot \mathbf{V}_i U_{ij+1/2}^n - A_i r U_i^n \right) \\
&\quad + q_i^{n+1}
\end{aligned}
\tag{12}
$$

where:

$$
\begin{aligned}
A_i &= \int N_i dR \\
\Delta t &= \text{timestep} \\
\beta &= \text{timeweighting} \\
&\quad \beta = 0 \text{ fully implicit} \\
&\quad \beta = 1 \text{ explicit} \\
&\quad \beta = 1/2 \text{ Crank-Nicolson} \\
U_i^{n+1} &= U(s_i, v_i, t^{n+1}) \\
\gamma_{ij} &= -\int_R \nabla N_i \cdot \mathbf{D} \cdot \nabla N_j dR \\
\eta_i &= \text{set of neighbours of node } i \\
q_i &= \text{source/sink term used for boundary conditions} \\
&\quad \text{and American constraints.} \\
U_{ij+1/2}^{n+1} &= \text{value of } U \text{ at the face between} \\
&\quad \text{node } i \text{ and node } j.
\end{aligned}
\tag{13}
$$

FIG. 1. *Finite volume surrounding node i. Points a and b are the centroids of their respective triangles. The line segments from a and b pass through the midpoint of the triangle edge $i-j$.*



We have also used mass lumping for the time derivative term. Other details concerning this discretization method can be found in [11, 13]. Note that A_i can be considered to be the area of the *cell* or finite volume surrounding node i. The finite volume surrounding node i is shown in Figure 1. The finite volume is constructed by joining the midpoint of each edge of a triangle to the centroid of the triangle [1, 18, 12]. The vector length $\vec{L_{ij}}$ in equation (12) is given by

$$\vec{L_{ij}} = \int_a^b \hat{n} ds \tag{14}$$

where the points a, b are shown in Figure 1, and $\hat{n}$ is the inward pointing normal to the face between node i and node j. An alternative choice of finite volume can be based on the perpendicular bisectors of triangle edges [12].

There are various choices for the terms $U_{ij+1/2}$, For example, second order central weighting for $U_{ij+1/2}$ is given by:

$$U_{ij+1/2} = \frac{U_i + U_j}{2} \tag{15}$$

while first order upstream weighting is given by

$$\begin{aligned} U_{ij+1/2} &= U_i \text{ if } \vec{L_{ij}} \cdot \mathbf{V}_i < 0 \\ &= U_j \text{ otherwise.} \end{aligned} \tag{16}$$

Note that equation (4) becomes first order hyperbolic as $v \to 0$. First order upstream weighting is usually too diffusive for accurate solutions, while central weighting may cause spurious oscillations in convection dominated regions. Recently, non-linear flux limiters have been used to obtain accurate solutions without causing oscillations. Essentially, these methods use a more accurate (usually second order) method as much as possible, but reduce to lower order accuracy only where necessary to avoid spurious oscillations [29, 30]. One popular method uses a van Leer limiter [25, 19, 4]. With reference to Figure 1, assume that node i is upstream of node j (the upstream directions are given by equation (16)). Point $2up$ is the value of U which is upstream of node i, interpolated using the two nearest nodes where U is known (see Figure 1). The value of $U_{ij+1/2}$ is then extrapolated to the face $(ij+1/2)$ using the values at U_i and U_{2up} [14]. A nonlinear limiter is applied to avoid spurious oscillations in the solution [2, 19, 12]. In this work we will use the van Leer limiter. Other possibilities include the smooth MUSCL limiter described in [2].

4. Solution of the Discrete Equations. The discrete equations (12) are in general non-linear. This is due to the use of a nonlinear flux limiter for the convection term, and also due to the application of the American constraint. The method used to apply this constraint, in an implicit fashion, will be described in a subsequent section. An approximate Newton iteration will be used to solve the discrete equations. The complete Jacobian is constructed with the exception of all derivatives with respect to the second upstream points U_{2up}, which are ignored. The iteration for a given timestep is deemed to have converged when

$$\max_i \frac{|(U_i^{n+1})^{k+1} - (U_i^{n+1})^k|}{\max(|(U_i^{n+1})^{k+1}|, |(U_i^n)|, 1.0)} < tol \tag{17}$$

where $(U_i^{n+1})^k$ is the k^{th} iterate for U_i^{n+1}. The Jacobian is solved using an incomplete LU [7, 8] preconditioned CGSTAB iteration [26]. An automatic timestep selection method is also used [22].

