THE SPARSE BASIS PROBLEM AND MULTILINEAR ALGEBRA*

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Abstract. Let A be a k by n underdetermined matrix. The sparse basis problem for the row space W of A is to find a basis of W with the fewest number of nonzeros. Suppose that all the entries of A are nonzero, and that they are algebraically independent over the rational number field. Then every nonzero vector in W has at least $n-k+1$ nonzero entries. Those vectors in W with exactly $n-k+1$ nonzero entries are the elementary vectors of W. A simple combinatorial condition that is both necessary and sufficient for a set of k elementary vectors of W to form a basis of W is presented here. A similar result holds for the null space of A where the elementary vectors now have exactly $k+1$ nonzero entries. These results follow from a theorem about nonzero minors of order m of the $(m-1)$st compound of an m by n matrix with algebraically independent entries, which is proved using multilinear algebra techniques. This combinatorial condition for linear independence is a first step towards the design of algorithms that compute sparse bases for the row and null space without imposing artificial structure constraints to ensure linear independence.

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1. Introduction. Many situations in computational linear algebra and numerical optimization require the computation of a sparse basis for the row space or the null space of a sparse, underdetermined matrix A. The *sparse row space basis problem* (hereafter the row space problem) is to compute a basis for the row space of A with the fewest number of nonzeros. Similarly, the *sparse null space basis problem* (hereafter the null space problem) is to compute a basis for the null space of A with the fewest number of nonzeros. It turns out that both these problems are computationally intractable: they are NP-hard [1, 8, 9]. Under a nondegeneracy assumption called the *matching property*, Hoffman and McCormick [5, 8] designed polynomial time algorithms to solve the row space problem. Sparsest null bases can be characterized by means of a matroid greedy algorithm [1, 9], yet the null space problem turned out to be harder than the row space problem; heuristic algorithms to compute sparse null bases were designed and implemented in [2, 4].

All algorithms known to us for computing sparse null bases have two components:

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a method to compute a sparse vector in the null space of the given matrix, and a mechanism for ensuring linear independence when previously computed null vectors are augmented with the new null vector. To keep the time complexity of null basis algorithms low, the latter is achieved by insisting that the null basis be a trapezoidal matrix, that is, a matrix of the form $\begin{bmatrix} B_1 & L \end{bmatrix}$ where L is either an identity matrix or a lower triangular matrix with nonzero diagonal elements. However, this might be a severe restriction on the structure of the null basis since there may be sparser null bases that are not trapezoidal.

The fundamental question that we consider is the following: Given an underdetermined matrix A whose nonzero elements are algebraically independent, is there a combinatorial condition that characterizes a set of linearly independent vectors in the row space (or null space) of A? By a combinatorial condition we mean a condition that uses only the zero-nonzero structure of the set of vectors. This question was raised as an unsolved problem in [9].[1] A solution to this problem will enable us to design algorithms for computing sparse bases for the row and null space without imposing artificial structure constraints to ensure linear independence.

Since we are concerned only with sparse bases, we can restrict our attention to *elementary vectors* of the subspace (Fulkerson [3], Rockafellar [10], and Tutte [12]). (This restriction is necessary to obtain a nontrivial solution of the problem.) Accordingly we now turn to a discussion of elementary vectors. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in the n-dimensional real vector space $\mathbf{R}^n$. The *support* of x is the subset of $\{1, 2, \ldots, n\}$ given by $\operatorname{supp}(x) = \{i : x_i \neq 0\}$. Now let W be a subspace of dimension k of $\mathbf{R}^n$. An *elementary vector* of W is a nonzero vector of W whose support is minimal, that is, does not properly contain the support of any nonzero vector of W. It is easy to verify that two elementary vectors of W with the same support are scalar multiples of each other and hence, up to scalar multiples, W has only finitely many elementary vectors. It is also easy to verify that the elementary vectors of W span W. It follows that a sparsest basis of W contains only elementary vectors. Thus it is natural to look for a basis of W among its elementary vectors.

Hence a more precise statement of the problem is to combinatorially characterize a set of linearly independent elementary vectors in the row space or the null space of an underdetermined matrix whose nonzero elements are algebraically independent. This problem turns out to be quite difficult, since the set of supports of the elementary vectors of a subspace W can have an intricate structure. However, we now consider a situation in which the set of supports of the elementary vectors has a simple structure, and in this case, we provide a combinatorial characterization of linear independence.

[1] We thank Steve Vavasis for rekindling our interest in this problem by raising it during the open problem session at the IMA workshop on Sparse Matrix Computations in October 1991.

Our proof of this result uses techniques from multilinear algebra.

Let A be a k by n matrix that is *nondegenerate* in the sense that every submatrix of A of order k is nonsingular. Then the support of each elementary vector in the row space of A has cardinality $n-k+1$ and each subset of $\{1,2,\ldots,n\}$ of cardinality $n-k+1$ is the support of some elementary vector (see the next section). Similarly the support of each vector in the null space of W has cardinality $k+1$ and each subset of $\{1,2,\ldots,n\}$ of cardinality $k+1$ is the support of some elementary vector. Even in the restrictive case in which W is the row space or null space of a nondegenerate matrix, it seems difficult to determine if a set of elementary vectors of W is linearly independent. The linear independence of elementary vectors of such subspaces W does not in general depend only on the supports of the elementary vectors. Thus we need a more restrictive assumption than nondegeneracy.

A k by n matrix A is *generic* if all of its kn elements are nonzero, and they form an algebraically independent set over the rational number field $\mathbf{Q}$. If A is generic over $\mathbf{Q}$, then obviously every submatrix of A of order k has a nonzero determinant. Hence, generic matrices are nondegenerate.

