Planar Mesh Refinement Cannot Be Both Local and Regular *

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Abstract

We show that two desirable properties for planar mesh refinement techniques are incompatible. Mesh refinement is a common technique for adaptive error control in generating unstructured planar triangular meshes for piecewise polynomial representations of data. Local refinements are modifications of the mesh that involve a fixed maximum amount of computation, independent of the number of triangles in the mesh. Regular meshes are meshes for which every interior vertex has degree 6. At least for some simple model meshing problems, optimal meshes are known to be regular, hence it would be desirable to have a refinement technique that, if applied to a regular mesh, produced a larger regular mesh. We call such a technique a regular refinement. In this paper, we prove that no refinement technique can be both local and regular. Our results also have implications for non-local refinement techniques such as Delaunay insertion or Rivara's refinement.

keywords unstructured mesh, triangulation, adaptive refinement

1 Introduction

In this paper, we show that two desirable properties for refinement techniques of two-dimensional, unstructured triangular mesh generation are incompatible. The better-known of these properties, locality, is that each refinement involves a fixed maximum amount of work; local refinement is reviewed in the second section of this paper. A regular mesh is a mesh in which all interior vertices have the same degree. A refinement technique is regular if its application to a regular mesh results in a regular mesh. Regularity is motivated primarily by the theory of optimal meshes as we elaborate in the third section. Our basic result is that a refinement cannot be both local and regular, as we prove in the fourth section. To establish this result, we prove in Theorem 1 that, in a simply-connected regular mesh, there is only one mesh topology for balls of vertices that are within a fixed path distance from a given vertex and that are interior to the mesh; *i.e.* all such interior balls are isomorphic.

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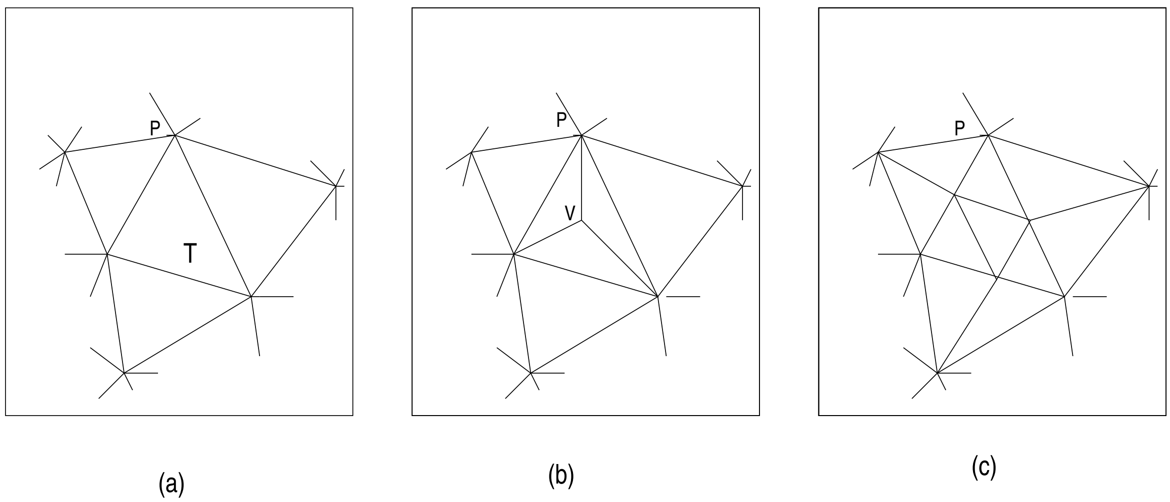

Figure 1: A typical interior triangle (a) and two simple local mesh refinements (b) and (c)

2 Mesh refinement

In the finite element method, and other contexts, data specified in a planar domain D is approximately represented by piecewise polynomial functions defined on a mesh of triangles on D. In this section, we give a general outline of the common technique for adaptively generating an unstructured triangular mesh suitable for such data that is the major context for the topic of this paper. Adaptive local refinement techniques modify the mesh to provide error control by examining each triangle of the mesh and computing an estimate of the error in the piecewise polynomial approximation. If the resulting error is too large, the mesh is modified either by adding new vertices, by modifying the triangle incidences of existing vertices, or both. See compendium [7].