5. American Options. American options are easily handled in a fully implicit fashion, through suitable definition of the source/sink term in equation (12). Two approaches will be discussed in this paper. Effectively, these are penalty methods for forcing the discrete problem to satisfy the early exercise constraint.

5.1. Constraint Switching. If an American option is to be priced, then we define two possible states for a node {ON, OFF}. The source term (equation (12)) is then defined as:

IF ($\text{state}_i = \text{ON}$) then

$$q_i^{n+1} = \frac{A_i}{\Delta t}(U_i^* - U_i^{n+1}) \times Large$$

ELSE

$$q_i^{n+1} = 0$$

ENDIF

(18)

where U_i^* is the value of the option if exercised immediately. For a vanilla American put, this is given by equation (8). In equation (18), $Large$ is a suitably defined large number.

After each nonlinear iteration, the state of each node can be switched:

```
IF ( state_i = ON )
      IF ( U_i^{n+1} > U_i^* ) then
            state_i := OFF
      ENDIF
ELSE
      IF ( U_i^{n+1} < U_i^* ) then
            state_i := ON
      ENDIF
ENDIF
```

(19)

Note that when $state_i$ = ON, then we must have $q_i^{n+1} > 0$ (since the American constraint adds value). Consequently, as $Large \to \infty$, then for nodes with $state_i$ = ON, then $U_i^{n+1} \to U_i^* - \epsilon$, where $\epsilon = 0(1/Large)$. This error in enforcing the constraint can be made arbitrarily small by making $Large$ sufficiently large.

The transition rules in equation (19) are based on the assumption that a minimum constraint is being imposed. In the case of an option with both maximum and minimum type constraints (e.g. callable convertible bonds [28]), there would be three possible states for a node, with the obvious changes to the transition rules.

5.2. Quadratic Source Term. If Newton iteration is used to solve the nonlinear discrete equations which result from use of the constraint switching method in (19), then the Jacobian has a discontinuous derivative at $U_i^{n+1} = U_i^*$, which might cause some difficulties. An alternative approach uses a smoother method of implementing the constraint. The source/sink term in equation (12) can be defined as:

$$q_i^{n+1} = \frac{A_i}{\Delta t}(\min(U_i^{n+1} - U_i^*, 0))^2 \times Large \tag{20}$$

where $Large$ is a suitably defined large number and U_i^* is the value of the option if exercised immediately.

Imagine solving the discrete equations with the source term (20), by a Newton iteration. Suppose the initial guess for the solution at U_i^{n+1} uses the value of U_i^n, and suppose that this value is above the value obtained by early exercise. Consequently, on the first iteration, the source term (20) is zero. If after the first iteration, $U_i^{n+1} > U_i^*$, then it is not optimal to exercise early, and the iteration terminates. However, if the first iteration produces $U_i^{n+1} < U_i^*$, then the source term becomes nonzero, and then forces another nonlinear iteration. Since the source term is positive, the next iteration will produce a larger value for U_i^{n+1}. The quadratic form for the source term will cause a monotonic approach to a value of $U_i^{n+1} = U_i^* - \epsilon$ with $\epsilon \ll 1$. The size of ϵ will be determined by the size of $Large$. The larger the value of this constant, the smaller ϵ, but in general the number of nonlinear iterations will increase as $Large$ increases in magnitude.

