PDE METHODS FOR PRICING BARRIER OPTIONS

R. Zvan
Department of Computer Science
rzvan@yoho.uwaterloo.ca

K.R. Vetzal
Centre for Advanced Studies in Finance
kvetzal@watarts.uwaterloo.ca

P.A. Forsyth
Department of Computer Science
paforsyth@yoho.uwaterloo.ca

University of Waterloo
Waterloo, ON
Canada N2L 3G1

July 1997

This work was supported by the National Sciences and Engineering Research Council of Canada, the Information Technology Research Center, funded by the Province of Ontario, and the Centre for Advanced Studies in Finance at the University of Waterloo.

Abstract

This paper presents an implicit method for solving PDE models of contingent claims prices with general algebraic constraints on the solution. Examples of constraints include barriers and early exercise features. In this unified framework, barrier options with or without American-style features can be handled in the same way. Either continuously or discretely monitored barriers can be accommodated, as can time-varying barriers. The underlying asset may pay out either a constant dividend yield or a discrete dollar dividend. The use of the implicit method leads to convergence in fewer time steps compared to explicit schemes. This paper also discusses extending the basic methodology to the valuation of two asset barrier options and the incorporation of automatic time stepping.

1 Introduction

The market for barrier options has been expanding rapidly. By one estimate, it has doubled in size every year since 1992 (Hsu (1997), p. 27). Indeed, as Carr (1995) observes, "standard barrier options are now so ubiquitous that it is difficult to think of them as exotic" (p. 174). There has also been impressive growth in the variety of barrier options available. An incomplete list of examples would include double barrier options, options with curved barriers, rainbow barriers (also called outside barriers, for these contracts the barrier is defined with respect to a second asset), partial barriers (where monitoring of the barrier begins only after an initial protection period), roll up and roll down options (standard options with two barriers: when the first barrier is crossed the option's strike price is changed and it becomes a knock-out option with respect to the second barrier), and capped options. There are also numerous applications of barrier-type options to various issues involving default risk (see for example Merton (1974), Boyle and Lee (1994), Ericsson and Reneby (1996), and Rich (1996) among many others).

The academic literature on the pricing of barrier options dates back at least to Merton (1973), who presented a closed-form solution for the price of a continuously monitored down-and-out European call. More recently, both Rich (1991) and Rubinstein and Reiner (1991) provide pricing formulas for a variety of standard European barrier options (i.e. calls or puts which are either up-and-in, up-and-out, down-and-in, or down-and-out). More exotic variants such as partial barrier options and rainbow barrier options have been explored by Heynen and Kat (1994a, 1994b, 1996) and Carr (1995). Expressions for the values of various types of double barrier options with (possibly) curved barriers are provided by Kunitomo and Ikeda (1992), Geman and Yor (1996), and Kolkiewicz (1997). Broadie and Detemple (1995) examine the pricing of capped options (of both European and American style). Quasi-analytic expressions for American options with a continuously monitored single barrier are presented by Gao, Huang, and Subrahmanyam (1996).

This is undeniably an impressive array of analytical results, but at the same time it must be emphasized that these results generally have been obtained in a setting which suffers from one or more of the following potential drawbacks. First, it is almost always assumed that the underlying asset price follows geometric Brownian motion, but there is some reason to suspect that this assumption may be undesirable.[1] Second, in most cases barrier monitoring is assumed to be continuous, but in practice it is often discrete (e.g. daily or weekly). As noted by Cheuk and Vorst (1996) among others, this can lead to significant pricing errors.[2] Third, any dividend payments made by the underlying asset are usually assumed to be continuous. While this may be reasonable in the case of foreign exchange options, it is less justifiable for individual stocks or even stock indices (see for example Harvey and Whaley (1992)). Fourth, in most cases it is not possible to value American-style securities. Fifth, if barriers change over time, they are assumed to do so as an exponential function of time. Aside from analytical convenience, there would not seem to be any compelling reason to impose this restriction. Finally, it should be noted that the availability of a closed-form solution does not necessarily mean that it is easy to compute. For example, the expression obtained by Heynen and Kat (1996) for the value of a discrete partial barrier option requires high dimensional numerical integration.

Factors such as these have led several authors to examine numerical methods for pricing barrier

[1] Boyle and Tian (1997) examine the pricing of barrier and lookback options using numerical methods when the underlying asset follows the CEV process and report significant pricing deviations from the lognormal model, after controlling for differences in volatility. They conclude that the issue of model specification is much more important in the case of path-dependent options than it is for standard options.

[2] Broadie, Glasserman and Kou (1995, 1996) provide an accurate approximation of discretely monitored barrier option values using continuous formulas with an appropriately shifted barrier. This approach works in the case of a single barrier when the underlying asset distribution is lognormal.

options. For the most part, the methods considered have been some form of binomial or trinomial tree.[3] Boyle and Lau (1994) and Reimer and Sandmann (1995) each investigate the application of the standard binomial model to barrier options. The basic conclusion emerging from these studies is that convergence can be very poor unless the number of time steps is chosen in such a way as to ensure that a barrier lies on a horizontal layer of nodes in the tree. This condition can be hard to satisfy in any reasonable number of time steps if the initial stock price is close to the barrier or if the barrier is time-varying.

Ritchken (1995) notes that trinomial trees have a distinct advantage over binomial trees in that "the stock price partition and the time partition are decoupled" (p. 19).[4] This allows increased flexibility in terms of ensuring that tree nodes line up with barriers, permitting valuation of a variety of barrier contracts including some double barrier options, options with curved barriers, and rainbow barrier options. However, Ritchken's method may still require very large numbers of time steps if the initial stock price is close to a barrier.

Cheuk and Vorst (1996) modify Ritchken's approach by incorporating a time-dependent shift in the trinomial tree, thus alleviating the problems arising with nearby barriers. They apply their model to a variety of contracts (e.g. discrete and continuously monitored down-and-outs, rainbow barriers, simple time-varying double barriers). However, even though there is considerable improvement over Ritchken's method for the case of a barrier lying close to the initial stock price, this algorithm can still require a fairly large number of time steps.

Boyle and Tian (1996) consider an explicit finite difference approach. They finesse the issue

[3] One exception is provided by Andersen (1996), who explores the use of Monte Carlo simulation methods.

[4] It should be emphasized that this statement is true only up to a point. Trinomial trees are a form of explicit finite difference method and as such are subject to a well-known stability condition which requires that the size of a time step be sufficiently small relative to the stock price grid spacing.

of aligning grid points with barriers by constructing a grid which lies right on the barrier and, if necessary, interpolating to find the option value corresponding to the initial stock price.

Figlewski and Gao (1997) illustrate the application of an "adaptive mesh" technique to the case of barrier options. This is another tree in the trinomial forest. The basic idea is to use a fine mesh (i.e. narrower stock grid spacing and, because this is an explicit type method, smaller time step) in regions where it is required (e.g. close to a barrier) and to graft the computed results from this onto a coarser mesh which is used in other regions. This is an interesting approach and would appear to be both quite efficient and flexible, though in their paper Figlewski and Gao only examine the relatively simple case of a down-and-out European call option with a flat, continuously monitored barrier. It also should be pointed out that restrictions are needed to make sure that points on the coarse and fine grids line up. The general rule is that halving the stock price grid spacing entails increasing the number of time steps by a factor of four.