In this paper we identify a necessary and sufficient condition that must be satisfied by the supports of the elementary vectors of the row space (respectively, null space) of a generic matrix in order that the elementary vectors be linearly independent. This condition leads to a polynomial algorithm for determining whether a set of elementary vectors in one of these two subspaces is a basis.

Let $\mathcal{J}=\{J_1,J_2,\ldots,J_t\}$ be t subsets of $\{1,2,\ldots,n\}$ each of cardinality $m-1$. Then $\mathcal{J}$ satisfies the *m-intersection property* provided

$$|\cap_{i\in P} J_i| \le m-|P| \quad (\forall P\subseteq\{1,2,\ldots,t\}, P\neq\emptyset). \tag{1.1}$$

The main results of this paper, as they apply to the row space and null space problems, are the following two theorems.

THEOREM 1.1. *Let A be a k by n matrix that is generic over $\mathbf{Q}$, and $\{I_1,I_2,\ldots,I_t\}$ denote a collection of $t\le k$ subsets of $\{1,2,\ldots,n\}$ each of cardinality $n-k+1$. Then the elementary vectors $x(I_1),x(I_2),\ldots,x(I_t)$ with supports $I_1,I_2,\ldots,I_t$, respectively, of the row space of A are linearly independent if and only if the set $\{\overline{I}_1,\overline{I}_2,\ldots,\overline{I}_t\}$ consisting of the complements of their supports satisfies the k-intersection property, that is,*

$$|\cap_{i\in P} \overline{I}_i| \le k-|P| \quad (\forall P\subseteq\{1,2,\ldots,t\}, P\neq\emptyset).$$

THEOREM 1.2. *Let A be a k by n matrix that is generic over $\mathbf{Q}$ and $\{I_1,I_2,\ldots,I_t\}$ denote a collection of $t\le n-k$ subsets of $\{1,2,\ldots,n\}$ each of cardinality $k+1$. Then the* *elementary vectors* $y(I_1), y(I_2), \ldots, y(I_t)$ *with supports* $I_1, I_2, \ldots, I_t$*, respectively, of the null space of* A *are linearly independent if and only if the set* $\{\overline{I}_1, \overline{I}_2, \ldots, \overline{I}_t\}$ *consisting of the complements of their supports satisfies the* $(n-k)$*-intersection property, that is,*

$$|\cap_{i \in P} \overline{I}_i| \leq n - k - |P| \quad (\forall P \subseteq \{1, 2, \ldots, t\}, P \neq \emptyset).$$

The combinatorial conditions given in these two theorems can be used to test the linear independence of a set of elementary vectors in polynomial time. We now show how this can be accomplished for the row space.

Let P be a nonempty subset of $\{1, \ldots, k\}$. The condition in Theorem 1.1 can be restated as

$$|\cup_{i \in P} I_i| \geq n - k + |P|,$$

since $|\cap_{i \in P} \overline{I_i}| + |\cup_{i \in P} I_i| = n$. Without loss of generality, assume that the rows in P are numbered $P = \{1, \ldots, p\}$. The last inequality yields

$$|\cup_{i \in P \setminus \{p\}} I_i \setminus I_p| \geq |P| - 1 \quad (\forall P \subseteq \{1, 2, \ldots, p\}, p \in P).$$

If we let X denote the k by n matrix with rows $x(I_1), x(I_2), \ldots, x(I_k)$, then this is the set of Philip Hall conditions for the submatrix $X[\{1, \ldots, p-1\}, \overline{I_p}]$ to have a row-perfect matching.

We can use the above condition to test the linear independence of a set of elementary vectors in the row space when a partial basis of $p-1$ rows is augmented by a newly computed row p. We assume inductively that the partial basis satisfies the k-intersection property. Now when the p-th row is added to the partial basis, we check whether the submatrix in the preceding paragraph has a row-perfect matching. If it does, then clearly every set $P' \subseteq P$ that includes p satisfies the k-intersection property. Also, every set $P' \subseteq P$ that does not include p satisfies the k-intersection property by the inductive hypothesis. Hence the k-intersection property for row space bases can be checked by solving k maximum matching problems. The matchings can be computed in $\mathcal{O}(k^{1.5}e)$ time, where e is the number of nonzeros in the sparse row basis.

Theorems 1.1 and 1.2 are consequences of a theorem (Theorem 2.1) about compound matrices, and we briefly review this matrix construction. Let X be a p by q matrix and let r be a positive integer with $r \leq p, q$. Let $\mathcal{S}_{r,p}$ denote the sequence of all subsets of $\{1, 2, \ldots, p\}$ of cardinality r ordered lexicographically. Similarly, let $\mathcal{S}_{r,q}$ denote the sequence of all subsets of $\{1, 2, \ldots, q\}$ of cardinality r ordered lexicographically. The r^{th}-*compound* of X is the $\binom{p}{r}$ by $\binom{q}{r}$ matrix $C_r(X)$ with rows indexed by $\mathcal{S}_{r,p}$ and columns indexed by $\mathcal{S}_{r,q}$ whose entry in the position corresponding to K in $\mathcal{S}_{r,p}$ and

L in $\mathcal{S}_{r,q}$ is the determinant $\det X[K, L]$ of the submatrix of X with row indices in K and column indices in L. An important fact about compounds is that the multiplicative property $C_r(XY) = C_r(X)C_r(Y)$ holds. In particular, if X is a square nonsingular matrix of order n and $Y = X^{-1}$, then $C_r(X)C_r(X^{-1}) = C_r(I_n) = I_N$, where $N \equiv \binom{n}{r}$, and hence $C_r(X)$ is nonsingular. Notice that if X is a square matrix of order n, then $C_{n-1}(X)$ is, up to multiplication of some of its rows and columns by -1, the adjoint of X.