5.3. Equivalence of Penalty Method and Linear Complementarity Formulation. Let the vector with components U_i^{n+1} be denoted by $\mathbf{U}^{n+1}$. Similarly

$$(\mathbf{q}^{n+1})_i = q_i^{n+1}$$
$$(\mathbf{U}^*)_i = U_i^*. \tag{21}$$

Let α be the the region of the computational domain D where:

$$q_i^{n+1} > 0 \quad \text{if} \quad (x_i, y_i) \in \alpha. \tag{22}$$

In other words, α is the region of D where it is optimal to exercise the option early. Let

$$\begin{aligned} (L\mathbf{U}^{n+1})_i &= \left(\frac{U_i^{n+1} - U_i^n}{\Delta t}\right) \\ &- \left\{(1-\beta)\left(\sum_{j\in\eta_i} \gamma_{ij}(U_j^{n+1} - U_i^{n+1}) + \sum_{j\in\eta_i} \vec{L_{ij}} \cdot \mathbf{V}_i U_{ij+1/2}^{n+1} - A_i r U_i^{n+1}\right)\right\} \\ &- \left\{\beta\left(\sum_{j\in\eta_i} \gamma_{ij}(U_j^n - U_i^n) + \sum_{j\in\eta_i} \vec{L_{ij}} \cdot \mathbf{V}_i U_{ij+1/2}^n - A_i r U_i^n\right)\right\}. \end{aligned} \tag{23}$$

For simplicity, we consider only the constraint switching method in the following. From equation (18) as $Large \to \infty$ we have

$$\begin{aligned} (U_i^{n+1} - U_i^*) &= -|O(\frac{1}{Large})| \quad (x_i, y_i) \in \alpha \\ &> 0 \qquad\qquad\qquad (x_i, y_i) \in D - \alpha \end{aligned} \tag{24}$$

From equations (18,23,24) it follows that

$$\begin{aligned} (L\mathbf{U}^{n+1})_i &= O(\frac{A_i}{\Delta t}) \qquad (x_i, y_i) \in \alpha \\ &= 0 \qquad\qquad (x_i, y_i) \in D - \alpha. \end{aligned} \tag{25}$$

Equations (24,25) then imply that

$$\begin{aligned} (\mathbf{U}^{n+1} - \mathbf{U}^*) &\geq -|O(\frac{1}{Large})| \qquad (x_i, y_i) \in D \\ L\mathbf{U} &\geq 0 \qquad\qquad\qquad (x_i, y_i) \in D \\ |(\mathbf{U}^{n+1} - \mathbf{U}^*) \cdot L\mathbf{U}| &\leq O(\frac{1}{Large})\frac{A_D}{\Delta t} \qquad (x_i, y_i) \in D \\ A_D &= \sum_i A_i. \end{aligned} \tag{26}$$

Therefore, as $Large \to \infty$, equation (26) can be regarded as an approximation to

$$\begin{aligned} (\mathbf{U}^{n+1} - \mathbf{U}^*) &\geq 0 \\ L\mathbf{U} &\geq 0 \\ (\mathbf{U}^{n+1} - \mathbf{U}^*) \cdot L\mathbf{U} &= 0 \end{aligned} \tag{27}$$

TABLE 1
Data for American put with stochastic volatility.

σ	.9
ρ	.1
κ	5.0
θ	.16
λ	0.0
r	.10
Time to expiry	.25 years
Exercise Price (E)	$10
U_i^*	$\max(E - s_i, 0)$

for all $(x, y) \in D$. This is a discrete form of the linear complementarity formulation of the American constraint [28]. The linear complementarity approach is an identical numerical problem to a discrete variational inequality [28]. Consequently, it is possible to demonstrate, in some cases, that a unique solution exists, and that the discrete solution converges to a solution having C^1 continuity across the early exercise boundary [10].

6. Clark and Parrot Problem [5]. An American put option with stochastic volatility was extensively studied in [5]. The data for this problem are given in Table 1.

Table 2 gives the values of the American Put computed on an 89×52 grid, and a refined grid formed by inserting a node between each node in the coarse grid (176×102). The timestep selector parameters [22] were halved for the fine grid as well. The values of *Large* and the convergence Newton iteration tolerance *tol* are given in Table 2. For comparison, the finest grid results from [5] are also given. A Crank-Nicolson timestepping method was used.