Each of these tree approaches may be viewed as some type of explicit finite difference method for solving a parabolic partial differential equation (PDE). In contrast, we propose to use an implicit method which has superior convergence (when the barrier(s) is close to the region of interest) and stability properties as well as offering additional flexibility in terms of constructing the spatial grid. The method also allows us to place grid points either near or exactly on barriers. In particular, we present an implicit method which can be used for PDE models with general algebraic constraints on the solution. Examples of constraints can include early exercise features as well as barriers. In this unified framework, barrier options with or without American constraints can be handled in the same way. Either continuously or discretely monitored barriers can be accommodated, as can time-varying barriers. The underlying asset may pay out either a constant dividend yield or a discrete dollar dividend. Note also that for an implicit method, the effects of an instantaneous change to boundary conditions (i.e. the application of a barrier) are felt immediately across the entire solution, whereas this is only true for grid points near the barrier for an explicit method. In other words, with an explicit method it will take several time steps for the effects of the constraint to propagate throughout the computational domain. Our proposed implicit method can achieve superior accuracy in fewer time steps.

The outline of the paper is as follows. Section 2 presents a detailed discussion of our methodology, including issues such as discretization and alternative means of imposing constraints. Section 3 provides illustrative results for a variety of cases. Extensions to the methodology are presented in Section 4. Section 5 concludes with a short summary.

2 Methodology

For expositional simplicity, we focus on the standard lognormal Black-Scholes setting. After performing a change of variables in order to convert the Black-Scholes PDE into a forward equation, we have

$$\frac{\partial V}{\partial t^*} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \tag{1}$$

where $t^* = T - t$, V denotes the value of the derivative security under consideration, S is the price of the underlying asset, σ is its volatility, and r is the continuously compounded risk free interest rate. We employ a discretization strategy which is commonly used in certain fields of numerical analysis such as computational fluid dynamics, though it appears to be virtually unknown in the finance literature. This is called a point-distributed finite volume scheme. For background details, the reader is referred to Roache (1972). Our reasons for choosing this approach are twofold: i) it is notationally simple (non-uniformly spaced grids can easily be described); and ii) it is readily adaptable to more complicated settings. In this setup the discrete version of equation (1) is given by:

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t^*} &= \theta F_{i-\frac{1}{2}}^{n+1} - \theta F_{i+\frac{1}{2}}^{n+1} + \theta f_i^{n+1} \\ &+ (1-\theta) F_{i-\frac{1}{2}}^n - (1-\theta) F_{i+\frac{1}{2}}^n + (1-\theta) f_i^n \end{aligned} \tag{2}$$

where V_i^{n+1} is the value at node i at time step $n+1$, Δt^* is the time step size, $F_{i-\frac{1}{2}}$ and $F_{i+\frac{1}{2}}$ are what is known in numerical analysis as flux terms, f_i is called a source/sink term and θ is a temporal weighting factor. To gain some intuition for this expression, think of the discrete grid as containing a number of cells. At the center of each cell i lies a particular value of the stock price, S_i. The change in value within cell i over a small time interval arises from three sources: i) the net flow into cell i from cell $i-1$; ii) the net flow into cell i from cell $i+1$; and iii) the change in value over the time interval due to discounting. In equation (2), the flux term $F_{i-\frac{1}{2}}$ captures the flow into cell i across the cell interface lying half-way between S_i and S_{i-1}. Similarly, the flux term $F_{i+\frac{1}{2}}$ captures the flow into cell $i+1$ from cell i across the interface midway between S_i and S_{i+1}. The change in value due to discounting is represented by the source/sink term f_i. The temporal weighting factor θ determines the type of scheme being used: fully implicit when $\theta = 1$, Crank-Nicolson when $\theta = \frac{1}{2}$, and fully explicit when $\theta = 0$.

For equation (1) the flux and source/sink terms at time level $n+1$ are defined as

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[-(\frac{1}{2}\sigma^2 S_i^2)\frac{(V_i^{n+1} - V_{i-1}^{n+1})}{\Delta S_{i-\frac{1}{2}}} - (rS_i)V_{i-\frac{1}{2}}^{n+1}\right] \tag{3}$$

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[-(\frac{1}{2}\sigma^2 S_i^2)\frac{(V_{i+1}^{n+1} - V_{i}^{n+1})}{\Delta S_{i+\frac{1}{2}}} - (rS_i)V_{i+\frac{1}{2}}^{n+1}\right] \tag{4}$$

where $\Delta S_i = \frac{1}{2}(S_{i+1} - S_{i-1})$, $\Delta S_{i+\frac{1}{2}} = S_{i+1} - S_i$, and

$$f_i^{n+1} = (-r)V_i^{n+1} \tag{5}$$

Corresponding definitions apply at time level n. Note that flux functions (3) and (4) allow for non-uniform grid spacing. This permits us to construct grids that have a fine spacing near the barriers and a coarse spacing away from the barriers.

The remaining terms to define in (3) and (4) are $V_{i-\frac{1}{2}}^{n+1}$ and $V_{i+\frac{1}{2}}^{n+1}$. These terms arise from the $(rS\frac{\partial V}{\partial S})$ term in the PDE. Generally, the Black-Scholes PDE can be solved accurately by treating this term using central weighting and we use this approach throughout this study.[5] In this case:

$$V_{i+\frac{1}{2}}^{n+1} = \frac{V_{i+1}^{n+1} + V_i^{n+1}}{2} \tag{6}$$

in equation (4). Furthermore, it is easy to verify that with central weighting the discretization given by equation (2) in the special case of a uniformly spaced grid is formally identical to the

[5] In more complex situations, more sophisticated methods may be required. For example, Zvan, Forsyth, and Vetzal (1996) demonstrate the use of one such alternative known as a flux limiter in the context of Asian options. Such methods may also be required if the interest rate is very high relative to the volatility. See Zvan, Forsyth, and Vetzal (1996, 1997) for further discussion.

standard type of discretization described in finance texts such as Hull (1993, section 14.7).[6]

Some interesting issues arise with respect to choice of the temporal weighting parameter θ. It is well-known that explicit methods may be unstable if the time step size is not sufficiently small relative to the stock grid spacing. On the other hand, both fully implicit and Crank-Nicolson methods are unconditionally stable. Both fully explicit and fully implicit methods are first-order accurate in time, whereas a Crank-Nicolson approach is second-order accurate in time. This seems to suggest that a Crank-Nicolson method might be the best choice, but in turns out that this is not correct in the case of barrier options. The reason is that applying a barrier can induce a discontinuity in the solution. A Crank-Nicolson method may then be prone to produce large and spurious numerical oscillations and very poor estimates of both option values and sensitivities (i.e. "the Greeks").

Zvan, Forsyth, and Vetzal (1996) have shown that in order to prevent the formation of spurious oscillations in the numerical solution, the following two conditions must be satisfied:

$$\Delta S_{i-\frac{1}{2}} < \frac{\sigma^2 S_i}{r} \tag{7}$$

and

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + r \tag{8}$$

Condition (7) is easily satisfied for most realistic parameter values for σ and r. Condition (8) is trivially satisfied when the scheme is fully implicit ($\theta = 1$). For a fully explicit or a Crank-Nicolson

[6] To be completely precise, there is a small difference in that it is traditional in finance to evaluate the rV term in the PDE at time level $n+1$ independent of θ. This permits the interpretation of an explicit method as a trinomial tree where valuation is done recursively using "risk-neutral probabilities" and discounting at the risk free rate.

scheme, condition (8) restricts the time step size as a function of the stock grid spacing. It is easily verified that the conditions which prevent oscillations in the fully explicit case are exactly the same as the commonly cited sufficient conditions which ensure that it is stable. Furthermore, even though a Crank-Nicolson approach is unconditionally stable, it can permit the development of spurious oscillations unless the time step size is no more than twice that required for a fully explicit method to be stable. Although a fully implicit scheme is only first-order accurate in time, it is our experience that the Black-Scholes PDE can be solved accurately using such a scheme. Hence, we chose to use a fully implicit method. This is advantageous because in order to obtain sufficiently accurate values for barrier options, the grid spacing near the barrier(s) generally needs to be fine. Thus, if a Crank-Nicolson method or a fully explicit scheme were used, the time step size would need to be prohibitively small in order to satisfy condition (8).

