Walking Streets Faster*

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Abstract

A fundamental problem in robotics is to compute a path for a robot from its current location to a given goal. In this paper we consider the problem of a robot equipped with an on-board vision system searching for a goal g in an unknown environment.

We assume that the robot is originally located at a point s on the boundary of a street polygon. A *street* is a simple polygon with two distinguished points s and g which are located on the polygon boundary and the part of the polygon boundary from s to g is weakly visible to the part from g to s and vice versa.

Our aim is to minimise the ratio of the length of the path traveled by the robot to the length of the shortest path from s to g. In analogy to on-line algorithms this value is called the *competitive ratio*. We present a series of strategies all of which follow the same general high level strategy. In the first part we present a class of strategies any of which can be shown to have a competitive ratio of $\pi + 1$. These strategies are robust under small navigational errors and their analysis is very simple.

In the second part we present the strategy *continuous lad* which is based on the heuristic optimality criterion of minimising the *l*ocal *a*bsolute *d*etour. We show an upper bound on the competitive ratio of *continuous lad* of ~ 2.03. Finally, we also present a hybrid strategy consisting of *continuous lad* and the strategy *Move-in-Quadrant*. We show that this combination of strategies achieves a competitive ratio of 1.73. This about halves the gap between the known $\sqrt{2}$ lower bound for this problem and the previously best known competitive ratio of 2.05.

*This research is supported by the DFG-Project "Diskrete Probleme", No. Ot 64/8-1.

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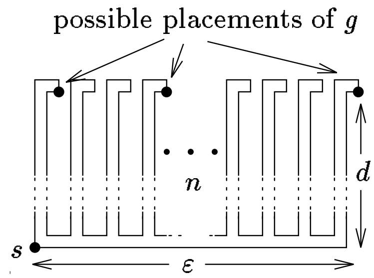

Figure 1: An example of a lower bound for searching in simple polygons.

1 Introduction

Finding a path from a starting location to a goal within a given scene is an important problem in robotics. Depending on the information available to the robot, e.g. if it has a map of its environment or knows the location of the goal, different techniques for planning a path can be applied. While most of the previous work has focussed on efficient algorithms for path planning if the robot knows its entire environment in advance, a more natural and realistic setting is to assume that the robot has only a partial knowledge of its surroundings.

In this paper we assume that the robot is equipped with an on-board vision system that provides the visibility map of its local environment. The search of the robot can be viewed as an on-line problem since it discovers its surroundings as it travels. Hence, one way to analyse the quality of a search strategy is to use the framework of competitive analysis as introduced by Sleator and Tarjan [17]. A search strategy is called *c-competitive* if the path traveled by the robot to find the goal is at most c times longer than a shortest path. The value c is called the *competitive ratio* of the strategy.

As Figure 1 illustrates, there is no strategy with a competitive ratio of $o(n)$ for scenes with arbitrary obstacles having a total of n vertices [3] even if we restrict ourselves to searching in a simple polygon. In this case, if an adversary places the target into the spike that is last explored by the strategy, then the robot will travel a distance of $\Omega(nd)$ while a shortest path has length at most $d + \varepsilon$. Hence, the competitive ratio is $\Omega(n)$ for large d and small ε.

Therefore, the on-line search problem has been studied previously in various contexts where the geometry of the obstacles is restricted. Papadimitrou and Yannakakis were the first to consider the case of traversing an unknown scene with rectangular obstacles in search of a goal whose location is known [16]. They show a lower bound of $\Omega(\sqrt{n})$ for the competitive ratio of any strategy. Later Blum, Raghavan, and Schieber provided a strategy that achieves this bound [3]. If the aspect ratio or the length of the longest side of the rectangles are bounded, better strategies are possible [4, 14].

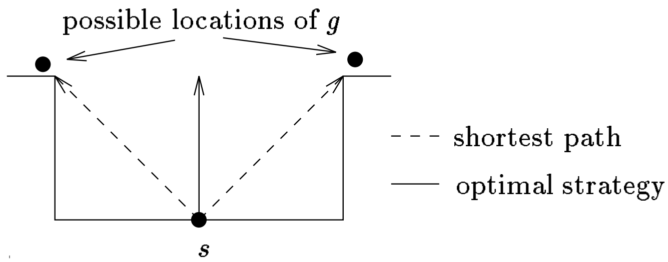

Figure 2: A lower bound for searching in rectilinear streets.

Kleinberg studies the problem of a robot searching inside a simple polygon for an unknown goal located on the boundary of the polygon [10]. He introduces the notion of *essential cuts* inside a polygon of which there may be considerably fewer than polygon vertices and gives an $O(m)$-competitive strategy for orthogonal polygons with m essential cuts.

Klein introduced the notion of a *street* as the first class of polygons which allow strategies with a constant competitive ratio even when the location of the goal is unknown [9]. In a street the starting point s and the goal g are located on the boundary of the polygon and the two polygonal chains from s to g are mutually weakly visible. Klein presents the strategy *lad* for searching in streets which is based on the idea of minimising the *l*ocal *a*bsolute *d*etour. He gives an upper bound on its competitive ratio of $1 + 3/2\pi$ (~ 5.71). Unfortunately, the analysis is quite involved, though it was recently improved by Icking to $1+\pi/2+\sqrt{1+\pi^2/4}$ (~ 4.44) [7]. Though Klein's strategy performs well in practice—he reports that no example has been found for which his strategy performs worse than 1.8—a tighter analysis remains elusive.

A strategy based on a different approach was presented by Kleinberg [10]. His strategy for searching in streets can be shown to have a competitive ratio of $2\sqrt{2}$ by a very simple analysis. A further improvement using ideas very similar to Kleinberg's achieves a competitive ratio of $\sqrt{1+(1+\pi/4)^2}$ (~ 2.05) [11]. Interestingly, for rectilinear streets Kleinberg shows that his strategy achieves an optimal competitive ratio of $\sqrt{2}$. The optimality is due to the trivial lower bound example shown in Figure 2. Here, if a strategy moves to the left or right before seeing g, then g can be placed on the opposite side, thus forcing the robot to travel more than $\sqrt{2}$ times the diagonal. Curiously enough, this is the only known lower bound even for arbitrarily oriented streets.

Finally, a more general class of polygons, called $\mathcal{G}$-streets, has been introduced by Datta and Icking that allows search strategies with a competitive ratio of 9.06 [5].

In this paper we present several strategies to traverse a street. All of them are similar to the original approach presented by Klein. The first strategy which can been viewed as a class of strategies, is called *Walk-in-Circles* and presents a very simple criterion for the robot to advance.

Its analysis is equally simple and gives a competitive ratio of $\pi + 1$. The second strategy that is presented is a slight modification of *lad* and is called *continuous lad*. We can show an upper bound of 2.03 on its competitive ratio.

The paper is organised as follows. In Section 2 we introduce the basic geometric concepts necessary for the rest of the paper. We also describe a "High Level Strategy"—first introduced by Klein [9]—that all presented algorithms follow and state some results about search strategies that follow this High Level Strategy. In Section 3 we then describe and analyse the family of strategies *Walk-in-Circles*. The second strategy, *continuous lad*, is presented and analysed in Section 4. Finally, in Section 5 we show that a hybrid strategy consisting of *continuous lad* and the strategy *Move-in-Quadrant* [11] achieves a competitive ratio of 1.73.

2 Preliminaries

Since we deal with point sets in the plane $\mathbb{E}^2$, we need the standard definitions of distance, norm, angle, etc. for points. If p, q, and r are three points in the plane, then we denote

(i) the L_2-distance between p and q by $d(p, q)$,

(ii) the L_2-norm of p by $\|p\|$,

(iii) the line segment between p and q by $\overline{pq}$, and

(iv) the counterclockwise angle between the line segment $\overline{qp}$ and the line segment $\overline{qr}$ at q by $\angle pqr$.

If $\mathcal{P}$ is a path in $\mathbb{E}^2$, we denote its length by $\lambda(\mathcal{P})$. Furthermore, if p and q are two points on $\mathcal{P}$, then we denote the part of $\mathcal{P}$ from p to q by $\mathcal{P}(p, q)$.

A *simple polygon* is a simple, closed curve that consists of the concatenation of line segments, called the *edges* of the polygon, such that no two consecutive edges are collinear. The end points of the edges are called the *vertices* of the polygon.

We consider a simple polygon P in the plane with n vertices and a robot inside P which is located at a start point s on the boundary of P. The robot has to find a path from s to the goal g. We denote the shortest path from s to g by $sp(s, g)$.

The search of the robot is aided by simple vision (i.e. we assume that the robot knows the visibility polygon of its current location). Furthermore, the robot retains all the information seen so far (in memory) and knows its starting and current position. We are, in particular, concerned with a special class of polygons called *streets* first introduced by Klein [9].

Definition 2.1 [9] *Let P be a simple polygon with two distinguished vertices, s and g, and let L and R denote the clockwise and counterclockwise, resp., oriented boundary chains leading from s to*

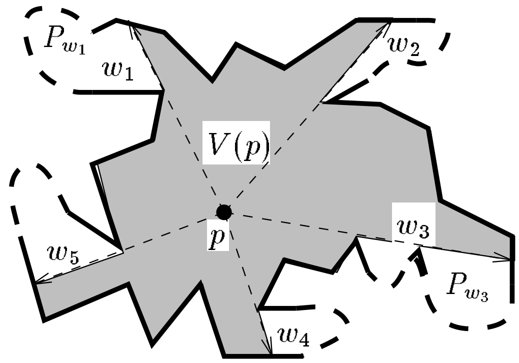

Figure 3: The visibility polygon $V(p)$ of p with windows $w_1, \ldots, w_5$.

g. If L and R are mutually weakly visible, i.e. if each point of L sees at least one point of R and vice versa, then (P, s, g) is called a street.

The only available information to the robot is its *visibility polygon.*

Definition 2.2 *Let P be a street with start point s and goal g. If p is a point of P, then the* visibility polygon *of p is the set of all points in P that are seen by p. It is denoted by $V(p)$.*

A window *of $V(p)$ is an edge of $V(p)$ that does not belong to the boundary of P (see Figure 3).*

A window w splits P into a number of subpolygons $P_1, \ldots, P_k$ one of which contains $V(p)$. We denote the union of the subpolygons that do not contain $V(p)$ by P_w.

The end point of a window w that is closer to p is called the *entrance point* of w. We assume that a window w has the orientation of the ray from p to entrance point of w. We say a window w is a *left window* if P_w is locally to the left of w w.r.t. the given orientation of w. A *right window* is defined similarly.

Let p be the current location of the robot and $\mathcal{P}_{sp}$ the path the robot followed from s to p. We assume that the robot knows the part of P that can be seen from $\mathcal{P}_{sp}$, i.e. the robot maintains the polygon $V(\mathcal{P}_{sp}) = \bigcup_{q \in \mathcal{P}_{sp}} V(q)$. We say a window w of $V(p)$ is a *true window* w.r.t. $\mathcal{P}_{sp}$ if P_w is not contained in $V(\mathcal{P}_{sp})$. We say two (true) windows w_1 and w_2 are *clockwise consecutive* if the clockwise oriented polygonal chain of $V(p)$ between w_1 and w_2 does not contain a (true) window different from w_1 and w_2. *Counterclockwise consecutive* is defined analogously.

If w_0 is the window of $V(p)$ that is intersected the first time by $\mathcal{P}_{sp}$, then it can be shown that all left true windows are clockwise consecutive and all right true windows are counterclockwise consecutive from w_0 [9, 10, 11]. Hence, if left true windows exist, then there is a clockwise-most left true window in $V(p)$ which we call the *left extreme true window* and denote by w^+. The *right* *extreme true window* w^- is defined similarly. The entrance point v^+ (v^-) of w^+ (w^-) is called the *left* (*right*) *extreme entrance point of* $V(p)$. It can be easily shown that g is contained in either P_{w^+} or P_{w^-} and that either v^+ or v^- belongs to $sp(s,g)$.

The algorithms we propose all follow the same high level strategy as described by Klein [9]. The general idea is that the robot moves from one point that is known to lie on $sp(s,g)$ to a point on $sp(s,g)$ that is closer to g by a sequence of moves as described below.

Algorithm High Level Strategy
Input: a street (P, s, g) and a path $\mathcal{P}_{sp}$ from s to the current position of the robot p;

while v^+ and v^- are both defined and g is not reached **do**
  Compute a path $\mathcal{P}_{pt}$ from p to some point t on $\overline{v^+v^-}$;
  Follow the path $\mathcal{P}_{pt}$ until one of the following events occurs:
    a) g becomes visible: the robot moves directly to g;
    b) P_{w^+} or P_{w^-} becomes visible:
      if P_{w^+} is visible
        then the robot moves to v^-;
        else the robot moves to v^+;
      end if
    c) v^+ (v^-) changes and the new extreme entrance point $\hat{v}^+$ $(\hat{v}^-)$ is not collinear with v^+ (v^-) and the robot position $\hat{p}$ (see Figure 4a): the robot moves to v^- (v^+);
    d) v^+ (v^-) changes and the new extreme entrance point $\hat{v}^+$ $(\hat{v}^-)$ is collinear with v^+ (v^-) and the robot position $\hat{p}$;
  Let p be the current robot position;
  Compute $V(p)$ and v^+ and v^- anew;
end while;

In [9] it is shown that the Cases a)–d) are the only possible events and that the actions described by the robot above in the Cases a)–c) lead to a point on $sp(s,g)$ that is closer to g. Note, in particular, that if t is reached, then an event of Category b) occurs. Furthermore, it is shown that as long as only events of Category d) occur, the sequences of points (v^+) and (v^-) form two reflex chains and that the visibility polygon of the robot weakly advances along these chains so that eventually one of the Events a)–c) occurs. An example of how the robot moves is given in Figure 4b.

The only detail left open by the above description is what path $\mathcal{P}_{pt}$ to choose. The particular method used to determine this choice is called a "low-level strategy" [9]. In the following we investigate two low-level strategies and analyse their performance.