The results are in general agreement with those in [5], but there are some differences. Note that in this work, the computations on the fine grid used smaller timesteps than the coarse grid results. Hence, the results in Table 2 (for this work) reflect both time and space truncation errors. In contrast, in [5], a constant timestep was used on all grids. As well, interpolation was used in [5] to obtain the values shown in Table 2 (a coordinate transformation was used in [5] to obtain discrete equations more suitable for a multigrid approach). These effects probably account for the differences between this work and the results in [5].

Table 3 compares the results for the above problem with various values for *tol* and *Large*. The coarse 89×52 grid was used for these computations. The constraint switching method (see Section 5.1) was used. This table should be viewed as comparing the effects of using different values of nonlinear convergence tolerance *tol* and *Large* (see equations (18 - 19)), for a given grid size and timestep sequence.

Note that if a tolerance of $tol = 10^{-k}$ is desired, then the value of *Large* should be $\simeq 10^k$. Examination of Table 3 shows that, as expected, there is no change in the solution to five figures for $k \geq 4$.

Table 2

Convergence of American put with stochastic volatility. Constraint switching method used with $tol = 10^{-5}$, $Large = 10^5$. Table values are the value of an American put with the stochastic volatility model, in dollars at the initial time $t^* = 0$ ($t = T$).

v	s				
	8.0	9.0	10.0	11.0	12.0
	89×52 grid				
.0625	2.0000	1.1078	0.5206	0.2142	0.0823
.25	2.0784	1.3338	0.7963	0.4485	0.2427
	177×103 grid, smaller timesteps				
.0625	2.0000	1.1076	0.5202	0.2138	0.0821
.25	2.0784	1.3337	0.7961	0.4483	0.2428
	Results in [5], finest grid				
.0625	2.0000	1.1080	0.5316	0.2261	0.0907
.25	2.0733	1.3290	0.7992	0.4536	0.2502

Table 4 shows similar results, but this time the quadratic source term (see Section 5.2) was used. For a quadratic penalty term, if an accuracy of $tol = 10^{-k}$ is desired, then $Large$ should be $\simeq 10^{2k}$. Although moderate accuracy can be obtained with the quadratic source term, difficulties were observed when requesting very tight convergence tolerances (note the non-convergence for $tol = 10^{-5}$ in Table 4).

The results shown here are representative of our observations for many problems. It appears that the constraint switching method is more efficient and reliable than the quadratic source method. This appears surprising at first glance, since the quadratic term would seem to be more easily solved using Newton iteration. However, the timesteps required for reasonable levels of time discretization error are quite small, so that the discontinuity in the derivative of the constraint switching source term does not appear to have serious consequences.

To isolate the effect of the American constraint on the nonlinear iterations, Table 5 shows the total number of nonlinear iterations required for solution of the above problem with various discretization methods.

If pure upstream weighting is used (equation (16)), then the only nonlinearity in the discrete equations is due to the American constraint. In this case, Table 5 indicates that about five nonlinear iterations per timestep is required to resolve the American constraint to five figure accuracy. In contrast, solving a European problem using the flux limiter requires about four iterations per timestep. Finally, use of the flux limiter with the American constraint requires almost the same number of total nonlinear iterations as with the American constraint with upstream weighting. Of course, the solutions to all these problems are not identical, so the comparisons are not perfectly valid. Nevertheless, it appears that the cost of using the DAE approach for the early exercise constraint, coupled with the flux-limited discretization, is not much more expensive than using the flux-limiter alone. However, if a European option is being priced using upstream weighting, then this is a purely linear problem, and only one iteration per timestep is necessary. Consequently, the cost of solving an American option with a

TABLE 3

Constraint switching method used with indicated values of Large and convergence tolerance tol. 89 × 52 grid used. Problem from [5].