Some refinement techniques are illustrated in Figure 1. If triangle T, as shown in Figure 1(a), is selected for refinement, a simple insertion of vertex V at its centroid would result in the configuration of Figure 1(b). A common alternative is to subdivide T into four smaller copies of itself as shown in Figure 1(c), which introduces the three midside points of T as new vertices in the refined mesh e.g. Bank, Sherman and Weiser, 1983, [3], Bank, 1990, [2].

One of the traditional 'mesh quality' objectives of mesh refinement has been to avoid introducing triangles with small (or large) angles[1]. When a new vertex is inserted into T using the technique of Figure 1(b), at least two new obtuse angles must be introduced into the mesh, which reduces the quality of the resulting mesh. If, however, the mesh is modified as in Figure 2 when V is inserted, no new obtuse angles are introduced in the configuration shown. In general, it is to be expected that suitable mesh refinement techniques will modify

[1]See, however, Babuska and Aziz ,1976, [1] and Simpson, 1994, [12].

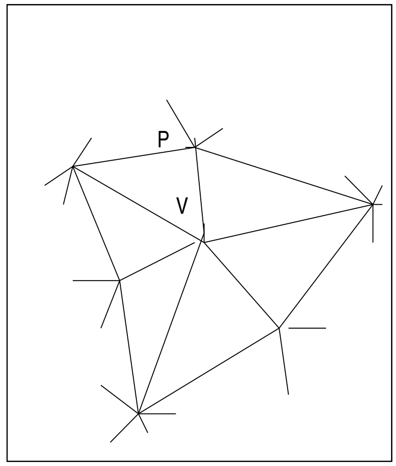

Figure 2: An alternative local refinement for insertion of V

a group of triangles; this can be seen in the splitting of the neighbours of T in Figure 1(c). These refinement schemes are particularly simple because they involve introducing a fixed number of vertices and modifying a fixed number of edges, so that they are relatively easy to implement and cheap to execute. More complex refinements include Delaunay insertions, Weatherill, 1988, [13], Chew, 1993, [4], or the recursive longest edge bisection approach of Rivara, 1984, [10], 1993 [11]. Our result has implications for these methods also despite the fact that they are not local.

We will use the term *refinement* for any technique for modifying a mesh which includes the insertion of at least one additional vertex and define the qualifiers *local* and *interior* with more precision and generality than is common. A refinement is a mesh valued function which takes as parameters the mesh, M, and some other finite set of inputs, I, (*e.g.* a specific triangle T of M as in the above examples). The resulting mesh will be designated using function notation as $R(M, I)$. We require all vertices of M to be included in $R(M, I)$ and $R(M, I)$ to have at least one additional vertex. Let $C_R(M, I)$ be the set of edges in $R(M, I)$ but not in M. If $\max_{M,I} C_R(M, I)$ is finite, we will call R a local refinement function. E.g. Suppose we were to define refinement functions, $R_b(M, I)$ or $R_c(M, I)$ which changed a mesh, M, shown in Figure 1(b) or Figure 1(c) for I as the single triangle T. Then $C_{R_b}(M, I) = 3$ for any mesh M, and $\max_{M,I} C_{R_c}(M, I) = 12$. However, in general, for refinements in which $R(M, I)$ is required to be a Delaunay triangulation $\max_{M,I} C_{R_c}(M, I) = \infty$ as can be seen from the family of meshes with $2m + 1$ vertices $V_k = (k, 0)$, $k{=}{-m}$ to m, $k \neq 0$ and $V_0 = (0, 100m)$, and input $I = V_{insert} = (0, 1)$.