The rest of this paper is organized as follows. In Section 2, first we show that the problem of linear independence of a set of elementary vectors (of the row space and null space) of a k by n nondegenerate matrix A is equivalent to the problem of determining whether the determinant of a certain submatrix of the $(k-1)$th compound matrix $C_{k-1}(A)$ of A is not zero. The entries of $C_{k-1}(A)$ are the determinants of all the submatrices of A of order $k-1$ arranged in lexicographical order of their set of row indices and of their set of column indices. If the determinant of this submatrix of $C_{k-1}(A)$ is nonzero, then we show that k-intersection property must be satisfied. However, to prove the converse for generic matrices, we must show that the k-intersection property implies that this determinant is not identically zero. Since the determinant of a submatrix of $C_{k-1}(A)$ is an expression involving determinants of submatrices of A of order $k-1$, we are faced with the task of showing that it is *not* a determinantal identity.[2] We conclude Section 2 by stating our main result (Theorem 2.1) about compound matrices. In Section 3 we discuss certain concepts in multilinear algebra, namely, tensor spaces and exterior vector spaces that are needed to obtain our results. In Section 4 we state our main theorem (Theorem 4.1) in multilinear algebra. In Section 5 we apply this theorem to prove Theorem 2.1. In Section 6 we give the proof of the main theorem. In Section 7 we make a few concluding remarks and state a conjecture.

2. Elementary vectors and matrix compounds. Let A be a k by n nondegenerate, real matrix and let W be the row space of A. Then each elementary vector of W contains exactly $k-1$ zeros and $n-k+1$ nonzeros. Moreover, given any subset I of $\{1, 2, \ldots, n\}$ of cardinality $n-k+1$ there is an elementary vector $x(I)$ of W whose support equals I. The nonzero coordinates of the vector $x(I)$ are given by

$$x(I)_j = (-1)^{p_j+1} \det A[:, \overline{I} \cup \{j\}] \quad (j \in I), \tag{2.1}$$

[2] One could argue that our task would have been a lot simpler if we had only to verify that a certain expression involving determinants of submatrices of A was a determinantal identity, that is, was equal to zero no matter what real values were substituted for the indeterminate entries of A. To show that an expression is not a determinant identity, one has to verify that one can choose real values for the indeterminate entries in order that the expression is not zero. One cannot expect to be able to construct these real values but only to show that they must exist.

where p_j equals the number of integers r in $\overline{I}$ that are less than j. Here $\overline{I}$ is the complement of I in $\{1, 2, \ldots, n\}$ and $A[:, \overline{I} \cup \{j\}]$ denotes the full-rowed submatrix of A of order k determined by the columns indexed by the integers in $\overline{I} \cup \{j\}$. To see that this defines a vector in the row space of A whose support is I, we expand the determinant in (2.1) by column j of A and obtain

$$x(I)_j = \sum_{i=1}^{k} (-1)^i \det A[\bar{i}, \overline{I}] a_{ij} \quad (j \in I) \tag{2.2}$$

where $\bar{i}$ denotes the complement of $\{i\}$ in $\{1, 2, \ldots, k\}$ and $A[\bar{i}, \overline{I}]$ is the submatrix of A determined by the rows and columns indexed by the integers in $\bar{i}$ and $\overline{I}$, respectively. For j in I, $x(I)_j$ is a linear combination of the elements in column j of A by (2.2). For j in $\overline{I}$, $x(I)_j$ is zero by (2.1), since it is the determinant of a matrix in which column j of A occurs twice. Thus $x(I)$ is a linear combination of the rows of A and hence belongs to the row space of A.

Let $x(I_1), x(I_2), \ldots, x(I_t)$ be t elementary vectors of W. For each vector $x(I_j)$ there exists a unique vector $y(I_j)$ in $\mathbf{R}^k$ such that

$$x(I_j) = y(I_j)A.$$

Moreover, since the rank of A is k, $x(I_1), x(I_2), \ldots, x(I_t)$ are linearly independent vectors in $\mathbf{R}^n$ if and only $y(I_1), y(I_2), \ldots, y(I_t)$ are linearly independent vectors in $\mathbf{R}^k$. Since $x(I_j)_i = 0$ for i in $\overline{I_j}$, the vector $y(I_j)$ is the unique (up to scalar multiples) nontrivial solution z in $\mathbf{R}^k$ of the $k-1$ equations

$$zA[:, \overline{I_j}] = 0.$$

Thus by Cramer's rule

$$y(I_j)_i = (-1)^i \det A[\bar{i}, \overline{I_j}] \quad (i = 1, 2, \ldots, k) \tag{2.3}$$

where, as before, $\bar{i}$ is the complement of $\{i\}$ in $\{1, 2, \ldots, k\}$. Hence

$$\begin{bmatrix} y(I_1)^T & y(I_2)^T & \cdots & y(I_t)^T \end{bmatrix}$$

is a k by t submatrix of the $(k-1)$st compound $C_{k-1}(A)$ of A. (More precisely, it is a k by t submatrix of $C_{k-1}(A)$ with row i multiplied by $(-1)^i$ for $i = 1, 2, \ldots, k$.) Note that $C_{k-1}(A)$ is a k by $\binom{n}{k-1}$ matrix. Summarizing, we have the following:

(I) The elementary vectors $x(I_1), x(I_2), \ldots, x(I_t)$ of the row space W of the k by n nondegenerate matrix A are linearly independent if and only if the k by t submatrix $C_{k-1}(A)[:, \{\overline{I_1}, \overline{I_2}, \ldots, \overline{I_t}\}]$ of $C_{k-1}(A)$ determined by its columns indexed by $\overline{I_1}, \overline{I_2}, \ldots, \overline{I_t}$ has rank equal to t. Equivalently, the elementary vectors $x(I_1), x(I_2), \ldots, x(I_t)$ are linearly independent if and only if not all of the determinants

$$\det C_{k-1}(A)[\{\overline{i_1}, \overline{i_2}, \ldots, \overline{i_t}\}, \{\overline{I_1}, \overline{I_2}, \ldots, \overline{I_t}\},$$
$$(1 \leq i_1 < i_2 < \cdots < i_t \leq k)$$

vanish.

