Robust Numerical Methods for PDE Models of Asian Options

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Abstract

We explore the pricing of Asian options by numerically solving the the associated partial differential equations. We demonstrate that numerical PDE techniques commonly used in finance for standard options are inaccurate in the case of Asian options and illustrate modifications which alleviate this problem. In particular, the usual methods generally produce solutions containing spurious oscillations. We adapt flux limiting techniques originally developed in the field of computational fluid dynamics in order to rapidly obtain accurate solutions. We show that flux limiting methods are total variation diminishing (and hence free of spurious oscillations) for non-conservative PDEs such as those typically encountered in finance, for fully explicit, and fully and partially implicit schemes. We also modify the van Leer flux limiter so that the second-order total variation diminishing property is preserved for non-uniform grid spacing.

1 Introduction

Asian options are securities with payoffs which depend on the average value of an underlying stock price over some time interval. Such options have proven to be much more difficult to value than regular stock options. Standard techniques tend to be impractical, inaccurate, or slow. For example, traditional binomial lattice methods require such enormous amounts of computer memory (owing to the necessity of keeping track of every possible path throughout the tree) that they are effectively unusable. Partial differential equation (PDE) methods, as traditionally implemented in the finance literature, are inaccurate (see Barraquand and Pudet (1996) for a discussion). Monte Carlo simulation works well for European-style options (see Kemna and Vorst (1990)), but is relatively slow. A number of approximations have appeared in the literature (e.g. Turnbull and Wakeman (1991), Vorst (1992), Levy (1992), Levy and Turnbull (1992)), which are again suitable only for European-style options. See also Geman and Yor (1993), who derive the Laplace transform of the European option price. Unfortunately, this transform is very difficult to invert.

With regard to American-style Asian options, there are even fewer alternatives. Hull and White (1993) propose a modification of the binomial method, but do not provide any proof of convergence. Neave (1994) uses a frequency distribution approach on a binomial lattice to derive approximate values for arithmetic average option values, but his method still requires calculations of order N^4, where N is the number of time-steps in the lattice. Barraquand and Pudet (1996) describe a *forward shooting grid* algorithm and prove that it is unconditionally convergent. We explore another possibility: a modified finite difference method. In general, the price of an Asian option can be found by solving a PDE in two space-like dimensions (see Ingersoll (1987) or Wilmott, Dewynne, and Howison (1993)). This PDE has the character of a two dimensional convection-diffusion problem with no diffusion in one of the spatial dimensions. As is well-known in computational fluid dynamics, standard centrally weighted methods for the convective term are prone to oscillatory solutions. Furthermore, as argued by Barraquand and Pudet (1996), standard finite difference methods (though generally faster than their proposed algorithm) are inaccurate because they introduce "spurious numerical diffusion" (p. 43).

In some cases the price of an Asian option can be modeled using a one-dimensional PDE. The two-dimensional PDE for a floating strike Asian option can be reduced to a one-dimensional PDE (see Ingersoll (1987) or Wilmott, Dewynne, and Howison (1993)). Recently, Rogers and Shi (1995) have formulated a one-dimensional PDE that can model the price of both floating and fixed strike Asian options. However, this PDE applies only to the case of European-style options and is particularly difficult to solve numerically since the diffusion term is very small for values of interest on the finite difference grid.

We demonstrate modifications to the common discretization methods which are designed to handle these problems. In particular, in both the two-dimensional and one-dimensional cases it is necessary to solve a problem with little or no diffusion (i.e. second-order derivative term) in a space dimension. The traditional approach in computational fluid dynamics would be to use first-order upstream weighting for the convective term to eliminate the oscillations caused by centrally weighted schemes (Roache, 1972). However, first-order upstream weighting results in solutions with excessive false diffusion. As an alternative, we employ a high order non-linear flux limiter for the convective terms. The resulting discrete non-linear algebraic equations are solved using full Newton iteration. In addition, we can also apply the American early exercise constraint to the algebraic system, and this can be handled in an implicit fully coupled manner. In cases where the model cannot be reduced to a problem in a single space dimension, the full two-dimensional problem must be solved. For example, the price of a fixed strike American-style Asian option must be found by solving the two-dimensional PDE. We apply the above methods (i.e. the flux limiter and full Newton iteration) to a full two-dimensional problem. In this case an iterative method, ILU-CGSTAB (D'Azevedo et al., 1992; van der Vorst, 1992), is used to solve the resulting Jacobian matrix.

The outline of the paper is as follows. Section 2 describes the option pricing models to be considered. Section 3 presents a discretization analysis for finite difference methods as applied to standard options. We concentrate on situations with extremely low volatility which, as noted above, are analogous to the case of Asian options. We illustrate the types of problems which can arise with commonly applied methods in finance and also how our modifications mitigate these difficulties, both in terms of option prices and hedging parameters. Section 4 presents applications to Asian options, and the paper concludes with a brief summary which is contained in Section 5.

2 The Models

We adopt the usual geometric Brownian motion model for the evolution of a stock price S:

$$dS = rSdt + \sigma S dB \tag{1}$$

where r denotes the risk free interest rate, σ is the volatility, and dB is a standard Brownian motion. Under the conventional assumptions of frictionless markets, the value at time t of a claim contingent on the stock price at subsequent time T may be represented as:

$$V(S(t),t) = \mathrm{e}^{-r(T-t)}\ \mathrm{E}_t\ [g(S(T),T)] \tag{2}$$

where $g(S(T),T)$ denotes the payoff function for the claim and E_t denotes expectation conditional on information available at time t. Familiar examples include European calls ($g(S(T),T) = \max(S(T) - K, 0)$) and puts ($g(S,T) = \max(K - S(T), 0)$) where K is the strike price of the option. It is well known that V solves the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3}$$

subject to the appropriate boundary conditions for the call or put. Analytic solutions for these cases were derived by Black and Scholes (1973). The early exercise feature for American put options can be incorporated by imposing the constraint

$$V(S(\tau),\tau) \geq \max(K - S(\tau), 0) \tag{4}$$

at each point in time τ over the life of the option. The hedging arguments underlying (3) are standard and may also be applied in the context of Asian options (see Ingersoll (1987) pp. 376-377 for a discussion). Such options depend on the arithmetic average of the stock price over some time interval. If we let

$$I(T) = \int_0^T S(\tau)d\tau$$

then the average is given by $A(T) = I(T)/T$. As noted by Ingersoll, the value of an Asian option is given by the following PDE with two space dimensions:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S\frac{\partial V}{\partial I} - rV = 0. \tag{5}$$

An equivalent formulation in terms of the average (A) rather than the running sum (I) is given in equation (3.4) of Barraquand and Pudet (1996):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + \frac{1}{t}(S - A)\frac{\partial V}{\partial A} - rV = 0. \tag{6}$$

Again, different terminal boundary conditions may be used to price various different types of securities. Examples include:

- floating strike call: $g(S(T), A(T), T) = \max(S(T) - A(T), 0)$
- floating strike put: $g(S(T), A(T), T) = \max(A(T) - S(T), 0)$
- fixed strike call: $g(S(T), A(T), T) = \max(A(T) - K, 0)$
- fixed strike put: $g(S(T), A(T), T) = \max(K - A(T), 0)$

The only known analytic solution is for the fixed strike case when $K = 0$. An early exercise constraint similar to (4) may be applied to value American-style Asian options. It is important to note that (5) has no diffusion term in the I direction and, similarly, (6) has no diffusion term in the A direction. This fact is the source of many numerical difficulties with standard finite difference methods.

As shown by Ingersoll, for floating strike options (5) may be reduced to a one-dimensional PDE by making the change of variables $R = S/I$. This is because the PDE and all of the relevant boundary conditions are linearly homogeneous in S and I. For fixed strike options, this homogeneity does not hold for the terminal value and so the reduction cannot be applied.

Analogous to (2), solutions to (5) and (6) may be represented as:

$$V(S(t), I(t), t) = \mathrm{e}^{-r(T-t)} \, \mathrm{E}_t \; [g(S(T), I(T), T)]$$

and

$$V(S(t), A(t), t) = \mathrm{e}^{-r(T-t)} \, \mathrm{E}_t \; [g(S(T), A(T), T)] \tag{7}$$

respectively. Rogers and Shi (1995) have recently formulated an alternative PDE based on the representation (7) and a scaling property of geometric Brownian motion. They define a new state variable

$$x = \frac{K - \int_0^t S(\tau)\mu(d\tau)}{S_t}$$

where μ is a probability measure with density $\rho(t)$ in $(0, T)$. For a fixed strike option, $\rho(t) = 1/T$. For a floating strike option, $K = 0$ and $\rho(t) = 1/T - \delta(T - t)$, where δ is a delta function. Rogers and Shi show that the value of an Asian option is governed by the following PDE:

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 W}{\partial x^2} - (\rho(t) + rx)\frac{\partial W}{\partial x} = 0 \tag{8}$$

The terminal conditions for a fixed strike call and floating strike put are

$$W(x, T) = \max(0, -x)$$

and

$$W(x, T) = \max(0, -x - 1),$$

respectively. The price of a fixed strike call with exercise price K and initial stock price S_0 is $S_0 W(\frac{K}{S_0}, 0)$. For a floating strike put, the price is $S_0 W(0, 0)$. In this setting, we have a one-dimensional PDE for both fixed and floating strike options. However, it cannot be applied in the case of American-style options (both the original representation (7) and the density $\rho(t)$ are defined according to exercise occurring only at maturity T).

Summing up, European-style Asian options may be valued using one-dimensional PDEs, either in the Rogers and Shi framework (for both fixed and floating strike options) or after a change of variables in (5) or (6) (only for floating strike options). This change also permits the pricing of American-style floating strike options in one-dimension. To value fixed strike options with early exercise opportunities, we must solve a two-dimensional PDE given by (5) or (6).

3 Discretization Analysis

Before addressing the issue of discretizing Asian option models, we will examine several discretization techniques for the Black-Scholes (1973) equation. As noted earlier, our main motivation is to show the types of problems which can arise when standard methods are used for problems with very low volatility as well as how our modifications may be used to control for these adverse effects. Although we demonstrate the problems for out-of-the-money European call options, the problems are pervasive for Asian option models (in-, at- and out-of-the-money).

3.1 European Options

The price of a European option can be determined by solving equation (3) subject to the appropriate terminal and boundary conditions. Equation (3) is a backward linear parabolic equation and may also be referred to as a convection-diffusion equation (Roache, 1972). The value of a European call option can be determined by solving (3) subject to the terminal condition

$$V(S(T), T) = \max(S(T) - K, 0)$$

and boundary conditions

$$V(0, t) = 0 \text{ and } V(S(t), t) \sim S(t) - K\mathrm{e}^{-r(T-t)} \text{ as } S(t) \to \infty.$$

To value a European put option, equation (3) must be solved subject to the terminal condition

$$V(S(T), T) = \max(K - S(T), 0)$$

and boundary conditions

$$V(0, t) = Ke^{-r(T-t)} \text{ and } V(S(t), t) \sim 0 \text{ as } S(t) \to \infty.$$

The Black-Scholes equation can be converted to a forward equation in time by substituting t with $t^* = T - t$ which evolves from expiration to the present. After performing the change of variables, equation (3) becomes

$$\frac{\partial V}{\partial t^*} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (-rS)\frac{\partial V}{\partial S} - rV. \tag{9}$$

which is in a form that is common in fluid dynamics. The term

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}$$

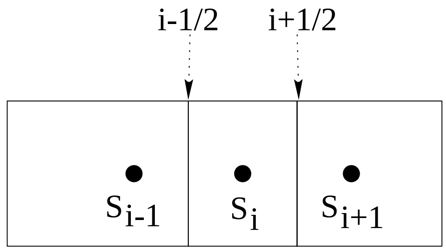

Figure 1: Schematic representation of the finite volume method.

is a parabolic diffusion term. The magnitude of the diffusion is given by $\frac{1}{2}\sigma^2 S^2$. In equation (9)

$$(-rS)\frac{\partial V}{\partial S}$$

is a first-order hyperbolic convective term. The convective term propagates information with a velocity of $-rS$. Since $rS \geq 0$, the information flows from the $S \to \infty$ boundary into the computational domain. If the velocity term is large compared to the diffusion term, then equation (9) is said to be *convection dominated.* Although equation (9) is formally parabolic, when it is convection dominated the numerical approximation behaves as if it was hyperbolic and is therefore much harder to solve accurately. For certain path-dependent options, such as Asian options, the problem of convection dominated PDEs can be especially severe.

Equation (9) can be discretized using the finite volume approach (see Figure 1 for a schematic representation and Roache (1972) for a derivation of the method). The resulting discretization with temporal weighting for the value at cell i at time-step $n+1$ written in general form is

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t^*} &= \theta F_{i-\frac{1}{2}}^{n+1} - \theta F_{i+\frac{1}{2}}^{n+1} + \theta f_i^{n+1} \\ &+ (1-\theta) F_{i-\frac{1}{2}}^{n} - (1-\theta) F_{i+\frac{1}{2}}^{n} + (1-\theta) f_i^{n}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \theta &= \text{temporal weighting } (0 \leq \theta \leq 1), \\ F_{i-\frac{1}{2}} &= \text{flux entering cell } i \text{ at interface } i - \tfrac{1}{2}, \\ F_{i+\frac{1}{2}} &= \text{flux leaving cell } i \text{ at interface } i + \tfrac{1}{2}, \\ f_i &= \text{source/sink term.} \end{aligned}$$

For a fully-implicit method we let $\theta = 1$, for $\theta = \frac{1}{2}$ we have the Crank-Nicolson method and for a fully-explicit method we let $\theta = 0$. The R.H.S. of equation (9) collects terms involving spatial derivatives into what are referred to in computational fluid dynamics as flux terms (denoted by F) and source/sink terms (which do not include spatial derivatives and are indicated by f). In particular

$$F_{i-\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[(-\frac{1}{2}\sigma^2 S_i^2)\frac{(V_i^{n+1} - V_{i-1}^{n+1})}{\Delta S_{i-\frac{1}{2}}} + (-rS_i)V_{i-\frac{1}{2}}^{n+1}\right], \tag{11}$$

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta S_i}\left[(-\frac{1}{2}\sigma^2 S_i^2)\frac{(V_{i+1}^{n+1} - V_{i}^{n+1})}{\Delta S_{i+\frac{1}{2}}} + (-rS_i)V_{i+\frac{1}{2}}^{n+1}\right] \tag{12}$$

and

$$f_i^{n+1} = (-r)V_i^{n+1}. \tag{13}$$

If we use a point-distributed finite volume scheme (i.e. cell interfaces are midway between adjacent nodes), then

$$\Delta S_i = \frac{S_{i+1} - S_{i-1}}{2}$$

and

$$\Delta S_{i+\frac{1}{2}} = S_{i+1} - S_i$$

in equation (12).