The appropriate strategy for imposing an algebraic constraint on the solution depends on the nature of the constraint. If the constraint is of a discrete nature (i.e. it holds at a point in time, not over an interval of time), such as a discretely monitored barrier, then it can be applied directly in an explicit manner. In other words, we compute the solution for a particular time level, apply the constraint if necessary, and move on to the next time level.[7] Consider the example of a discretely monitored down-and-out option with no rebate. If time level $n+1$ corresponds to a monitoring date, we first compute V^{n+1} and then apply the constraint:

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H \\ V_i^{n+1} & \text{otherwise} \end{cases} \tag{9}$$

[7]Note that this is exactly the way that the early exercise feature for American options has been traditionally handled in finance applications.

where H is the initial level of the barrier, h is a positive function which allows the barrier to move over time, and α^{n+1} is an arbitrary parameter. Note that for constant barriers h is always equal to one. Similarly, for a discretely monitored double knock-out option we compute V^{n+1} and if necessary apply the constraint:

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1}) H_{lower} \text{ or } S_i \geq h(t^{n+1}, \beta^{n+1}) H_{upper} \\ V_i^{n+1} & \text{otherwise} \end{cases} \tag{10}$$

where H_{lower} (H_{upper}) is the initial level of the lower (upper) barrier, and β^{n+1} is an arbitrary parameter.

If the constraint under consideration is not of a discrete nature, then it may be better to use an alternative strategy which imposes the constraint in an implicit fully coupled manner. This ensures that the constraint holds over a time interval (from time level n to $n+1$), not only at one point in time. Suppose for example that we want to value some kind of barrier call option with continuous early exercise opportunities. The constraint is $V_i^{n+1} \geq \max(S_i - K, 0)$. Zvan, Forsyth, and Vetzal (1996) demonstrate how to impose this in an implicit fully coupled manner. Instead of solving the discrete system given by (2) we solve

$$\frac{\Phi_i^{n+1} - V_i^n}{\Delta t^*} = F_{i-\frac{1}{2}}^{n+1}(V_{i-1}^{n+1}, V_i^{n+1}) - F_{i+\frac{1}{2}}^{n+1}(V_i^{n+1}, V_{i+1}^{n+1}) + f_i^{n+1}(V_i^{n+1}) \tag{11}$$

by constructing a Jacobian matrix and using full Netwon iteration, where for call options $V_i^{n+1} = \max(\Phi_i^{n+1}, S_i - K, 0)$.

Similar to American options, options with continuously applied barriers can be valued using

(11) where for down-and-out barrier options

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H \\ \Phi_i^{n+1} & \text{otherwise} \end{cases} \tag{12}$$

American barrier options where the barriers are applied continuously can be valued by incorporating the early-exercise feature into constraint (12) as follows:

$$V_i^{n+1} = \begin{cases} 0 & \text{if } S_i \leq h(t^{n+1}, \alpha^{n+1})H \\ \max(\Phi_i^{n+1}, S_i - K, 0) & \text{otherwise} \end{cases} \tag{13}$$

The importance of evaluating a constraint implicitly or explicitly appears to depend on the constraint itself. Zvan, Forsyth, and Vetzal (1996) report very little difference either way in computed values for standard American put options. However, as will be shown below, there can be a significant advantage to using the implicit fully coupled approach in the case of barrier constraints.

Finally, to handle cases where the underlying asset pays a discrete dollar dividend we use the jump condition (Willmott, Dewynne, and Howison (1993)) $V(S_i, t_+^*) = V(S_i - D, t_-^*)$ with linear interpolation, where D is the discrete dividend, and t_+^* and t_-^* are times just before and after the ex-dividend date, respectively. The case of a constant dividend yield can be dealt with in the usual way.

3 Results

This section presents a set of illustrative results. We focus on knock-out options with zero rebate in order to maximize comparability with existing published results. European knock-in option values may be calculated either directly or by using the fact that the sum of a knock-out option and the corresponding knock-in option generates a standard European option, at least when the rebate is zero. If the rebate is non-zero, or if the knock-in option is American style, the methods described by Reimer and Sandmann (1995) in the binomial context may be applied.

Results for European down-and-out call options where the barrier is applied continuously and discretely are contained in Table 1. The results are for cases where the barrier is close to the point of interest. That is, $H = 99.9$ and $S = 100$. Although the continuous application of the constant barrier effectively establishes a boundary condition at the same point throughout the life of the option, discretization (11) and constraint (12) were used to obtain the numerical solution for this case in order to maintain generality.

Continuous		Daily		Weekly	
PDE	Analytic	PDE	C & V	PDE	C & V
0.16	0.16	1.51	1.51	3.00	2.96

Table 1: European down-and-out call values when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $K = 100$ and $S = 100$. C & V denotes results obtained by Cheuk and Vorst (1996).

The results in Table 1 were obtained using non-uniform grids. A grid spacing of $\Delta S = 0.1$ near the barrier and $\Delta t^* = 0.05$ were used when the barrier was continuously applied. For the barrier applied daily $\Delta t^* = 0.0005$ and $\Delta S = 0.01$ near the barrier. A grid spacing of $\Delta S = 0.01$ near the barrier was used for the barrier applied weekly with $\Delta t^* = 0.0025$. The PDE results in Table 1 can be considered accurate to within \$0.01, since reduction of ΔS and Δt^* changed the solution by less than \$0.005. Table 1 indicates that the PDE method converges to a slightly higher value than obtained by Cheuk and Vorst (1996) for options were the barrier is applied weekly. This issue will be addressed later in the paper.

As noted by Cheuk and Vorst (1996), Table 1 illustrates that there is a considerable difference between continuous monitoring and discrete monitoring, even with daily monitoring. It is clearly inappropriate to use continuous models in the case of discrete barriers.