If we assume that the matrix A is generic over $\mathbf{Q}$, then by taking $t = k$ we see that the problem of determining whether a set of k elementary vectors of the subspace W (the row space of the k by n generic matrix A over $\mathbf{Q}$) is a basis of W is equivalent to the problem of determining whether the determinant of a submatrix of order k of the $(k-1)$st compound $C_{k-1}(A)$ does not vanish identically (that is, is *not* an identity satisfied by the determinants of the submatrices of order $k-1$ of k by n real matrices).

Considerations similar to the above apply to the null space U of the matrix A. Assume again that A is nondegenerate. Then the supports of elementary vectors of U are exactly the subsets I of $\{1, 2, \ldots, n\}$ of cardinality $k+1$. Indeed by Cramer's rule again, it follows that for each subset I of $\{1, 2, \ldots, n\}$ of cardinality $k+1$ the elementary vector $y(I)$ of U with support I satisfies

$$y(I)_i = (-1)^i \det A[:, I \setminus \{i\}] \quad (i \in I).$$

Let $y(I_1), y(I_2), \ldots, y(I_t)$ be t elementary vectors of U. There exists an $n-k$ by n matrix B with rank equal to $n-k$ such that the row space of B equals U. Suppose that some submatrix of B of order $n-k$ has a zero determinant. Then after elementary row operations we may assume that some row of B has at least $n-k$ zeros. Since $AB^T = O$ this implies that some set of k columns of A is linearly dependent contradicting the nondegeneracy of A. We conclude that the matrix B is also nondegenerate. Let $z(I)$ be the unique vector in $\mathbf{R}^{n-k}$ such that $y(I) = z(I)B$. The vectors $y(I_1), y(I_2), \ldots, y(I_t)$ are linearly independent if and only if $z(I_1), z(I_2), \ldots, z(I_t)$ are linearly independent. The vector $z(I_j)$ is the unique (up to scalar multiples) nontrivial solution v of

$$vB[:, \overline{I_j}] = 0.$$

Using Cramer's rule as above we conclude the following:

(II) The elementary vectors $y(I_1), y(I_2), \ldots, y(I_t)$ of the null space U of the k by n nondegenerate matrix A are linearly independent if and only if the $n-k$ by t submatrix of $C_{n-k-1}(B)$ determined by its columns indexed by $\{\overline{I_1}, \overline{I_2}, \ldots, \overline{I_t}\}$ has rank equal to t. Equivalently, the elementary vectors $y(I_1), y(I_2), \dots, y(I_t)$ are linearly independent if and only if not all of the determinants

$$\det C_{n-k-1}(B)[\{\overline{i_1}, \overline{i_2}, \dots, \overline{i_t}\}, \{\overline{I_1}, \overline{I_2}, \dots, \overline{I_t}\}],$$
$$(1 \le i_1 < i_2 < \cdots < i_t \le n-k)$$

vanish.

If A is generic over $\mathbf{Q}$, then by taking $t = n-k$ we see that the problem of determining whether a set of $n-k$ elementary vectors of the null space U of A is a basis of U is equivalent to the problem of determining whether the determinant of a full-rowed submatrix of order $n-k$ of the $(n-k-1)$st compound of the matrix B does not vanish identically.

Now let A denote an m by n real matrix. Let $J_1, J_2, \dots, J_t$ be $t \le m$ subsets of $\{1, 2, \dots, n\}$ each of cardinality $m-1$. We consider the m by t (full-rowed) submatrix

$$C_{m-1}(A)[:, \{J_1, J_2, \dots, J_t\}] \tag{2.4}$$

of the $(m-1)$st compound of A. If for some $i \ne j$ we have $J_i = J_j$, then two columns of (2.4) are identical and hence the matrix has linearly dependent columns. If $t > m$ then (2.4) has more columns than rows and hence has linearly dependent columns. We generalize these observations by showing that if the t sets $J_1, J_2, \dots, J_t$ do not satisfy the m-intersection property, then the columns of (2.4) are linearly dependent.

Assume that $p \ge 2$ of the sets, say $J_1, J_2, \dots, J_p$, satisfy

$$J_1 \cap J_2 \cap \cdots \cap J_p = J \text{ where } |J| = q \ge m-p+1.$$

First suppose that the columns of A with index in J are linearly dependent. Then the matrix $A[:, J_1]$ has linearly dependent columns and hence its rank is at most $m-2$. We may multiply A with nonsingular matrices corresponding to elementary row operations without changing linearly independent sets of columns of A. By the multiplicative property of compounds, the same observation can be made for compound matrices of A. Hence we may assume that the last two rows of $A[:, J_1]$ are zero rows. This implies that the column of $C_{m-1}(A)$ with index J_1 is a zero column and hence (2.4) has linearly dependent columns.

Now suppose that the columns of A with index in J are linearly independent. Using the multiplicative property of compounds again we may assume that

$$A = \left[\begin{array}{c|c} I_q & X \\ \hline O & F \end{array}\right]$$

where I_q is the identity matrix of order q, O is an $m-q$ by q zero matrix and F is an $m-q$ by $n-q$ matrix. Let Z be the m by p submatrix of (2.4) corresponding to the index sets $J_1, J_2, \ldots, J_p$. Let $J_i' = J_i \setminus J$ $(i = 1, 2, \ldots, p)$. The submatrix of Z determined by its last $m-q$ rows equals

$$C_{m-1}(A)[\{\overline{q+1}, \ldots, \overline{m}\}, \{J_1, J_2, \ldots, J_p\}] = C_{m-q-1}(F)[:, \{J_1', J_2', \ldots, J_p'\}].$$

By the Laplace expansion for determinants along a set of rows it follows that for each j between 1 and q, the row of Z indexed by $\overline{j}$ is a linear combination of its last $m-q$ rows. Hence the rank of Z is at most

$$m - q \leq m - (m - p + 1) = p - 1.$$

Thus the columns of Z, and hence the columns of (2.4), are linearly dependent if $J_1, J_2, \ldots, J_t$ do not satisfy the m-intersection property.