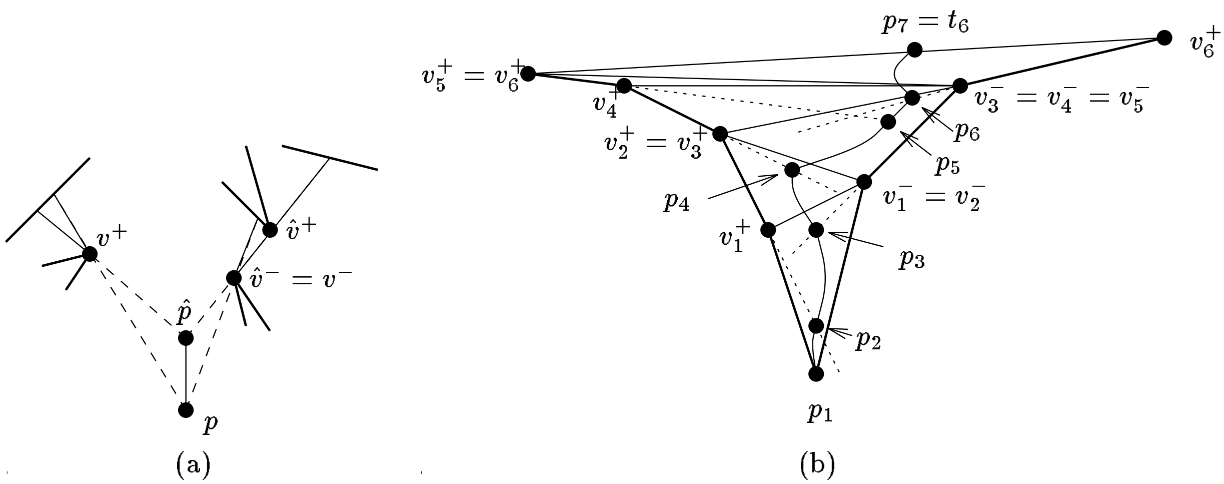

Figure 4: (a) As the robot moves to $\hat{p}$ the left extreme entrance point "jumps" from v^+ to $\hat{v}^+$ and the robot moves directly to $v^- = \hat{v}^-$. (b) An example of the execution of the *High Level Strategy* where only events of Category d) occur. Note that at $p_7 = t_7$ an event of Category b) occurs.

Note that the two other previously described strategies [10, 11] do not follow the above high-level strategy. Instead, as long as the angle $\angle v^+pv^-$ is small, they choose an arbitrary ray between v^+ and v^- and follow this ray until either an events in one of the categories a) or c) occurs or the angle $\angle v^+pv^-$ becomes large enough ($\geq \pi/2$) while ignoring the events of categories b) and d).

2.1 Definitions and Preliminary Results about Low-Level Strategies

A first observation we can make about the high level strategy is that if one of the Cases a)–c) occurs, then we know which of v^+ or v^- belongs to $sp(s,g)$ and, hence, the competitive ratio of the strategy is given by ratio of the length of the path that the robot travels between two points p and q on $sp(s,g)$ and the shortest path $sp(s,g)(p,q)$ from p to q.

So in the following we assume that the robot starts out at a point $p_1 \in sp(s,g)$ and encounters a number of events of Category d). Each of these events correspond to a point p_i, $i \geq 2$, at which new left and right extreme entrance points v_i^+ and v_i^- as well as a new path $\mathcal{P}_i$ from p_i to a point t_i on $\overline{v_i^+ v_i^-}$ are computed. Let d_i^+ be the distance from p_i to v_i^+ and d_i^- be the distance from p_i to v_i^-.

Given p_i, the point p_{i+1} is defined as the first point on $\mathcal{P}_i$ such that either the left or the right extreme entrance point of $V(p_{i+1})$ is different from v_i^+ or v_i^-, respectively (see Figure 5). At the point p_{i+1} the robot computes a new target point and a new path $\mathcal{P}_{i+1}$.

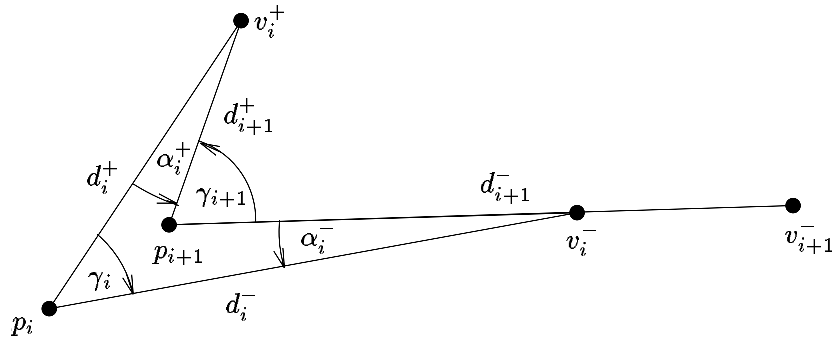

Figure 5: p_{i+1} is the first point on $\mathcal{P}_i$ where the left or right extreme entrance point changes.

We denote the angle $\angle p_{i+1} v_i^+ p_i$ by α_i^+ and the angle $\angle p_{i+1} v_i^- p_i$ by α_i^-. The angle $\angle v_i^+ p_i v_i^-$ is denoted by γ_i. We can make the following elementary observation about the angles γ_{i+1} and γ_i.

Observation 2.1 $\gamma_{i+1} = \gamma_i + \alpha_i^+ + \alpha_i^-$.

Let $a_i^+ = d_i^+ - (d_{i+1}^+ - d(v_i^+, v_{i+1}^+))$ and $a_i^- = d_i^- - (d_{i+1}^- - d(v_i^-, v_{i+1}^-))$. Note that either $d(v_i^+, v_{i+1}^+) = 0$ or $d(v_i^-, v_{i+1}^-) = 0$. Furthermore, note that the distance of p_{i+1} to v_i^+ is $d_i^+ - a_i^+$ and the distance of p_{i+1} to v_i^- is $d_i^- - a_i^-$. Let $\mathcal{V}_i^+$ be the shortest path from p_1 to v_i^+, and $\mathcal{V}_i^-$ the the shortest path from p_1 to v_i^-.

If the distance of the robot position to v_i^+ and v_i^- does not increase as the robot moves on $\mathcal{P}_i$, for all $1 \le i \le k$, then the length of $\mathcal{V}_i^+$ or $\mathcal{V}_i^-$ can be expressed as d_i^+ plus the sum of the a_i^+ or d_i^- plus the sum of the a_i^-, respectively.

Lemma 2.2 *If, for all $1 \le j \le k$, $d(p_{j+1}, v_j^+) \le d_j^+$ and $d(p_{j+1}, v_j^-) \le d_j^-$, then we have, with the above definitions,*

$$\lambda(\mathcal{V}_i^+) = \sum_{j=1}^{i-1} a_j^+ + d_i^+ \qquad \textit{and} \qquad \lambda(\mathcal{V}_i^-) = \sum_{j=1}^{i-1} a_j^- + d_i^-.$$

Proof: We only show the lemma for $\lambda(\mathcal{V}_i^+)$. The proof is by induction. Since $d(p_0, v_1^+) = d_1^+$ by definition, the claim holds for $i = 1$. Recall that $a_i^+ + d_{i+1}^+ = d_i^+ + d(v_i^+, v_{i+1}^+)$ and $\lambda(\mathcal{V}_{i+1}^+) = \lambda(\mathcal{V}_i^+) + d(v_i^+, v_{i+1}^+)$ since the vertices $v_1^+, \ldots, v_{i+1}^+$ form a reflex chain. Hence,

$$\lambda(\mathcal{V}_{i+1}^+) \;=\; \lambda(\mathcal{V}_i^+) + d(v_i^+, v_{i+1}^+) = \sum_{j=1}^{i-1} a_j^+ + d_i^+ + d(v_i^+, v_{i+1}^+) = \sum_{j=1}^{i-1} a_j^+ + d_{i+1}^+ + a_i^+ = \sum_{j=1}^{i} a_j^+ + d_{i+1}^+$$

which completes the proof. □

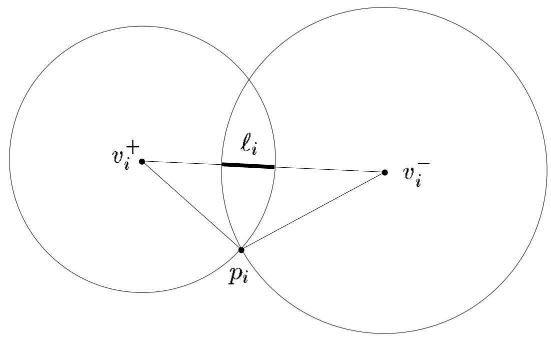

Figure 6: The subsegment of target points.

3 Strategy Walk-in-Circles

In this section we present a family of strategies for the problem of searching in an unknown street. We show that a robot using a strategy from this family follows a path that is at most $\pi + 1$ times longer than the shortest possible path. We then use this new strategy as part of a hybrid method to obtain an equally simple strategy of slightly more complex analysis with a competitive ratio of $\frac{1}{2}\sqrt{\pi^2 + 4\pi + 8} \sim 2.76$. More importantly, we show that the $\pi + 1$ strategy is robust under small navigational errors.

Let ℓ_i be the subsegment of $\overline{v_i^+ v_i^-}$ which consists of the points t such that $d(v_i^+, t) \leq d(v_i^+, p_i)$ and $d(v_i^-, t) \leq d(v_i^-, p_i)$ (see Figure 6). The algorithm chooses a point t_i in the target segment ℓ_i and moves in a straight line towards it. If a new window appears, the robot recomputes ℓ_i according to the updated points v_{i+1}^- and v_{i+1}^+, and the new position p_{i+1}, until the goal is found (see Figure 7).

In the analysis we only consider the case where the goal turns out to be on the right side. This is without loss of generality since the local target selection strategy is invariant under reflections.

The length of the path traversed by the robot is determined by the sum of the length of all segments $\overline{p_i p_{i+1}}$, i.e. $\sum_{i=1}^{k-1} d(p_i, p_{i+1})$, where k is the number of extreme entrance points seen by the robot until one of the events of Category a)–c) occurs. Let q be the intersection point of the ray with origin in v_i^- going through p_{i+1} and the circle centred at v_i^- passing through p_i (see Figure 7). By the triangle inequality, we have $d(p_i, p_{i+1}) \leq d(p_i, q) + d(q, p_{i+1})$.

In turn the length of $d(p_i, q)$ is bounded by the length of the circular arc $\widehat{p_i q}$. Recall that $\alpha_i^- = \angle p_{i+1} v_i^- p_i = \angle q v_i^- p_i$. The length of the circular arc $\widehat{p_i q}$ is given by $\alpha_i^- \cdot d(p_i, v_i^-) = \alpha_i^- \cdot d_i^-$.

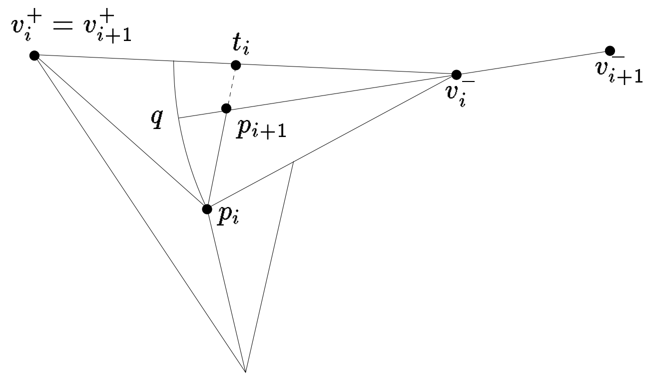

Figure 7: A single step in the strategy.

Thus,

$$\sum_{i=1}^{k-1} d(p_i, p_{i+1}) \;\le\; \sum_{i=1}^{k} (d(p_i, q) + d(q, p_{i+1})) \le \sum_{i=1}^{k} \left(\alpha_i^- \cdot d_i^- + d(q, p_{i+1})\right),$$

and the competitive ratio is determined by

$$\frac{\sum_{i=1}^{k} d(p_i, p_{i+1}) + d(p_{k+1}, v_k^-)}{\lambda(\mathcal{V}_k^-)} \;\le\; \frac{\sum_{i=1}^{k} \alpha_i^- \cdot d_i^- + \sum_{i=1}^{k} d(q, p_{i+1}) + d(p_{i+1}, v_k^-)}{\lambda(\mathcal{V}_k^-)}.$$

Recall that $\lambda(\mathcal{V}_k^-)$ is the length of the shortest path from the p_1 to v_k^-. Since we assume that v_k^- is located on $sp(s,g)$, $\mathcal{V}_k^-$ is a part of $sp(s,g)$.

Note that since t_i is contained in the circle C_i with centre v_i^- and radius $d(v_i^-, p_i)$ by the definition of ℓ_i, the whole line segment $\overline{p_i t_i}$ is contained in C_i and, therefore, $d(v_i^-, p_{i+1}) \le d(v_i^-, p_i)$. The definition of q now yields $d(q, p_{i+1}) = d(q, v_i^-) - d(p_{i+1}, v_i^-) = a_i^-$ and Lemma 2.2 implies that $\sum_{j=1}^{k} a_j^- + d(p_{k+1}, v_k^-) = \sum_{j=1}^{k-1} a_j^- + d_k^- \le \lambda(\mathcal{V}_k^-)$ and $d_i^- \le \lambda(\mathcal{V}_k^-)$, for all $1 \le i \le k$; hence, we obtain

$$\sum_{i=1}^{k} \alpha_i^- d(p_i, v_i^-) + \sum_{i=1}^{k} d(q, p_{i+1}) + d(p_{k+1}, v_k^-) \;=\; \sum_{i=1}^{k} \alpha_i^- d_i^- + \sum_{i=1}^{k-1} a_i^- + d_k^- \le \left(\sum_{i=1}^{k} \alpha_i^- + 1\right) \lambda(\mathcal{V}_k^-)$$

As it was noted by Klein (see proof of Lemma 2.7 in [9]), the angle $\angle v_i^+ p_i v_i^-$ never exceeds π. Observation 2.1 implies that $\angle v_{i+1}^+ p_{i+1} v_{i+1}^- \ge \angle v_i^+ p_i v_i^- + \alpha_i^-$, and thus $\pi \ge \angle v_k^+ p_k v_k^- \ge \angle v_0^+ p_0 v_0^- + \sum_i \alpha_i^-$. From which we obtain the following upper upper bound for the competitive ratio of the algorithm

$$\frac{\sum_i d(p_i, p_{i+1})}{\lambda(\mathcal{V}_k^-)} \;\le\; \pi + 1.$$

Theorem 3.1 *A robot moving traveling under the strategy* Walk-in-Circles *has a* $\pi+1$ *competitive ratio.*

As the target in each step i is selected from the interval ℓ_i this provides a margin of navigational error for the robot. That is, the strategy is robust under small constant bias of compass heading. The tolerance of the strategy is proportional to the aspect ratio of the shortest vs. longest edge encountered and the smallest distinguished angle between left or right extreme entrance points.