Note that the flux functions (11) and (12) allow for non-uniform grid spacing. Thus, we can construct grids which will make the numerical computations more efficient by having a fine grid spacing near and at the exercise price and a coarse grid away from the exercise price.

We will first examine handling the convective term $V_{i+\frac{1}{2}}^{n+1}$ in equation (12) using the following central weighting scheme

$$V_{i+\frac{1}{2}}^{n+1} = \frac{V_{i+1}^{n+1} + V_i^{n+1}}{2}$$

which has second-order accuracy for uniform grids. To ensure that solutions produced using central weighting are free of spurious oscillations, we must satisfy the Peclet condition (Shyy, 1994)

$$\frac{1}{\Delta S_{i-\frac{1}{2}}} > \frac{r}{\sigma^2 S_i} \tag{14}$$

and the additional condition

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + r \tag{15}$$

for all cells i. For a precise definition of "spurious oscillations", and a derivation of (14) and (15), refer to Appendix A.

When equation (3) is convection dominated (i.e. when r is large relative to σ) the grid spacing necessary to satisfy conditions (14) and (15) becomes prohibitively fine. Note that if $S_0 = 0$, then $\Delta S_{i-\frac{i}{2}} = S_i$ at $i = 1$ in equation (14). Thus, equation (14) at $i = 1$ becomes

$$\frac{1}{S_1} > \frac{r}{\sigma^2 S_1}. \tag{16}$$

Condition (16) implies that $\frac{\sigma^2}{r} > 1$, which may not be satisfied, independent of how fine a grid spacing is used. However, this does not present a problem in practice since the convective flux leaving cell 1 is very small because the velocity is only $-rS_1$.

Figure 2 contains plots of the price, delta, and gamma of a European call with one year to maturity when $K = 15$, $r = 0.15$ and $\sigma = 0.01$. The value was calculated using the Crank-Nicolson method with a uniform grid spacing of $\Delta S = 0.1$ and $\Delta t^* = 0.01$, and central weighting for the convection term. The solution is oscillatory because the grid spacing violated the Peclet condition. Although the oscillations are small for the option value, they increase significantly for the sensitivities. Such an interest rate/volatility structure is clearly unrealistic. However, this example was chosen because, in general, the price of a continuously averaged Asian option can be modeled by a two-dimensional PDE with no diffusion in one of the dimensions. Hence, this choice of parameters serves to illustrate the difficulties involved in solving Asian option problems.

It is sometimes suggested in the finance literature that a log transform be performed on the Black-Scholes equation (Brennan and Schwartz, 1978; Hull and White, 1990). Brennan and Schwartz obtained

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial y^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial V}{\partial y} - rV = 0, \tag{17}$$

after performing the following substitution of variables $y = \ln(S)$ in equation (3). Using the finite volume discretization (10) after converting equation (17) to a forward PDE, the flux function becomes

$$F^{n+1}_{i+\frac{1}{2}} = \frac{1}{\Delta y_i}\left[(-\frac{1}{2}\sigma^2)\frac{(V^{n+1}_{i+1} - V^{n+1}_i)}{\Delta y_{i+\frac{1}{2}}} + (-r + \frac{\sigma^2}{2})V^{n+1}_{i+\frac{1}{2}}\right].$$

The log transformed PDE has the convenient property that the convection and diffusion coefficients are constant, which simplifies the numerical solution somewhat. Brennan and Schwartz state that explicit methods are generally unstable when applied to equation (3)

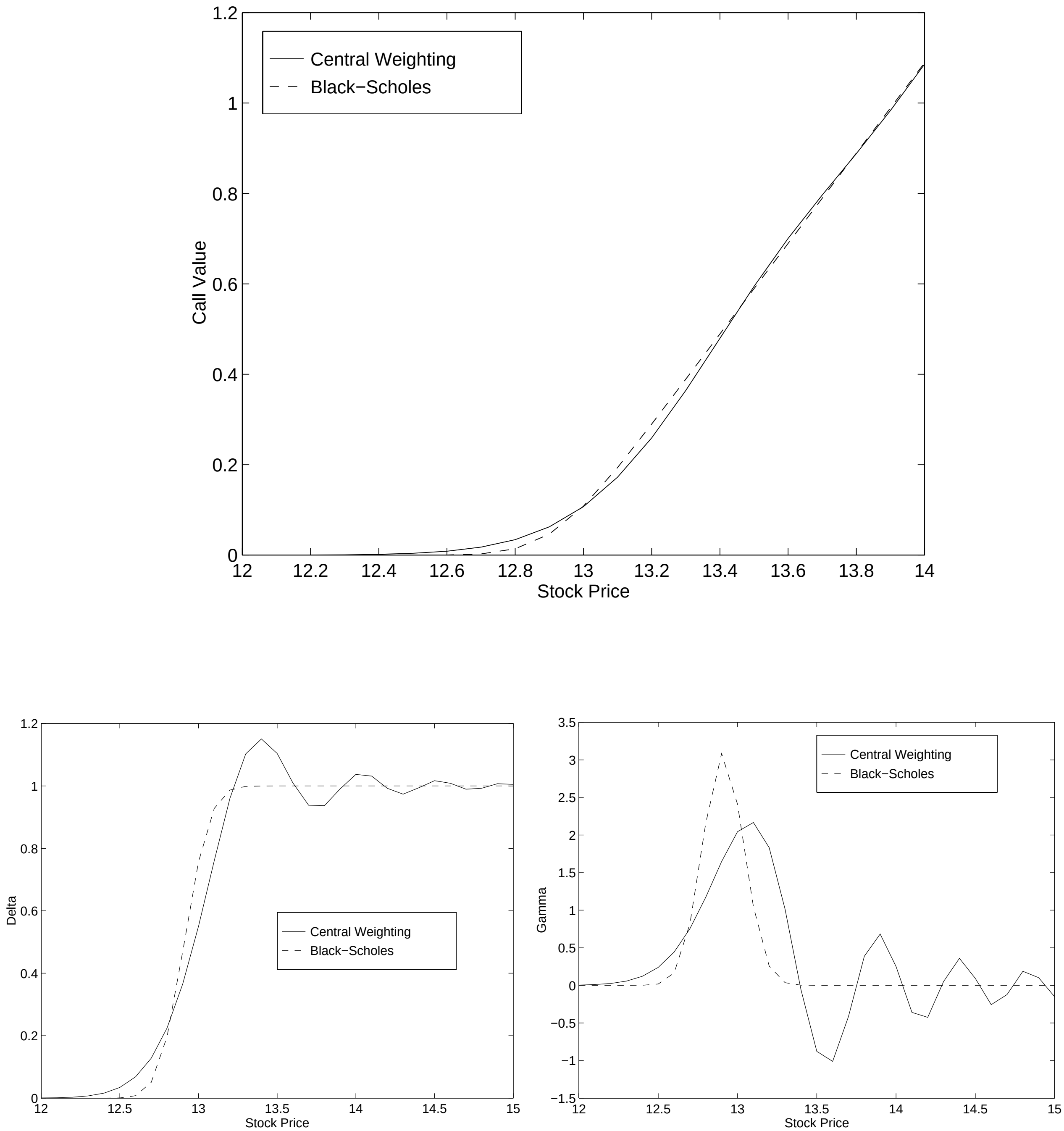

Figure 2: European call price, delta, and gamma when $K = 15$, $r = 0.15$, $\sigma = 0.01$ and $T - t = 1.0$. Calculated using central weighting with $\Delta S = 0.1$, $\Delta t^* = 0.01$ and $\theta = \frac{1}{2}$. Plotted against the Black-Scholes analytical solution.

directly, and that the log transformation allows for the direct application of explicit methods to equation (17). These statements are, in fact, somewhat misleading. If one ensures that the conditions (14) and (15) are met, a fully explicit method will be stable and free of oscillations when solving (3). Conditions must also be met to prevent spurious oscillations when the transformed PDE is being solved. For equation (17) we must meet the Peclet condition

$$\frac{1}{\Delta y_{i-\frac{1}{2}}} > \frac{\left|r - \frac{\sigma^2}{2}\right|}{\sigma^2} \tag{18}$$

and the additional condition

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2}{2}\left(\frac{1}{\Delta y_{i-\frac{1}{2}}\Delta y_i} + \frac{1}{\Delta y_{i+\frac{1}{2}}\Delta y_i}\right) + r. \tag{19}$$

Notice that for the log transformation conditions (18) and (19) are constant for uniform grid spacings, unlike conditions (14) and (15) which vary over the grid for equation (3). The log transformation appears to eliminate the problem of not being able to satisfy the Peclet condition (14) as $S \to 0$ if $\frac{\sigma^2}{r} \leq 1$, when the PDE is posed in the (S,t) domain. However, the log transformation effectively does not solve the problem as $S \to 0$ since this would imply that $y \to -\infty$. Of course, in practice this problem is avoided because a finite computational domain is used. It is also interesting to note that the effective ΔS spacing is very small for small values of y. Figure 3 demonstrates that oscillations can also occur when the log transformed equation is solved using a centrally weighted convection scheme, and conditions (18) and (19) are not met.

The Peclet condition (14) can be re-written as

$$\frac{(rS_i)\Delta S_{i-\frac{1}{2}}}{\frac{1}{2}\sigma^2 S_i^2} < 2, \tag{20}$$

where the L.H.S. is the cell Peclet number (Shyy, 1994). If the grid spacing is not sufficiently fine when the convection term dominates the diffusion term (i.e. when r is large relative to σ), the cell Peclet number will exceed condition (20). To eliminate the need for excessively fine grid spacing the true diffusion can be augmented by additional numerical diffusion. One approach to supplement the true diffusion with numerical diffusion which has been used in computational fluid dynamics is first-order upstream weighting (Roache, 1972).

The first-order upstream weighting scheme for equation (9) is

$$V_{i+\frac{1}{2}}^{n+1} = V_{up}^{n+1} = \begin{cases} V_i^{n+1} & \text{if } -rS \geq 0, \\ V_{i+1}^{n+1} & \text{otherwise} \end{cases}$$

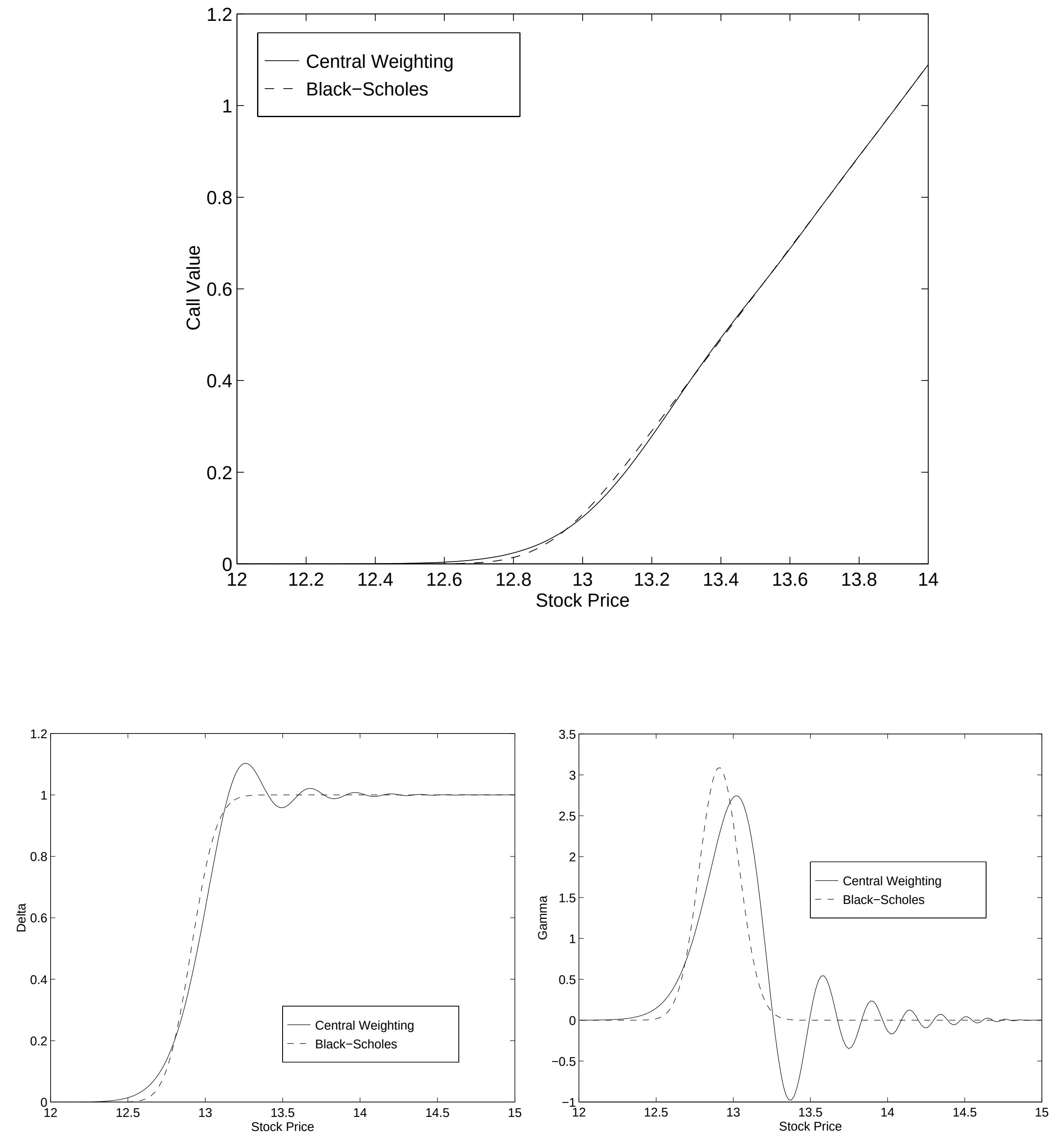

Figure 3: European call price, delta, and gamma when $K = 15$, $r = 0.15$, $\sigma = 0.01$ and $T - t = 1.0$. Log transformed PDE solved using central weighting with $\Delta y = 0.0013$, $\Delta t^* = 0.04$ and $\theta = \frac{1}{2}$. Plotted against the Black-Scholes analytical solution.

in equation (12). In this case, since $rS \geq 0$, $V_{up}^{n+1} = V_{i+1}^{n+1}$. Conversely, V_i^{n+1} is termed the downstream value. Upstream weighting corresponds to a fluid flow problem, where information should only flow from upstream to downstream cells.