Our main result about compound matrices asserts that for generic matrices, the converse holds as well.

THEOREM 2.1. *Let A be a m by n matrix that is generic over* $\mathbf{Q}$*. Let $J_1, J_2, \ldots, J_t$ be t subsets of $\{1, 2, \ldots, n\}$ each of cardinality $m-1$. Then the rank of the m by t submatrix of the $(m-1)$st compound $C_{m-1}(A)$ given by*

$$C_{m-1}(A)[:, \{J_1, J_2, \ldots, J_t\}] \tag{2.5}$$

equals t if and only if $J_1, J_2, \ldots, J_t$ satisfy the m-intersection property.

In the next section we discuss the multilinear algebra that we use in order to show that if A is generic over $\mathbf{Q}$ and $J_1, J_2, \ldots, J_t$ are subsets of cardinality $m-1$ that satisfy the m-intersection property

$$| \cap_{i \in P} J_i | \leq m - |P| \quad (P \subseteq \{1, 2, \ldots, t\}) \tag{2.6}$$

then the columns of (2.5) are linearly independent.

Theorem 2.1 is proved in Section 5.

3. Tensor and exterior spaces. We refer the reader to Marcus ([6] and [7]) for the basic multilinear algebra discussed in this section. As already pointed out, our task is made more complicated by the fact that we have to show that a certain expression is not a determinantal identity. The multilinear algebra is needed (apparently) to show the existence of certain numbers without actually being able to construct them.

Let W be a n-dimensional vector space over $\mathbf{R}$. The *tensor product* of W with itself is the n^2-dimensional real vector space $W \otimes W$ spanned by the decomposable tensors $x \otimes y$ with x and y in W. The tensor product is an abstract algebraic construction. If W equals $\mathbf{R}^n$, and $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ are vectors in W, then a concrete realization of $x \otimes y$ is the outer product $x^T y$. In this case $W \otimes W$ is the vector space spanned by the outer products of vectors in W.

The m^{th} *tensor power* of W is the n^m-dimensional real vector space

$$\otimes^m W = W \otimes \cdots \otimes W \ (m\ W\text{'s})$$

spanned by all of the decomposable tensors $w_1 \otimes \cdots \otimes w_m$ where $\{w_1, \ldots, w_m\} \subseteq W$. The essential facts to keep in mind about the tensor power $\otimes^m W$ are:

(1) the map

$$(w_1, \ldots, w_m) \to w_1 \otimes \cdots \otimes w_m$$

is multilinear: for instance,

$$(cw_1' + dw_1'') \otimes w_2 \cdots \otimes w_m = c(w_1' \otimes w_2 \otimes \cdots w_m) + d(w_1'' \otimes w_2 \otimes \cdots \otimes w_m)$$

for all real numbers c and d and all vectors $w_1', w_1'', w_2, \ldots, w_m$ in W,

(2) $c(w_1 \otimes w_2 \otimes \cdots \otimes w_m) = (cw_1) \otimes w_2 \otimes \cdots \otimes w_m = \cdots = w_1 \otimes w_2 \otimes \cdots \otimes (cw_m)$ for all real numbers c and all vectors $w_1, w_2, \ldots, w_m$ in W, and

(3) if $\{x_1, x_2, \ldots, x_n\}$ is a basis of W then the set of n^m vectors

$$\{x_{i_1} \otimes \cdots \otimes x_{i_m} : 1 \leq i_1, \ldots, i_m \leq n\}$$

is a basis of $\otimes^m W$.

An inner product $(\cdot, \cdot)$ on W induces an inner product on $\otimes^m W$ by defining

$$(w_1 \otimes \cdots \otimes w_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} (w_i, v_i)$$

and extending linearly.[3]

The *wedge product* of vectors $w_1, \ldots, w_m$ is the element of $\otimes^m W$ defined by

$$w_1 \wedge \cdots \wedge w_m = \sum_{\sigma} \operatorname{sign}(\sigma) w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(m)}$$

where the summation extends over all permutations σ of $\{1, 2, \ldots, m\}$ and $\operatorname{sign}(\sigma)$ is $+1$ if σ is an even permutation and -1 otherwise. If $w_1, w_2, \ldots w_m$ are the row vectors of an m by n matrix B, then $C_m(B)$ is a concrete realization of $w_1 \wedge \cdots \wedge w_m$. The subspace of $\otimes^m W$ spanned by all the wedge products of m vectors of W is the mth *exterior space*[4] over W and is denoted by $\wedge^m W$. The essential facts to keep in mind about the exterior space $\wedge^m W$ are:

[3] All of this applies to the complex number field provided we use a unitary inner product.

[4] It is also called the mth *Grassmann space* over W and the mth *skew-symmetric space* over W.

(i) if $\{y_1, y_2, \ldots, y_n\}$ is a basis of W then the set of vectors

$$\{y_{i_1} \wedge \cdots \wedge y_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}$$

is a basis of $\wedge^m W$ (in particular, these vectors are linearly independent) and

$$\dim \wedge^m W = \binom{n}{m},$$

(ii) $w_1 \wedge \cdots \wedge w_m = 0$ if and only if the vectors $w_1, \ldots, w_m$ are linearly dependent, and

(iii) if U is a subspace of W of dimension m with a basis $u_1, \ldots, u_m$, then $\{u_1 \wedge \cdots \wedge u_m\}$ is the subspace $\wedge^m U$ of $\wedge^m W$ of dimension 1.