3.1 A Hybrid Method

From the analysis above it is clear that the competitive ratio of strategy *Walk-in-Circles* is directly dependent on the total "turn" angle $\Phi = \sum_i \alpha_i^-$. As it was pointed out, Φ is smaller than π minus the initial angle $\angle v_1^+ p_1 v_1^-$. This implies that, if the initial angle is large, the strategy gives a better competitive ratio.

In this section we consider a hybrid method, in which a strategy similar to that proposed by Kleinberg [10] is followed for initial angles $\angle v_1^+ p_1 v_1^-$ smaller than $\pi/2$ and the strategy *Walk-in-Circles* is used for angles larger than $\pi/2$.

Hybrid Strategy.

Cases a)–c) are as in the *High Level Strategy.*

Case d) If $\angle v_1^+ p_1 v_1^- \le \pi/2$ then the robot moves on the line perpendicular to $\overline{v_1^+ v_1^-}$.[1] As the robot advances it updates the vertices v_i^+ and v_i^- as the windows seen change. When either of $\angle v_i^+ p\, p_1$ or $\angle v_i^- p\, p_1 = \pi/2$, where p is the current position of the robot, it switches to strategy *Walk-in-Circles*, with p as starting point.

Case e) If $\angle v_1^+ p_1 v_1^- \ge \pi/2$ then the robot uses strategy *Walk-in-Circles.*

From the analysis of strategy *Walk-in-Circles*, it follows that Cases a)–c) and e) have a competitive ratio of at most $\pi/2+1$. Case d) requires a more careful analysis.

If, as in the previous section, we assume that the goal lies on the right side, then the optimal trajectory is given by $d(p_1, v_1^-) + \sum_i d(v_i^-, v_{i+1}^-)$. Let j be the index of the reflex vertex in which the robot switched strategies. Notice that $\angle v_j^- p v_j^+$ is now bigger equal to $\pi/2$.

Lemma 3.2 *The distance traversed by the robot up to the point where it switches strategy is bounded by* $d(p_1,p) \le \sqrt{d(p_1,p_j^{\pm})^2 - d(p,p_j^{\pm})^2}$ *on either side.*

Proof: For the vertex forming the right angle, the lemma follows trivially from the Theorem of Pythagoras. On the opposing vertex, say as in Figure 8, the law of the cosines states $d(p_1, v_j^-)^2 =$

[1] If line l_1 perpendicular to the line through v_1^+ and v_1^- that goes through p_1 does not intersect $\overline{v_1^+ v_1^-}$, then the robot follows the side of the triangle (p_1, v_1^+, v_1^-) that is closer to l_1. This does not change the analysis.

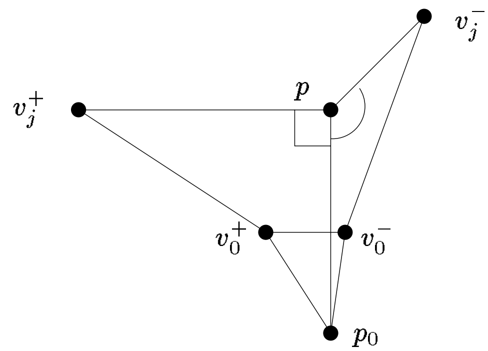

Figure 8: A hybrid strategy.

$d(p_1,p)^2+d(p,v_j^-)^2-2\,d(p_1,p)\,d(p,v_j^-)\cos(\angle p_1pv_j^-)$; which implies $d(p_1,v_j^-)^2 \geq d(p_1,p)^2+d(p,v_j^-)^2$ as $\angle p_1pv_j^- \geq \pi/2$, from which the lemma follows. □

As the robot applies strategy *Walk-in-Circles* as if p was the starting point, we have that the length of the distance traversed by it from p onwards is bounded by $(\pi/2+1)\left(d(p,v_j^-)+\sum_{i=j}^{k-1} d(v_i^-,v_{i+1}^-)\right)$. Thus the competitive ratio is given by $\mathcal{R}/\lambda(\mathcal{V}_{k-1}^-)$ where,

$$\mathcal{R} \;=\; \sqrt{d(p_1,v_j^-)^2-d(p,v_j^-)^2}+(\pi/2+1)\left(d(p,v_j^-)+\sum_{i=j}^{k-1} d(v_i^-,v_{i+1}^-)\right).$$

Let $\lambda(\mathcal{V}_{k-1}^-)' = d(p_1,v_j^-)+\sum_{i=j}^{k-1} d(v_i^-,v_{i+1}^-)$. Since $\lambda(\mathcal{V}_{k-1}^-) \geq \lambda(\mathcal{V}_{k-1}^-)'$, then $\mathcal{R}/\lambda(\mathcal{V}_{k-1}^-) \leq \mathcal{R}/\lambda(\mathcal{V}_{k-1}^-)'$. Without loss of generality, we can assume that $d(p_1,v_j^-)=1$. If $\mathcal{R}/\lambda(\mathcal{V}_{k-1}^-)' \leq (\pi/2+1+r)$, for some $r\geq 0$, then $\mathcal{R} \leq (\pi/2+1+r)\,\lambda(\mathcal{V}_{k-1}^-)'$, which implies

$$\sqrt{1-d(p,v_j^-)^2}+(\pi/2+1)\,d(p,v_j^-) \leq (\pi/2+1+r)+r\sum_{i=j}^{k-1} d(v_i^-,v_{i+1}^-).$$

Since $\sum_{i=j}^{k-1} d(v_i^-,v_{i+1}^-)$ can be arbitrarily small, for the expression above to be satisfied we need $\pi/2+1-\sqrt{1-d(p,v_j^-)^2}-(\pi/2+1)\,d(p,v_j^-) \geq -r$. Let $f(x)=\pi/2+1-\sqrt{1-x^2}-(\pi/2+1)\,x$. This function has an absolute minimum in the domain of interest at $x_{min}=(\pi+2)/\sqrt{\pi^2+4\pi+8}$ with $f(x_{min})=\pi/2+1-\frac{1}{2}\sqrt{\pi^2+4\pi+8}$, from which the fact that $r\geq \frac{1}{2}\sqrt{\pi^2+4\pi+8}-\pi/2-1$ follows. Since the competitive ratio $\mathcal{R}/\lambda(\mathcal{V}_{k-1}^-)$ is bounded by $\pi/2+1+r$, we have the following theorem.

Theorem 3.3 *The* Hybrid Strategy *has a competitive ratio of at most* $\frac{1}{2}\sqrt{\pi^2+4\pi+8}$.

The value $\frac{1}{2}\sqrt{\pi^2+4\pi+8}$ is approximately 2.76.

4 *lad* and Beyond

In this section we consider a new strategy which is similar in spirit to the first strategy that was proposed to traverse streets [9]. In [9] the strategy *lad* is presented which is based on the idea of minimising the *l*ocal *a*bsolute *d*etour. The importance of *lad*—apart from being the first strategy proposed—lies in the fact that it is the only strategy that uses a heuristic optimality criterion to guide the robot. All other strategies that have been presented have no comparable feature. The well-chosen heuristic and its excellent performance in practice make *lad* a very attractive strategy. Unfortunately, it seems that it is exactly this property that makes *lad* also extremely difficult to analyse. As mentioned before the best performance guarantee is $1 + \pi/2 + \sqrt{1 + \pi^2/4}$ (~ 4.44) which seems to be a very loose bound considering that the competitive ratio of the strategy observed in practice is less than 1.8 [9].

In the following we present a slight variant of *lad* which we call *continuous lad* that also follows the paradigm of minimising the local absolute detour but whose analysis turns out to be much simpler and tighter. It can be shown to achieve a competitive ratio of ~ 2.03 which is slightly better than the best performance guarantee of ~ 2.05 known so far [11]. We start out with some additional definitions and observations.

4.1 The Strategy *lad*

We give a short description of the rationale behind *lad* as well as its definition, so as to stress both the differences and similarities between it and *continuous lad.* Consider a point p_1 that is known to belong to the shortest path from s to g and assume that v_1^+ and v_1^- are defined. The robot computes a point t_1 on $\overline{v_1^+ v_1^-}$ and chooses the line segment $\overline{p_1 t_1}$ as its route of travel. The local absolute detour if v_1^+ lies on the path from s to g is given by the distance which the robot travels from p_1 to v_1^+ via t_1 which is $d(p_1, t_1) + d(t_1, v_1^+)$ minus the length of the shortest path $d(p_1, v_1^+)$ from p_1 to v_1^+. Similarly, the absolute detour for v_1^- is given by $d(p_1, t_1) + d(t_1, v_1^-) - d(p_1, v_1^-)$. The strategy *lad* now chooses t_1 on $\overline{v_1^+ v_1^-}$ so that it minimises the maximum of both local absolute detours, i.e., the local absolute detour is minimised if

$$d(p_1, t_1) + d(t_1, v_1^+) - d(p_1, v_1^+) \quad = \quad d(p_1, t_1) + d(t_1, v_1^-) - d(p_1, v_1^-), \tag{1}$$

where $d(v_1^+, t_1) + d(t_1, v_1^-) = d(v_1^+, v_1^-)$. Solving for $d(v_1^+, t_1)$ yields

$$d(v_1^+, t_1) \quad = \quad \frac{d(p_1, v_1^+) - d(p_1, v_1^-) + d(v_1^+, v_1^-)}{2}. \tag{2}$$

In general, if the robot has not been able to decide whether v_i^+ or v_i^- belongs to the shortest path from s to g after i steps, it chooses a new target point t_i on $\overline{v_i^+ v_i^-}$ and the line segment $\mathcal{P}_i = \overline{p_i t_i}$ to travel from its current position p_i to t_i. Let $\mathcal{Q}_i$ be the path of the robot from p_1 to p_i

and recall that $\mathcal{V}_i^+$ is the shortest path from p_1 to v_i^+ and $\mathcal{V}_i^-$ the shortest path from p_1 to v_i^-. If v_i^+ lies on the shortest path from s to g, then the local absolute detour is given by the distance the robot travels from p_1 to v_i^+ which is $\lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i) + d(t_i, v_i^+)$ minus the length of the shortest path $\lambda(\mathcal{V}_i^+)$ from p_1 to v_i^+. A similar statement holds if v_i^- belongs to $sp(s, g)$. Hence, the maximum local absolute detour is minimised if

$$\lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i) + d(t_i, v_i^+) - \lambda(\mathcal{V}_i^+) \quad = \quad \lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i) + d(t_i, v_i^-) - \lambda(\mathcal{V}_i^-) \tag{3}$$

and the point t_i on $\overline{v_i^+ v_i^-}$ is given by

$$d(v_i^+, t_i) \quad = \quad \frac{\lambda(\mathcal{V}_i^+) - \lambda(\mathcal{V}_i^-) + d(v_i^+, v_i^-)}{2}. \tag{4}$$

Note that $\lambda(\mathcal{V}_i^+) = \sum_{j=1}^{i-1} d(v_{j+1}^+, v_j^+)$ and $\lambda(\mathcal{V}_i^-) = \sum_{j=1}^{i-1} d(v_{j+1}^-, v_j^-)$ where we define $v_1^+ = v_1^- = p_1$.

4.2 Walk-in-Circles and *lad*

It is tempting to try to use the ideas of the analysis of the strategy Walk-in-Circles to analyse the strategy *lad*, as this would imply a bound of ~ 4.14 on it. In fact, in most cases *lad* chooses a target point t_i located inside the circles defined by Walk-in-Circles. However, as the following example shows, there are some rare cases for which *lad* chooses points outside the circles of reference.

For the analysis of the strategy Walk-in-Circles to apply we need $d(v_i^+, p_i) \geq d(v_i^+, t_i)$. Consider the polygon in Figure 9. In this case $p_1 = (0, -1)$, $v_1^+ = (-1/2, 0)$, and $v_1^- = (y+1, y)$ with $y = -1/2 + \frac{\sqrt{10}}{5} + \sqrt{-1 + \sqrt{10}}\sqrt{10}$.

For this value of y, the target point t_1 is located on the y-axis. Let the point v_2^- be located in the ray from the origin trough v_1^-, and such that $d(v_1^-, v_2^+) = \epsilon$, for some arbitrarily small $\epsilon > 0$. In such a case, the new target point t_2 is arbitrarily close to t_1 as nothing else has changed. However, as the robot is located on the pedal point of the line perpendicular to the trajectory and passing through v_1^+, the distance to v_1^+ can only increase as the robot continues moving on towards the point $t_2 \approx t_1$. Thus t_2 is not contained in the circle entered at v_2^+ and passing through p_2 as required by the analysis Walk-in-Circles.

This points to the fact that the *lad* strategy is difficult to analyse, while not so complex strategies yield better bounds.