Let $d_M(P, Q)$ be the path distance in mesh M from vertex P to vertex Q, *i.e.* the minimum number of edges for any path from P to Q. Let $B_M(P, r)$ be the submesh of M containing all vertices, Q, such that $d_M(P, Q) \leq r$; i.e. $B_M(P, r)$ is the ball of vertices of path distance r from P. The general idea of an interior refinement by a refinement function R is that the effects of the refinement should be confined to balls in the interior of M and $R(M, I)$. We define R applied to M and I to be an *interior* refinement of M if there is a ball $B_M(P, r)$ such that

$$R(M, I) \; = \; R(B_M(P, r), I) \; + \; (M \setminus B_M(P, r)) \tag{1}$$

$$R(B_M(P, r), I) \text{ is contained in some interior ball of } R(M, I).$$

Here we interpret $R(B_M(P, r), I)$ as a submesh of $R(M, I)$ with the same boundary vertices as $B_M(P, r)$. As one would expect, a local refinement function (with $\max_{M,I} C_R(M, I) = k$) applied in a ball $B_M(P, r)$ which is at a sufficient distance from the boundary of M (i.e. $d_M(B_M(P, r), \partial M) > k$) produces an an interior refinement of M.

3 Regular meshes and optimal mesh generation

A triangular mesh is *regular* if six edges and six triangles are incident on each interior mesh vertex [2]. The primary motivation for our interest in regular meshes is their connection with optimally efficient meshes as elaborated further below. Regular meshes provide some advantages of simplicity and approximation accuracy for finite element methods, and other computations using meshes for piecewise polynomial data representation. Frey and Field, 1991, [8], describe a method for improving a mesh that is not regular by modifying its topology to bring its vertex degrees closer to six. Marcum and Weatherill, 1995, [9], comment on some advantages regular meshes can provide for turbulent viscous flow computation. In practice, however, most unstructured meshes are not regular. The result of this paper may provide one reason why.

Mesh generation tasks can be specified formally in several ways as constrained optimization problems, Simpson, 1994, [12]. In general, there are neither criteria to recognize nor methods for computing meshes that meet these optimality specifications. In practice, good suboptimal meshes that are inexpensive to generate are satisfactory and most mesh generation techniques, including mesh refinement, can be viewed as heuristics that produce suboptimal meshes. For simple idealized cases, however, it is possible to produce provably optimal meshes, e.g. D'Azevedo, 1991, [5] and D'Azevedo and Simpson, 1991, [6]. One striking feature of these optimal meshes is that they are regular.

The elementary local refinement techniques that we used as examples above, when applied to a regular mesh, destroy this regularity. Hence these refinement techniques appear to reduce the resemblance of the modified mesh to an optimal one. In Figure 1(a), the triangle T is shown as having each of its three vertices of degree 6. After the insertion of V according to the technique shown in Figure 1(b), the original vertices of T have degree 7, and the new vertex, V, has degree 3. If the insertion is done according to Figure 2, then the newly inserted V has degree 6, but each of the vertices now connected to V has degree either 5 or 7.

We will call a refinement *regular* if, when it is applied to a regular mesh M, with some input I, it produces a larger regular mesh. The general question which motivated this paper

[2] We try to conform to standard unstructured mesh terminology. Conformity is complicated in the case of the term 'regular' because it is commonly used to describe several different mesh properties. In particular, the property referred to by "regular" in [3] is commonly referred to by "conforming", as per [11].

is whether it is possible to devise a refinement technique that is both regular and local, in the sense of the previous section. We prove the stronger result that a refinement $R(M,I)$ applied to a regular mesh M on a simply-connected domain cannot be both a regular refinement and an interior refinement, for any choice of I.

4 Establishing the Result

Let S be a standard regular triangulation of the plane; *i.e.* the mesh with triangles whose vertices are (i,j), $(i+1,j)$ and either $(i+1,j+1)$ or $(i,j-1)$, for integers i and j. Let S_1 be a simply-connected finite submesh of S. Then the degree of a boundary vertex of S_1 must between 2 and 5 inclusive. We will refer to the sequence of the degrees of the boundary vertices of S_1 taken in counter-clockwise order, starting at an arbitrary boundary vertex, as the degree sequence for S_1. In Lemma 1, we claim that S_1 is uniquely determined, up to isomorphism, by its degree sequence.

Lemma 1 *A simply-connected finite submesh of S is determined up to isomorphism by its degree sequence.*

The proof by induction on the boundary path length is straightforward.