The artificial diffusion is introduced through the truncation error of the one-sided differencing (Roache, 1972). The scheme is first-order accurate for uniform grids. The accuracy deteriorates for non-uniform grids because the discretization does not produce true one-sided differences. This was not an issue for our examples, which used uniform grids. To prevent oscillations from forming when upstream weighting is used we must meet only the following condition

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + \frac{rS_i}{\Delta S_i} + r. \tag{21}$$

The derivation of (21) is analogous to the derivation of (14) and (15) in Appendix A. Figure 4 demonstrates how the solutions are no longer oscillatory. Unfortunately, it is also apparent that first-order upstream weighting produces solution profiles that are too diffuse.

To produce oscillation free solutions without the excessive diffusion of first order upstream weighting we examined the non-linear van Leer flux limiter (Sweby, 1984; Blunt and Rubin, 1992). For the van Leer flux limiter

$$V_{i+\frac{1}{2}}^{n+1} = V_{up}^{n+1} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2}(V_{down}^{n+1} - V_{up}^{n+1}) \tag{22}$$

in equation (12), where

$$q_{i+\frac{1}{2}}^{n+1} = \frac{V_{up}^{n+1} - V_{2up}^{n+1}}{S_{2up} - S_{up}} \bigg/ \frac{V_{down}^{n+1} - V_{up}^{n+1}}{S_{up} - S_{down}} \tag{23}$$

(this formulation allows for non-uniform grids; refer to Appendix C for a derivation) and

$$\phi(q_{i+\frac{1}{2}}^{n+1}) = \frac{\left|q_{i+\frac{1}{2}}^{n+1}\right| + q_{i+\frac{1}{2}}^{n+1}}{1 + \left|q_{i+\frac{1}{2}}^{n+1}\right|}.$$

In equation (23) V_{2up}^{n+1} is the second upstream point. That is, if V_{i+1}^{n+1} is the upstream point to node i, then V_{i+2}^{n+1} is the second upstream point. Conceptually, the scheme only adds numerical diffusion at points where the gradient is steep. The scheme is second-order accurate away from regions which are augmented by numerical diffusion, and has the property that it is total variation diminishing (TVD). A scheme is TVD when

$$TV(V^{n+1}) \leq TV(V^n),$$

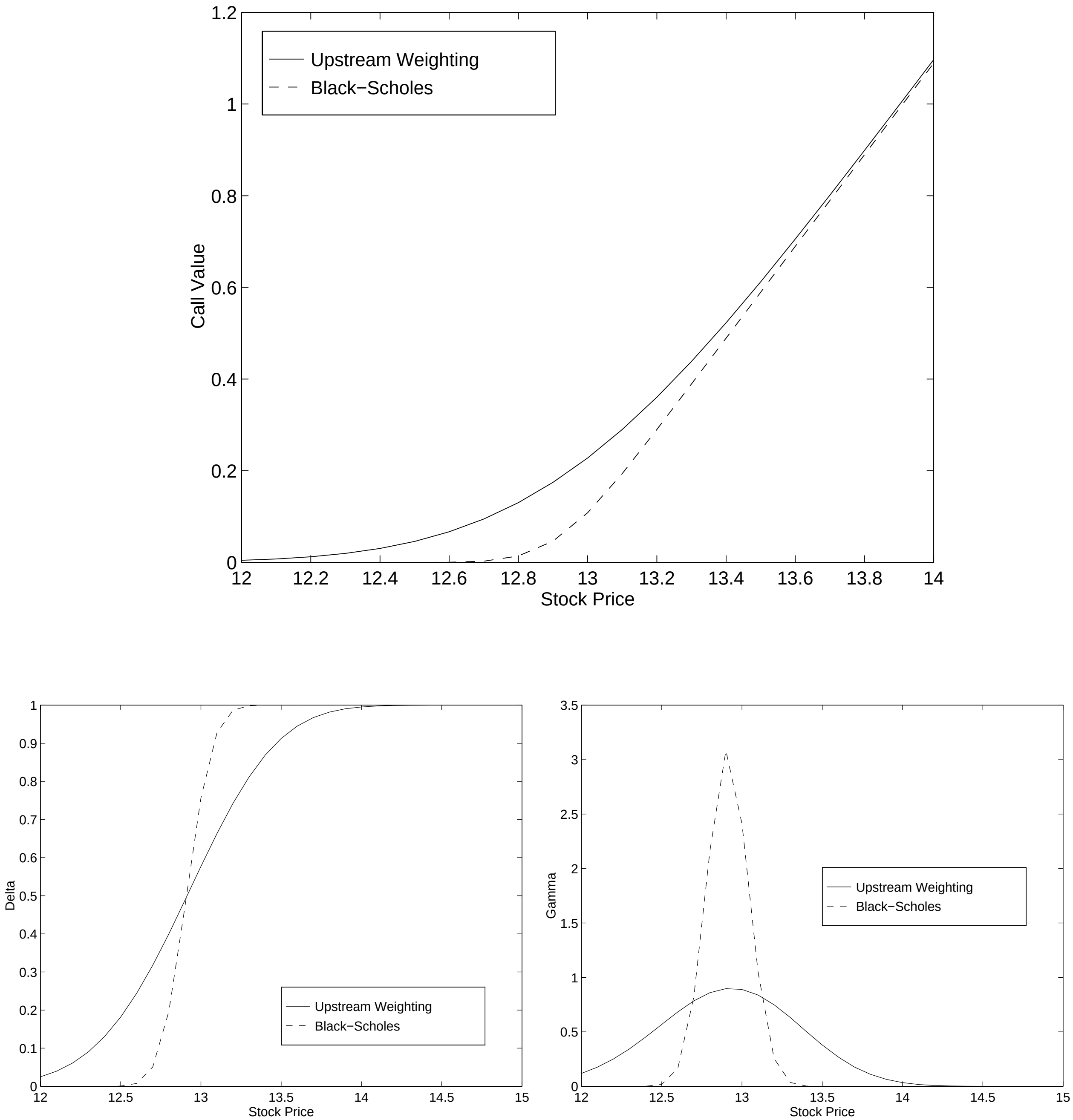

Figure 4: European call price, delta, and gamma when $K = 15$, $r = 0.15$, $\sigma = 0.01$ and $T - t = 1.0$. Calculated using first-order upstream weighting with $\Delta S = 0.1$, $\Delta t^* = 0.01$ and $\theta = \frac{1}{2}$. Plotted against the Black-Scholes analytical solution.

where $TV(V^{n+1})$ is the total variation of the solution and is defined as

$$TV(V^{n+1}) = \sum_i \left| V_{i+1}^{n+1} - V_i^{n+1} \right| .$$

Thus, if a scheme is TVD the solution cannot contain oscillations, otherwise the total variation would increase. Stability and convergence proofs of TVD methods for conservation laws can be found in LeVeque (1990). An important component in convergence proofs is a bound on the variation of the solution (Sweby, 1984). In Appendix B we show that these methods are also TVD for non-conservative PDEs, such as equation (3). For a detailed analysis of the TVD conditions for equation (22), refer to Appendix C. Note that if a limiter is not used, as in Dewynne and Wilmott (1995), then the solution cannot be guaranteed to be oscillation free unless the grid is very fine.

Figure 5 demonstrates that the van Leer flux limiter produced oscillation free solutions without the excessive smearing of first-order upstream weighting. Since the flux limiter is non-linear the solutions were obtained using full Newton iteration. The van Leer flux limiter should be used with the Crank-Nicolson method when equation (3) is convection dominated. In this situation, a fully-implicit method generates a smeared solution, as Figure 6 illustrates. This is due to the fact that the Crank-Nicolson method is second-order accurate in time, while the fully-implicit method is only first-order accurate in time.

3.2 American Options

The value of an American put option must meet the condition

$$V(S(\tau), \tau) \geq \max(K - S(\tau), 0) \tag{24}$$

at all points in time τ over the life of the contract. Wilmott, Dewynne and Howison (1993) have shown that an American option can be valued by solving

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} \leq rV \tag{25}$$

subject to constraint (24). If we consider solving the discrete system with two unknowns at each node, Φ_i^{n+1} and V_i^{n+1} where V_i^{n+1} is the value of the American option, then equations (24) and (25) can be posed in discrete form as

$$\begin{aligned} \frac{\Phi_i^{n+1} - V_i^n}{\Delta t^*} &= \theta F_{i-\frac{1}{2}}^{n+1}(V_{i-1}^{n+1}, V_i^{n+1}, V_{i+1}^{n+1}) \\ &- \theta F_{i+\frac{1}{2}}^{n+1}(V_i^{n+1}, V_{i+1}^{n+1}, V_{i+2}^{n+1}) + \theta f_i^{n+1}(V_i^{n+1}) \\ &+ (1-\theta) F_{i-\frac{1}{2}}^{n}(V_{i-1}^{n}, V_i^{n}, V_{i+1}^{n}) \\ &- (1-\theta) F_{i+\frac{1}{2}}^{n}(V_i^{n}, V_{i+1}^{n}, V_{i+2}^{n}) + (1-\theta) f_i^{n}(V_i^{n}) \end{aligned} \tag{26}$$

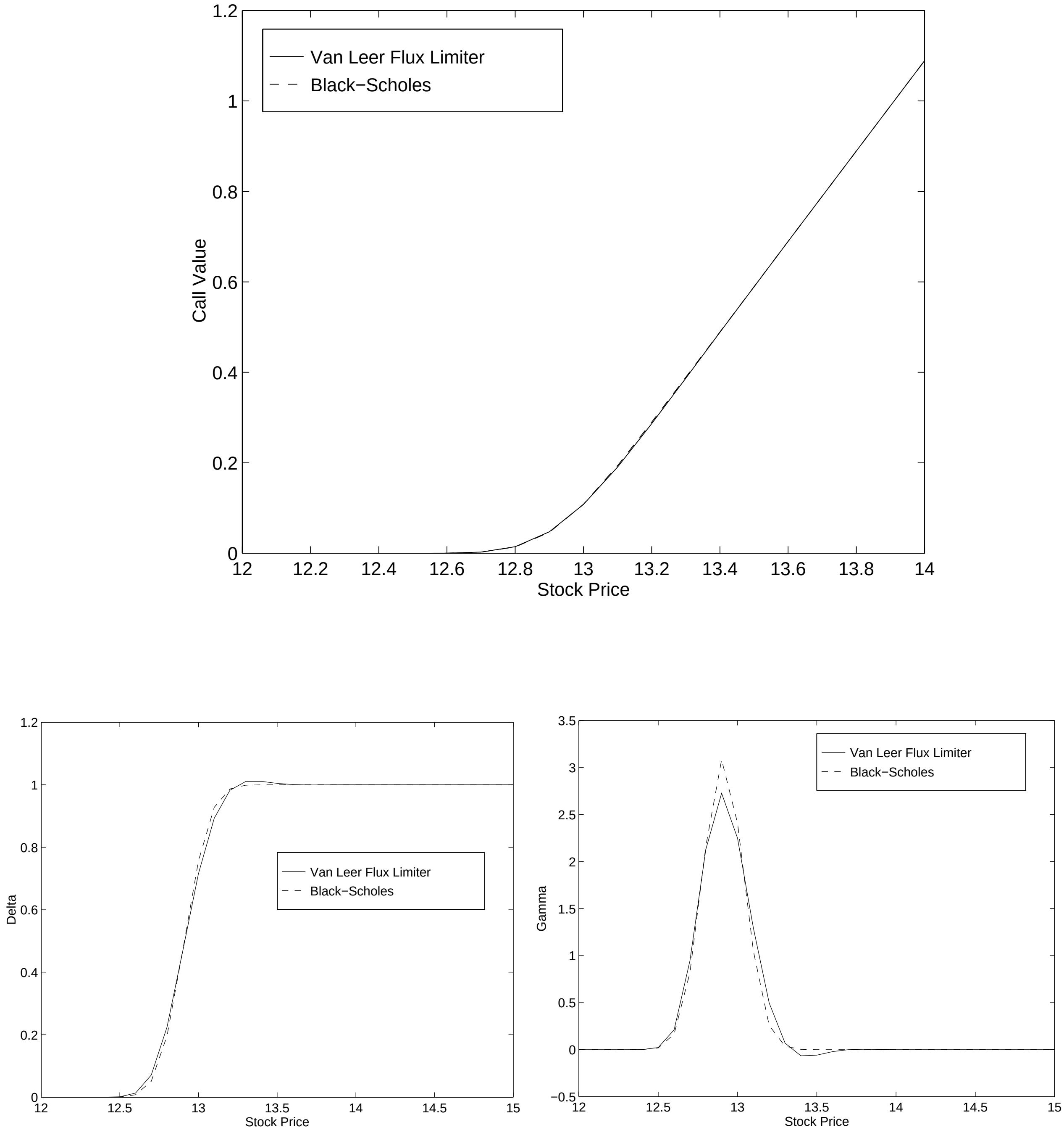

Figure 5: European call price, delta, and gamma when $K = 15$, $r = 0.15$, $\sigma = 0.01$ and $T - t = 1.0$. Calculated using the van Leer flux limiter with $\Delta S = 0.1$, $\Delta t^* = 0.01$ and $\theta = \frac{1}{2}$. Plotted against the Black-Scholes analytical solution.