Using the definition of wedge product we calculate that the induced inner product on the exterior space $\wedge^m W$ satisfies

$$(3.1) \qquad \begin{aligned} (u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_m) &= m!(u_1 \otimes \cdots \otimes u_m, v_1 \wedge \cdots \wedge v_m) \\ &= m! \det \begin{bmatrix} (u_1, v_1) & \cdots & (u_1, v_m) \\ \vdots & \ddots & \vdots \\ (u_m, v_1) & \cdots & (u_m, v_m) \end{bmatrix}. \end{aligned}$$

Hereafter we shall denote any matrix of the form as the one appearing in (3.1) by specifying its (i,j)th element:

$$\Big[\ (u_i, v_j)\ \Big] \quad (\text{for } i, j = 1, \ldots, m).$$

If U and V are two subspaces of W of dimension m, then it follows from (ii) and (iii) that for bases $\{u_1, \ldots, u_m\}$ of U and $\{v_1, \ldots, v_m\}$ of V, whether or not $(u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_m)$ equals zero is independent of the choice of the bases $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ of V. *For convenience we denote any of these inner products* $(u_1 \wedge \cdots \wedge u_m, v_1 \wedge \cdots \wedge v_m)$ *by* $[U, V]$. The *orthogonal complement* of a subspace V of W is denoted by $V^\perp$.

LEMMA 3.1. *Let U and V be subspaces of W of dimension m. Then the following are equivalent:*

(*a*) $[U, V] \neq 0$,

(*b*) $U^\perp \cap V = \{0\}$,

(*c*) $U \cap V^\perp = \{0\}$.

Proof. Let $u_1, \ldots, u_m$ be a basis of U. If there were a nonzero vector v_1 in $U^\perp \cap V$, then extending v_1 to a basis $v_1, \ldots, v_m$ of V we see that the determinant in (3.1) is zero and hence $[U,V]=0$. Therefore (a) implies (b). Now assume that (b) holds and consider the vector space $U_i = \langle u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_m \rangle$ spanned by all but the ith basis vector u_i. It follows from (b) that $\dim U_i^{\perp} \cap V = 1$, since we can obtain a vector in this subspace by subtracting appropriate multiples of the vectors in a basis of U_i from u_i. Let v_i be any nonzero vector in $U_i^{\perp} \cap V$. By (b) again we conclude that $(u_i, v_i) \neq 0$. We can repeat this argment to conclude that $(u_i, v_i) \neq 0$, for $i = 1, \ldots, m$. Hence the determinant in (3.1), $\prod_{j=1}^{m}(u_j, v_j) \neq 0$ and thus (a) holds. Since $[U,V]=[V,U]$, (b) and (c) are equivalent and the lemma follows. □

4. A theorem in multilinear algebra. We now formulate a theorem concerning exterior spaces that enables us to solve our original problems concerning bases for the row space and null space of a generic matrix. In the next section we show how this theorem and a combinatorial lemma can be used to prove Theorems 1.1 and 1.2. In the final section we prove the multilinear algebra theorem. It will be convenient to use the language of projective geometry and algebraic varieties to describe the theorem.

We obtain an equivalence relation on points in $\mathbf{R}^{N+1}$ by defining two points $x = (x_0, \ldots, x_N)$ and $x' = (x_0', \ldots, x_N')$ to be *equivalent* if there is a real constant λ such that $x = \lambda x'$. Then N-dimensional projective space over the real field $\mathbf{P}^N(\mathbf{R})$ is the set of equivalence classes of this relation on $\mathbf{R}^{N+1} \setminus \{0\}$, and $(x_0, \ldots, x_N)$ are the homogeneous coordinates of x. Note that the projective dimension is one less than the number of coordinates.

Let

$$U_0 = \{p : p = (x_0, \ldots, x_N) \in \mathbf{P}^N(\mathbf{R}) \text{ and } x_0 \neq 0\}.$$

Then the map Φ taking $(x_1, \ldots, x_N) \in \mathbf{R}^N$ to $(1, x_1, \ldots, x_N) \in \mathbf{P}^N(\mathbf{R})$ is a one-to-one correspondence between $\mathbf{R}^N$ and U_0 because given $p = (x_0, \ldots, x_N) \in U_0$, we can multiply by $(1/x_0)$ to obtain an equivalent point and then compute the inverse map from U_0 to $\mathbf{R}^N$. Thus we can identify U_0 with $\mathbf{R}^N$. If $H = \{p \neq 0 : p = (0, x_1, \ldots, x_N)\}$ 'the hyperplane at infinity', then N-dimensional projective space has the representation $\mathbf{P}^N(\mathbf{R}) = U_0 \cup H$, i.e., it consists of $\mathbf{R}^N$ augmented with the hyperplane at infinity.

A *variety* is the solution set of a system of multivariate polynomials $p_1=0, \ldots, p_s = 0$ in the variables $x_0, \ldots, x_N$. It is a *projective variety* if each p_i is a homogeneous polynomial, i.e., each term in p_i has the same total degree.

Let W be an inner product space of dimension n over $\mathbf{R}$. Let m be an integer with $1 \leq m \leq n$. The set of all subspaces X of W of dimension m are the points of a projective variety $\mathcal{W}_m$. Choose an m by n matrix E whose rows form a basis of X, and consider the map $X \mapsto C_m(E)$ that maps the subspace X to the set of $\binom{n}{m}$ determinants of all submatrices of order m of E. This is a well-defined, injective map from the set of m-dimensional subspaces of W to real projective space $\mathcal{P}$ of (projective) dimension $\binom{n}{m}-1$. The $\binom{n}{m}$ homogeneous coordinates are called the *Plücker coordinates* of X, and they satisfy certain quadratic relations called the *Plücker relations*. If we choose another matrix F whose rows form a basis of X, then the effect is to multiply the *Plücker coordinates* of X by a common nonzero scale factor. The projective variety $\mathcal{W}_m$ consists of all points that satisfy the *Plücker relations*, and is known as the *Grassmann variety*.