4.3 The Strategy *continuous lad*

In the strategy *continuous lad* the robot also follows a path from p_i to t_i where t_i is determined by Equation 4; however, the robot does not move on a straight line segment. Instead, it moves on a path $\mathcal{P}_i$ such that for *each point* p on $\mathcal{P}_i$ the local absolute detour is minimised. Instead of being a line segment, $\mathcal{P}_i$ is now part of a hyperbola. Although this slight modification may seem

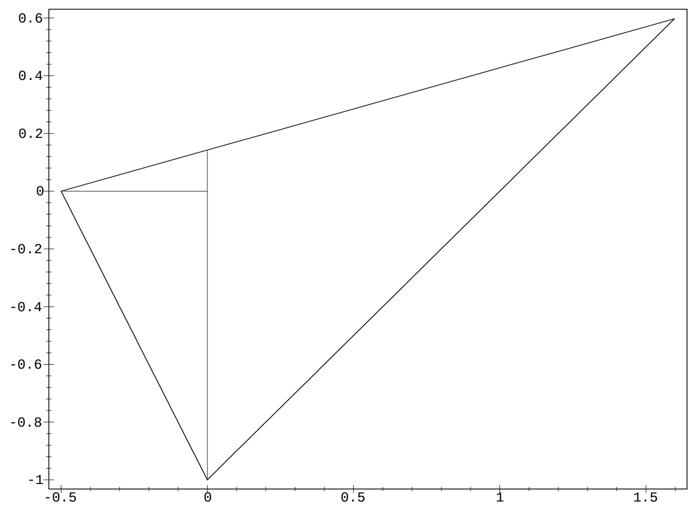

Figure 9: A polygon with a *lad*-walk inside.

to complicate the analysis further, it, in fact, allows to prove a much tighter upper bound on the competitive ratio for *continuous lad* than for *lad*. Note also that although the strategies seem to be almost identical, the points p_i at which the left or right extreme entrance points change for *lad* and *continuous lad* can be quite far apart.

We assume that the robot travels along a path $\mathcal{P}_i$ from p_i to p_{i+1} such that every point p on $\mathcal{P}_i$ satisfies Equation 3 if we replace t_i by p and $\mathcal{P}_i$ by $\mathcal{P}_i(p_i, p)$. If the robot follows the strategy *continuous lad*, then $a_i^+ = a_i^-$ and the location of t_i is only determined by d_i^+ and d_i^-.

Lemma 4.1 *If the robot travels on a path $\mathcal{P}_i$ such that for all $p \in \mathcal{P}_i$,*

$$\lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i(p_i, p)) + d(p, v_i^+) - \lambda(\mathcal{V}_i^+) \quad = \quad \lambda(\mathcal{Q}_i) + \lambda(\mathcal{P}_i(p_i, p)) + d(p, v_i^-) - \lambda(\mathcal{V}_i^-),$$

then $a_i^+ = a_i^- > 0$.

Proof: The proof is by induction on i. For $i = 1$, we have $\lambda(\mathcal{V}_1^+) = d(p_1, v_1^+)$ and $\lambda(\mathcal{V}_1^-) = d(p_1, v_1^-)$ and if we set $p = p_2$, then the above equation immediately yields

$$a_1^- = d(p_1, v_1^-) - d(p_2, v_1^-) = d(p_1, v_1^+) - d(p_2, v_1^+) = a_1^+.$$

Since the robot moves into the interior of the triangle (p_1, v_1^+, v_1^-), it is easy to see that $a_1^+ > 0$. Just consider the circles around v_1^+ and v_1^- with radius $d(p_1,v_1^+)$ and $d(p_1,v_1^-)$, respectively. So now assume the claim is true, for all $1 \leq i \leq k-1$. Since $d(p_{i+1}, v_i^+) = d_i^+ - a_i^+ < d_i^+$, for all $1 \leq i \leq k-1$, Lemma 2.2 holds and $\lambda(\mathcal{V}_k^+) = \sum_{j=0}^{k-1} a_j^+ + d_k^+$. Similarly, we have $\lambda(\mathcal{V}_k^-) = \sum_{j=0}^{k-1} a_j^- + d_k^-$. By the induction hypothesis $\sum_{j=0}^{k-1} a_j^+ = \sum_{j=0}^{k-1} a_j^-$ and the above equation again yields

$$a_k^- = d_k^- - d(p_{k+1}, v_k^-) = d_k^+ - d(p_{k+1}, v_k^+) = a_k^+.$$

The inequality $a_k^+ > 0$ follows as in the case $i = 1$. □

4.4 Analysis of the Strategy *continuous lad*

In the following we assume that the robot travels on a path $\mathcal{P}_i$ such that, for all points p_{i+1} on $\mathcal{P}_i$ the distances a_i^+ and a_i^- are the same, i.e. $a_i^+ = a_i^- = a_i$. We analyse a step of the High-Level-Algorithm which consists of k consecutive events of Category d) and one event in one of the Categories a)–c). As a first step we compute an upper bound on the length of the path that is given by the line segments connecting the points p_i to p_{i+1}. In a second step we then show how to extend this analysis to $\mathcal{Q}_k$.

We present two bounds for the length of the path connecting the points p_j, $1 \leq j \leq k$. The first bound gives a good approximation if the angle γ_i is small and the second bound approximates large angles.

Lemma 4.2 *If $a_i^+ = a_i^-$, then $d(p_i, p_{i+1}) \leq a_i^+ / \cos(\gamma_{i+1}/2)$.*

Proof: Let p_{i+1} be a point such that $a_i^+ = a_i^-$ and u_i^+ the point on $\overline{p_i v_i^+}$ at distance a_i^+ from p_i. The point u_i^- is defined analogously. Consider the quadrilateral Q formed by p_i, u_i^+, u_i^-, and p_{i+1} as shown in Figure 10a.

The location of p_{i+1} is completely determined by the angles α_i^+, α_i^-, and γ_i. The angle of Q formed at u_i^+ is $(\pi + \alpha_i^+)/2$ and at u_i^- it is $(\pi + \alpha_i^-)/2$. Since $\alpha_i^+ + \alpha_i^- + \gamma_i = \gamma_{i+1}$, we can choose α_i^+ and α_i^- in order to maximise the distance of p_{i+1} to p_i. Let $\theta_i = \angle u_i^+ p_{i+1} u_i^-$. Note that $\theta_i = 2\pi - \gamma_i - (\pi + \alpha_i^+)/2 - (\pi + \alpha_i^-)/2 = \pi - \gamma_i - \alpha_i^+/2 - \alpha_i^-/2$.

In order to compute the angle that maximises $d(p_i, p_{i+1})$ we consider the situation in Figure 10b. Let $\delta_1 = \angle u_i^- u_i^+ p_{i+1}$ and $\delta_2 = \angle p_{i+1} u_i^- u_i^+$. Hence, $\delta_1 + \delta_2 = \pi - \theta_i$, where the angle θ_i is fixed. Furthermore, we introduce a coordinate system such that the origin is located at u_i^+, $u_i^- = (1, 0)$, and p_i is located on the line $L = \{(x, y) \mid x = 1/2\}$. Let C be the circle that passes through u_i^-, p_{i+1}, and u_i^+ and c its centre. The path of all points with $\delta_1 + \delta_2 = \pi - \theta_i$ is the arc A of C from u_i^- to u_i^+ that contains p_{i+1} (see [15, Sec. 16, Th. 4]).

We claim that $d(p_i, p_{i+1})$ is maximal for $\delta_1 = \delta_2$. Let q be the topmost point of the arc A, i.e., $\delta_1 = \delta_2$ if $p_{i+1} = q$. We note that c is located on the line L. If c is above p_i, then the circle with centre p_i and radius $d(p_i, q)$ contains C and, hence, q is the point with maximal distance to p_i.

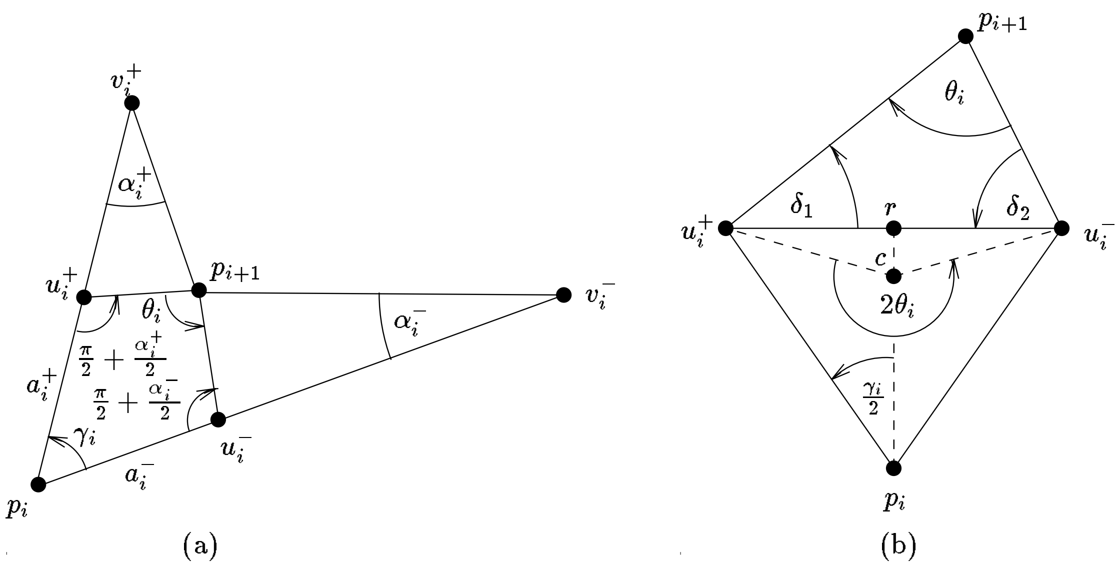

Figure 10: (a) Illustrating the proof of Lemma 4.2. (b) Maximising the distance from p_i to p_{i+1} by choosing δ_1 and δ_2.

We claim that c is above p_i. Let r be the point $(1/2, 0)$. In order to show that c is above p_i we compute the distances $d(p_i, r)$ and $d(c, r)$. The angle $\angle rp_iu_i^+$ is obviously $\gamma_i/2$. Hence, $d(p_i, r) = 1/2 \cot(\gamma_i/2)$. By [15, Sec. 16, Th. 2] the angle $\angle u_i^+ cu_i^-$ equals $2\theta_i$ and, hence, the angle $\angle rcu_i^+$ equals $(1/2)(2\pi - 2\theta_i) = \gamma_i + \alpha_i^+/2 + \alpha_i^-/2$; this yields $d(c, r) = 1/2 \cot(\gamma_i + \alpha_i^+/2 + \alpha_i^-/2) < 1/2 \cot(\gamma_i/2) = d(p_i, r)$ as claimed.

Therefore, we can assume $\alpha_i^+ = \alpha_i^-$ and we have the configuration displayed in Figure 11. Since

$$\cos\left(\frac{\alpha_i^+}{2}\right) = \frac{x}{a_i^+} \qquad \text{and} \qquad \cos\left(\frac{\alpha_i^+ + \gamma_i}{2}\right) = \frac{x}{d(p_i, p_{i+1})},$$

we obtain

$$d(p_i, p_{i+1}) = \frac{\cos(\alpha_i^+/2) a_i^+}{\cos((\gamma_i + \alpha_i^+)/2)}.$$

With $(\alpha_i^+ + \gamma_i)/2 \leq \gamma_{i+1}/2 < \pi/2$ and $\cos(\alpha_i^+/2) \leq 1$ the claim follows. □

For large angles we make use of the observation that $d(p_{i+1}, v_i^+) \leq d_i^+$ and $d(p_{i+1}, v_i^-) \leq d_i^-$ by Lemma 4.1. Hence, *continuous lad* belongs to the class of *Walk-in-Circles* Strategies and the distance between p_i and p_{i+1} is less than or equal to $\alpha_i^+ d_i^+ + a_i$ which we state as a lemma for further reference.

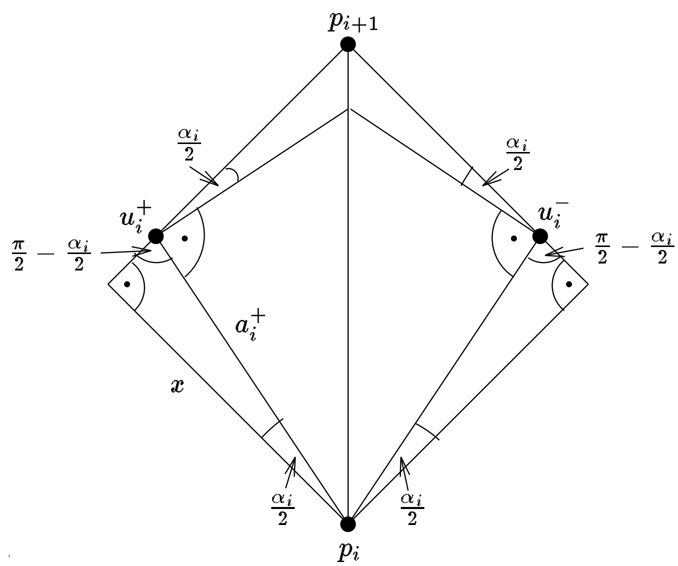

Figure 11: Choosing γ_i.

Lemma 4.3 *If $a_i^+ = a_i^-$, for all $1 \leq i \leq k$, then*

$$d(p_{i+1}, p_i) \quad \leq \quad \min\{\alpha_i^+ d_i^+, \alpha_i^- d_i^-\} + a_i.$$

We now can analyse the competitive ratio of *continuous lad*. To do so, we first consider the path $\mathcal{P}_i'$ that consists of the line segments connecting the points p_i. For the analysis we split the execution of the strategy into two parts. In the first part we consider the length of the path of the robot until the angle between v_i^+ and v_i^- is equal to some angle γ and in the second part we consider the length of the remaining path of the robot. In order to see that it is possible to chose γ to be any value that is greater than or equal to γ_0, let $p_i(t)$ denote the position of the robot at time t while traveling on $\mathcal{P}_i$. We argue that $\gamma(t) = \angle v_i^- p_i(t) v_i^+$ is a continuous and monotonously increasing function as the robot travels on $\mathcal{P}_i$.