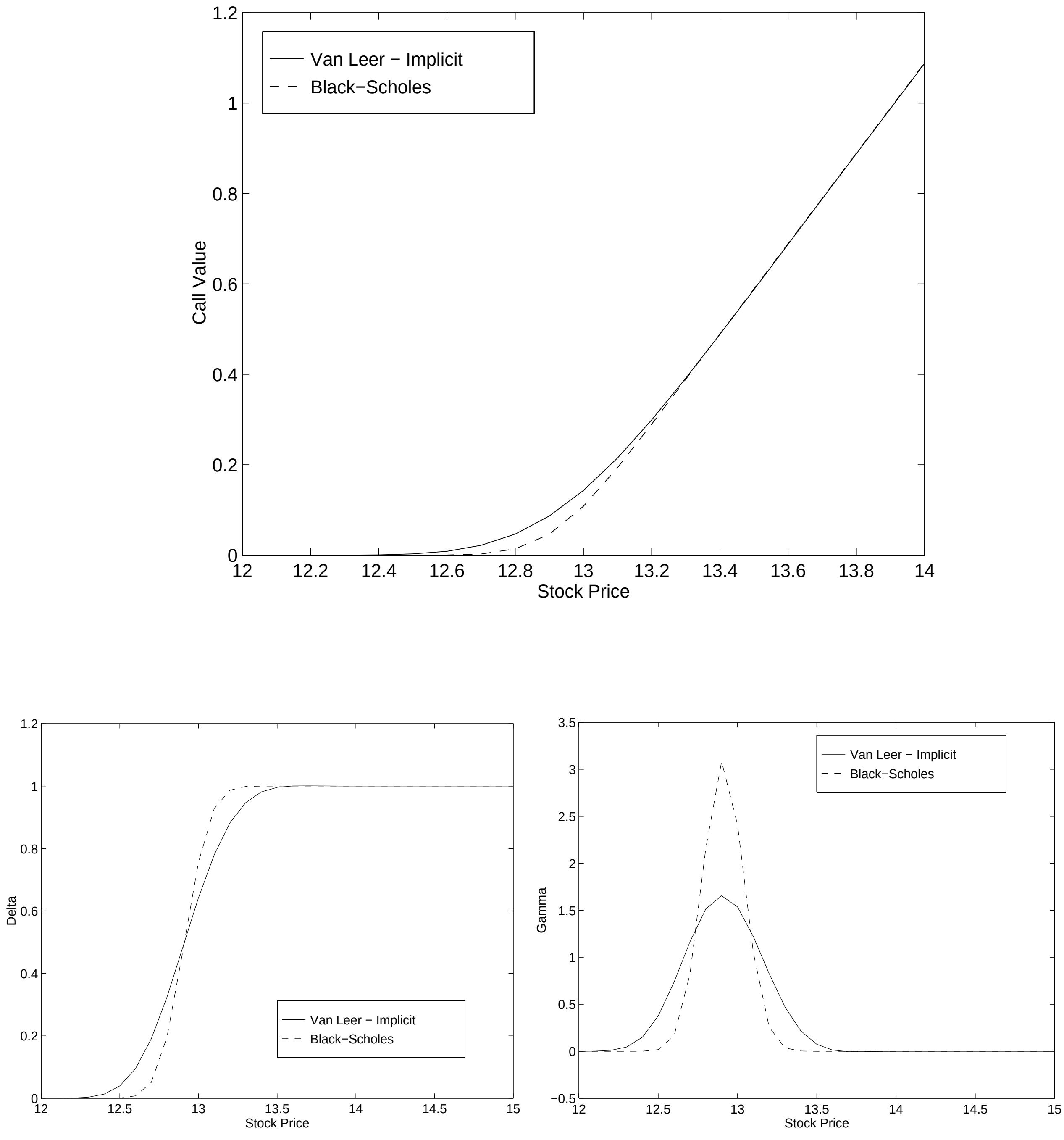

Figure 6: European call price, delta, and gamma when $K = 15$, $r = 0.15$, $\sigma = 0.01$ and $T - t = 1.0$. Calculated using the van Leer flux limiter with $\Delta S = 0.1$, $\Delta t^* = 0.01$ and $\theta = 1$. Plotted against the Black-Scholes analytical solution.

K	Months	Binomial	Van Leer
35	1	0.08	0.08
	4	0.70	0.69
	7	1.22	1.21
40	1	1.31	1.30
	4	2.48	2.47
	7	3.17	3.16
45	1	5.06	5.06
	4	5.71	5.70
	7	6.24	6.23

Table 1: American put values when $r = 0.05$ and $\sigma = 0.3$. The van Leer limiter was used with $\theta = \frac{1}{2}$, $\Delta t^* = 0.0019$ and a non-uniform spatial grid with $\Delta S = 0.5$ near and at the exercise price. The binomial results were obtained from Geske and Shastri (1985).

where for a put option

$$V_i^{n+1} = \max(\Phi_i^{n+1}, K - S_i, 0) \tag{27}$$

Note that the L.H.S. of equation (26) can be written as

$$\frac{V_i^{n+1} - V_i^n}{\Delta t^*} + \frac{\Phi_i^{n+1} - V_i^{n+1}}{\Delta t^*}$$

and since by (27)

$$\frac{\Phi_i^{n+1} - V_i^{n+1}}{\Delta t^*} \leq 0.$$

Thus, equation (26) is a discrete form of the inequality (25).

If the system is non-linear, the constraint is applied at each Newton iteration. Table 1 contains results obtained using the constraint in an implicit fully coupled manner with the van Leer limiter to price American put options. As the table demonstrates, the prices generated are virtually identical to those produced by the binomial method.

The traditional approach seen in the finance literature when using PDEs to value American options is to apply the American constraint explicitly (Brennan and Schwartz, 1977; Geske and Shastri, 1985; Hull, 1993). That is, equation (3) is solved and after each time-step the constraint is applied to the solution. This differs from the implicit fully coupled method which solves equation (25) directly. However, we only noticed differences in the rate of convergence for large time-steps. That is, the implicit application of the constraint required fewer non-linear iterations when the time-step was large. These time-steps were too large to achieve convergence to an accurate solution. For suitable time-steps, we found no difference in the rate of convergence. Thus, in a practical sense, there is no difference between applying the constraint implicitly or explicitly. Although we have never experienced stability problems using practical time-step sizes, there is always a possibility that the explicit application of the constraint will lead to instability. Consequently, we will apply the American constraint implicitly. The additional computational cost of applying the constraint implicitly is negligible.

4 Continuous Arithmetic Asian Options

Based on the results from section 3, we will now examine treating convection using the van Leer flux limiter in discretizations of one- and two-dimensional PDE models of Asian options.

4.1 One-Dimensional Models

One-dimensional models for pricing continuously averaged arithmetic Asian options have been derived by Ingersoll (1987) and Rogers and Shi (1995). The Ingersoll model cannot be used to price fixed strike options. However, it can be used to price American-style floating strike options (Wilmott et al., 1993).

Although the Rogers and Shi (1995) model cannot handle the early exercise feature, it can be used to price both floating and fixed strike options. Consequently, we chose to examine this model. After converting (8) to a forward equation, we have the following flux function

$$F_{i+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x_i}\left[(-\frac{1}{2}\sigma^2 x_i^2)\frac{(W_{i+1}^{n+1} - W_i^{n+1})}{\Delta x_{i+\frac{1}{2}}} + (\rho^{n+1} + r x_i) W_{i+\frac{1}{2}}^{n+1}\right]$$

and source term $f_i^{n+1} = 0$. It should be noted that x will take on negative values and thus, $(\rho^{n+1} + r x_i)$ may take on both negative and non-negative values. The discretization of $W_{i+\frac{1}{2}}^{n+1}$ must take this into account.

Tables 2 and 3 contain the results obtained by using the van Leer flux limiter to value fixed and floating strike Asian options, respectively. Rogers and Shi (1995) examined a number of different volatility/interest rate structures. Tables 2 and 3 contain only the results for $r = 0.15$, their most difficult case. Note that the time-step size for these examples was selected on the basis of several trial runs, which indicated that decreasing the time step size further had no discernible effect on the results to four significant figures.

The results demonstrate that sufficiently accurate values can be obtained for most volatilities in under an average of 10 seconds for both fixed and floating strike Asian options. Sufficiently accurate results were obtained for all volatilities in under an average of 16 seconds.

σ	K	Δx			Bounds	
		0.025	0.010	0.005	Lower	Upper
0.05	95	11.002	11.093	11.094	11.094	11.114
	100	6.609	6.779	6.793	6.794	6.810
	105	2.778	2.758	2.748	2.744	2.761
0.10	90	15.381	15.399	15.399	15.399	15.445
	100	6.991	7.035	7.030	7.028	7.066
	110	1.427	1.405	1.410	1.413	1.451
0.20	90	15.637	15.646	15.643	15.641	15.748
	100	8.405	8.411	8.409	8.408	8.515
	110	3.536	3.551	3.554	3.554	3.661
0.30	90	16.524	16.517	16.514	16.512	16.732
	100	10.203	10.210	10.210	10.208	10.429
	110	5.713	5.727	5.729	5.728	5.948
Mean Exec. Time (sec.)		6.90	13.23	23.02		

Table 2: Fixed strike Asian call values when $r = 0.15$, $S_0 = 100$ and $T - t = 1$. The van Leer limiter was used with $\theta = \frac{1}{2}$ and $\Delta t^* = 0.0025$. A non-uniform spatial grid was used, where Δx denotes the spacing in the region $[-0.25, 1.25]$. Mean execution times are for runs performed on a DEC Alpha. The bounds were obtained from Rogers and Shi (1995).

By sufficiently accurate, we mean that the values are no more than 0.05% of S_0 outside of the bounds derived by Rogers and Shi. It should be noted that equation (8) will become convection dominated as $x \to 0$, and for short maturities since $\rho(t) = \frac{1}{T}$ or $\rho(t) = \frac{1}{T} - \delta(T - t)$. Although Tables 2 and 3 do not contain results for short maturity options, we will see in section 4.2 (see Table 4) that accurate results can be obtained for short maturities using the van Leer limiter.

Figure 7 compares results that we obtained using the van Leer limiter and a centrally weighted scheme with Rogers and Shi's (1995) results which were obtained using the method of lines. The same coarse spatial grid was used for all the PDE runs. Note that the discretization which uses the van Leer limiter converges more rapidly than either central weighting or the method of lines. For this low volatility case the bounds that Roger and Shi derived are very tight. In fact, the bounds were plotted as cross hairs because the upper and lower bounds appear at the same point when plotted. It is interesting to note that sophisticated PDE methods must be used if the problem is convection dominated, while the PDE problem is easily solved when the volatility is large.

σ	Δx			Bounds	
	0.025	0.010	0.005	Lower	Upper
0.10	0.123	0.217	0.244	0.252	0.415
0.20	1.659	1.700	1.708	1.710	2.187
0.30	3.587	3.606	3.609	3.609	4.604
Mean Exec. Time (sec.)	9.25	15.98	27.26		

Table 3: Floating strike Asian put values when $r = 0.15$, $S_0 = 100$ and $T - t = 1$. The van Leer limiter was used with $\theta = \frac{1}{2}$ and $\Delta t^* = 0.0025$. A non-uniform spatial grid was used, where Δx denotes the spacing in the region $[-1.25, 0.25]$. Mean execution times are for runs performed on a DEC Alpha. The bounds were obtained from Rogers and Shi (1995).

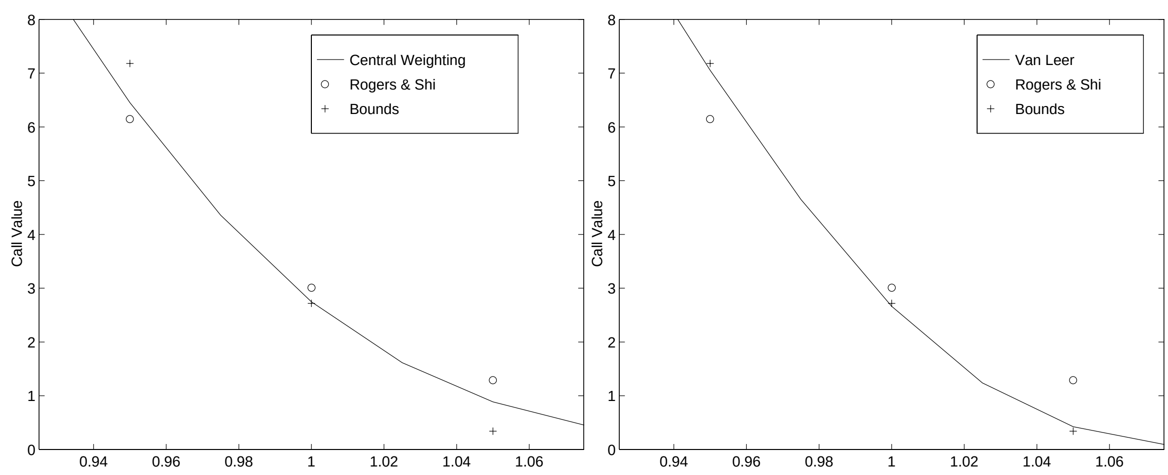

Figure 7: Fixed strike Asian option values when $r = 0.05$, $\sigma = 0.05$ and $T - t = 1.0$. Note that at the point in time plotted on the graphs, $x = \frac{K}{S_0}$. The solutions were computed using central weighting and the van Leer flux limiter with $\Delta x = 0.025$, $\Delta t^* = 0.0025$ and $\theta = \frac{1}{2}$. The Rogers and Shi (1995) results were calculated using the method of lines. The bounds were obtained from Rogers and Shi.