A *subvariety* of $\mathcal{W}_m$ is a variety that is a nonempty subset of the subspaces in $\mathcal{W}_m$. A subvariety of $\mathcal{W}_m$ is *proper* provided that it does not contain at least one subspace of W.

Let X denote a subspace of W of dimension m. By property (i) of exterior spaces, $\wedge^{m-1}X$ is a subspace of $\otimes^{m-1}W$ of dimension m. By property (ii) $\wedge^m(\wedge^{m-1}X)$ is a subspace of $\otimes^{m(m-1)}W$ of dimension 1. Let $U_1, U_2, \ldots, U_m$ be m subspaces of W of dimension $m-1$. Then each $\wedge^{m-1}U_i$ is a subspace of $\otimes^{m-1}W$ of dimension one, and $(\wedge^{m-1}U_1) \wedge (\wedge^{m-1}U_2) \wedge \cdots \wedge (\wedge^{m-1}U_m)$ is a subspace of $\otimes^{m(m-1)}W$ of dimension zero or one. The subspaces $U_1, U_2, \ldots, U_m$ satisfy the *dimension m-intersection property* provided that

$$\dim \cap_{i \in P} U_i \leq m - |P| \quad (\forall P \subseteq \{1, 2, \ldots, m\}, P \neq \emptyset) \tag{4.1}$$

Clearly the dimension m-intersection property is the analogue for subspaces of the m-intersection property for subsets.

We now come to the main theorem, the proof of which is given in the final section.

THEOREM 4.1. *Let W be an inner product space over $\mathbf{R}$ of dimension n, let m be an integer with $2 \leq m \leq n$, and let $U_1, U_2, \ldots, U_m$ be m subspaces of W of dimension $m-1$. Define $\mathcal{W}_m(U_1, U_2, \ldots, U_m)$ to be the set of all subspaces X of W of dimension m satisfying*

$$[\wedge^m(\wedge^{m-1}X), (\wedge^{m-1}U_1) \wedge (\wedge^{m-1}U_2) \wedge \cdots \wedge (\wedge^{m-1}U_m)] = 0. \tag{4.2}$$

Then $\mathcal{W}_m(U_1, U_2, \ldots, U_m)$ is a proper subvariety of $\mathcal{W}_m$ if and only if U_1, …, U_m satisfy the dimension m-intersection property.

In other words, the theorem states that there exists an m-dimensional subspace X of W for which (4.2) is not satisfied if and only if U_1, …, U_m satisfy the dimension m-intersection property.

Let X have a basis $x_1, \ldots, x_m$, and for $i = 1, \ldots, m$, let X_i be a subspace of X spanned by x_1, …, x_{i-1}, x_{i+1}, …, x_m. Then making use of (3.1), we can express the inner product in (4.2) as

$$\begin{aligned} &[(\wedge^{m-1}X_1) \wedge (\wedge^{m-1}X_2) \wedge \cdots \wedge (\wedge^{m-1}X_m), \\ &\qquad (\wedge^{m-1}U_1) \wedge (\wedge^{m-1}U_2) \wedge \cdots \wedge (\wedge^{m-1}U_m)] \\ &= \det\Big[\ [X_i, U_j]\ \Big] \quad (\text{for } i, j = 1, \ldots, m). \end{aligned} \tag{4.3}$$

We will make use of this representation of the inner product in the remaining sections of the paper.

5. Proofs of Theorems 1.1,1.2 and 2.1. Before applying Theorem 4.1 to compound matrices, we prove the following lemma that may be of interest in its own right.

LEMMA 5.1. *Let $I_1, I_2, \ldots, I_t$ be $t < m$ subsets of $\{1,2,\ldots,n\}$ each of cardinality $m-1$, and assume that the m-intersection property*

$$| \cap_{i \in P} I_i | \leq m - |P| \tag{5.1}$$

holds for all nonempty subsets P of $\{1,2,\ldots,t\}$. Then there exist $m-t$ subsets $I_{t+1},\ldots,I_m$ of $\{1,2,\ldots,n\}$ of cardinality $m-1$ such that (5.1) *holds for all nonempty subsets P of $\{1,2,\ldots,m\}$.*

Proof. It suffices to show that there exists a subset I_{t+1} of $\{1,2,\ldots,n\}$ of cardinality $m-1$ such that (5.1) holds for all nonempty subsets P of $\{1,2,\ldots,t+1\}$. If $|\cap_{i\in P} I_i| < m - |P|$ for all subsets P of $\{1,2,\ldots,t\}$ with $|P| \geq 2$, then we may choose I_{t+1} to be any subset of $\{1,2,\ldots,n\}$ of cardinality $m-1$ different from $I_1, I_2, \ldots, I_t$.

Hence consider the situation when there exists a subset P with $|P| \geq 2$ that satisfies the m-intersection property (5.1) as an equality. We show that then $\{1,2,\ldots,t\}$ can be partitioned into maximal subsets that satisfy (5.1) as equalities.