We assume that the robot has reached the point p_i and the angle $\angle v_i^- p_i v_i^+$ is γ_i. At p_i the robot chooses a point t_i on the line segment $\overline{v_i^+ v_i^-}$ and moves on the path $\mathcal{P}_i$ from p_i to t_i until a new left or right extreme entrance point becomes visible. We assume that $p_i(0) = p_i$ and $p_i(1) = t_i$. Since $d(p_i(t), v_i^+)$ and $d(p_i(t), v_i^-)$ both decrease monotonically and continuously with t, the angles $\alpha_i^+(t) = \angle p_i v_i^+ p_i(t)$ and $\alpha_i^-(t) = \angle p_i(t) v_i^- p_i$ are continuous and monotonously increasing functions of t and, therefore, $\gamma_i(t) = \gamma_i + \alpha_i^+(t) + \alpha_i^-(t)$ is also a continuous and monotonously increasing function of t. Hence, if $\gamma_0 \leq \gamma \leq \gamma_k$, there is one $0 \leq i_0 \leq k$ and one $0 \leq t_0 \leq 1$ with $\gamma_{i_0}(t_0) = \gamma$. We can assume in the following that the robot follows the Strategy *continuous lad* until either one of the extreme entrance points is undefined or $\gamma_{i_0}(t_0) = \gamma$, for some i_0 and t_0. We insert the point

$p_{i_0+1} = p_{i_0}(t_0)$ into the sequence of points $(p_1, \ldots, p_k)$ so that now there are $k+1$ points p_i. If $\gamma_0 \geq \gamma$, then we define $p_{i_0+1} = p_0$. In both cases we have $\gamma_{i_0+1} \geq \gamma$.

In order to analyse the competitive ratio of *continuous lad* we need to estimate the length of the path $\mathcal{P}_i$ that is the concatenation of parts of hyperbolas. Recall that if $\Lambda(\mathcal{P}_i)$ is the set of all finite sequences of points on $\mathcal{P}_i$ that occur in order, i.e. $\Lambda(\mathcal{P}_i) = \{(r_1, \ldots, r_m) \mid m \geq 1, r_j \in \mathcal{P}_i, \text{for all } 1 \leq j \leq m, \text{ and } r_{j+1} \text{ occurs after } r_j \text{ on } \mathcal{P}_i\}$, then $\lambda(\mathcal{P}_i) = \sup\{\sum_{j=1}^{m-1} d(r_j, r_{j+1}) \mid (r_1, \ldots, r_m) \in \Lambda(\mathcal{P}_i)\}$. We will make use of this definition, to estimate the length of $\mathcal{P}_i$.

So consider a sequence $(r_1, \ldots, r_m) \in \Lambda(\mathcal{P}_i)$. Since we are interested in obtaining a supremum and adding points only increases $\sum_{j=1}^{m-1} d(r_j, r_{j+1})$, we can assume that $(p_1, \ldots, p_k)$ is a subsequence of $(r_1, \ldots, r_m)$. We define as before v_j^+ to be the left entrance point of $V(r_j)$, $d_j^+ = d(r_j, v_j^+)$, and $a_j^+ = d_j^+ - (d_{j+1}^+ - d(v_i^+, v_{i+1}^+))$. We define v_j^-, d_j^-, a_j^-, γ_j, etc. analogously. Note that with the above definitions Lemmas 2.2 and 4.1 still hold which implies that Lemmas 4.2 and 4.3 hold as well.

So let $\mathcal{R}_m$ be the path connecting the points $r_1, \ldots, r_m$ by line segments. We now observe that if the angle γ_j at r_j is less than or equal to γ, then Lemma 4.2 yields that $d(r_{j-1}, r_j) \leq a_{j-1}/\cos(\gamma/2)$. Otherwise, Lemma 4.3 yields that $d(r_j, r_{j+1}) \leq \alpha_j^+ d_j^+ + a_j^+$ and $d(r_j, r_{j+1}) \leq \alpha_j^- d_j^- + a_j^-$. If j_0 is the index, such that $\gamma_{j_0+1} = \gamma$, then we obtain the following analysis of the length of $\mathcal{R}_m$ where we assume w.l.o.g. that $v_{m-1}^+ \in sp(s, g)$.

$$\begin{aligned}
\sum_{j=1}^{m-1} d(r_j, r_{j+1}) + d_m^+ &= \sum_{j=1}^{j_0} d(r_j, r_{j+1}) + \sum_{j=j_0+1}^{m-1} d(r_j, r_{j+1}) + d_m^+ \\
&\leq \sum_{j=1}^{j_0} \frac{a_j^+}{\cos(\gamma/2)} + \sum_{j=j_0+1}^{m-1} (\alpha_j^+ d_j^+ + a_j^+) + d_m^+ \\
&\leq \sum_{j=1}^{j_0} \frac{a_j^+}{\cos(\gamma/2)} + \left(\lambda(\mathcal{V}_m^+) - \sum_{j=1}^{j_0} a_j^+\right) \sum_{j=j_0+1}^{m-1} \alpha_j^+ + \sum_{j=j_0+1}^{m-1} a_j^+ + d_m^+ \\
&\leq \frac{1}{\cos(\gamma/2)} \sum_{j=1}^{j_0} a_j^+ + (\pi - \gamma + 1)\left(\lambda(\mathcal{V}_m^+) - \sum_{j=1}^{j_0} a_j^+\right) \\
&\leq \max\{1/\cos(\gamma/2), \pi - \gamma + 1\} \lambda(\mathcal{V}_m^+)
\end{aligned}$$

So let the angle γ be chosen such that the maximum of $\{1/\cos(\gamma/2), \pi - \gamma + 1\}$ is minimised, i.e., that $1/\cos(\gamma/2) = \pi - \gamma + 1$. By numerical evaluation we obtain that $\gamma \sim 2.111$. Hence, $(\pi - 2.111) + 1$ (~ 2.03) is an upper bound on the length of the path $\mathcal{R}_m$ that connects the points r_j by straight line segments. Since $(r_1, \ldots, r_m)$ is chosen arbitrarily from $\Lambda(\mathcal{P}_i)$, the supremum of $\{\sum_{j=1}^{m-1} d(r_j, r_{j+1}) \mid (r_1, \ldots, r_m) \in \Lambda(\mathcal{P}_i)\}$ is also bounded by 2.03. We state this as a theorem.

Theorem 4.4 *The strategy* continuous lad *has a competitive ratio of at most* 2.03.

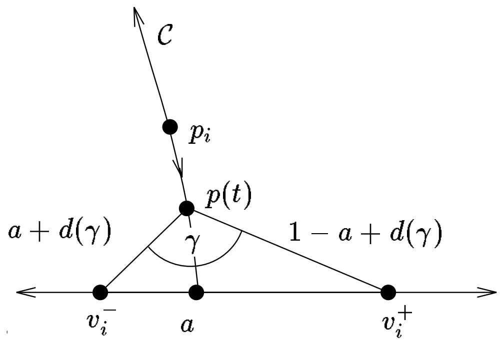

Figure 12: The path $\mathcal{C}$.

Although we obtain an estimate of the length of the path $\mathcal{Q}_k$ in this way, it is still interesting to look at how much is lost by traveling on a hyperbola instead of a straight line. In the next section we show that the length of such a path is at most $3/4\sqrt{2}$ (~ 1.061) times longer than the length of a line segment.

4.5 Bounding the Length of $\mathcal{P}_i$

The optimal strategy that minimises the local absolute detour would be to travel from point p_i to point p_{i+1} on a straight line segment. A robot using *continuous lad*, however, travels on a hyperbola. In this section we give an upper bound on ratio between the length of the path $\mathcal{P}_i$ that consists of all points p with $d(v_i^+, p_i) - d(v_i^+, p) = d(v_i^-, p_i) - d(v_i^-, p)$ and the length of a line segment between the start point and some other point of $\mathcal{P}_i$. This indicates how much is lost in *continuous lad* by traveling on a hyperbola.

If we assume that the path is parameterised over $[0, 1]$ and that the position of the robot for $t \in [0, 1]$ is given by $p(t)$ and the length of the path is given by $\mathcal{P}_i(t)$, then we are interested in measuring the ratio of the length of $\mathcal{P}_i(t)$ w.r.t. to $d(p_i, p(t))$.

So we want to compute the maximum of the ratio $\mathcal{P}_i(t)/d(p_i, p(t))$, over all $t \in [0, 1]$ and all possible triangles (p_i, v_i^+, v_i^-). In order to compute this ratio we take the following view of $\mathcal{P}_i$. If we fix v_i^+, v_i^- and the point t_i on $\overline{v_i^+ v_i^-}$ where $\mathcal{P}_i$ ends, then the set of possible starting points p_i lies on a semi-infinite path $\mathcal{C}$ of which $\mathcal{P}_i$ is the part from p_i to some point p_{i+1} closer to t_i. Hence, it suffices to consider $\mathcal{C}$. If we assume a coordinate system with the origin at v_i^- and with v_i^+ on the positive x-axis, set the length of $\overline{v_i^+ v_i^-}$ to be 1, and the distance of the end point t_i of $\mathcal{C}$ to v_i^- to be $a \in [0, 1]$ (see Figure 12), then the points (x, y) on $\mathcal{C}$ are given by the solutions to the following set of equations.

$$x^2 + y^2 = (a+d)^2 \qquad \text{and} \qquad (1-x)^2 + y^2 = (1-a+d)^2$$

which has the solutions

$$x = a + (2a-1)d \qquad \text{and} \qquad y = 2\sqrt{a(1-a)d(1+d)}.$$

If we parameterise $\mathcal{C}$ by d, then

$$\begin{aligned}
p(d) &= \begin{pmatrix} a+(2a-1)d \\ \sqrt{a(1-a)d(1+d)} \end{pmatrix} \qquad \text{and} \\
\mathcal{C}(d_0, d_1) &= \int_{d_0}^{d_1} \left\| \frac{d}{d\delta} p(\delta) \right\|_2 d\delta \\
&= \int_{d_0}^{d_1} \sqrt{(2a-1)^2 + a(1-a)\left(4 + \frac{1}{\delta(\delta+1)}\right)}\, d\delta \\
&= \int_{d_0}^{d_1} \sqrt{1 + \frac{a(1-a)}{\delta(\delta+1)}}\, d\delta
\end{aligned}$$

where $\mathcal{C}(d_0, d_1)$ denotes the length of the path from $p(d_0)$ to $p(d_1)$. Note that since $x(a,d)$ is linear in d and

$$\begin{aligned}
\frac{\partial y}{\partial d}(a,d) &= \frac{\sqrt{a(1-a)}(2d+1)}{\sqrt{d(d+1)}} \geq 0 \\
\frac{\partial^2 y}{(\partial d)^2}(a,d) &= -\frac{1}{2}\frac{\sqrt{a(1-a)}}{(d(d+1))^{3/2}} \leq 0,
\end{aligned}$$

$\mathcal{C}$ is the graph of a concave, monotone function.

We are interested in the ratio

$$\frac{\mathcal{C}(d_0, d_1)}{d(p(d_0), p(d_1))} = \frac{\int_{d_0}^{d_1} \sqrt{1 + \frac{a(1-a)}{\delta(\delta+1)}} d\delta}{\sqrt{((2a-1)(d_1-d_0))^2 + \left(2\sqrt{a(1-a)}\left(\sqrt{d_1(1+d_1)} - \sqrt{d_0(1+d_0)}\right)\right)^2}}. \tag{5}$$

If we substitute $\varepsilon = \delta/(a(1-a))$ in the right hand side of Equation 5, then

$$d\delta = a(1-a)d\varepsilon.$$

With $e_1 = d_1/(a(1-a))$ we obtain

$$\mathcal{C}(d_0, d_1) = a(1-a)\int_{e_0}^{e_1} \sqrt{1 + \frac{1}{\varepsilon(a(1-a)\varepsilon + 1)}} d\varepsilon.$$

Note that the above substitution is injective for d, i.e., , for each d and each $0 < a < 1$, there is a e with $e = d/(a(1-a))$. Hence, an upper bound on the ratio if we substitute e for d, for all $e \geq 0$ and $0 < a < 1$ also yields an upper bound on the ratio in Equation 5.

Since $d_1 = a(1-a)e_1$, we have

$$\begin{aligned} d(p(d_0), p(d_1)) &= d(p(a(1-a)e_0), p(a(1-a)e_1)) \\ &= a(1-a)\sqrt{8a(1-a)e_1e_0 + 4(e_1+e_0) + (e_1e_0)^2 - 8\sqrt{e_1}\sqrt{1+a(1-a)e_1}\sqrt{e_0}\sqrt{1+a(1-a)e_0}}. \end{aligned}$$

We first show that $d(p(a(1-a)e_1), p(a(1-a)e_0))/(a(1-a))$ has a minimum at $a = 1/2$. To see this consider the function

$$f(a) = a(1-a)e_1e_0 - \sqrt{e_1}\sqrt{1+a(1-a)e_1}\sqrt{e_0}\sqrt{1+a(1-a)e_0}.$$

With

$$\frac{df}{da}(a) = (1-2a)\left(e_1e_0 - \frac{1}{2}\frac{\sqrt{e_1e_0}\sqrt{1+a(1-a)e_0}e_1}{\sqrt{1+a(1-a)e_1}} - \frac{1}{2}\frac{\sqrt{e_1e_0}\sqrt{1+a(1-a)e_1}e_0}{\sqrt{1+a(1-a)e_0}}\right)$$

it can be easily seen that $\frac{df}{da}(a) = 0$ if and only if $a = 1/2$, $\frac{df}{da}(a) < 0$, for $a < 0$, and $\frac{df}{da}(a) > 0$, for $a > 0$; hence, $a = 1/2$ is a minimum and we obtain

$$\frac{\mathcal{C}(a(1-a)e_0, a(1-a)e_1)}{d(p(a(1-a)e_0), p(a(1-a)e_1))} \leq \frac{\int_{e_0}^{e_1}\sqrt{1+\frac{1}{\varepsilon(a(1-a)\varepsilon+1)}}d\varepsilon}{\sqrt{4(e_1+e_0)+e_1^2+e_0^2-2\sqrt{e_1e_0(e_1+4)(e_0+4)}}}.$$

The numerator of the above quotient achieves its maximum for $a \to 0$ or $a \to 1$. Hence,

$$\begin{aligned} \max_{a\in[0,1]} \frac{\mathcal{C}(a(1-a)e_0, a(1-a)e_1)}{d(p(a(1-a)e_0), p(a(1-a)e_1))} &\leq \frac{\int_{e_0}^{e_1}\sqrt{1+\frac{1}{\varepsilon}}d\varepsilon}{\sqrt{4(e_1+e_0)+e_1^2+e_0^2-2\sqrt{e_1e_0(e_1+4)(e_0+4)}}} \\ &= \frac{\sqrt{e_1}\sqrt{e_1+1}+\frac{1}{2}\ln\left(e_1+\frac{1}{2}+\sqrt{e_1}\sqrt{e_1+1}\right)-\sqrt{e_0}\sqrt{e_0+1}-\frac{1}{2}\ln\left(e_0+\frac{1}{2}+\sqrt{e_0}\sqrt{e_0+1}\right)}{\sqrt{4(e_1+e_0)+e_1^2+e_0^2-2\sqrt{e_1e_0(e_1+4)(e_0+4)}}}. \end{aligned}$$

In the following we show that $\mathcal{C}(d_0, d_1)/d(p(d_0), p(d_1))$ is bounded by $3/4\sqrt{2}$ (~ 1.061), for all $0 < d_0 \leq d_1$.