s					
8.0	10.0	12.0	8.0	10.0	12.0
v=.0625			v=.25		
$tol = 10^{-3}$, $Large = 10^3$ Total nonlinear iterations = 100					
2.0000	0.5206	0.0823	2.0783	0.7963	0.2427
$tol = 10^{-4}$, $Large = 10^4$ Total nonlinear iterations = 117					
2.0000	0.5206	0.0823	2.0784	0.7963	0.2427
$tol = 10^{-5}$, $Large = 10^5$ Total nonlinear iterations = 125					
2.0000	0.5206	0.0823	2.0784	0.7963	0.2427
$tol = 10^{-6}$, $Large = 10^6$ Total nonlinear iterations = 133					
2.0000	0.5206	0.0823	2.0784	0.7963	0.2427
$tol = 10^{-8}$, $Large = 10^8$ Total nonlinear iterations = 154					
2.0000	0.5206	0.0823	2.0784	0.7963	0.2427

TABLE 4

Quadratic source method used with indicated values of Large and convergence tolerance tol. 89 × 52 grid used. Problem from [5].

s					
8.0	10.0	12.0	8.0	10.0	12.0
v=.0625			v=.25		
$tol = 10^{-3}$, $Large = 10^6$ Total nonlinear iterations = 98					
1.9999	0.5205	0.0823	2.0783	0.7963	0.2427
$tol = 10^{-4}$, $Large = 10^8$ Total nonlinear iterations = 140					
2.0000	0.5206	0.0823	2.0783	0.7963	0.2427
$tol = 10^{-5}$, $Large = 10^{10}$ Total nonlinear iterations = $****$					
Not Converged					

TABLE 5

Constraint switching with upstream weighting and flux limiter, convergence tolerance 10^{-5}, $Large = 10^5$. For upstream weighting, the only nonlinearity is due to the American constraint. Problem from [5].

Method	Number of Nonlinear iterations	Number of timesteps
American upstream	122	27
European with Flux Limiter	104	27
American Flux Limiter	125	27

flux-limited discretization is about five times greater than for a European option with upstream weighting, for this quite severe convergence criteria.

7. An American Chooser Based on European Barrier Options. A chooser option gives the holder the right to either a call or a put at maturity [16]. The payoff for a chooser is given by:

$$\begin{aligned}
\text{Payoff} \;=\; & \max(C(s,v,T_C - T, E_C), P(s,v,T_P - T, E_P)) \\
& C = \text{ Value of Call} \\
& P = \text{ Value of Put} \\
& s = \text{ asset price} \\
& v = (\text{volatility})^2 \\
& E_C = \text{ Call exercise price} \\
& E_P = \text{ Put exercise price} \\
& T_C = \text{ Maturity date of Call} \\
& T_P = \text{ Maturity date of Put} \\
& T = \text{ Maturity date of Chooser.}
\end{aligned}$$

The value of a chooser is determined by solving for a put over the period $T_P \rightarrow T$, and then solving for a call over the period $T_C \rightarrow T$. The payoff $U(s_i, v_i, t^* = T) = U_i^0$ for the chooser at node i is then given by

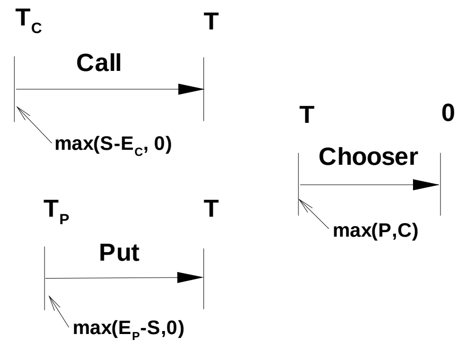
$$U_i^0 = \max(P_i, C_i). \tag{28}$$

This provides the initial condition for equation (4), which is then solved over the life of the chooser option. This is illustrated in Figure 2.

In this example, we will also be specifying discrete dollar dividends. Discrete dollar dividends are easily handled with an unstructured grid. If t^+ and t^- represent the times just before and after the dividend dates (recall that $t = T - t^*$), then

$$U(s, v, t^+) = U(s - D^*, v, t^-) \tag{29}$$

FIG. 2. *Schematic of Chooser option. Put P and call C are solved over different periods (T_C, T_P), with exercise prices E_C, E_P, for underlying asset price S. Then the maximum of the put and call values at each node gives the terminal payoff of the chooser, which is then solved over the life of the chooser.*



where D is the dividend payment and

$$D^* = \min(D, s). \tag{30}$$

Equation (30) prevents the unrealistic phenomenon of dividend payments being larger than the asset price. The value of $U(s - D^*, v, t^-)$ is interpolated using linear interpolation on the triangular mesh.