Figure 1 demonstrates the oscillatory solution obtained using the Crank-Nicolson method to

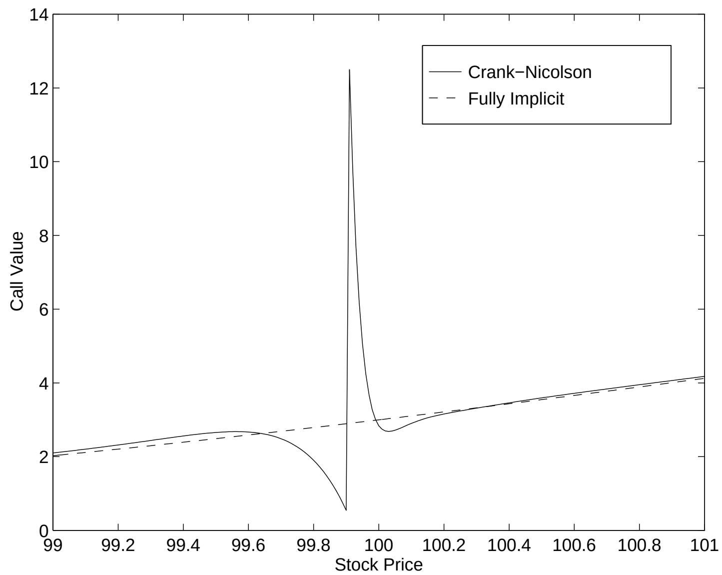

Figure 1: European down-and-out call option with a constant barrier applied weekly calculated using Crank-Nicolson and fully implicit schemes when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H = 99.9$ and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.01$ near the barrier was used and $\Delta t^* = 0.0025$.

value a European down-and-out call where the barrier is applied weekly. The grid spacing and time step size are identical to that used to obtain accurate solutions with a fully implicit scheme. The oscillations result because condition (8) was not satisfied. In order to satisfy condition (8) in the region of the barrier when a Crank-Nicolson scheme is used, the time step size must be less than 5.00×10^{-7}. This time step size is several orders of magnitude smaller than the time step size of $\Delta t^* = 2.50 \times 10^{-3}$ needed to obtain accurate results using a fully implicit scheme. Note that if a fully explicit scheme were used the stable time step size is less than 2.50×10^{-7}.

Barrier Application	European		Dividend	American
	PDE	C & V		
Continuously	2.04	2.03	1.92	5.25
Daily	2.48	2.48	2.32	5.90
Weekly	3.01	2.99	2.80	6.37

Table 2: Double knock-out call values with continuously and discretely applied constant barriers when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{lower} = 95$, $H_{upper} = 125$, $K = 100$ and $S = 100$. C & V denotes results obtained by Cheuk and Vorst (1996). *Dividend* denotes European option values where the underlying asset pays a discrete dividend of \$2 at $T - t = 0.25$. *American* denotes values for options that are continuously early-exercisable where the underlying asset does not pay dividends.

We also point out that oscillations are a potential problem with the Cheuk and Vorst (1996) algorithm, at least in some circumstances. As noted by Cheuk and Vorst, if the time step size is too large, then their tree probabilities can be negative. In such cases, their algorithm is not guaranteed to prevent oscillations.

We next consider double knock-out call options. Table 2 contains results for cases where the barriers are applied continuously and discretely. In Table 2, the results for the continuously applied barriers were obtained using a uniform spacing of $\Delta S = 0.5$ with $\Delta t^* = 0.0025$. The results for the discretely applied barriers were obtained using a non-uniform grid spacing of $\Delta S = 0.01$ near the barriers. The time step size was $\Delta t^* = 0.00025$ and $\Delta t^* = 0.001$ for barriers applied daily and weekly, respectively. Reduction of ΔS and Δt^* changed the PDE results in Table 2 by less than \$0.005.

In Table 2 we also include results for cases where the underlying asset pays a discrete dividend of \$2 at $T - t = 0.25$ and where there is no dividend paid but the option is American. They also show that the early exercise premia for the American cases are very large. Note that (at least in the continuously monitored case) this is due to the presence of the upper barrier — by Proposition

Down-and-out	$\Delta S = 0.1$ $\Delta t^* = 0.02$	$\Delta S = 0.05$ $\Delta t^* = 0.01$	$\Delta S = 0.025$ $\Delta t^* = 0.005$	$\Delta S = 0.0125$ $\Delta t^* = 0.0025$	$\Delta S = 0.00625$ $\Delta t^* = 0.00125$
	2.91	2.96	2.98	3.00	3.00
Double knock-out	$\Delta S = 0.5$ $\Delta t^* = 0.01$	$\Delta S = 0.25$ $\Delta t^* = 0.005$	$\Delta S = 0.125$ $\Delta t^* = 0.0025$	$\Delta S = 0.0625$ $\Delta t^* = 0.00125$	$\Delta S = 0.03125$ $\Delta t^* = 0.000625$
	2.97	2.99	3.00	3.01	3.01

Table 3: Successive grid refinements demonstrating convergence for European down-and-out and double knock-out call options with barriers applied weekly when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $K = 100$, and $S = 100$. For the down-and-out case, $H = 99.9$. For the double knock-out case, $H_{lower} = 95$ and $H_{upper} = 125$. ΔS denotes the grid spacing near the barrier(s).

5 c) of Reimer and Sandmann (1995), a continuously monitored American down-and-out call on a non-dividend-paying stock will not be optimally exercised early if the barrier is lower than the strike price.

As Tables 1 and 2 indicate, the PDE method generally converges to the converged values obtained by Cheuk and Vorst (1996). However, as seen in Tables 1 and 2, we found that the PDE method converged to slightly higher values (see Table 3) from those reported by Cheuk and Vorst for options with barriers that are applied weekly. We suspect that this may be due to a difference in when the barrier(s) is applied because of a difference in the definition of a weekly time interval. We defined a week to be 5 days of a 250 day year.

Figure 2 is a plot of the oscillations that result when the Crank-Nicolson method is used for a double knock-out barrier with the same grid spacing and time step size as was used to produce sufficiently accurate results with a fully implicit scheme. Again, condition (8) was violated, which resulted in severe oscillations near the barriers for the Crank-Nicolson method. Figure 3 is a plot of a European double knock-out option where the underlying asset pays a discrete dividend of \$2 at $T - t = 0.25$, and the barriers are applied weekly. Notice that the dividend case produces lower

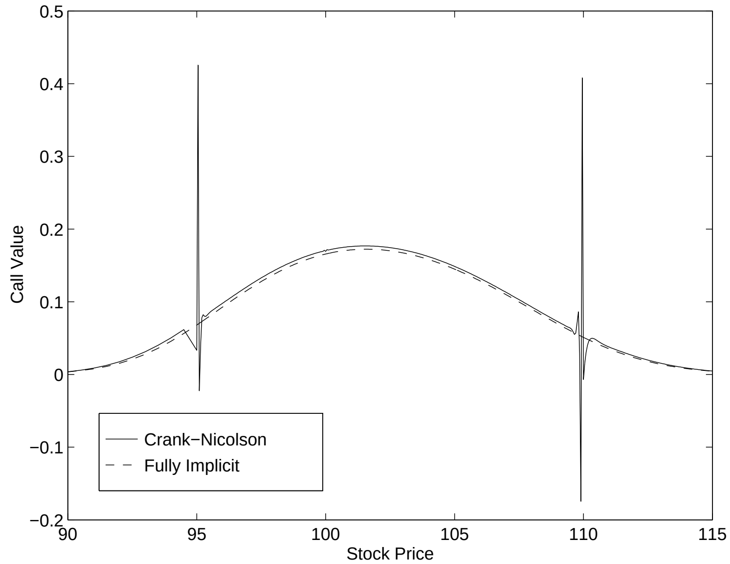

Figure 2: European double knock-out call option with a constant barrier applied weekly calculated using Crank-Nicolson and fully implicit schemes when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{lower} = 95$, $H_{upper} = 110$ and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.05$ near the barrier was used and $\Delta t^* = 0.002$.