4.2 Two-Dimensional Models

When American-style fixed strike Asian options are to be priced, we cannot use the one-dimensional models outlined in section 4.1 (Wilmott et al., 1993; Barraquand and Pudet, 1996). In these cases a full two-dimensional PDE must be solved. We chose to examine equation (6) because numerical experiments using equations (5) and (6) indicated that fewer nodes were needed to achieve equivalent accuracy for equation (6) when compared to (5). This was due to the fact that the significant part of the computational domain appears to be smaller for formulations using the average as opposed to the running sum. One may note that at $t = 0$ a singularity exists in equation (6) because of the $\frac{1}{t}(S - A)\frac{\partial V}{\partial A}$ term. However, this can be avoided if we assume that $S = A$ at $t = 0$. Thus, equation (6) simply becomes the Black-Scholes equation at $t = 0$.

After converting equation (6) to a forward PDE, the finite volume discretization is

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t^*} &= \theta F_{i-\frac{1}{2},j}^{n+1} - \theta F_{i+\frac{1}{2},j}^{n+1} \\ &+ \theta F_{i,j-\frac{1}{2}}^{n+1} - \theta F_{i,j+\frac{1}{2}}^{n+1} + \theta f_{i,j}^{n+1} \\ &+ (1-\theta)F_{i-\frac{1}{2},j}^{n} - (1-\theta)F_{i+\frac{1}{2},j}^{n} \\ &+ (1-\theta)F_{i,j-\frac{1}{2}}^{n} - (1-\theta)F_{i,j+\frac{1}{2}}^{n} + (1-\theta)f_{i,j}^{n}, \end{aligned}$$

where

$$F_{i,j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta A_{i,j}}\left[\frac{1}{t}(A_{i,j} - S_{i,j})V_{i,j+\frac{1}{2}}^{n+1}\right].$$

As was the case in section 4.1, in order to ensure that the upstream points are defined appropriately the discretization of $V_{i,j+\frac{1}{2}}^{n+1}$ using the van Leer limiter must take into account the fact that $\frac{1}{t}(A_{i,j} - S_{i,j})$ will take on negative and non-negative values. Also, if at j_{max} there exist $A_{i,j_{max}}$ which are less than $S_{i,j_{max}}$, then an appropriate boundary condition must be imposed at those points. $F_{i+\frac{1}{2},j}^{n+1}$ and $f_{i,j}^{n+1}$ are similar to (12) and (13), respectively.

Since the PDE is two-dimensional, we used incomplete LU decomposition and the stabilized conjugate gradient method (ILU-CGSTAB) to solve the resulting system of equations (D'Azevedo et al., 1992; van der Vorst, 1992). We also incorporated a time-step selector (Forsyth and Sammon, 1986) into our solver.

We found that the quality of the solution for certain terminal conditions is highly dependent upon the grid spacing. We conjecture that the problem arises because equation (6) is similar to the turning point problem, which has known difficulties (Ascher et al., 1988; Tang, 1992). This is only an issue in the case of European floating strike calls and puts. One must ensure that the $S : A$ aspect ratio for grid spacings used in these cases is always $k : 1$, where

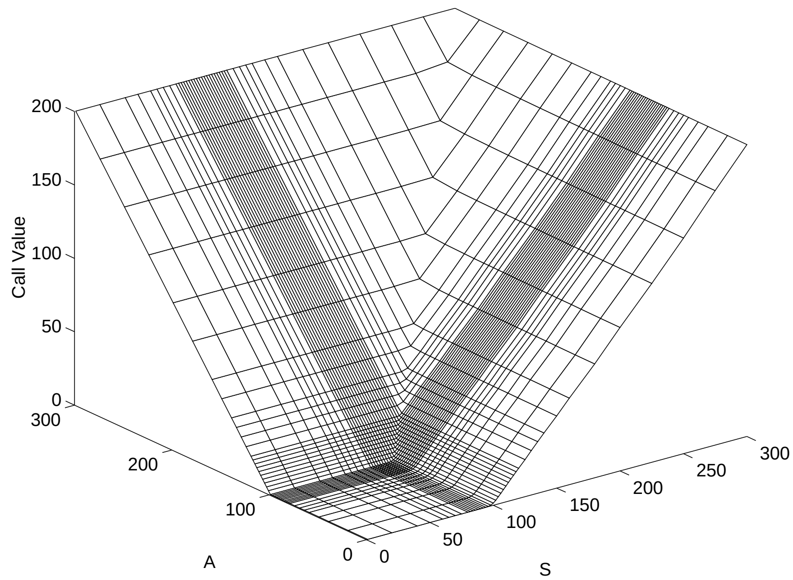

Figure 8: American fixed strike call value when $K = 100$, $r = 0.10$, $\sigma = 0.10$, and $T - t = 0.25$. Calculated using the van Leer flux limiter with $\theta = \frac{1}{2}$.

k is an arbitrary integer greater than or equal to 1. If one is interested in the values and sensitivities over a large region of the grid, it would be best to use a uniform grid spacing in the A dimension that achieves sufficient accuracy. We emphasize, however, that this issue is easily avoided entirely since these cases can be handled using a one-dimensional model.

Tables 4, 5 and 6 contain the results of using the van Leer flux limiter to solve equation (6). Table 4 contains the results for European and American fixed strike call options. The results for zero strike Asian call options, which have an analytical solution, are contained in Table 5. Table 6 contains the results for European and American floating strike puts. The parameters were chosen to allow a comparison with Barraquand and Pudet's (1996) results. The grid spacing was chosen to achieve an accuracy of at least 0.10% of S_0. To determine the accuracy, the computed lower bound and the solution (using a fine grid) to the one-dimensional Roger and Shi (1995) model were taken to be the true solutions. Figures 8 and 9 are plots of an American fixed strike call and American floating strike put, respectively.

Barraquand and Pudet (1996) correctly observe that an explicit centrally weighted scheme for equation (6) is unstable. Note that equation (6) is convection dominated in the A direction because there is no diffusion effect in this dimension. In particular, the convective term in the A dimension becomes very large as $t \to 0$. Barraquand and Pudet also note that implicit centrally weighted schemes will generally produce unsatisfactory results because of the numerical diffusion introduced by this first-order accurate in time scheme. More im-

σ	T-t	K	European				American	
			Lower	1-D	2-D	B & P	2-D	B & P
0.10	0.25	95	6.118	6.114	6.133	6.132	6.646	6.546
		100	1.851	1.841	1.793	1.869	1.903	1.967
		105	0.148	0.162	0.162	0.151	0.161	0.152
	0.50	95	7.220	7.216	7.244	7.248	7.687	7.632
		100	3.104	3.064	3.052	3.100	3.180	3.212
		105	0.714	0.718	0.726	0.727	0.733	0.735
	1.00	95	9.285	9.286	9.316	9.313	9.662	9.616
		100	5.255	5.254	5.261	5.279	5.398	5.394
		105	2.294	2.295	2.314	2.313	2.340	2.336
0.20	0.25	95	6.476	6.461	6.501	6.500	7.521	7.371
		100	2.932	2.923	2.928	2.960	3.224	3.219
		105	0.947	0.958	0.971	0.966	1.009	1.001
	0.50	95	7.891	7.890	7.921	7.793	8.908	8.805
		100	4.505	4.502	4.511	4.548	4.901	4.893
		105	2.211	2.206	2.229	2.241	2.337	2.337
	1.00	95	10.295	10.294	10.309	10.336	11.295	11.218
		100	7.042	7.041	7.042	7.079	7.548	7.521
		105	4.509	4.508	4.519	4.539	4.742	4.729
0.40	0.25	95	8.102	8.097	8.123	8.151	9.548	9.447
		100	5.168	5.164	5.175	5.218	5.846	5.826
		105	3.063	3.061	3.082	3.106	3.349	3.347
	0.50	95	10.346	10.344	10.357	10.425	11.997	10.927
		100	7.572	7.570	7.574	7.650	8.527	8.519
		105	5.371	5.371	5.384	5.444	5.899	5.913
	1.00	95	13.721	13.716	13.721	13.825	15.749	15.649
		100	11.121	11.120	11.115	11.213	12.497	12.439
		105	8.910	8.911	8.912	8.989	9.825	9.790
Mean Execution Time (sec.)					45.64		52.05	

Table 4: American and European fixed strike call values when $r = 0.10$ and $S_0 = 100$. Lower and 1-D refer to the Rogers and Shi (1995) lower bound and one-dimensional PDE, respectively. B & P refers to the results obtained by Barraquand and Pudet (1996). 2-D refers to the value obtained by solving the two-dimensional PDE using the van Leer limiter with $\theta = \frac{1}{2}$ and a non-uniform spatial A x S grid of 41 x 45. Δt^* was set to one day, two days and three days for maturities of three, six and twelve months, respectively. Mean execution times are for runs performed on a DEC Alpha.

σ	T-t	Analytic	2-D
0.10	0.25	98.763	98.770
	0.50	97.547	97.558
	1.00	95.175	95.186
0.20	0.25	98.763	98.769
	0.50	97.547	97.552
	1.00	95.175	95.166
0.40	0.25	98.763	98.760
	0.50	97.547	97.541
	1.00	95.175	95.186

Table 5: Analytic and numerical solution of the two-dimensional PDE model for the zero strike Asian call when $r = 0.10$ and $S_0 = 100$. The van Leer limiter was used with $\theta = \frac{1}{2}$ and a non-uniform spatial A x S grid of 41 x 45. Δt^* was set to one day, two days and three days for maturities of three, six and twelve months, respectively. The Analytical results were obtained from Barraquand and Pudet (1996).

σ	T-t	European				American	
		Lower	1-D	2-D	B & P	2-D	B & P
0.10	0.25	0.628	0.636	0.582	0.632	1.359	1.194
	0.50	0.666	0.668	0.640	0.671	1.601	1.494
	1.00	0.598	0.598	0.589	0.614	1.952	1.799
0.20	0.25	1.714	1.719	1.679	1.724	2.867	2.773
	0.50	2.147	2.123	2.093	2.135	3.806	3.684
	1.00	2.457	2.449	2.423	2.498	4.932	4.812
0.40	0.25	3.971	3.971	3.926	3.981	6.111	5.996
	0.50	5.242	5.244	5.185	5.260	8.361	8.223
	1.00	6.674	6.678	6.603	6.785	11.352	11.229
Mean Execution Time (sec.)				38.35		43.14	

Table 6: American and European floating strike put values when $r = 0.10$ and $S_0 = 100$. Lower and 1-D refer to the Rogers and Shi (1995) lower bound and one-dimensional PDE, respectively. B & P refers to the results obtained by Barraquand and Pudet (1996). 2-D refers to the value obtained by solving the two-dimensional PDE using the van Leer limiter with $\theta = \frac{1}{2}$ and a non-uniform spatial A x S grid of 41 x 45. Δt^* was set to one day, two days and three days for maturities of three, six and twelve months, respectively. Mean execution times are for runs performed on a DEC Alpha.

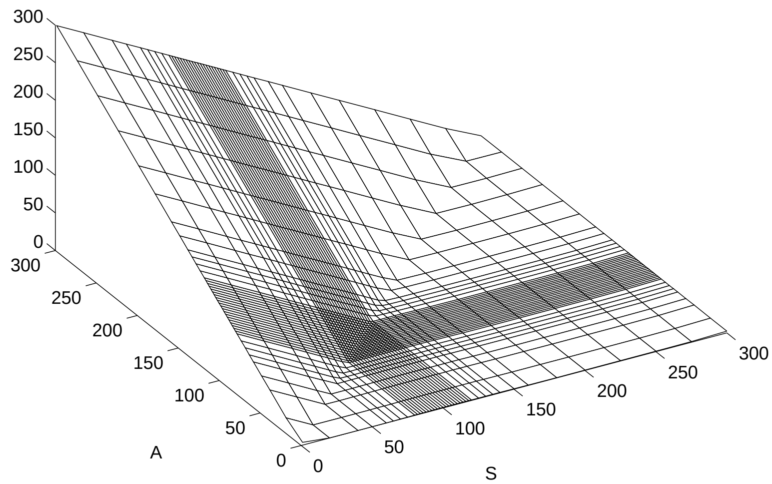

Figure 9: American floating strike put value when $r = 0.10$, $\sigma = 0.10$, and $T - t = 0.25$. Calculated using the van Leer flux limiter with $\theta = \frac{1}{2}$.

portantly, Barraquand and Pudet fail to mention that solutions generated using a centrally weighted scheme for equation (6) cannot be ensured to be free of oscillations. For example, Figure 10 demonstrates the severe oscillations that resulted from pricing a European-style fixed strike call option using central weighting. The grid spacing used was identical to that used to obtain our results with the van Leer limiter. Using the van Leer flux limiter in conjunction with the Crank-Nicolson scheme gives us a method that is oscillation free and second-order accurate in time. The careful reader will note that as $t \to 0$ the CFL conditions established in Appendix C cannot be satisfied. However, this does not appear to have affected the quality of our results. If one wanted to ensure that the method was TVD even as $t \to 0$, the flux in the A direction can be switched from a Crank-Nicolson scheme to a fully implicit scheme for specific cells when the CFL condition is violated. Although this will result in additional numerical diffusion, we found that the values differed by no more than \$0.05 from the results reported here.