In order to see this we show that

$$\begin{aligned} D(e_0, e_1) &= 3/4\sqrt{2}\sqrt{4(e_1+e_0)+e_1^2+e_0^2-2\sqrt{e_1e_0(e_1+4)(e_0+4)}} - \sqrt{e_1}\sqrt{e_1+1} - \\ &\quad \frac{1}{2}\ln\left(e_1+\frac{1}{2}+\sqrt{e_1}\sqrt{e_1+1}\right)+\sqrt{e_0}\sqrt{e_0+1}+\frac{1}{2}\ln\left(e_0+\frac{1}{2}+\sqrt{e_0}\sqrt{e_0+1}\right) \geq 0. \end{aligned}$$

We consider the derivative of D w.r.t. e_0.

$$D_{e_0}(e_0,e_1) = \frac{3\sqrt{2}}{8}\frac{4-4\frac{\sqrt{e_1(\frac{1}{4}e_1+1)(\frac{1}{4}e_0+1)}}{\sqrt{e_0}}-\frac{\sqrt{e_0e_1(\frac{1}{4}e_1+1)}}{\sqrt{\frac{1}{4}e_0+1}}+2e_0}{\sqrt{4e_0+e_1^2-8\sqrt{e_0e_1(\frac{1}{4}e_1+1)(\frac{1}{4}e_0+1)}+4e_1+e_0^2}}+\sqrt{1+\frac{1}{e_0}}.$$

In the following we compute the roots of D_{e_0} w.r.t. e_0. Let

$$\begin{aligned} D_1(e_0,e_1) &= \frac{3\sqrt{2}}{8}\left(4-4\frac{\sqrt{e_1(\frac{1}{4}e_1+1)(\frac{1}{4}e_0+1)}}{\sqrt{e_0}}-\frac{\sqrt{e_0e_1(\frac{1}{4}e_1+1)}}{\sqrt{\frac{1}{4}e_0+1}}+2e_0\right)+ \\ &\quad \sqrt{1+\frac{1}{e_0}}\sqrt{4e_0+e_1^2-8\sqrt{e_0e_1\left(\frac{1}{4}e_1+1\right)\left(\frac{1}{4}e_0+1\right)}+4e_1+e_0^2} \\ &= \frac{1}{16}\left(\frac{-6\sqrt{2}\sqrt{e_0}\sqrt{e_0+4}+3\sqrt{2}\sqrt{e_1}\sqrt{e_1+4}e_0+6\sqrt{2}\sqrt{e_1}\sqrt{e_1+4}-3\sqrt{2}e_0^{3/2}\sqrt{e_0+4}}{\sqrt{e_0(e_0+4)}}-\right. \\ &\quad \left.\frac{4\sqrt{e_0+1}\sqrt{4e_0+e_1^2-2\sqrt{e_1}\sqrt{e_1+4}\sqrt{e_0}\sqrt{e_0+4}+4e_1+e_0^2}\sqrt{e_0+4}}{\sqrt{e_0(e_0+4)}}\right). \end{aligned}$$

If $D_{e_0}(e_0,e_1)=0$, then $D_1(e_0,e_1)16\sqrt{e_0(e_0+4)}=0$. So assume $D_1(e_0,e_1)16\sqrt{e_0(e_0+4)}=0$. Then,

$$\begin{aligned} &-6\sqrt{2}\sqrt{e_0}\sqrt{e_0+4}+3\sqrt{2}\sqrt{e_1}\sqrt{e_1+4}e_0+6\sqrt{2}\sqrt{e_1}\sqrt{e_1+4}-3\sqrt{2}e_0^{3/2}\sqrt{e_0+4}= \\ &\quad 4\sqrt{e_0+1}\sqrt{4e_0+e_1^2-2\sqrt{e_1}\sqrt{e_1+4}\sqrt{e_0}\sqrt{e_0+4}+4e_1+e_0^2}.\sqrt{e_0+4} \end{aligned}$$

By squaring both sides and subtracting the terms on the right hand side from the terms on the left hand side we obtain

$$\begin{aligned} D_2(e_0,e_1) &= 2e_0^4-4\sqrt{e_1}\sqrt{e_1+4}e_0^{5/2}\sqrt{e_0+4}+8e_1e_0^2-32e_1e_0+16e_0^{3/2}\sqrt{e_0+4}\sqrt{e_1}\sqrt{e_1+4}- \\ &\quad 8e_1^2e_0+2e_1^2e_0^2+32e_1+32e_0+8e_1^2-24e_0^2-16\sqrt{e_1}\sqrt{e_1+4}\sqrt{e_0}\sqrt{e_0+4}. \end{aligned}$$

Clearly, if $D_1(e_0,e_1)=0$, then $D_2(e_0,e_1)=0$. We again split the terms on the two sides and obtain

$$\begin{aligned} &2e_0^4+8e_1e_0^2-32e_1e_0-8e_1^2e_0+2e_1^2e_0^2+32e_1+32e_0+8e_1^2-24e_0^2= \\ &\quad 4\sqrt{e_1}\sqrt{e_1+4}e_0^{5/2}\sqrt{e_0+4}-16e_0^{3/2}\sqrt{e_0+4}\sqrt{e_1}\sqrt{e_1+4}+16\sqrt{e_1}\sqrt{e_1+4}\sqrt{e_0}\sqrt{e_0+4}. \end{aligned}$$

Again we square and subtract the terms on the right hand side from the terms on the left hand side. This yields the following.

$$D_3(e_0,e_1) = 4\left(896e_1e_0^2-512e_1e_0-384e_0^3+256e_1^2+256e_0^2+144e_0^4+64e_1e_0^4+608e_1^2e_0^2-\right.$$

$$640e_1^2e_0 - 2e_1^2e_0^6 + 8e_1^2e_0^5 - 8e_1e_0^6 + 32e_1^2e_0^4 - 256e_1^2e_0^3 + 192e_1^3e_0^2 - 256e_1^3e_0 + 8e_1^3e_0^4 - 64e_1^3e_0^3 + 128e_1^3 + 32e_0^5 + 16e_1^4 - 24e_0^6 + e_0^8 + 32e_1e_0^5 - 512e_1e_0^3 + 24e_1^4e_0^2 - 32e_1^4e_0 + e_1^4e_0^4 - 8e_1^4e_0^3\Big) .$$

Note that

$$D_3(e_0, e_1) = 4(e_0 - 2)^4(e_1 + 4 + e_0)^2(e_1 - e_0)^2.$$

Hence, the only values of e_0 for which $D_{e_0}(e_0, e_1)$ can vanish are $e_0 = 2$ and $e_0 = e_1$. Therefore, the minimum of $D(e_0, e_1)$ is either assumed for $e_0 = 2$ or $e_0 = e_1$ or at the boundaries of the domain of D which are at $e_0 = 0$ or $e_0 = e_1$.

For $e_0 = e_1$ we have $D(e_0, e_0) = 0$. It can be easily seen that $D(2, e_1)$ only vanishes for $e_1 = 2$ and that $D(0, e_1) > 0$, for $e_0 > 0$. Hence, $D(e_0, e_1) \geq 0$ and $\mathcal{C}(d_0, d_1)/d(p(d_0), p(d_1))$ is bounded by $3/4\sqrt{2}$, for all $0 \leq d_0 \leq d_1$ as claimed.

5 Changing the Strategy

If we take a closer look at the analysis of Strategy *continuous lad*, then we notice that the ratio obtained for small angles is much tighter than the bound on large angles. Unfortunately, it is not obvious how to improve the analysis. However, there is another option. Since the robot can measure the angle between v_i^+ and v_i^- at its position, it is possible to change the strategy once a certain threshold is reached. We assume that the robot has encountered k events of Category d) in *continuous lad* and switches to a new strategy at point p_k.

In the following we consider the Strategy *Move-in-Quadrant* which was already presented in [11] but we provide a tighter analysis if the angle γ_k is larger than $\pi/2$.

In order to present the strategy we need the notion of a projection of a point. The *orthogonal projection* p' of a point p onto a line segment l is defined as the point of l that is closest to p.

Strategy Move-in-Quadrant
Input: A point p_k in P such that the angle $\gamma_k = \angle v_k^- p_k v_k^+$ is $\geq \pi/2$;
  $i := k$;
  while v_i^+ and v_i^- of $V(p_i)$ are defined **do**
    Move to the orthogonal projection p_{i+1} of p_k onto the line segment l_i from v_i^+ to v_i^-;
    Compute the points v_{i+1}^+ and v_{i+1}^- of the visibility polygon $V(p_{i+1})$ of p_{i+1};
    $i := i + 1$;
  end while;

The correctness of the strategy has been proven in [11]. Note that the Strategy *Move-in-Quadrant*

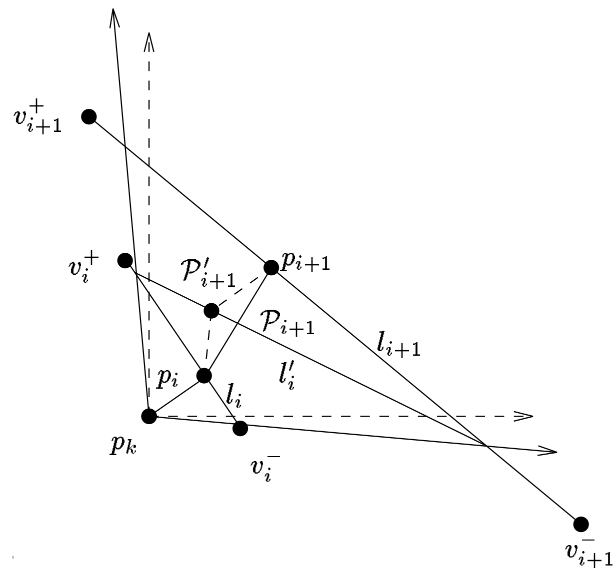

Figure 13: Introducing a new segment between l_i and l_{i+1}.

also follows the schema of the High-Level-Strategy except that events of Category d) are replaced by events of Category d′): $t_i = p_{i+1}$ is reached.

5.1 Analysis of the Strategy *Move-in-Quadrant*

In the following we assume that the Strategy *Move-in-Quadrant* has stopped after $m - k$ iterations. As before the shortest path goes either through v_i^+ or v_i^- [11].

Recall that γ_k is defined to be the angle $\angle v_k^- p_k v_k^+$ which we assume to be greater than or equal to $\gamma_k \geq \pi/2$. We introduce a coordinate system where p_k is the origin. Let Δr_k be the angle between the x-axis and the line segment $\overline{p_k v_k^-}$ and δ_k^+ the angle between the y-axis and the line segment $\overline{p_k v_k^+}$. We choose the orientation of the axes so that $\delta_k^+ = \Delta r_k$ and define $\delta_k = \delta_k^+ = \Delta r_k$.

Now suppose that we have arrived at point p_i and move to point p_{i+1} in the next iteration. To simplify the analysis, we consider the line segment l'_i from the intersection point of $\overline{v_i^+ v_i^-}$ with the line through p_k and v_k^+ to the intersection point of $\overline{v_{i+1}^+ v_{i+1}^-}$ with the line through p_k and v_k^- as shown Figure 13.

The line segment l'_i is located between l_i and l_{i+1}. If we consider the path $\mathcal{P}'_i$ from p_k to p_i that visits the orthogonal projections of p_k onto the line segments l_j and l'_j in order, for $k \leq j \leq i$, then the length of $\mathcal{P}'_i$ is obviously greater than or equal to the length of $\mathcal{P}_i$. Furthermore, $\mathcal{P}_i$ and $\mathcal{P}'_i$ share the same start and end point. Hence, for the simplicity of exposition we assume in the

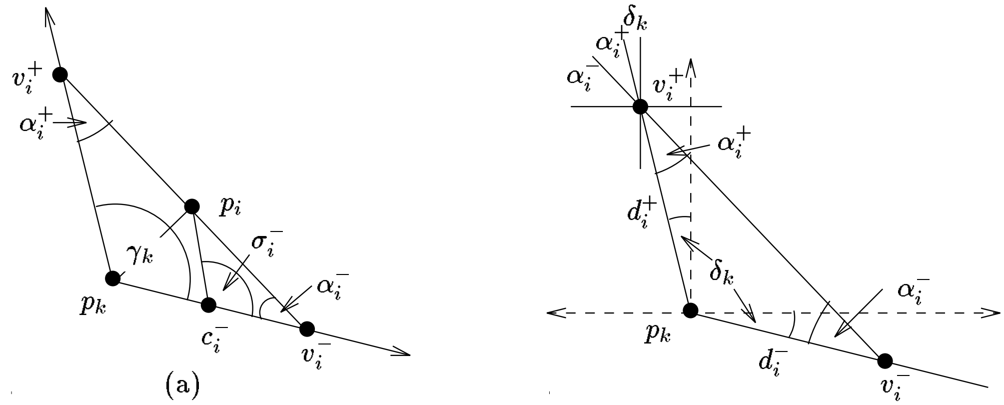

Figure 14: (a) The relationship between the angles α_i^+, α_i^-, γ_k, and $\sigma_i^- = \pi - 2\alpha_i^-$. (b) Computing $d(p_k, v_i^-)$ if $d(p_k, v_i^+) = 1$.

following that v_i^+ and v_i^- are located on the line from p_k to v_k^+ and v_k^-, respectively, and that either $v_i^+ = v_{i+1}^+$ or $v_i^- = v_{i+1}^-$.