We give an example for an American chooser option written on a European put and call (European options cannot be exercised early). The European put and call have double knockout barriers, which are observed weekly. More formally, the knockout barriers are defined as

$$\begin{aligned} U(s, v, t^+) &= U(s, v, t^-) \\ & \quad \text{if } 80 \le s \le 110 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{31}$$

where t^+, t^- are the times just after and just before application of the barrier. Note that equation (31) imposes a jump discontinuity on the solution after each barrier observation date. Barriers are used to reduce the cost of an option, which is desirable for purchasers of the option if they believe that the underlying asset is likely to trade only within a restricted range. The data for the European put and call are given in Table 6. The data for the chooser, with initial condition given by equation (28) is given in Table 7. No barriers are applied to the chooser option.

The early exercise constraint is implemented using constraint switching, with the value of U_i^* (equation (18)) given by

$$U_i^* = U_i^0 \tag{32}$$

TABLE 6

Data for the stochastic volatility put and call, which are the basis for the chooser option. These are European options with discretely observed barriers.

σ	.5
ρ	-.5
κ	.2
θ	.04
λ	0.0
r	.05
Time to expiry	.5 years
Exercise Price: Put	$100
Exercise Price: Call	$90
Dividend	$1.00 quarterly
Knockout barriers at	$80 , $110
Barriers observed	weekly
Early Exercise	No

TABLE 7

Data for the stochastic volatility chooser. This is an American chooser (i.e. it can be exercised at any time).

σ	.5
ρ	-.5
κ	.2
θ	.04
λ	0.0
r	.05
Time to expiry	1.0 years
Dividend	$1.00 quarterly
Early Exercise	Yes

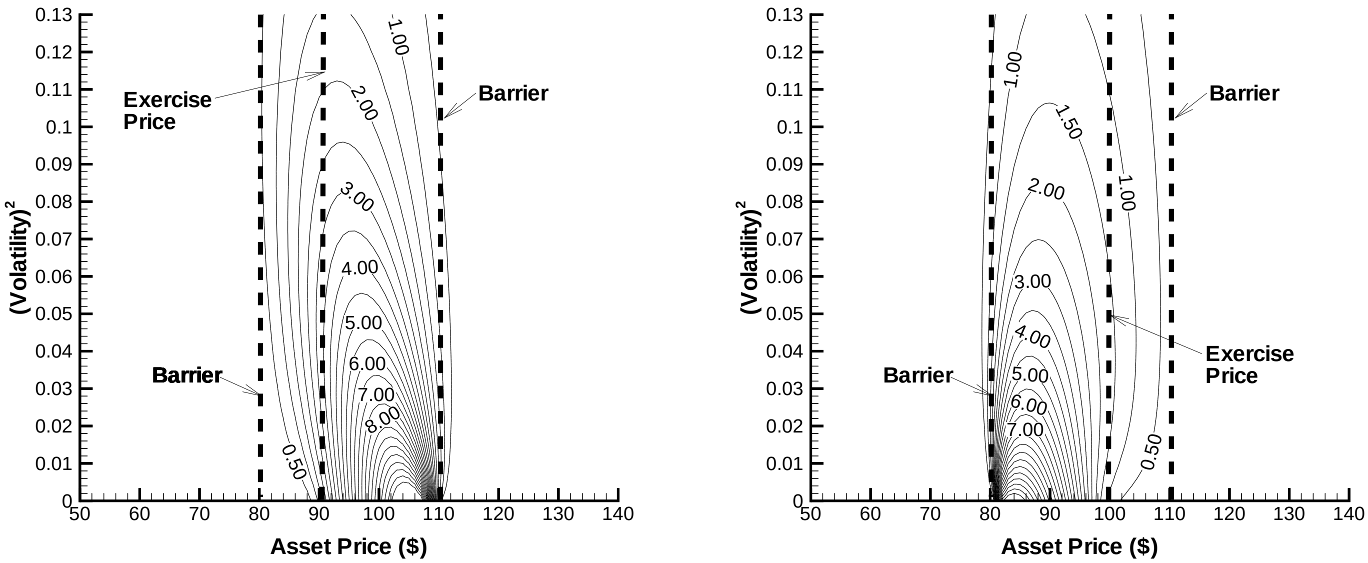


FIG. 3. *Value of European put and call at $t^* = T$. The put and call have discrete double knockout barriers at \$80 and \$110. Left: call, right: put.*