Barraquand and Pudet (1996) state that because the correlated variables A and S are taken to be independent in equation (6), numerical schemes for solving equation (6) will not necessarily converge to a solution. However, as the results in Tables 4, 5 and 6 demonstrate, we did not experience any failures to converge to an appropriate solution during our study. A stronger demonstration of convergence using successive grid refinements is contained in Table 7.

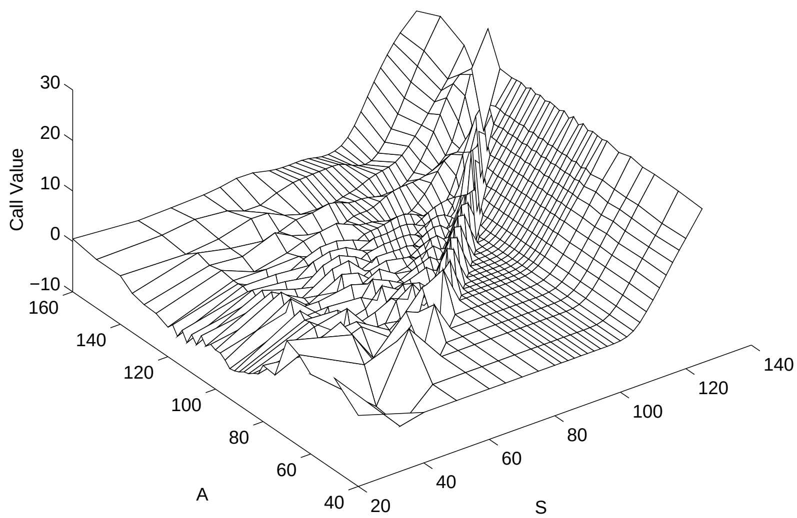

Figure 10: European fixed strike call option when $K = 100$, $r = 0.10$, $\sigma = 0.10$, and $T - t = 0.25$. Calculated using central weighting with $\theta = \frac{1}{2}$.

	K	Spatial Grid (A x S)		
		41 x 45	81 x 89	161 x 177
European	95	6.501	6.482	6.480
	100	2.928	2.937	2.936
	105	0.971	0.952	0.951
American	95	7.521	7.474	7.465
	100	3.224	3.228	3.222
	105	1.009	0.993	0.992

Table 7: Successive grid refinements demonstrating the convergence of the two-dimensional PDE for a fixed strike Asian option when $S_0 = 100$, $r = 0.10$, $\sigma = 0.20$ and $T - t = 0.25$.

T-t	Mean Exec. Time (sec.)		
	B & P	1-D	2-D
0.5	6	4	34
1.0	54	7	67
2.0	485	15	118

Table 8: Mean execution times for obtaining European fixed strike call values. B & P refers to the forward shooting grid algorithm. 1-D and 2-D refer to results of similar accuracy obtained using one-dimensional and two-dimensional PDEs, respectively. Runs were performed on a DEC Alpha.

The results in Tables 4, 5 and 6 demonstrate that the desired level of accuracy of 0.10% of S_0 can be obtained in all cases in under an average (for all maturities considered) of 53 seconds. We implemented Barraquand and Pudet's (1996) forward shooting grid algorithm and found that the desired level of accuracy could be obtained in under an average (for all maturities considered) of 32 seconds. More specifically, 90 time-steps were required for the three and six month maturities using time-step sizes of one and two days, respectively. The average time required for 90 time-steps was under 21 seconds. Although the parameters were the same as those used by Barraquand and Pudet, our time of 21 seconds is slightly higher than the Barraquand and Pudet result of 15 seconds. By performing the runs on a different DEC Alpha from the one used for our analysis, we were able to obtain an average time of 15 seconds. For the one year maturity with a time-step size of three days, 120 time-steps were required. In this case the average time required was under 54 seconds.

The substantial increase in time required is due to the fact that the forward shooting grid algorithm has a time complexity of $\mathcal{O}(N^3)$, where N is the number of time-steps. For a given grid size, the time complexity for a PDE method is linear, that is $\mathcal{O}(T)$. Thus, as the number of required time-steps increases, the time required for the Barraquand and Pudet algorithm will grow dramatically. Table 8 demonstrates the cubic time complexity of the forward shooting grid algorithm and the linear time complexity (for a given grid size) of PDE methods. For example, to value a two year fixed strike call option with a time-step size of three days required an average of 485 seconds using the Barraquand and Pudet algorithm. This compares to an average of 118 seconds for the two-dimensional PDE and 15 seconds for the one-dimensional PDE. Both methods have polynomial spatial complexities with degrees (of two for the one asset case) that grow with the number of assets, and thus can only be used to solve problems with a low number of spatial dimensions.

5 Conclusions

The naive application of central differences to the numerical treatment of certain partial differential equations can result in a number of difficulties. We have demonstrated that treating convection using central differences in the discretization of PDEs with low diffusion relative to convection, such as Asian option models, can produce solutions containing spurious oscillations. As noted by Barraquand and Pudet (1996), explicit centrally weighted schemes are unstable when applied to equations (5) and (6). Furthermore, one cannot ensure that implicit centrally weighted schemes will be free of oscillations when applied to equations (5) and (6).

To remedy the problem of spurious oscillations produced by centrally weighted schemes while still maintaining high order accuracy, we employed the use of a high order flux limiter. We treated convection using the second-order accurate van Leer flux limiter. The limiter is second-order accurate away from regions with steep gradients where it augments the true diffusion with numerical diffusion. The van Leer limiter has the property that it is total variation diminishing and thus produces oscillation free solutions. Using the van Leer limiter in conjunction with the Crank-Nicolson scheme gives us a method that is second-order accurate in time and oscillation free.

We have demonstrated that the application of the van Leer limiter to one-dimensional Asian option models (European and American floating strike, and European fixed strike) leads to the rapid computation of accurate solutions (i.e. within an average of 10 seconds for most volatility/interest rate structures for maturities of up to one year). For the most extreme volatility/interest rate structures an accurate solution can be obtained within an average of 16 seconds. When the full two-dimensional model must be used, as is the case for American fixed strike options, the computation time naturally increases. However, accurate solutions can be computed in under an average of 53 seconds (DEC Alpha) for the maturities that we considered. Accurate solutions were obtained for both European- and American-style Asian options using the two-dimensional model.

PDE methods have $\mathcal{O}(n^d)$ spatial complexities, where n is the number of cells in a dimension and d is the number of dimensions. Thus, PDE methods can only be used to solve problems with a low number of spatial dimensions. This is also the case for Barraquand and Pudet's (1996) forward shooting grid algorithm. However, the forward shooting grid algorithm has a cubic time complexity compared to the linear time complexity of PDE methods. Thus, the forward shooting grid algorithm is less desirable for large numbers of time-steps. For example, the pricing and hedging of long-term (e.g. 5 year) Asian options is a problem of practical interest to insurance companies selling guarantees on investment annuity products.

The application of the van Leer limiter is not limited only to PDE pricing models for

Asian options. It can be applied to other financial PDE models that have the problem of convection dominance. Furthermore, since the method is non-linear it can be easily extended to solve non-linear option models such as that of Peszek (1995).

Appendices

A Prevention of Spurious Oscillations

In this appendix we will derive the conditions under which centrally weighted schemes for equation (9) will not produce spurious oscillations. The same arguments can be used to derive conditions for other schemes, such as, first-order upstream weighting.

Using the point-distributed finite volume discretization

$$\frac{V_i^{n+1} - V_i^n}{\Delta t^*} = F_{i-\frac{1}{2}}^{n+1} - F_{i+\frac{1}{2}}^{n+1} + f_i^{n+1}$$

with central weighting in space and fully implicit time-stepping for equation (9) gives us

$$\begin{aligned}\frac{V_i^{n+1} - V_i^n}{\Delta t^*} &= \frac{1}{\Delta S_i}\left[(-\frac{1}{2}\sigma^2 S_i^2)\frac{(V_i^{n+1} - V_{i-1}^{n+1})}{\Delta S_{i-\frac{1}{2}}} + (-rS_i)\frac{(V_i^{n+1} + V_{i-1}^{n+1})}{2}\right] \\ &- \frac{1}{\Delta S_i}\left[(-\frac{1}{2}\sigma^2 S_i^2)\frac{(V_{i+1}^{n+1} - V_i^{n+1})}{\Delta S_{i+\frac{1}{2}}} + (-rS_i)\frac{(V_{i+1}^{n+1} + V_i^{n+1})}{2}\right] - rV_i^{n+1}.\end{aligned}$$

Letting $k_i = \frac{1}{2}\sigma^2 S_i^2$ and $a_i = rS_i$, and simplifying

$$\begin{aligned}V_i^{n+1} - V_i^n &= \left(\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{a_i \Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n+1} \\ &+ \left(-\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} - r\Delta t^*\right) V_i^{n+1} \\ &+ \left(\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{a_i \Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n+1}. \qquad (28)\end{aligned}$$

After incorporating a temporal weighting factor, θ (where $\theta = 1$ is a fully implicit scheme, $\theta = \frac{1}{2}$ is the Crank-Nicolson method and $\theta = 0$ is a fully explicit scheme), equation (28) becomes

$$V_i^{n+1} - V_i^n = \theta\left(\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{a_i \Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n+1}$$

$$
\begin{aligned}
&+ \quad \theta\left(-\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - \frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} - r\Delta t^*\right) V_i^{n+1} \\
&+ \quad \theta\left(\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} + \frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n+1} \\
&+ \quad (1-\theta)\left(\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - \frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n} \\
&+ \quad (1-\theta)\left(-\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - \frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} - r\Delta t^*\right) V_i^{n} \\
&+ \quad (1-\theta)\left(\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} + \frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n}. \qquad (29)
\end{aligned}
$$

Regrouping terms in equation (29) gives

$$
\begin{aligned}
&- \quad \left(\theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - \theta\frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n+1} \\
&+ \quad \left(1 + \theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} + \theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} + \theta r\Delta t^*\right) V_i^{n+1} \\
&- \quad \left(\theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} + \theta\frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n+1} \\
&= \\
&\quad\quad \left((1-\theta)\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n} \\
&+ \quad \left(1 - (1-\theta)\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} - (1-\theta)r\Delta t^*\right) V_i^{n} \\
&+ \quad \left((1-\theta)\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}} + (1-\theta)\frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n},
\end{aligned}
$$

and then rearranging

$$
\begin{aligned}
&- \quad \left(\theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} - \theta\frac{a_i\Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n+1} \\
&+ \quad (1 + \theta r\Delta t^*)\, V_i^{n+1} + \left(\theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}} + \theta\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}}\right) V_i^{n+1}
\end{aligned}
$$

$$
\begin{aligned}
- & \left(\theta \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \theta \frac{a_i \Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n+1} \\
= & \\
& \left((1-\theta)\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{a_i \Delta t^*}{2\Delta S_i}\right) V_{i-1}^{n} \\
+ & \left(1-(1-\theta) r \Delta t^*\right) V_i^n + \left(-(1-\theta)\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right) V_i^n \\
+ & \left((1-\theta)\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + (1-\theta)\frac{a_i \Delta t^*}{2\Delta S_i}\right) V_{i+1}^{n}. \qquad (30)
\end{aligned}
$$

In order to isolate the behavior of the solution from the spatially independent exponential decay (which is due to the $-rV$ term in equation (9)), we will eliminate the exponential decay term by substituting

$$
V_i^{n+1} = \beta_i^{n+1} \left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)^n , \qquad (31)
$$

where the superscript n for $\left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)$ is an exponent. We will determine the conditions which result in non-oscillatory behavior for β_i^{n+1} in the following.