Let L_i be the length of the path $\mathcal{P}_i$ traveled by the robot from p_k to reach p_i; let α_i^- be the angle $\angle p_i v_i^- p_k$, and d_i^- the distance $d(p_k,v_i^-)$ (see Figure 14a and b). Similarly, let α_i^+ be the angle $\angle p_k v_i^+ p_i$ and d_i^+ the distance $d(p_k,v_i^+)$. We define the angle α_i as $\min\{\pi/2 - \alpha_i^+, \pi/2 - \alpha_i^-\}$ and the distance d_i as $\min\{d_i^+, d_i^-\}$. Note that $\pi/2 - \alpha_i^+ + \pi/2 - \alpha_i^- = \gamma_k$ and, therefore, $\alpha_i^+ + \alpha_i^- + 2\delta_k = \pi/2$ or $\pi/2 - \alpha_i^+ = \alpha_i^- + 2\delta_k$ and $\pi/2 - \alpha_i^- = \alpha_i^+ + 2\delta_k$. In particular, $\alpha_i = \pi/2 - \alpha_i^+$ if and only if $d_i = d_i^+$.

Our approach to analyse our strategy is based on the idea of a potential function Q_i [11]. It is our aim to show that $L_i + Q_i \leq (\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2})d_i$, for all $k \leq i \leq m$, where we define $Q_i = \alpha_i d_i$. So suppose the robot has reached the point p_i and $L_i \leq (\gamma_k/2 + \cot\gamma_k/2 - \alpha_i)d_i$ and d_i is equal to the distance between p_0 and v_i^-. For simplicity of description we assume that the distance from p_k to v_i^+ is 1.

For the distance d_i^- we obtain

$$d_i^- = \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}. \tag{6}$$

Equation 6 can be seen as follows. Consider the triangle displayed in Figure 14b.

v_i^- is the intersection point of the line with slope $\tan\delta_k$ through p_k and the line with slope $\tan(\pi/2 + \alpha_i^+ + \delta_k)$ through v_i^+. Hence, the coordinates of v_i^- are given by the solution to the

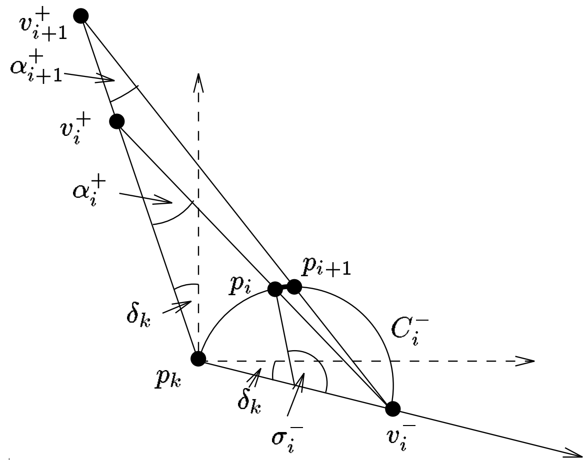

Figure 15: Case 1 if the robot moves from p_i to p_{i+1}.

equation

$$\lambda \begin{pmatrix} \cos\delta_k \\ -\sin\delta_k \end{pmatrix} = \begin{pmatrix} -\sin\delta_k \\ \cos\delta_k \end{pmatrix} + \mu \begin{pmatrix} \cos(\frac{\pi}{2} + \alpha_i^+ + \delta_k) \\ \sin(\frac{\pi}{2} + \alpha_i^+ + \delta_k) \end{pmatrix}$$

which yields the solution

$$\lambda = \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)} \text{ and } \mu = -\frac{\cos(2\delta_k)}{\cos(\alpha_i^+ + 2\delta_k)}$$

for μ and λ. Hence,

$$v_i^- = \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)} \begin{pmatrix} \cos\delta_k \\ -\sin\delta_k \end{pmatrix}$$

and $d(p_k, v_i^-)$ is given by

$$d(p_k, v_i^-) \quad = \quad \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}. \tag{7}$$

The robot moves now from p_i to p_{i+1}. Since $d_i = d_i^-$, $\alpha_i = \pi/2 - \alpha_i^- \leq \pi/2 - \alpha_i^+$. We distinguish three cases.

Case 1 $\alpha_{i+1}^+ < \alpha_i^+ \leq \alpha_i^- \leq \alpha_{i+1}^-$.
Hence, $d_{i+1}^- = d_i^-$. Note that p_{i+1} is on the circle C_i^- with centre at $c_i^- = (\cos\delta_k, -\sin\delta_k)d_i^-/2$ and radius $d_i^-/2$ (see Figure 15). Let σ_i^- be the angle $\angle p_i c_i^- v_i^+$. Note that $\sigma_i^- = \pi - 2\alpha_i^- = 2\alpha_i$. The arc A_i^- of C_i^- from p_i to p_{i+1} has length

$$(\sigma_i^- - \sigma_{i+1}^-)d_i^-/2 = 2(\alpha_i - \alpha_{i+1})d_i^-/2 = (\alpha_i - \alpha_{i+1})d_i^-.$$

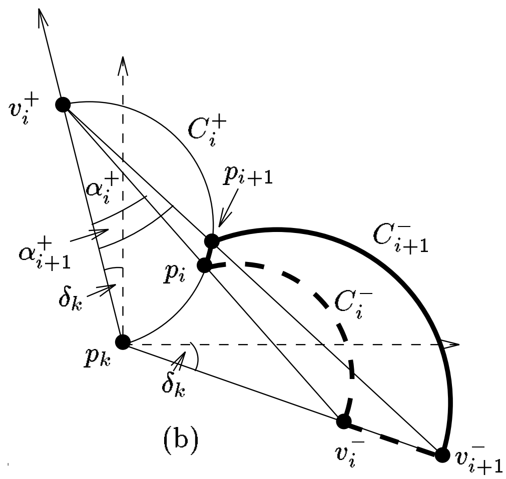

Figure 16: Case 2 if the robot moves from p_i to p_{i+1}.

Clearly, the line segment $\overline{p_i p_{i+1}}$ is shorter than the arc A_i^-. Hence,

$$\begin{aligned} L_{i+1} &= L_i + d(p_i, p_{i+1}) \\ &\leq \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2}\right) d_i - \alpha_i d_i + (\alpha_i - \alpha_{i+1}) d_i \\ &= \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2}\right) d_{i+1} - \alpha_{i+1} d_{i+1}. \end{aligned}$$

Case 2 $\alpha_i^+ < \alpha_{i+1}^+ \leq \alpha_{i+1}^-$ (see Figure 16).
Hence, $d_{i+1} = d_i^- = \sin\alpha_{i+1}^+ / \cos(\alpha_{i+1}^+ + 2\delta_k)$ by Equation 7. Note that p_{i+1} is on the circle C_i^+ with centre at $c_i^+ = 1/2(-\sin\delta_k, \cos\delta_k)$ and radius 1/2. The arc A_i^+ of C_i^+ from p_i to p_{i+1} has length $1/2(\sigma_i^+ - \sigma_{i+1}^+) = \alpha_{i+1}^+ - \alpha_i^+$. Clearly, the line segment $\overline{p_i p_{i+1}}$ is shorter than the arc A_i^+. Hence,

$$\begin{aligned} L_{i+1} &= L_i + d(p_i, p_{i+1}) \\ &\leq \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2}\right) d_i - \alpha_i d_i + \alpha_{i+1}^+ - \alpha_i^+ \end{aligned}$$

We want to show that

$$\left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_i\right) d_i + (\alpha_{i+1}^+ - \alpha_i^+) \quad \leq \quad \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_{i+1}\right) d_{i+1} \tag{8}$$

or

$$\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} \quad \geq \quad \frac{\alpha_{i+1} d_{i+1} - \alpha_i d_i + \alpha_{i+1}^+ - \alpha_i^+}{d_{i+1} - d_i}$$

with $\gamma_k - \pi/2 \le \alpha_i \le \alpha_{i+1} \le \pi/4 - \delta_k$. Let $\beta_i = \alpha_{i+1}^+ - \alpha_i^+$; hence, $\alpha_{i+1} = \pi/2 - \alpha_{i+1}^- = \alpha_{i+1}^+ + 2\delta_k = \alpha_i^+ + \beta_i + 2\delta_k$. If we now define

$$f(\alpha_i, \beta_i, \delta_k) = \frac{\beta_i + \frac{(\alpha_i^+ + \beta_i + 2\delta_k)\sin(\alpha_i^+ + \beta_i)}{\cos(\alpha_i^+ + \beta_i + 2\delta_k)} - \frac{(\alpha_i^+ + 2\delta_k)\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}}{\frac{\sin(\alpha_i^+ + \beta_i)}{\cos(\alpha_i^+ + \beta_i + 2\delta_k)} - \frac{\sin\alpha_i^+}{\cos(\alpha_i^+ + 2\delta_k)}},$$

then we want to prove that $f(\alpha, \beta, \delta) \le \gamma/2 + \cot(\gamma/2)$, where $\gamma = \pi/2 + 2\delta$, for all $(\alpha, \beta, \delta) \in \Delta = \{(x, y, z) \mid x \ge 0, y \ge 0, z \ge 0, x + y + z \le \pi/4\}$ since we assume that $\alpha_{i+1}^+ \le \alpha_{i+1}^-$, i.e. $\alpha_{i+1}^+ \le \pi/2 - \alpha_{i+1}^+ - 2\delta_k$ or $\alpha_i^+ + \beta_i + \delta_k \le \pi/4$.

As a first step we show that

$$\begin{aligned} \frac{\partial f}{\partial \alpha}(\alpha, \beta, \delta) = \quad & 1 - \beta\left(2\frac{\sin(2\alpha + \beta + 2\delta)}{\sin\beta} + \frac{1}{2}\frac{\cos(2\alpha + \beta + 2\delta)}{\sin\beta\cos 2\delta} + \frac{\sin(2\alpha + \beta)}{\sin\beta\cos 2\delta} + \right. \\ & \left. \frac{\cos(2\alpha + \beta)}{\sin\beta} - \frac{1}{2}\frac{\cos(2\alpha + \beta - 2\delta)}{\sin\beta\cos 2\delta}\right) \\ & \ge 0, \end{aligned}$$

for all $(\alpha, \beta, \delta) \in \Delta$. To see this consider

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial f}{\partial \alpha}(\alpha, \beta, \delta)\sin\beta\right) = \quad -2\frac{\beta}{\cos 2\delta}(\cos(2\alpha + \beta + 4\delta) + \sin(2\alpha + \beta + 2\delta)).$$

The equation

$$\cos(2\alpha + \beta + 4\delta) + \sin(2\alpha + \beta + 2\delta) = 0$$

has as only solution $2\alpha + \beta = -\pi/4 - 3\delta$ in the range $2\alpha + \beta \in [-\pi/2, \pi/2]$. Hence, for $(\alpha, \beta, \delta) \in \Delta$,

$$\frac{\partial}{\partial \alpha}\left(\frac{\partial f}{\partial \alpha}(\alpha, \beta, \delta)\sin\beta\right) \le 0$$

and the minimum value of $\frac{\partial f}{\partial \alpha}(\alpha, \beta, \delta)\sin\beta$ is achieved for the maximum value of α which is $\pi/4 - \beta - \delta$. If $\Delta' = \{(x, y) \mid x \ge 0, y \ge 0, x + y \le \pi/4\}$, then

$$\min_{(\alpha,\beta,\delta)\in\Delta} \frac{\partial f}{\partial \alpha}(\alpha, \beta, \delta) = \min_{(\beta,\delta)\in\Delta'} \frac{\partial f}{\partial \alpha}\left(\frac{\pi}{4} - \beta - \delta, \beta, \delta\right)$$

Let $f_\alpha(\beta, \delta) = \frac{\partial f}{\partial \alpha}\left(\frac{\pi}{4} - \beta - \delta, \beta, \delta\right)$. We want to show $f_\alpha(\beta, \delta) \ge 0$. To see this consider

$$\frac{\partial}{\partial \beta}(f_\alpha(\beta, \delta)\sin\beta\cos 2\delta) = \quad (1 - \sin 2\delta)(\sin\beta + \beta\cos\beta) + \beta\sin\beta\cos\delta \ge 0.$$

Hence, $f_\alpha(\beta, \delta)$ is monotone in β and

$$\begin{aligned} \min_{(\beta,\delta)\in\Delta'} f_\alpha(\beta, \delta) &= \lim_{\beta\to 0} f_\alpha(\beta, \delta) \\ &= \lim_{\beta\to 0} \frac{\beta}{4\sin\beta}\left(\frac{\sin(\beta - 2\delta)}{\cos 2\delta} - \frac{\cos(\beta + 4\delta)}{\cos 2\delta} + 2\sin\beta + 2\cos(\beta + 2\delta) + \right. \\ &\qquad \left. 3\frac{\cos\beta}{\cos 2\delta} - 3\frac{\sin\beta + 2\delta}{\cos 2\delta}\right) \\ &= 0. \end{aligned}$$

Thus, we have shown that f is monotone in α and, therefore,

$$\max_{(\alpha,\beta,\delta)\in\Delta} f(\alpha,\beta,\delta) = \max_{(\beta,\delta)\in\Delta'} f\left(\frac{\pi}{4}-\beta-\delta,\beta,\delta\right)$$

If we define $g(\beta,\delta) = f\left(\frac{\pi}{4}-\beta-\delta,\beta,\delta\right)$, then

$$\frac{\partial}{\partial\beta}\left(\frac{\partial g}{\partial\beta}(\beta,\delta)\frac{1}{2}\frac{1}{\cos 2\delta(\cos 2\beta-1)}\right) = \frac{1}{2}\sin(4\delta)\cos(2\beta) - \cos 2\delta\cos 2\beta - \frac{1}{2}\sin 4\delta + \cos 2\delta$$

which is equal to 0 if and only if $\cos(2\beta)=1$ or $\sin(4\delta)/2 = \cos 2\delta$, the latter of which only holds for $\delta=\pi/4$. Furthermore, since $0\le\beta\le\pi/4$, the former holds only if $\beta=0$. Therefore, we can see easily that $\frac{\partial}{\partial\beta}\left(\frac{\partial g}{\partial\beta}(\beta,\delta)\frac{1}{2}\frac{1}{\cos 2\delta(\cos 2\beta-1)}\right)\le 0$ and, thus, $\frac{\partial g}{\partial\beta}(\beta,\delta)/2(\cos 2\delta(\cos 2\beta-1))$ is monotonously decreasing in β.