where U_i^0 is given in equation (28).

These problems were solved on a 123×76 grid. For the European put and call, a fully implicit timestepping method was used. This is necessary to avoid spurious oscillations, as discussed in [30] when pricing discrete barrier options. Crank-Nicolson timestepping was used for the chooser computation.

Figure 3 shows the results for the put and call at $t^* = T$ (see Figure 2). This data is used for the initial condition for the chooser (equation (28)) and the American constraint (equations(32)).

Figure 4 gives the results for the chooser option at the initial time ($t^* = 0$). For comparison, the results are also given for a chooser based on the same initial data, but without the American early exercise feature. Note the regions near $V = .04$, Asset Price =\$ 95, where the American chooser has significantly more value than the European version. Grid and timestep reduction studies show that the discretization errors in the region of interest are <\$0.01.

The optimal early exercise regions (at $t^* = 0$ and $t^* = 0.5$) are shown in Figure 5. These regions are determined from

$$\begin{aligned} U_i &< U_i^* - \epsilon \\ & \epsilon \ll 1. \end{aligned} \tag{33}$$

In these cases, the optimal early exercise regions are multiply connected, which causes no particular difficulty for the penalty method of satisfying the American constraint.

8. Conclusions. The American early exercise constraint for option pricing problems can be viewed as simply transforming the original convection diffusion equation

FIG. 4. *Value of a chooser option, written on a European put and call at $t^* = 0$. The put and call have double knockout barriers at \$80 and \$110. Left: American chooser, right: European chooser.*