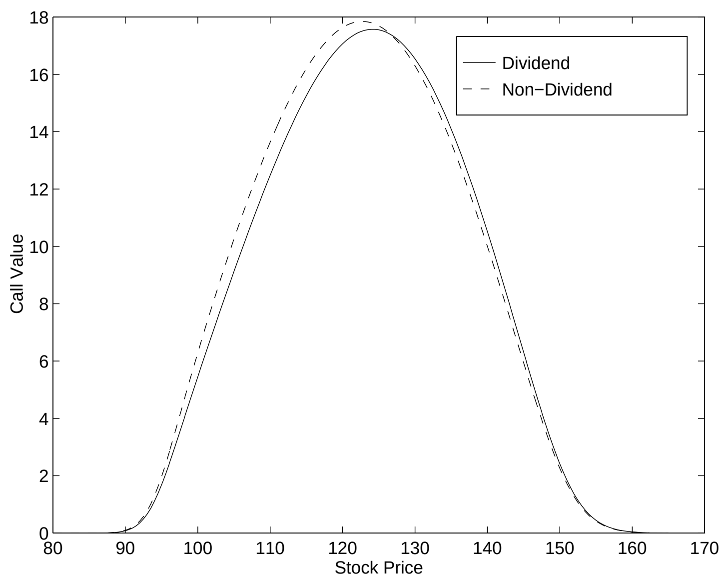

Figure 3: European double knock-out call options with a constant barrier applied weekly where the underlying asset does not pay a dividend and where the underlying asset pays a discrete dividend (no dividend protection) of \$2 at $T - t = 0.25$, when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{lower} = 95$, $H_{upper} = 150$ and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.05$ near the barrier was used and $\Delta t^* = 0.002$.

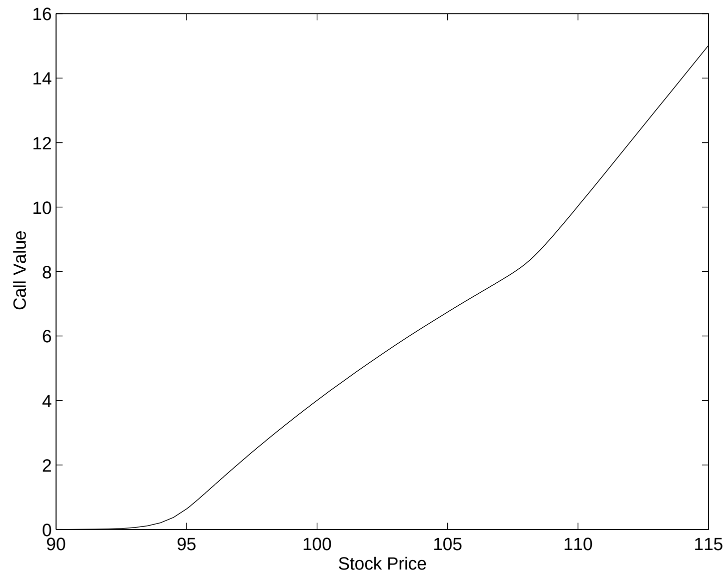

Figure 4: American (continuously early-exercisable) double knock-out call option with a constant barrier applied daily when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{lower} = 95$, $H_{upper} = 110$, and $K = 100$. A non-uniform spatial grid with $\Delta S = 0.05$ near the barrier was used and $\Delta t^* = 0.002$.

values than the non-dividend case, unless the stock price is relatively close to the upper barrier. This reflects the reduced probability of crossing the upper barrier due to the dividend. A plot of an American double knock-out option where the barriers are applied daily is contained in Figure 4. Clearly, discrete monitoring has a large impact. With continuous monitoring, the option would be worthless for all stock price values less than \$95 or above \$110. The positive value in the region below \$95 is due to the probability of the stock climbing back above the boundary before the next day.

It is interesting to note that to obtain accurate solutions for the double knock-out options with continuously applied barriers considered here, only a relatively large grid spacing of $\Delta S = 0.5$ was needed. This is due to the fact that the continuous application of the constant barriers effectively establishes boundary conditions at the same points throughout the life of the option, and because

σ	Continuous				Weekly	
	Outward		Inward		Outward	Inward
	PDE	K & I	PDE	K & I		
0.20	35.17	35.13	24.74	24.67	38.51	29.85
0.30	24.99	24.94	14.12	14.02	32.99	21.65
0.40	14.88	14.81	7.22	7.17	24.39	14.80

Table 4: European double knock-out call values for continuously and discretely applied time-varying barriers when $r = 0.05$, $T - t = 0.25$, $H_{lower} = 800$, $H_{upper} = 1200$, $K = 1000$ and $S = 1000$. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$. K & I denotes results obtained by Kunitomo and Ikeda (1992).

the underlying PDE is the Black-Scholes equation. An analogous situation exists for down-and-out options with continuous barriers considered here. However, a finer grid spacing near the barrier was used for such options because the barrier was close to the region of interest.

Although the grids for the examples with constant barriers considered here were constructed such that a node fell directly on the barrier, we found that it was not actually necessary to do so if the grid spacing was fine. However, if a large grid spacing was being used, then it was necessary to place a node right on the barrier or substantial pricing errors could result.

Table 4 contains results for European double knock-out options with time-varying continuous and weekly barriers where $h(t^{n+1}, \alpha^{n+1}) = e^{\alpha^{n+1}t^{n+1}}$ and $h(t^{n+1}, \beta^{n+1}) = e^{\beta^{n+1}t^{n+1}}$. For inward moving barriers $\alpha^{n+1} = 0.1$ and $\beta^{n+1} = -0.1$. For outward moving barriers $\alpha^{n+1} = -0.1$ and $\beta^{n+1} = 0.1$. Note that discretely applied time-varying barriers can be viewed as step barriers. A grid spacing of $\Delta S = 0.5$ and $\Delta t^* = 0.001$ was chosen in order to obtain option values that differed by no more than 0.01% of the exercise price from the results obtained by Kunitomo and Ikeda (1992) for the case of continuously applied barriers. The large impact of discrete monitoring is once readily again apparent, particularly for higher values of σ.

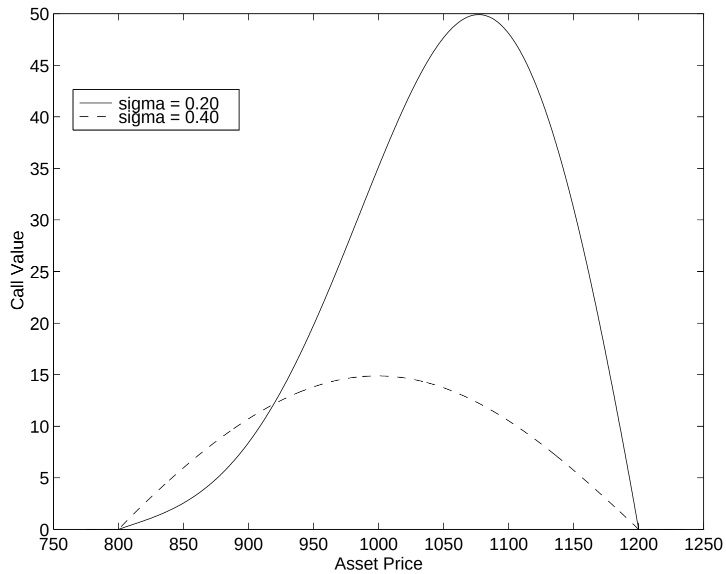

Figure 5: European double knock-out call options when $\sigma = 0.20$ and $\sigma = 0.40$, $r = 0.05$, $T - t = 0.25$, $H_{lower} = 800$, $H_{upper} = 1200$ and $K = 1000$. The barriers are outward moving and continuously applied. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$.