It can be easily checked that

$$\lim_{\beta\to 0}\frac{\partial g}{\partial\beta}(\beta,\delta)=0$$

and, therefore, $\frac{\partial g}{\partial\beta}(\beta,\delta)\le 0$, for all $0\le\beta\le\pi/4-\delta$. This in turn implies that g is monotonously decreasing in β and

$$\begin{aligned}
\max_{(\alpha,\beta,\delta)\in\Delta} f(\alpha,\beta,\delta) &= \max_{\beta\in[0,\pi/4-\delta]} g(\beta,\delta) \\
&= \lim_{\beta\to 0}\frac{\pi}{4}+\delta+\frac{\beta}{2}+\frac{\beta\cos(\beta+2\delta)}{2\sin\delta}+\frac{\beta\cos\beta}{\sin\beta\cos 2\delta}-\frac{3\beta\sin(\beta+2\delta)}{4\sin\beta\cos 2\delta}+ \\
&\quad \frac{\beta\sin(\beta-2\delta)}{4\sin\beta\cos 2\delta}-\frac{\beta\cos(\beta+4\delta)}{4\sin\beta\cos 2\delta} \\
&= \frac{\pi}{4}+\delta+\cot\left(\frac{\pi}{4}+\delta\right) \\
&= \frac{\gamma}{2}+\cot\left(\frac{\gamma}{2}\right)
\end{aligned}$$

where $\gamma=\pi/2+2\delta$ as claimed.

Case 3 $\alpha_i^+\le\alpha_{i+1}^+$ and $\alpha_{i+1}^-\le\alpha_{i+1}^+$.
Hence, $d_{i+1}=d_{i+1}^+=1$ and $\alpha_{i+1}=\pi/2-\alpha_{i+1}^+$ (see Figure 17). Note that p_{i+1} is again on the circle C_i^+ with centre at $c_i^+=1/2(-\sin\delta,\cos\delta)$ and radius 1/2. As in Case 2 the arc A_i^+ of C_i^+ from p_i to p_{i+1} has length $1/2(\sigma_i^+-\sigma_{i+1}^+)=\alpha_{i+1}^+-\alpha_i^+$ and the line segment $\overline{p_ip_{i+1}}$ is shorter than the arc A_i^+. We obtain

$$L_{i+1} = L_i + d(p_i,p_{i+1}) \le \left(\frac{\gamma_k}{2}+\cot\frac{\gamma_k}{2}-\alpha_i\right)d_i^- + (\alpha_{i+1}^+-\alpha_i^+)$$

We split A_i^+ into two arcs A_i' and A_i'' where A_i' is the arc from p_i to the diagonal of the first quadrant and A_i' is the arc from the diagonal of the first quadrant to p_{i+1}. The arc A_i' is paid for by the

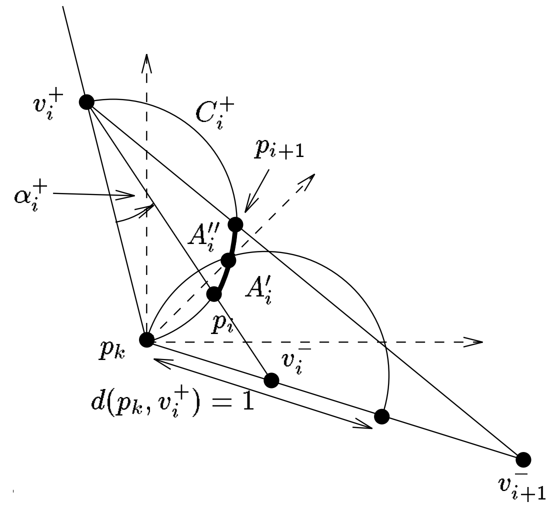

Figure 17: Case 3 if the robot moves from p_i to p_{i+1}.

increase $d_{i+1}^- - d_i^-$ while the arc A_i'' just reduces the potential. More precisely, we have

$$
\begin{aligned}
&\left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_i\right) d_i^- + (\alpha_{i+1}^+ - \alpha_i^+) = \\
= \;& \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_i\right) d_i^- + \left(\left(\frac{\pi}{4} - \delta_k\right) - \alpha_i^+\right) + \left(\alpha_{i+1}^+ - \left(\frac{\pi}{4} - \delta_k\right)\right) \\
\leq \;& \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \left(\frac{\pi}{2} - \left(\frac{\pi}{4} - \delta_k\right)\right)\right) \cdot 1 + \left(\alpha_{i+1}^+ - \frac{\pi}{4} + \delta_k\right) \\
= \;& \frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \frac{\pi}{4} - \delta_k + \alpha_{i+1}^+ - \frac{\pi}{4} + \delta_k \\
= \;& \left(\frac{\gamma_k}{2} + \cot\frac{\gamma_k}{2} - \alpha_{i+1}\right) d_{i+1}^+.
\end{aligned}
$$

Here, the last inequality can be easily seen if we set $\alpha_{i+1}^+ = \pi/4 - \delta_k$ in Inequality (8). We then have $\alpha_{i+1} = \pi/2 - (\pi/4 - \delta_k)$ since $\alpha_{i+1}^+ + \alpha_{i+1}^- + 2\delta_k = \pi/2$ and $\alpha_{i+1} = \pi/2 - \alpha_{i+1}^-$. This proves the claim.

In fact, we can show the following lemma.

Lemma 5.1 *For all $k \leq i \leq m$,*

$$
\frac{\gamma_k}{2} + \cot\left(\frac{\gamma_k}{2}\right) \geq \max\left\{\frac{L_i + d(p_i, v_i^+)}{d(p_k, v_i^+)}, \frac{L_i + d(p_i, v_i^-)}{d(p_k, v_i^-)}\right\}
$$

Proof: To see that $L_i + d(p_i, v_i^-) \leq (\gamma_k/2 + \cot(\gamma_k/2))d(p_0, v_i^-)$ we add a point v_{i+1}^+ on the y-axis at infinity. Then, $v_{i+1}^- = v_i^-$ and the line segment $\overline{v_{i+1}^- v_{i+1}^+}$ is parallel to the line segment

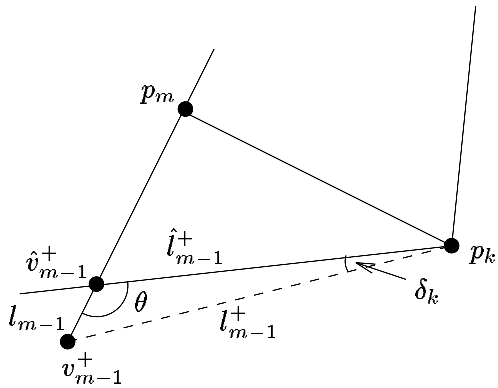

Figure 18: Bounding the final competitive ratio.

$\overline{p_k v_i^+}$; therefore, $p_{i+1} = v_i^-$. Since the functions considered in the above analysis are continuous, the analysis still holds and $L_i + d(p_i, v_i^-) = L_i + d(p_i, p_{i+1}) = L_{i+1} \leq (\gamma_k/2 + \cot(\gamma_k/2))d_{i+1} = (\gamma_k/2 + \cot(\gamma_k/2))d(p_0, v_i^-)$. An analogous construction yields the result for v_i^+. □

5.2 The Final Ratio

In order to obtain the final competitive ratio for one step we have to take into account that the robot has to move to either v^+_{m-1} or v^-_{m-1}. If v^-_m is undefined, then v^+_{m-1} belongs to the shortest path from s to g. Lemma 5.1 gives an upper bound on the maximum distance the robot travels in order to reach $\hat{v}^+_{m-1}$ in Figure 18 which is located on the line through p_k and v_k^+.

Let l_{m-1} be the line segment between v^+_{m-1} and $\hat{v}^+_{m-1}$ and θ the angle between $\hat{l}^+_{m-1}$ and l_{m-1}. The length of l^+_{m-1} grows monotonously with θ if the lengths of $\hat{l}^+_{m-1}$ of l_{m-1} are fixed. Hence, the maximum ratio of $(c\lambda(\hat{l}^+_{m-1}) + \lambda(l_{m-1}))/\lambda(l^+_{m-1})$ is achieved for the minimum angle θ which is $\theta = \gamma_k = \pi/2 + 2\delta_k$. Let the length of $\hat{l}^+_{m-1}$ be d_1 and the length of l_{m-1} be d_2. Hence, the maximum distance traveled by the robot from p_k to v^+_{m-1} is bounded by

$$F(\delta_k) = \max \frac{c(\delta_k)d_1 + d_2}{\sqrt{d_1^2 + d_2^2 - 2d_1 d_2 \cos(\frac{\pi}{2} + 2\delta_k)}}.$$

where $c(\delta) = \pi/4 + \delta + \cot(\pi/4 + \delta)$. This maximum is achieved at

$$d_2 = \max\{0, d_1 \frac{1 - c(\delta_k)\ sin2\delta_k}{c(\delta_k) - \sin 2\delta_k}\}$$

and yields a value of

$$F(\delta_k) = \frac{c(\delta_k) + \frac{1-c(\delta_k)\sin 2\delta_k}{c(\delta_k)-\sin 2\delta_k}}{\sqrt{1+\left(\frac{1-c(\delta_k)\sin 2\delta_k}{c(\delta_k)-\sin 2\delta_k}\right)^2 + 2\frac{(1-c(\delta_k)\sin 2\delta_k)\sin 2\delta_k}{c(\delta_k)-\sin 2\delta_k}}}.$$

The same analysis applies if v_m^+ is undefined.

If we combine this result with Lemma 4.2, we obtain the following upper bound on the distance traveled by the robot if the shortest path from s to g goes through v_{m-1}^+. Recall that $\mathcal{P}_i$ is the path the robot follows from point p_i to p_{i+1}, where we set $p_{m+1} = v_{m-1}^+$.

$$\begin{aligned}
\sum_{j=0}^{m} \lambda(\mathcal{P}_i) &= \sum_{j=0}^{k-1} \lambda(\mathcal{P}_i) + \sum_{j=k}^{m} d(p_j, p_{j+1}) \\
&\leq \frac{1}{\cos(\gamma_k/2)} \sum_{j=0}^{k} a_j^+ + F(\delta_k)\left(\lambda(\mathcal{V}_{m-1}^+) - \sum_{j=0}^{k} a_j^+\right) \\
&\leq \max\left\{\frac{1}{\cos(\gamma_k/2)}, F(\delta_k)\right\} \lambda(\mathcal{V}_{m-1}^+)
\end{aligned}$$

with $\gamma_k = \pi/2 + 2\delta_k$. Again the minimum competitive ratio is achieved if both the terms in the maximum are equal. This yields a value of 1.91 for γ_k and a competitive ratio of ~ 1.73. We state this result as a theorem.

Theorem 5.2 *If a robot uses the strategy* continuous lad *until the angle* $\gamma = \angle v^- p v^+$ *where* p *is the robot position and* v^+ *and* v^- *are the extreme entrance points of* $V(p)$ *equals* 1.91 *and then switches to the strategy* Move-in-Quadrant, *then the competitive ratio of the path traveled by the robot is at most* 1.73.

6 Conclusions

We have presented a number of strategies for a robot to search in a street if it is given the visibility map of its local surroundings. All the strategies presented follow that same "high level strategy" as outlined by Klein [9]. We use the framework of competitive analysis to show performance guarantees on our algorithms.

In the first part we introduce a family of strategies called *Walk-in-Circles*. The strategies are extremely simple and have an equally simple analysis. It also turns out that *Walk-in-Circles* is robust under small navigational errors.

In the second part we consider the strategy *continuous lad* that is very closely related to the strategy *lad* [9]. It employs the optimality criterion of minimising the local absolute detour. The resulting path the robot follows is concatenation of parts of hyperbolas. Though the path generated by the strategy is fairly complicated, its analysis turns out to be much simpler than the analysis of the strategy *lad* which uses the same optimality criterion by chooses to travel on straight line segments.

The strategy *continuous lad* has a relatively good competitive ratio of 2.03. Surprisingly, this strategy combined with the strategy *Move-in-Quadrant*—which has the best previously known competitive ratio of 2.05—results in a hybrid strategy with a competitive ratio of 1.73.

Often the idealistic assumption that a robot can follow a precomputed path without deviation is violated by real life robots. An interesting open problem is, therefore, if it is possible for a robot to traverse a scene with a predetermined maximal navigational error per unit traversed at a predetermined competitive ratio. Also the gap between the lower bound of $\sqrt{2}$ and the upper bound of 1.73 for search strategies in streets is still significant and needs to be improved.

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