If all simply-connected regular meshes were isomorphic to submeshes of S, then a proof by contradiction that a refinement could not be both regular and interior would be easy, as follows. Let S_1 be a submesh of S isomorphic to M. Then the number of vertices in M is the same as in S_1, which is determined by the degree sequence of the boundary of S_1. If we then were to assume that $R(M,I)$ is regular and has the same boundary as M, but has more vertices, we would have a contradiction. However, it is straightforward to construct simply-connected regular meshes which are not isomorphic to submeshes of S.

In Theorem 1, we establish that for any simply-connected regular mesh M, there is a ball $B_M(P,r)$, interior to M, that is isomorphic to $B_S((0,0),r)$. This result then allows us to make a similar if somewhat more technically involved argument about the non-existence of a regular, interior, refinement.

Theorem 1 is based on two lemmas presented below. In Lemma 2, we observe that closed paths in a planar mesh M can be mapped to closed paths in the standard mesh S. This observation allows us to prove in Lemma 3 a crucial property of the boundaries of any regular submesh N that is isomorphic to a part of S: i.e. any edges between boundary vertices of N that are present in a supermesh must have images in S.

The proofs of these results use combinatorial properties of plane-embedded graphs, which we shall review here. In a plane embedding of a graph, each vertex receives a prescribed (counterclockwise) ordering of its neighbours; the collection of these orderings for all the vertices determines the topology of the embedding. Given the order of neighbours at each vertex, one may follow consecutive edges around a face. Graphs derived from regular triangulations of simply-connected domains have three pertinent combinatorial properties. First, all of the faces but one (the exterior face, containing the point at infinity) have three edges. Second, the graph is biconnected; equivalently, no vertex appears twice in the sequence of vertices on the exterior face. Third, any simple cycle in the graph encloses a unique subgraph (including the cycle as its boundary), this subgraph corresponds to a regular triangulation of

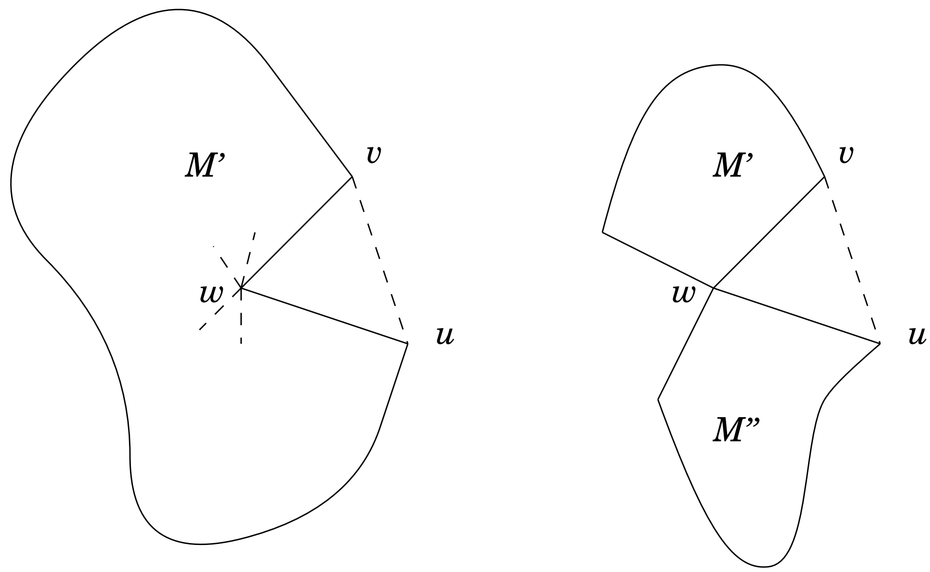

Figure 3: The two cases: w is interior or boundary.

a simply connected subdomain. In what follows, we shall use the terms "mesh" and "graph" interchangeably. We shall term a mesh "valid" if it satisfies the conditions above.