Figure 5 is a plot of European double knock-out options with differing volatilities where the barriers are outward moving and continuously applied. Note that the option value may or may not be increasing in volatility. The intuition for this is that higher volatility implies an increased probability of a relatively high payoff but also a greater chance of crossing a barrier. A plot of European double knock-out options with inward and outward moving barriers is contained in Figure 6. As we would expect, shrinking the distance between the barriers causes a large drop in the initial option value, especially for stock price values midway between the barriers.

Figure 7 demonstrates the difference in value between an option with continuously applied outward moving barriers and an option with outward moving barriers that are applied weekly.

The convergence of the method for pricing time-varying barrier options is demonstrated in Table 5. Table 6 contains option values where the barriers are applied in an implicit fully coupled manner

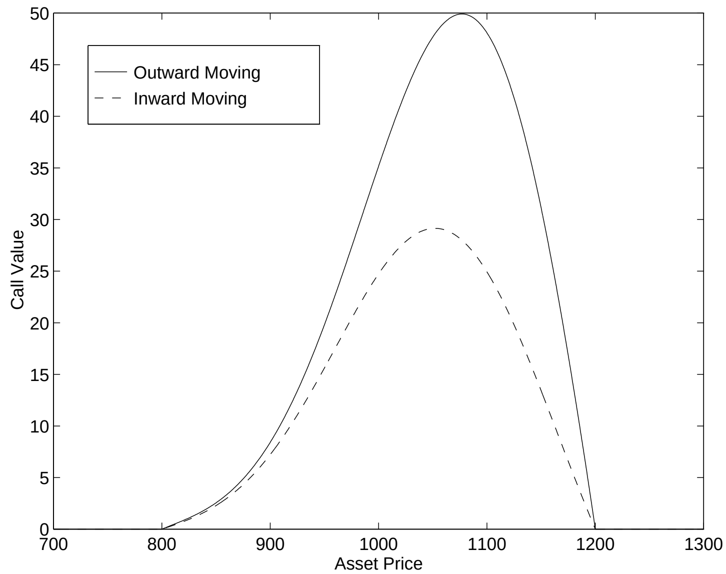

Figure 6: European double knock-out call options with outward and inward moving continuously applied barriers when $r = 0.05$, $\sigma = 0.20$, $T - t = 0.25$, $H_{lower} = 800$, $H_{upper} = 1200$ and $K = 1000$. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$.

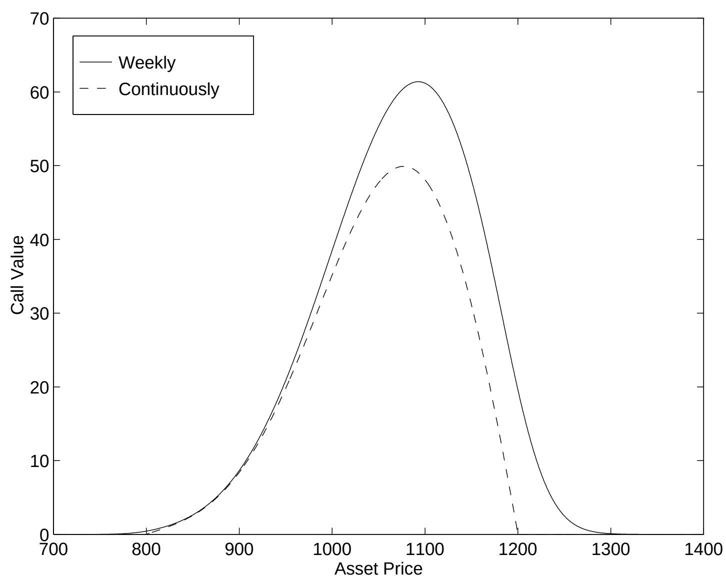

Figure 7: European double knock-out call options with weekly and continuously applied outward moving barriers when $r = 0.05$, $\sigma = 0.20$, $T - t = 0.25$, $H_{lower} = 800$, $H_{upper} = 1200$ and $K = 1000$. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$.

Barrier Movement	$\Delta S = 0.5$ $\Delta t^* = 0.001$	$\Delta S = 0.25$ $\Delta t^* = 0.0005$	$\Delta S = 0.125$ $\Delta t^* = 0.00025$	$\Delta S = 0.05$ $\Delta t^* = 0.0001$	K & I
Outward	14.88	14.84	14.83	14.81	14.81
Inward	7.25	7.21	7.19	7.18	7.17

Table 5: Successive grid refinements demonstrating convergence for European double knock-out calls with continuously applied time-varying barriers when $r = 0.05$, $\sigma = 0.4$, $T - t = 0.25$, $H_{lower} = 800$, $H_{upper} = 1200$, $K = 1000$ and $S = 1000$. K & I denotes results obtained by Kunitomo and Ikeda (1992).

σ	Outward			Inward		
	Explicit	Implicit	K & I	Explicit	Implicit	K & I
0.20	34.25	35.17	35.13	25.97	24.74	24.67
0.30	25.11	24.99	24.94	15.82	14.12	14.02
0.40	15.75	14.88	14.81	8.84	7.22	7.17

Table 6: Explicit and implicit application of continuously applied time-varying barriers for European double knock-out calls when $r = 0.05$, $T - t = 0.25$, $H_{lower} = 800$, $H_{upper} = 1200$, $K = 1000$ and $S = 1000$. A uniform spatial grid with $\Delta S = 0.5$ was used and $\Delta t^* = 0.001$. K & I denotes results obtained by Kunitomo and Ikeda (1992).

or explicitly. The table demonstrates that the implicit fully coupled application of the constraint for barrier options leads to more rapid convergence.

4 Extensions

4.1 Automatic Time Stepping

Although the results in Section 3 were obtained using constant time stepping, such problems lend themselves quite readily to automatic time stepping. This is because, say for discrete barrier options, the constant time step (which must be determined by trial and error) is limited by the fact that small time steps are needed immediately after the application of a barrier(s). Automatic time stepping procedures will cut the time step (if necessary) immediately after the application of a barrier(s) and then increase it as the solution becomes smooth according to some specified error criterion.

One such automatic time stepping procedure is that of Sammon and Rubin (1986). Sammon and Rubin derived a method for fully implicit schemes where the global time truncation error will be less than or equal to a specified target error. In their method

$$\Delta t^{n+1} = 2\epsilon / \sqrt{\|\frac{\partial^2 \overrightarrow{V}^n}{\partial (t^n)^2}\|_\infty}, \tag{14}$$

where ϵ is the target global time truncation error. In equation (14)

$$\frac{\partial^2 \overrightarrow{V}^n}{\partial (t^n)^2} = \frac{1}{\Delta t^n} \left[\frac{\partial \overrightarrow{V}^n}{\partial (t^n)} - \frac{\partial \overrightarrow{V}^{n-1}}{\partial (t^{n-1})} \right],$$

where

$$\frac{\partial \overrightarrow{V}^n}{\partial (t^n)} = -\mathbf{J}^{-1} \frac{\partial \overrightarrow{\gamma}^n}{\partial (t^n)},$$

$\mathbf{J}$ is the Jacobian and $\frac{\partial \overrightarrow{\gamma}^n}{\partial (t^n)} = -\frac{\overrightarrow{V}^n - \overrightarrow{V}^{n-1}}{(\Delta t^n)^2}$ (see Mehra, Hadjitofi and Donnelly (1982)). Note that if the fluxes are dependent on t^n, as is the case when the barriers are time-varying, then the flux functions and source terms should be included in the calculation of $\frac{\partial \overrightarrow{\gamma}^n}{\partial (t^n)}$.