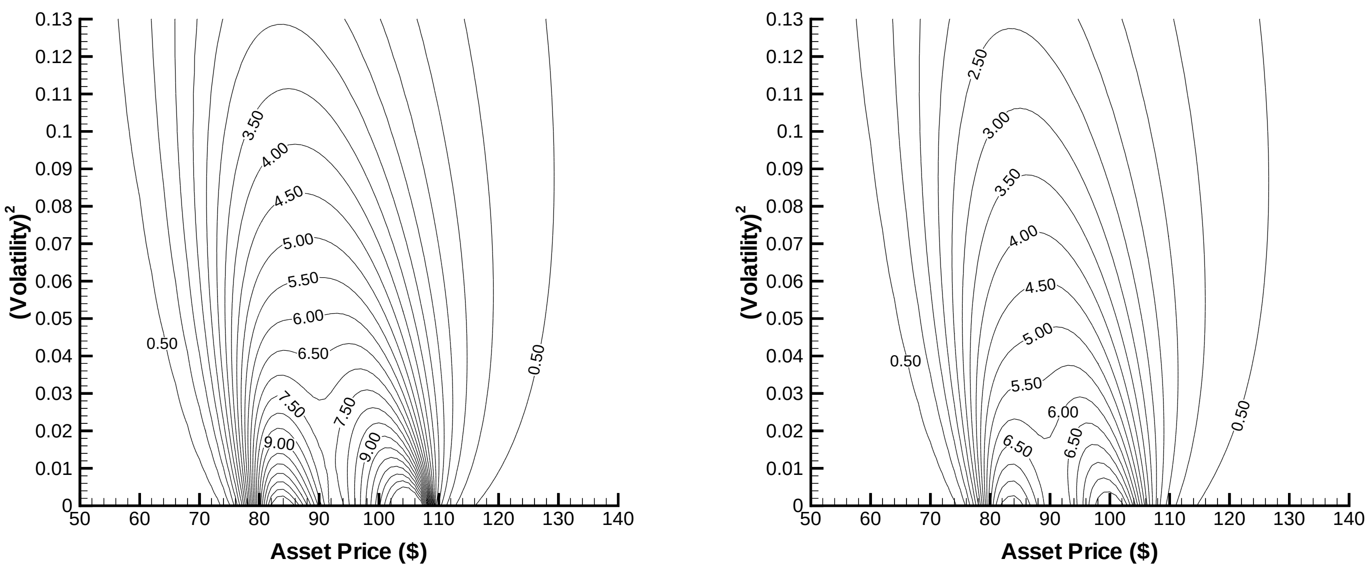
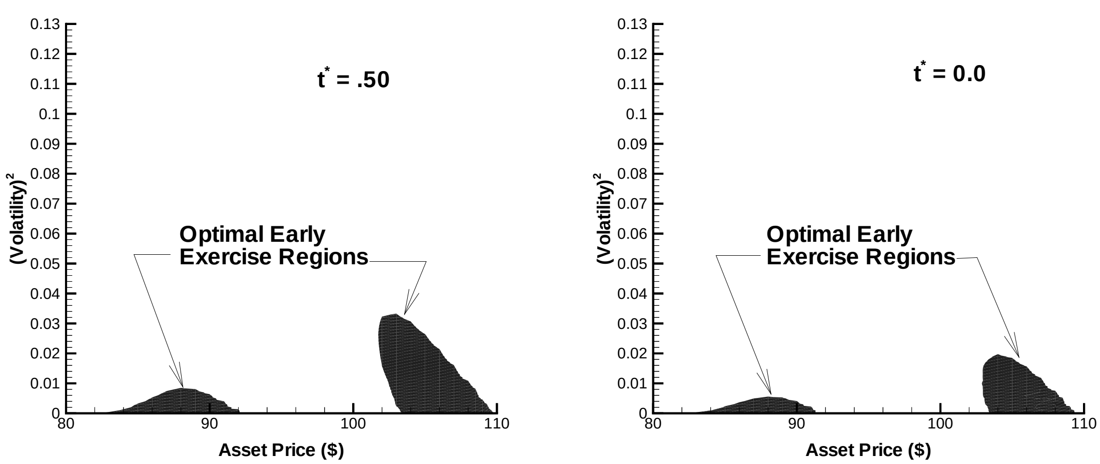


FIG. 5. *Optimal early exercise regions for the American chooser option, left: $t^* = 0.5$; right: $t^* = 0.0$.*



into a differential algebraic equation. Since option pricing constraints are typically nonlinear, the resulting set of nonlinear discretized equations can be solved by approximate Newton iteration. This approach allows for the use of modern, robust methods for iterative solution of the Jacobian matrix.

There are various ways to impose the early exercise constraint. A smoothly differentiable penalty method was compared with a constraint switching technique. The constraint switching method does not have a continuous derivative at points where the constraint is switched. Somewhat surprisingly, the constraint switching method was superior to the smooth penalty technique.

The constraint switching method for computing American options was demonstrated on some option pricing problems based on a stochastic volatility model (which gives rise to a problem in two space-like dimensions). Even very complex American constraints (e.g. an American chooser written on discrete barrier options with discrete dividends) with multiply connected early exercise regions were easily handled.

The method used here to impose early exercise constraint is very straightforward to implement. Other types of constraints (e.g. callable convertible bonds) are easily modelled. As long as an efficient sparse matrix solution method is used, there are no restrictions on using this technique for higher dimensional problems. Note that the computationally intensive part of these computations, the solution of the sparse Jacobian, is completely decoupled from the details of any particular model, which permits the use of modern sparse matrix software.

Since most stochastic models of the underlying assets for option pricing will result in a convection-diffusion problem, and virtually any type of constraint can be forced using a suitable definition of the discrete source/sink term, this means that it is possible to construct a modular library for pricing a wide variety of options. This is because the basic discrete equations are formally identical for a large number of different types of options. Use of modern object-oriented approaches to software development thus permit the user to develop complex new pricing models simply by writing a small number of virtual functions.

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