Any path in M, considered as a sequence of edges (u_1, u_2), (u_2, u_3), ..., (u_k, u_{k+1}), has an image in the standard mesh, as follows. Map the first edge to an arbitrary edge (v_1, v_2) of the standard mesh. Map each succeeding edge into the standard mesh according to the placement of the edge in the order of the edges around its endpoint; that is, if (u_i, u_{i+1}) is the jth edge following (u_{i-1}, u_i) around point u_i, then map (u_i, u_{i+1}) to the jth edge following (v_{i-1}, v_i) in the order around v_i. (If $j > 6$, wrap around v_i as often as necessary.) We will denote the image of a path p under this mapping by $I(p)$; the image of any edge e or vertex v is denoted $I(e)$ or $I(v)$.

Lemma 2 *Let p be the boundary path of M, taken either clockwise or counter-clockwise starting from any edge. If p returns to its start, with $(u_k, u_{k+1}) = (u_1, u_2)$, then the image path $I(p)$ also returns to its start, with $(v_k, v_{k+1}) = (v_1, v_2)$.*

Proof. We use induction on the number of edges in M. If M is a triangle, the image path is also a triangle, satisfying the required condition.

Now suppose that the result holds for all meshes with less than r edges, and let M have r edges. Select any boundary edge $e = (u, v)$. Since M is biconnected, e forms part of a triangle uvw. We consider two cases. In case one, w is not a boundary point of M, and $M' = M - e$ is a valid mesh. In case two, $M - e$ has two biconnected components; let M' be the component containing v and M'' the component containing u. Both components are valid meshes and contain w.

For case one, consider the boundary walk in M' that starts with edge (w, v) mapped to $((0,0),(1,0))$. By the induction hypothesis, the image of this walk returns to its start. Hence the penultimate edge (u, w) has image $I(u, w) = (x, (0,0))$ for some point x; since w has degree six, we must have $x = (1, -1)$.

Nodes u and v have degree one higher in M than in M'. Therefore, in an image of the boundary path of M, the edges that are also boundary edges of M' receive the same image they did in the walk around M, and the second occurrence of e has the same image as the first, as required.

For case two, consider one walk in each component, starting from (w, v) mapped to $((0,0),(1,0))$ and (u, w) mapped to $((1,-1),(0,0))$. Each image returns to its start, by induction.

A walk in M starting from $e = (u, v)$ mapped to $((1,-1),(1,0))$ will have the same image as the walk in M' until they reach w, with image $I(w) = (0,0)$. Since the degree of w in M is the sum of its degrees in M' and in M'', the walk in M will continue by following the walk in M''. When it reaches u, the final image of (u, v) will be $((1,-1),(1,0))$, as required. □

Lemma 3 *Let M be a valid mesh. Suppose that N is a valid submesh of M and I is an isomorphism of N into a submesh of S. For any two vertices a and b on the boundary of N, if (a, b) is an edge of M, then $(I(a), I(b))$ is an edge of S.*

Proof. The vertices a and b divide ∂N into two pieces P_1 and P_2. Assume without loss of generality that the cycle $P_1, (a, b)$ encloses the path P_2. Consider the walk from b along the reversal of P_2 to a and returning to b. This walk is a cycle in M and hence encloses a submesh N' of M; by Lemma 2 applied to N', the walk's image path in S returns to its origin. In particular, $I(b)$ and $I(a)$ are adjacent in S, as required. □

For any point P in a mesh M, let $B(P, r)$ be the ball of radius r around P, that is, the set of vertices in M at distance at most r from P together with the edges between pairs of such vertices. The ball $B(P, 2)$ in the standard mesh is illustrated in Figure 4. In the standard mesh, the ball $B(P, r)$ has six boundary vertices of degree three and $6r - 6$ boundary vertices of degree four, as one can easily establish by induction. We now show that in any mesh, a ball that does not reach the boundary must be isomorphic to this standard ball.

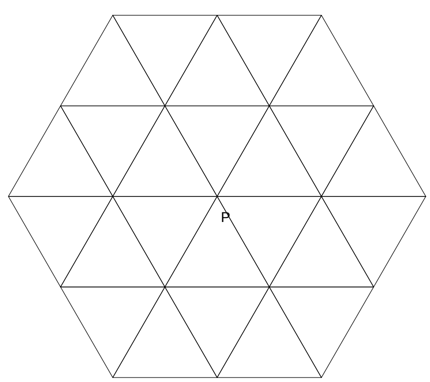

Figure 4: The ball $B(P, 2)$ in the standard mesh.