When using (14), a time step size must be specified for the initial two time steps and the two time steps immediately following the application of a barrier(s) (since $\frac{\partial^2 \overrightarrow{V}^n}{\partial (t^n)^2}$ is meaningless at such points in time). In practice a small time step size is specified for the first two steps. The time step selector will then increase the time step significantly if appropriate. Also, $\|\frac{\partial^2 \overrightarrow{V}^n}{\partial (t^n)^2}\|_\infty$ need not be

Barrier Application	ϵ				Converged Solution
	0.10	0.04	0.02	0.01	
Continuously	2.10	2.06	2.05	2.05	2.04
Daily	2.55	2.55	2.53	2.51	2.51
Weekly	3.08	3.03	3.02	3.01	3.01

Table 7: Double knock-out call values with continuously and discretely applied constant barriers computed using automatic time stepping when $r = 0.10$, $\sigma = 0.2$, $T - t = 0.5$, $H_{lower} = 95$, $H_{upper} = 125$, $K = 100$ and $S = 100$. ϵ denotes the specified global time truncation error.

computed over the entire domain, but only for the area surrounding the region of interest.

Table 7 contains results obtained using (14). In Table 7 *Converged Solution* refers to the converged option values obtained using constant time step sizes (see Table 2) where the solution is accurate to within \$0.01. The spatial grids used for the results obtained with automatic time stepping were the same as those used for obtaining the converged option values. Table 7 demonstrates that the actual errors are generally less than the specified global time truncation errors (ϵ). Thus, the method produces time step sizes that are slightly conservative, which is consistent with Sammon and Rubin (1986) since ϵ in (14) is an upper bound for the error. Note that automatic time stepping can be used when valuing options with continuously applied barriers or general options. For such options, the time step size will be increased as the solution profiles smoothen.

Although the computation of $\frac{\partial \overrightarrow{\gamma}^n}{\partial (t^n)}$ requires an additional matrix solve, this does not introduce a great amount of additional overhead since the Jacobian has already been constructed. In fact, computational savings can be gained when the time step size can grow to be sufficiently large (for example, when longer term general options or discrete barrier options where the barrier(s) is applied infrequently are being valued).

4.2 Two Asset Barrier Options

The above methods can be applied to pricing barrier options written on two assets. The general form for an option which is a function of two factors is

$$V_{t^*} + \mathbf{U} \cdot \nabla V = \nabla \cdot \mathbf{D} \cdot \nabla V - rV, \tag{15}$$

where the form of $\mathbf{U}$ and $\mathbf{D}$ are determined by the precise nature of a given model. Suppose a two asset option is to be valued, and the option value is given by $V = V(S_1, S_2, t^*)$ where S_1 and S_2 are the prices of two traded assets. If S_1 and S_2 evolve according to

$$\begin{aligned} dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dz_1, \\ dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dz_2 \end{aligned}$$

where dz_1 and dz_2 are Wiener processes with $E(dz_1 dz_2) = \rho dt$, then V is given by equation (15) with

$$\mathbf{D} = \frac{1}{2} \begin{pmatrix} S_1^2 \sigma_1^2 & S_1 S_2 \rho \sigma_1 \sigma_2 \\ S_1 S_2 \rho \sigma_1 \sigma_2 & S_2^2 \sigma_2^2 \end{pmatrix}$$

and

$$\mathbf{U} = - \begin{pmatrix} S_1(r - \sigma_1^2 - \rho \sigma_1 \sigma_2 / 2) \\ S_2(r - \sigma_2^2 - \rho \sigma_1 \sigma_2 / 2) \end{pmatrix}.$$

Equation (15) can be discretized using a finite element approach as described in Forsyth, Zvan and Vetzal (1997).

Table 8 gives the parameters for a two asset double knock-out pricing problem. The barriers are defined as

$$V(S_1, S_2, t^*_{app}) = \begin{cases} V(S_1, S_2, t^*_{app}) & \text{if } 90 \leq S_1, S_2 \leq 120, \\ 0 & \text{otherwise} \end{cases}$$

where t^*_{app} is the application date of the barriers, which are applied weekly.[8] The payoff for this

σ_1	0.40
σ_2	0.20
ρ	-0.50
r	0.05
Time to maturity	0.25 years
Exercise Price (K)	\$100
Barriers Applied	Weekly

Table 8: Parameters for the two asset double knock-out barrier problem.

example problem is based on the *worst of* the two assets. For a call this would be

$$V(S_1, S_2, 0) = \max(\min(S_1, S_2) - K, 0).$$

As mentioned earlier, Crank-Nicolson time weighting results in large oscillations for this problem, so a fully implicit method was used. This problem was computed using the finite element method on an 81×81 grid. A fine grid run using a 161×161 grid with smaller time steps showed

[8] Note that our methodology allows us to apply barriers to either asset or both assets. Previous work by Ritchken (1995) and Cheuk and Vorst (1996) examines two dimensional problems but where barriers are only applied to one of the assets.

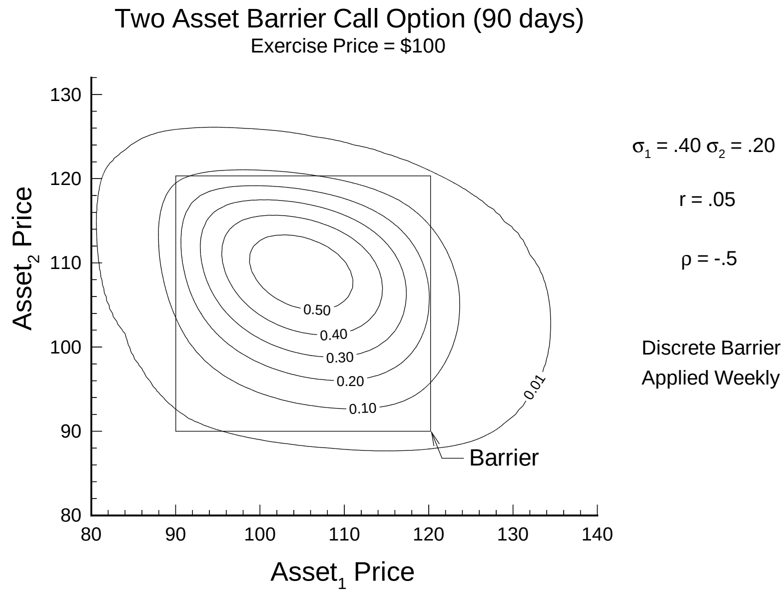

Figure 8: Two asset barrier call option on the worst of two assets when $K = \$100$, $\sigma_1 = 0.40$, $\sigma_2 = 0.20$ and $\rho = -0.50$. The barrriers are applied weekly.

that these results are accurate to within \$0.01 (the largest errors being right at the barrier). Figure 8 shows the contours of constant value for an option with a time to maturity of 0.25 years. Figure 9 shows similar results, except that $\sigma_1 = \sigma_2 = 0.50$ and $\rho = 0.50$. A comparison of the figures reveals how the option value is affected by the volatility and correlation parameters.