Substituting (31) into equation (30) gives

$$
\begin{aligned}
- & \left(\theta \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \theta \frac{a_i \Delta t^*}{2\Delta S_i}\right) \beta_{i-1}^{n+1} \left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)^n \\
+ & \left(1+\theta r \Delta t^*\right) \beta_i^{n+1} \left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)^n \\
+ & \left(\theta \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} + \theta \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right) \beta_i^{n+1} \left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)^n \\
- & \left(\theta \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \theta \frac{a_i \Delta t^*}{2\Delta S_i}\right) \beta_{i+1}^{n+1} \left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)^n \\
= & \\
& \left((1-\theta)\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{a_i \Delta t^*}{2\Delta S_i}\right) \beta_{i-1}^{n} \left(\frac{1-(1-\theta) r \Delta t^*}{1+\theta r \Delta t^*}\right)^{n-1}
\end{aligned}
$$

$$+ \quad (1-(1-\theta)r\Delta t^*)\,\beta_i^n \left(\frac{1-(1-\theta)r\Delta t^*}{1+\theta r\Delta t^*}\right)^{n-1}$$

$$+ \quad \left(-(1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^n \left(\frac{1-(1-\theta)r\Delta t^*}{1+\theta r\Delta t^*}\right)^{n-1}$$

$$+ \quad \left((1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + (1-\theta)\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i+1}^n \left(\frac{1-(1-\theta)r\Delta t^*}{1+\theta r\Delta t^*}\right)^{n-1}. \qquad (32)$$

By letting $\alpha = (\frac{1-(1-\theta)r\Delta t^*}{1+\theta r\Delta t^*})$ and dividing by $1+\theta r\Delta t^*$ in equation (32) we obtain

$$- \quad \left(\theta\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \theta\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i-1}^{n+1}\left(\frac{\alpha^n}{1+\theta r\Delta t^*}\right)$$

$$+ \quad \beta_i^{n+1}\alpha^n + \left(\theta\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} + \theta\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^{n+1}\left(\frac{\alpha^n}{1+\theta r\Delta t^*}\right)$$

$$- \quad \left(\theta\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \theta\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i+1}^{n+1}\left(\frac{\alpha^n}{1+\theta r\Delta t^*}\right)$$

$$=$$

$$\left((1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i-1}^n\left(\frac{\alpha^{n-1}}{1+\theta r\Delta t^*}\right)$$

$$+ \quad \beta_i^n\alpha^n + \left(-(1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - (1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^n\left(\frac{\alpha^{n-1}}{1+\theta r\Delta t^*}\right)$$

$$+ \quad \left((1-\theta)\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + (1-\theta)\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i+1}^n\left(\frac{\alpha^{n-1}}{1+\theta r\Delta t^*}\right).$$

Dividing by α^n gives

$$- \quad \left(\frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{\theta}{(1+\theta r\Delta t^*)}\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i-1}^{n+1}$$

$$+ \quad \left(1 + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^{n+1}$$

$$- \quad \left(\frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i\Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{a_i\Delta t^*}{2\Delta S_i}\right)\beta_{i+1}^{n+1}$$

$$=$$

$$\left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) \beta^n_{i-1}$$

$$+ \quad \left(1 - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} \right) \beta^n_i$$

$$+ \quad \left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) \beta^n_{i+1}.$$

Regrouping terms

$$\left(1 + \frac{\theta}{(1+\theta r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} \right) \beta^{n+1}_i$$

$$=$$

$$\left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) \beta^n_{i-1}$$

$$+ \quad \left(1 - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} \right) \beta^n_i$$

$$+ \quad \left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) \beta^n_{i+1}$$

$$+ \quad \left(\frac{\theta}{(1+\theta r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{\theta}{(1+\theta r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) \beta^{n+1}_{i-1}$$

$$+ \quad \left(\frac{\theta}{(1+\theta r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) \beta^{n+1}_{i+1}. \tag{33}$$

If we require all the coefficients of β in (33) to be positive, then we must ensure that

$$\left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) > 0 \tag{34}$$

and

$$\left(\frac{\theta}{(1+\theta r\Delta t^*)} \frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{\theta}{(1+\theta r\Delta t^*)} \frac{a_i \Delta t^*}{2\Delta S_i} \right) > 0 \tag{35}$$

in (33). After recalling that $k_i = \frac{1}{2}\sigma^2 S_i^2$ and $a_i = rS_i$, and simplifying, condition (34) becomes

$$\frac{1}{\Delta S_{i-\frac{1}{2}}} > \frac{r}{\sigma^2 S_i}. \tag{36}$$

Condition (36) is also known as the Peclet condition. Note that if condition (36) is met, then both (34) and (35) will be satisfied. We also must meet the additional condition

$$\left(1 - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right) > 0. \tag{37}$$

After substituting $k_i = \frac{1}{2}\sigma^2 S_i^2$ and $a_i = rS_i$, and simplifying, condition (37) becomes

$$\frac{1}{(1-\theta)\Delta t^*} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + r. \tag{38}$$

If conditions (36) and (38) are met, then all the coefficients of β in (33) are positive and we can employ the maximum principle. By defining $\beta_i^{\max}$ such that

$$\beta_i^{\max} = \max(\beta_{i-1}^n, \beta_i^n, \beta_{i+1}^n, \beta_{i-1}^{n+1}, \beta_{i+1}^{n+1}),$$

then equation (33) can be written as

$$\begin{aligned}
&\left(1 + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^{n+1} \\
\leq \\
&\left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{a_i \Delta t^*}{2\Delta S_i}\right)\beta_i^{\max} \\
+ &\left(1 - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^{\max} \\
+ &\left(\frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{(1-\theta)}{(1-(1-\theta)r\Delta t^*)}\frac{a_i \Delta t^*}{2\Delta S_i}\right)\beta_i^{\max} \\
+ &\left(\frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} - \frac{\theta}{(1+\theta r\Delta t^*)}\frac{a_i \Delta t^*}{2\Delta S_i}\right)\beta_i^{\max} \\
+ &\left(\frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{a_i \Delta t^*}{2\Delta S_i}\right)\beta_i^{\max}.
\end{aligned} \tag{39}$$

Simplifying

$$\left(1 + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i-\frac{1}{2}}} + \frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i \Delta t^*}{\Delta S_i \Delta S_{i+\frac{1}{2}}}\right)\beta_i^{n+1}$$

$$\leq$$

$$\left(1+\frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i-\frac{1}{2}}}+\frac{\theta}{(1+\theta r\Delta t^*)}\frac{k_i\Delta t^*}{\Delta S_i\Delta S_{i+\frac{1}{2}}}\right)\beta_i^{\max}.$$

Thus,

$$\beta_i^{n+1}\leq\beta_i^{\max}. \tag{40}$$

Similarly, if we employ the minimum principle by defining

$$\beta_i^{\min}=\min(\beta_{i-1}^n,\beta_i^n,\beta_{i+1}^n,\beta_{i-1}^{n+1},\beta_{i+1}^{n+1}),$$

then

$$\beta_i^{n+1}\geq\beta_i^{\min}. \tag{41}$$

Hence, (40) and (41) imply that no new local maxima or minima can occur in the numerical solution for β_i^{n+1}, which is a precise definition of a non-oscillatory solution. Since $V_i^{n+1}=\beta_i^{n+1}\left(\frac{1-(1-\theta)r\Delta t^*}{1+\theta r\Delta t^*}\right)^n$ and $1>(1-\theta)r\Delta t^*$ by condition (38), the solution for V_i^{n+1} is β_i^{n+1} multiplied by a spatially independent positive decay term. Thus, the solution for V_i^{n+1} will not contain spurious oscillations if conditions (36) and (38) are met.

B TVD Schemes for PDEs in Non-Conservative Form

Flux limiting schemes are often expressed as fully explicit schemes in the literature (van Leer, 1974; Sweby, 1984) for PDEs that are in the following form

$$\frac{\partial V}{\partial t}=-\frac{\partial(aV)}{\partial S}, \tag{42}$$

where a is the convective velocity. Equation (42) is said to be in conservative form. Blunt and Rubin (1992) have shown that partially implicit and fully implicit schemes are TVD for conservative equations. In this appendix we will show that fully explicit, partially implicit and fully implicit flux limiting schemes are TVD for PDEs in non-conservative form, and derive the criteria under which these schemes are TVD. The arguments are similar to those found in Blunt and Rubin (1992).

Consider the scalar convection equation

$$\frac{\partial V}{\partial t}=-a(S)\frac{\partial V}{\partial S}, \tag{43}$$

which is in non-conservative form. We make the simplifying assumption that we are solving (43) on an infinite region, so that the effect of boundary conditions may be ignored. This assumption is usually made in TVD analysis (Sweby, 1984; Blunt and Rubin, 1992). For flux limiting schemes, $V_{i+\frac{1}{2}}^{n+1}$ is often expressed as (Blunt and Rubin, 1992; Yang and Przekwas, 1992)

$$V_{i+\frac{1}{2}}^{n+1} = V_{up}^{n+1} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2}(V_{down}^{n+1} - V_{up}^{n+1}), \tag{44}$$

where $\phi(q_{i+\frac{1}{2}}^{n+1})$ is the limiter function and

$$q_{i+\frac{1}{2}}^{n+1} = \frac{V_i^{n+1} - V_{i-1}^{n+1}}{V_{i+1}^{n+1} - V_i^{n+1}}. \tag{45}$$

Using a flux limiting scheme, the fully implicit finite volume discretization of (43) for $a \geq 0$ is

$$\begin{aligned} \frac{V_i^{n+1} - V_i^n}{\Delta t} &= \frac{a_i}{\Delta S_i}\left[V_{i-1}^{n+1} + \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2}(V_i^{n+1} - V_{i-1}^{n+1})\right] \\ &- \frac{a_i}{\Delta S_i}\left[V_i^{n+1} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2}(V_{i+1}^{n+1} - V_i^{n+1})\right]. \end{aligned}$$

Simplifying

$$V_i^{n+1} - V_i^n = \frac{a_i \Delta t}{\Delta S_i}\left[(V_{i-1}^{n+1} - V_i^{n+1}) + \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2}(V_i^{n+1} - V_{i-1}^{n+1}) - \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2}(V_{i+1}^{n+1} - V_i^{n+1})\right],$$

which is equivalent to

$$V_i^{n+1} - V_i^n = -\lambda_i \Delta V_{i-1}^{n+1}\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2}\left(\frac{V_{i+1}^{n+1} - V_i^{n+1}}{V_i^{n+1} - V_{i-1}^{n+1}}\right)\right],$$

where $\lambda_i = \frac{a_i \Delta t}{\Delta S_i}$ and $\Delta V_{i-1}^{n+1} = (V_i^{n+1} - V_{i-1}^{n+1})$. Noting that $q_{i+\frac{1}{2}}^{n+1} = \frac{V_i^{n+1} - V_{i-1}^{n+1}}{V_{i+1}^{n+1} - V_i^{n+1}}$

$$V_i^{n+1} - V_i^n = -\lambda_i \Delta V_{i-1}^{n+1}\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2q_{i+\frac{1}{2}}^{n+1}}\right].$$

Incorporating a temporal weighting factor, θ (where $\theta = 1$ is a fully implicit scheme and $\theta = 0$ is a fully explicit scheme),

$$\begin{aligned} V_i^{n+1} - V_i^n &= -\lambda_i \Delta V_{i-1}^{n+1}\theta\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2q_{i+\frac{1}{2}}^{n+1}}\right] \\ &- \lambda_i \Delta V_{i-1}^{n}(1-\theta)\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n})}{2q_{i+\frac{1}{2}}^{n}}\right]. \end{aligned} \tag{46}$$

Similarly,

$$\begin{aligned} V_{i+1}^{n+1} - V_{i+1}^{n} &= -\lambda_{i+1}\Delta V_i^{n+1}\theta\left[1 - \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2} + \frac{\phi(q_{i+\frac{3}{2}}^{n+1})}{2q_{i+\frac{3}{2}}^{n+1}}\right] \\ &- \lambda_{i+1}\Delta V_i^{n}(1-\theta)\left[1 - \frac{\phi(q_{i+\frac{1}{2}}^{n})}{2} + \frac{\phi(q_{i+\frac{3}{2}}^{n})}{2q_{i+\frac{3}{2}}^{n}}\right]. \end{aligned} \quad (47)$$

Letting

$$c_{i-1}^{n} = \lambda_i(1-\theta)\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n})}{2q_{i+\frac{1}{2}}^{n}}\right]$$

and

$$c_{i-1}^{n+1} = \lambda_i\theta\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2q_{i+\frac{1}{2}}^{n+1}}\right],$$

then equations (46) and (47) can be written as

$$V_i^{n+1} - V_i^{n} = -c_{i-1}^{n+1}\Delta V_{i-1}^{n+1} - c_{i-1}^{n}\Delta V_{i-1}^{n} \quad (48)$$

and

$$V_{i+1}^{n+1} - V_{i+1}^{n} = -c_i^{n+1}\Delta V_i^{n+1} - c_i^{n}\Delta V_i^{n}, \quad (49)$$

respectively. Subtracting (48) from (49)

$$\Delta V_i^{n+1} - \Delta V_i^{n} = -c_i^{n+1}\Delta V_i^{n+1} - c_i^{n}\Delta V_i^{n} + c_{i-1}^{n+1}\Delta V_{i-1}^{n+1} + c_{i-1}^{n}\Delta V_{i-1}^{n}.$$

$$(1 + c_i^{n+1})\Delta V_i^{n+1} = (1 - c_i^{n})\Delta V_i^{n} + c_{i-1}^{n+1}\Delta V_{i-1}^{n+1} + c_{i-1}^{n}\Delta V_{i-1}^{n}.$$

If we impose that

$$1 \geq c_i^{n} \geq 0 \quad (50)$$

and

$$c_i^{n+1} \geq 0 \quad (51)$$

for all i, then

$$(1 + c_i^{n+1})\left|\Delta V_i^{n+1}\right| \leq (1 - c_i^{n})\left|\Delta V_i^{n}\right| + c_{i-1}^{n+1}\left|\Delta V_{i-1}^{n+1}\right| + c_{i-1}^{n}\left|\Delta V_{i-1}^{n}\right|.$$

Summing over all i and noting that $\sum_i c_i^n \left|\Delta V_i^n\right| = \sum_i c_{i-1}^n \left|\Delta V_{i-1}^n\right|$ gives us

$$\sum_i \left|\Delta V_i^{n+1}\right| \leq \sum_i \left|\Delta V_i^n\right|.$$

Thus, the scheme will be TVD if conditions (50) and (51) are met. Hence, we require that

$$1 \geq \lambda_i(1-\theta)\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^n)}{2} + \frac{\phi(q_{i+\frac{1}{2}}^n)}{2q_{i+\frac{1}{2}}^n}\right] \geq 0 \tag{52}$$

and

$$\lambda_i\theta\left[1 - \frac{\phi(q_{i-\frac{1}{2}}^{n+1})}{2} + \frac{\phi(q_{i+\frac{1}{2}}^{n+1})}{2q_{i+\frac{1}{2}}^{n+1}}\right] \geq 0 \tag{53}$$

for all i and n, for the scheme to be TVD. Fully implicit (i.e. $\theta = 1$) flux limiting schemes will be TVD if we ensure that condition (53) is met. If in addition to condition (53) we ensure that condition (52) is met, then fully explicit (i.e. $\theta = 0$) and Crank-Nicolson (i.e. $\theta = \frac{1}{2}$) flux limiting schemes will be TVD. Thus, if conditions (52) and (53) are satisfied, the scheme will be TVD for non-conservative PDEs. In fact, these conditions are similar to those required to ensure that flux limiting schemes are TVD for PDEs in conservative form (Blunt and Rubin, 1992). Note that equation (42) does not contain a diffusion term, thus the conditions will be overly stringent for equations such as (9).