Theorem 1 *Let P be any interior point of a valid mesh M. Then for all r, either $B(P, r)$ contains a boundary point of M or $B(P, r)$ is isomorphic to the ball of radius r in the standard mesh.*

Proof. We proceed by induction on r. For the base case $r = 1$, we note that P has six neighbours forming six triangles each with a corner at P. Hence $B(P, 1)$ is isomorphic to the standard mesh.

Suppose that $B(P, r)$, $r \geq 1$, has only interior vertices of M and is isomorphic to the standard mesh. In particular, the boundary of $B(P, r)$ has six vertices of degree three and $6r - 6$ vertices of degree four, joined into a cycle by $6r$ edges.

If $B(P, r + 1)$ contains a boundary point, then $B(P, r')$ contains a boundary point for all $r' \geq r + 1$ and there is nothing more to prove.

Suppose $B(P, r + 1)$ is interior to M. Each of the vertices of the boundary of $B(P, r)$ has degree six in M; thus there are $12r - 6$ edges between boundary vertices of $B(P, r)$ and vertices in $B(P, r + 1) \setminus B(P, r)$. These new edges form $6r$ triangles (one on each edge of $\partial B(P, r)$) and 6 other edges. Additional triangles fill in between these triangles and single edges. If $B(P, r + 1) \setminus B(P, r)$ has $6r + 6$ distinct vertices, then the isomorphism of $B(P, r)$ extends to $B(P, r + 1)$.

We now show that $B(P, r+1) \setminus B(P, r)$ cannot have fewer than $6r + 6$ vertices. Suppose to the contrary; *i.e.*, suppose two triangles on edges (a_1, a_2) and (b_1, b_2) of $\partial B(P, r)$ have the same point Q as apex. Let N be the submesh of M formed from $B(P, r)$ by adding the point Q and the two edges (Q, a_1) and (Q, a_2). Then by the induction hypothesis N is isomorphic to a submesh of S. By Lemma 3, edges (Q, b_1) and (Q, b_2) of M must have images in S. No such images exist, however. Thus Q cannot exist and $B(P, r + 1) \setminus B(P, r)$ must have exactly $6r + 6$ distinct vertices.

Finally, Lemma 3 implies that $B(P, r + 1)$ has no additional edges not mentioned above. Hence $B(P, r + 1)$ is isomorphic to a submesh of S, as required. □

5 Conclusions

Returning to the primary objective of this paper, we can now assert that there is no hope of finding a clever local refinement algorithm that preserves regularity in a mesh. Further, we can assert that for nonlocal refinement techniques like Delaunay insertion, or Rivara's recursive longest edge bisection, no particular instance applied to a regular mesh can make a modification of the mesh which is both regular and confined to the interior of the mesh.

To support these assertions, let us hypothesize to the contrary the existence of a regular mesh M and a refinement function R that, for some input I, makes an interior, reqular refinement of M. The claim that the refinement is interior indicates two things as per (1). One is the existence of an interior ball, $B_M(P, r)$, in M such that the vertices on the boundary of $B_M(P, r)$ occur in $R(M, I)$ with the same degree sequence. Moreover, the difference between M and $R(M, I)$ is confined to $R(B_M(P, r), I)$ as a submesh of $R(M, I)$. The other implication of R being interior is that $R(B_M(P, r), I)$ lies in some interior ball of $R(M, I)$. By Theorem 1 and the assumption that R is regular, $R(B_M(P, r), I)$ is isomorphic to a submesh, S_1, of S and $B_M(P, r)$ is isomorphic to a submesh, S_2, of S. But, since the degree sequences of the boundaries of $R(B_M(P, r), I)$ and $B_M(P, r)$ are identical, S_1 and S_2 must be isomorphic (Lemma 1) and in particular contain the same number of vertices. This leads to a contradiction that $R(M, I)$ is a refinement of M and hence $R(B_M(P, r), I)$ must contain at least one more vertex than $B_M(P, r)$.

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