5 Conclusions

We have described an implicit PDE approach to the pricing of barrier options and illustrated its application to a variety of different types of these contracts. We have shown that a Crank-Nicolson approach, though stable, can produce very poor answers. We have also demonstrated that applying barrier constraints in an implicit fully coupled manner can lead to more rapid convergence.

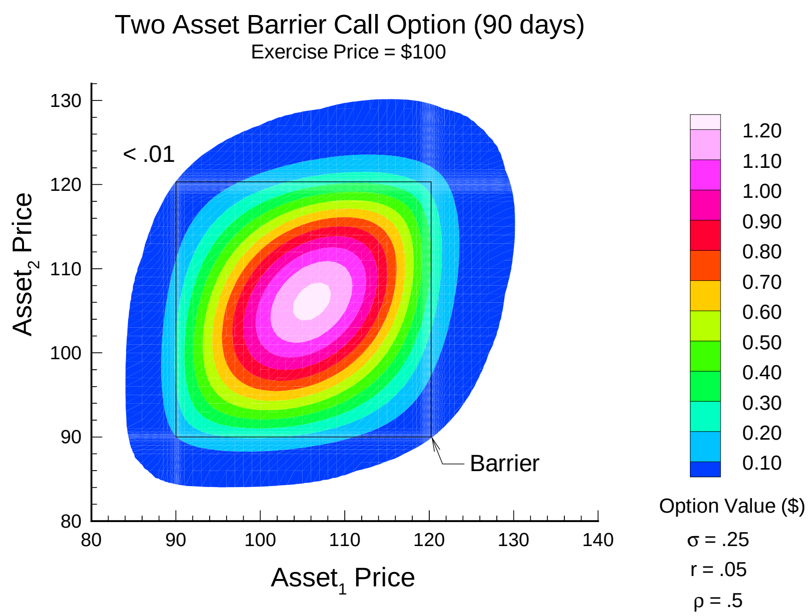

Figure 9: Two asset barrier call option on the worst of two assets when $K = \$100$, $\sigma_1 = \sigma_2 = 0.50$ and $\rho = 0.50$. The barriers are applied weekly.

Furthermore, due to the very small grid spacing required near the barrier(s) (in order to obtain accurate solutions), the time step size restrictions for an explicit method are very severe. Examples in this work show that an accurate explicit method would require time steps four orders of magnitude smaller than a fully implicit scheme (which admittedly has more computational overhead per time step).

References

Andersen, L.B.G. (1996). Monte Carlo simulation of barrier and lookback options with continuous or high-frequency monitoring of the underlying asset. Working paper, General Re Financial Products Corp., New York.

Boyle, P.P. and I. Lau (1994). Bumping up against the barrier with the binomial method. *Journal of Derivatives* 1 (Summer), 6–14.

Boyle, P.P. and I. Lee (1994). Deposit insurance with changing volatility: An application of exotic options. *Journal of Financial Engineering* 3, 205–227.

Boyle, P.P. and Y. Tian (1996). An explicit finite difference approach to the pricing of barrier options. Working paper, Centre for Advanced Studies in Finance, University of Waterloo.

Boyle, P.P. and Y. Tian (1997). Pricing path-dependent options when the underlying asset follows the CEV process. Working paper, Centre for Advanced Studies in Finance, University of Waterloo.

Broadie, M. and J. Detemple (1995). American capped call options on dividend-paying assets. *Review of Financial Studies* 8, 161–192.

Broadie, M., P. Glasserman and S. Kou (1995). A continuity correction for discrete barrier options. Working paper, Columbia Business School.

Broadie, M., P. Glasserman and S. Kou (1996). Connecting discrete and continuous path-dependent options. Working paper, Columbia Business School.

Carr, P. (1995). Two extensions to barrier option valuation. *Applied Mathematical Finance* 2, 173–209.

Cheuk, T.H.F. and T.C.F. Vorst (1996). Complex barrier options. *Journal of Derivatives* 4 (Fall), 8–22.

Ericsson, J. and J. Reneby (1996). Options on corporate securities: A compound barrier option approach. Working paper, Stockholm School of Economics.

Figlewski, S. and B. Gao (1997). The adaptive mesh model: A new approach to efficient option pricing. Working paper, Stern School of Business, New York University.

Forsyth, P., R. Zvan and K. Vetzal (1997). A general finite element approach for PDE option pricing. Proceedings of Numerical Methods in Finance, Toronto.

Gao, B., J. Huang and M.G. Subrahmanyam (1996). An analytical approach to the valuation of American path-dependent options. Working paper, Stern School of Business, New York University.

Geman, H. and M. Yor (1996). Pricing and hedging double-barrier options: A probabilistic approach. *Mathematical Finance* 6, 365–378.

Harvey, C.R. and R.E. Whaley (1992). Dividends and S&P 100 index option valuation. *Journal of Futures Markets* 12, 123–137.

Heynen, P. and H. Kat (1994a). Crossing barriers. *RISK* 7 (June), 46–51.

Heynen, P. and H. Kat (1994b). Partial barrier options. *Journal of Financial Engineering* 3, 253–274.

Heynen, P. and H. Kat (1996). Discrete partial barrier options with a moving barrier. *Journal of Financial Engineering* 5, 199–209.

Hsu, H. (1997). Surprised parties. *RISK* 10 (April), 27–29.

Hull, J.C. (1993). *Options, Futures, and Other Derivative Securities.* Second edition. Prentice Hall, Englewood Cliffs, New Jersey.

Kolkiewicz, A.W. (1997). Pricing and hedging more general double barrier options. Working paper, Department of Statistics and Actuarial Science, University of Waterloo.

Kunitomo, N. and M. Ikeda (1992). Pricing options with curved boundaries. *Mathematical Finance* 2, 275–298.

Mehra, R.K., M. Hadjitofi and J.K. Donnelly (1982). An automatic time step selector for reservoir models. SPE 10496, presented at the 6th Symposium on Reservoir Simulation.

Merton, R.C. (1973). Theory of rational option pricing. *Bell Journal of Economics and Management Science* 4, 141–183.

Merton, R.C. (1974). On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance* 29, 449–470.

Reimer, M. and K. Sandmann (1995). A discrete time approach for European and American barrier options. Working paper, Department of Statistics, Rheinische Friedrich-Wilhelms-Universität, Bonn.

Rich, D. (1991). The mathematical foundations of barrier option-pricing theory. *Advances in Futures and Options Research* 7, 267–311.

Rich, D. (1996). The valuation and behavior of Black-Scholes options subject to intertemporal default risk. *Review of Derivatives Research* 1, 25–59.

Ritchken, P. (1995). On pricing barrier options. *Journal of Derivatives* 3 (Winter), 19–28.

Roache, P. (1972). *Computational Fluid Dynamics.* Hermosa, Albuquerque, New Mexico.

Rubinstein, M. and E. Reiner (1991). Breaking down the barriers. *RISK* 4 (September), 28–35.

Sammon, P.H. and B. Rubin (1986). Practical control of time step selection in thermal simulation. *SPE Reservoir Engineering* (March), 163–170.

Willmott, P., J. Dewynne and J. Howison (1993). *Option Pricing: Mathematical Models and Computation.* Oxford Financial Press, Oxford.

Zvan, R., P.A. Forsyth and K.R. Vetzal (1996). Robust numerical methods for PDE models of Asian options. Technical Report CS-96-28, Department of Computer Science, University of Waterloo.

Zvan, R., P.A. Forsyth and K.R. Vetzal (1997). Oscillations. Mimeo, Department of Computer Science, University of Waterloo.