C The Flux Limiter Function

In this appendix we will first examine the properties that flux limiter functions, ϕ, possess for uniform grids. We will then modify the van Leer flux limiter function to account for non-uniform grids. The analysis for uniform grid spacing is similar to that in Blunt and Rubin (1992). We include the analysis for uniform grids in order to provide the reader with sufficient background to understand the analysis for non-uniform grid spacing. The examination of flux limiter functions for non-uniform grids has not, to the best of our knowledge, appeared in the literature. As in Appendix B, the results in this appendix will pertain to scalar convection PDEs, such as, equations (42) and (43). However, the results can be extended to other PDEs, such as, equation (9).

For simplicity and clarity we may at times omit superscripts and/or subscripts when referring to the flux limiter argument (i.e. q) defined in equation (45) and node values.

C.1 Uniform Grid Spacing

In appendix B we derived conditions (52) and (53) which flux limiting schemes must meet in order to be TVD. If we impose the conditions $\phi(q) \geq 0$ and $\frac{\phi(q)}{q} \geq 0$, then condition (53) can be restated as

$$\left(1 - \frac{\phi(q)}{2}\right) \geq 0. \tag{54}$$

Thus, if

$$0 \leq \phi(q) \leq 2 \tag{55}$$

then condition (54) will be satisfied and fully implicit (i.e. $\theta = 1$) flux limiting schemes will be TVD. Noting that $\phi(q) \geq 0$ and $\frac{\phi(q)}{q} \geq 0$, then condition (52) can be restated as

$$\frac{1}{\lambda} \geq (1-\theta)\left(1 + \frac{\phi(q)}{2q}\right), \tag{56}$$

where λ, defined in Appendix B, is the CFL number (Roache, 1972; Shyy, 1994). If

$$0 \leq \frac{\phi(q)}{q} \leq 2, \tag{57}$$

then condition (56) will be satisfied for fully explicit schemes (i.e. $\theta = 0$) when $\lambda \leq \frac{1}{2}$. For Crank-Nicolson schemes (i.e. $\theta = \frac{1}{2}$) condition (56) will be satisfied when $\lambda \leq 1$.

Equations (55) and (57) define a region in which the flux limiter function must lie in order for the scheme to be TVD. The shaded region in figure 11 denotes the TVD region (Sweby, 1984). Note that the conditions $\phi(q) \geq 0$ and $\frac{\phi(q)}{q} \geq 0$ imply that $\phi(q)$ vanishes when $q < 0$. Referring to figure 11, along the line $\phi(q) = 1$ the flux limiting scheme (44) reverts to a second-order accurate centrally weighted scheme. That is,

$$V_{i+\frac{1}{2}} = \frac{V_{i+1} + V_i}{2}$$

when $\phi(q) = 1$. Along the line $\phi(q) = q$ the flux limiting scheme reverts to

$$V_{i+\frac{1}{2}} = V_i + \left(\frac{V_i - V_{i-1}}{2}\right),$$

which is a second-order accurate two-point upstream weighted scheme.

We do not only require that the scheme be TVD, but that it be second-order spatially accurate whenever possible. This can be achieved by making the flux limiting scheme a

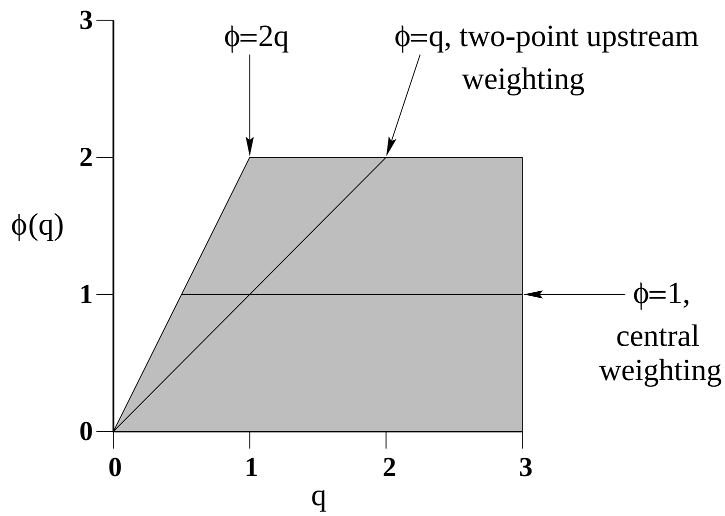

Figure 11: TVD region for uniform grid spacing.

weighted convex average of a centrally weighted scheme and a two-point upstream weighted scheme (Sweby, 1984). Thus, if the limiter function is in the region shown in figure 12, then the scheme will be second-order accurate. One such function is the van Leer flux limiter

$$\phi(q) = \frac{|q| + q}{1 + |q|} \tag{58}$$

(van Leer, 1974; Sweby, 1984). Note that when $q \leq 0$ the scheme reverts to a first-order accurate upstream weighted scheme.

C.2 Non-Uniform Grid Spacing

For non-uniform grids, a point-distributed flux limiting scheme will not revert to a two-point upstream scheme when $\phi(q) = q$, but rather when $\phi(q) = q\gamma$, where

$$\gamma = \frac{S_{i+1} - S_i}{S_i - S_{i-1}}$$

for $q_{i+\frac{1}{2}}$ given by equation (45). We will later see that this implies that we must modify the flux limiter function in order to ensure that the scheme will be second-order accurate whenever possible. Furthermore, we must first expand the region that we require to be TVD. Figure 13 demonstrates that the line representing the two-point upstream scheme will fall outside the TVD region for certain magnitudes of grid size changes. Specifically, if the magnitude of the grid size change, γ, is greater than or equal to 2, the two-point upstream scheme will fall outside the TVD region.

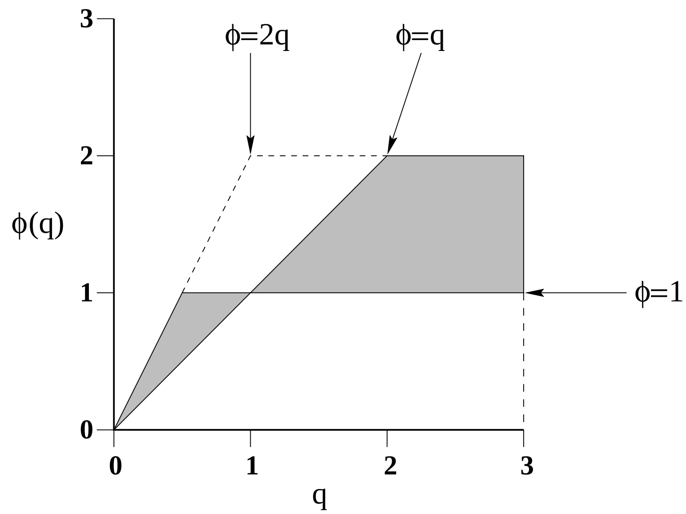

Figure 12: Second-order TVD region for uniform grid spacing.

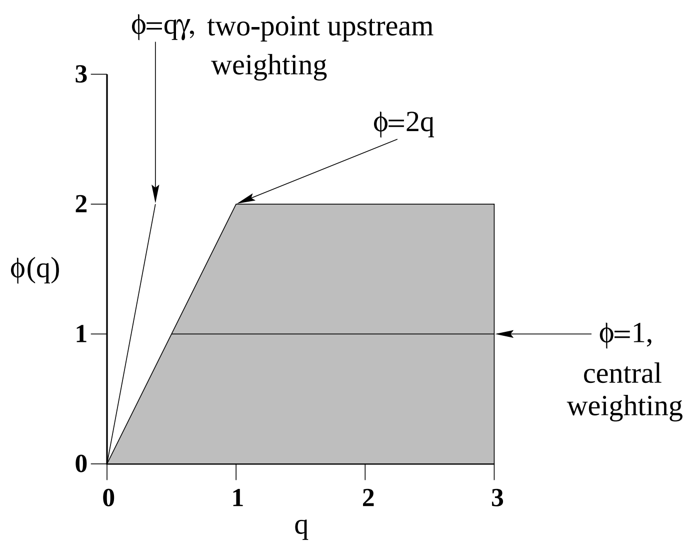

Figure 13: TVD region for uniform grid spacing with lines representing centrally weighted and two-point upstream schemes for non-uniform grid spacing. Where $\gamma = \frac{S_{i+1}-S_i}{S_i-S_{i-1}}$ for $q_{i+\frac{1}{2}}$ given by equation (45). The case where $\gamma > 2$ is illustrated.

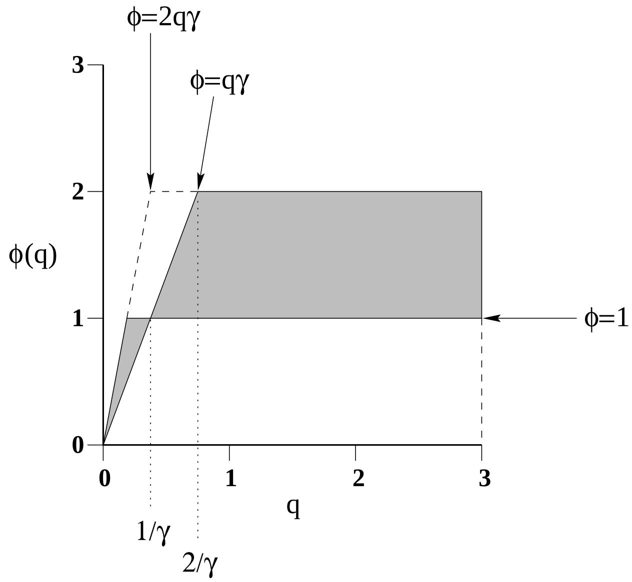

Figure 14: Second-order TVD region for non-uniform grid spacing. Where $\gamma = \frac{S_{i+1}-S_i}{S_i-S_{i-1}}$ for $q_{i+\frac{1}{2}}$ given by equation (45).

To enlarge the TVD region we need not modify (55), but we must alter (57). One possible alteration is

$$0 \leq \frac{\phi(q)}{q} \leq 2\gamma. \tag{59}$$

Assuming that condition (59) holds, Figure 14 is a graphical representation of the second-order region that we require to be TVD. If we were to use the van Leer limiter (58) the flux limiting scheme (44) will still be TVD, but the limiter will not pass through the second-order region shown in figure 14. Thus, we must construct a limiter function for non-uniform grids. A possible candidate for a new limiter function is

$$\phi(q) = \frac{|q| + q}{\frac{1}{\gamma} + |q|}, \tag{60}$$

which was obtained by modifying the van Leer limiter. To establish (60) as a limiter function for non-uniform grids, we must show that it passes through the region denoted in figure 14 that we require to be second-order TVD. That is, we must ensure that

$$\begin{array}{rll} i) & q\gamma \leq \frac{2q}{\frac{1}{\gamma}+q} \leq 2q\gamma, & \text{for } 0 \leq q < \frac{1}{\gamma} \\ ii) & 1 \leq \frac{2q}{\frac{1}{\gamma}+q} \leq q\gamma, & \text{for } \frac{1}{\gamma} \leq q < \frac{2}{\gamma} \\ iii) & 1 \leq \frac{2q}{\frac{1}{\gamma}+q} \leq 2, & \text{for } \frac{2}{\gamma} \leq q. \end{array}$$

Condition (i) simplifies to $\frac{1}{\gamma} \geq q \geq 0$, which is satisfied for $0 \leq q < \frac{1}{\gamma}$. Condition (ii) is equivalent to $2q \geq \frac{1}{\gamma} + q \geq \frac{2}{\gamma}$, which is satisfied since $q \geq \frac{1}{\gamma}$ for $\frac{1}{\gamma} \leq q < \frac{2}{\gamma}$. Condition (iii) simplifies to $q \geq \frac{1}{\gamma} \geq 0$, which is true for $\frac{2}{\gamma} \leq q$. Thus, the limiter function (60) passes through the second-order region that we require to be TVD.

Note that, we can force the original van Leer limiter (58) through the second-order region that we require to be TVD for non-uniform grids by modifying the function argument (45). That is, instead of modifying (58) we can modify $q_{i+\frac{1}{2}}$, such that,

$$q_{i+\frac{1}{2}} = \frac{V_i - V_{i-1}}{V_{i+1} - V_i}(\gamma).$$

However, this cannot be generalized to schemes that are not point-distributed or for other limiter functions.

Finally, we must establish the CFL conditions (Roache, 1972; Shyy, 1994) under which the flux limiting scheme (44), using the modified limiter (60), will be TVD. In other words, we must satisfy condition (56) assuming (59) holds. Since we did not alter (55) when we modified the limiter function, a fully implicit scheme using (60) will always be TVD. However, because we altered (57) when we modified the limiter, fully explicit and partially implicit schemes will be TVD under different CFL conditions from those required for uniform grids. Condition (56) will now be satisfied if

$$\frac{1}{\lambda} \geq (1 - \theta)(1 + \gamma),$$

because of property (59). Thus, the CFL conditions required to ensure that fully explicit and partially implicit flux limiting schemes will be TVD are dependent upon the magnitude of the grid size changes. For example, if we limit the magnitude of grid size changes to 2, that is $\gamma \leq 2$, then a fully explicit scheme will be TVD for $\lambda \leq \frac{1}{3}$, and a Crank-Nicolson scheme will be TVD for $\lambda \leq \frac{2}{3}$. Conversely, fully explicit and partially implicit schemes will be TVD for a given λ by restricting the magnitude of grid size changes.

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