Let P and Q be two *nondisjoint* subsets of $\{1,2,\ldots,t\}$ satisfying $|\cap_{i\in P} I_i| = m-|P|$ and $|\cap_{i\in Q} I_i| = m-|Q|$, respectively. Write $X \equiv \cap_{i\in P} I_i$ and $Y \equiv \cap_{i\in Q} I_i$. Then applying the identity $|X \cap Y| = |X| + |Y| - |X \cup Y|$ we obtain

$$| \cap_{i\in P\cup Q} I_i| = | \cap_{i\in P} I_i| + | \cap_{i\in Q} I_i| - |(\cap_{i\in P} I_i) \cup (\cap_{i\in Q} I_i)|.$$

Since $\cap_{i\in P} I_i, \cap_{i\in Q} I_i \subseteq \cap_{i\in P\cap Q} I_i$, we see that

$$(\cap_{i\in P} I_i) \cup (\cap_{i\in Q} I_i) \subseteq \cap_{i\in P\cap Q} I_i.$$

Putting it all together, we obtain

$$\begin{aligned} m - |P\cup Q| &\geq | \cap_{i\in P\cup Q} I_i| \\ &= | \cap_{i\in P} I_i| + | \cap_{i\in Q} I_i| - |(\cap_{i\in P} I_i) \cup (\cap_{i\in Q} I_i)| \\ &= m - |P| + m - |Q| - |(\cap_{i\in P} I_i) \cup (\cup_{i\in Q} I_i)| \\ &\geq m - |P| + m - |Q| - |(\cap_{i\in P\cap Q} I_i)| \\ &\geq m - |P| + m - |Q| - (m - |P\cap Q|) \\ &= m - |P \cup Q|. \end{aligned}$$

Therefore

$$| \cap_{i\in P\cup Q} I_i| = m - |P\cup Q|.$$

It follows that there exists a partition $\{P_1, P_2, \ldots, P_\ell\}$ of $\{1, 2, \ldots, t\}$ into $\ell \geq 1$ sets such that (5.1) holds with equality for each P_i and

$$| \cap_{i \in Q} I_i| = m - |Q| \text{ implies that } Q \subseteq P_i \text{ for some } i.$$

We proceed to show how the set I_{t+1} may be chosen in this situation. Let x be any element of $\cap_{i \in P_1} I_1$, and choose I_{t+1} to be any subset of $m-1$ elements of $\{1, 2, \ldots, n\}$ such that

$$I_{t+1} \cap (\cap_{i \in P_1} I_i) = \cap_{i \in P_1} I_i \setminus \{x\}.$$

Since $|I_{t+1}| = m-1$ and by the choice of I_{t+1} we have $|I_{t+1} \cap (\cap_{i \in P_1} I_i)| = m - |P_1| - 1$, I_{t+1} contains exactly $|P_1|$ elements not in $\cap_{i \in P_1} I_i$. To prove that (5.1) holds for all nonempty subsets P of $\{1, 2, \ldots, t+1\}$, it suffices to show that for each nonempty subset Q of $\{1, 2, \ldots, t\}$ for which $| \cap_{i \in Q} I_i| = m - |Q|$, we have

$$\cap_{i \in Q} I_i \not\subseteq I_{t+1}. \tag{5.2}$$

Case 1: $Q \subseteq P_1$. Then

$$x \in \cap_{i \in P_1} I_i \subseteq \cap_{i \in Q} I_i \text{ and } x \notin I_{t+1}$$

imply that (5.2) holds.
Case 2: $Q \subseteq P_j$ for some $j \neq 1$. Then using (5.1) and the fact that P_1 is maximal with respect to the property that $| \cap_{i \in P_1} I_i| = m - |P_1|$, we obtain

$$q := |(\cap_{i \in P_1} I_i) \cap (\cap_{i \in Q} I_i| = | \cap_{i \in P_1 \cup Q} I_i| \leq m - |P_1| - |Q| - 1.$$

Hence

$$|(\cap_{i \in Q} I_i) \setminus (\cap_{i \in P_1} I_i)| = m - |Q| - q \geq |P_1| + 1.$$

Now by construction, I_{t+1} contains exactly $|P_1|$ elements not in $\cap_{i \in P_1} I_i$. Since $\cap_{i \in Q} I_i$ contains at least $|P_1| + 1$ elements not in $\cap_{i \in P_1} I_i$, there exists an element y in $\cap_{i \in Q} I_i$ that is not an element of I_{t+1}. This completes the proof. □

Proof of Theorem 2.1: In Section 2 we showed that the m-intersection property is a necessary condition for the matrix (2.5) to have full row rank. Now suppose that the m-intersection property holds. It follows from Lemma 5.1 that it suffices to prove that the rank of the matrix (2.5) equals m when $t = m$. Thus assume that $t = m$, that is, that (2.5) is a square matrix. Since the entries of A are algebraically independent over $\mathbf{Q}$ and since the determinant of the matrix (2.5) is a polynomial in the entries of

A with integer coefficients, it suffices to show that this determinant is not identically zero. Let $e_1, e_2, \ldots, e_n$ be the standard basis of $\mathbf{R}^n$, and let U_k denote the subspace spanned by $\{e_i : i \in J_k\}$ $(k = 1, 2, \ldots, t)$. We shall write the standard basis of U_k as $\{e_1^k, \ldots, e_{m-1}^k\}$. Since $\{J_1, J_2, \ldots, J_m\}$ satisfy the m-intersection property, it follows easily that $\{U_1, U_2, \ldots, U_m\}$ satisfy the dimension m-intersection property. By Theorem 4.1 there exists a subspace X of $\mathbf{R}^n$ of dimension m such that (4.2) does not hold.

Let B be an m by n matrix whose rows $x_1, \ldots, x_m$ form a basis of X. Now

$$\begin{aligned} &[X_i, U_k] \\ &= \left(x_1 \wedge \cdots \wedge x_{i-1} \wedge x_{i+1} \wedge \cdots \wedge x_m, e_1^k \wedge \cdots \wedge e_{m-1}^k\right) \\ &= \det\left[\ (x_j, e_\ell^k)\ \right] \quad (\text{for } j = 1, \ldots, i-1, i+1, \ldots, m, \quad \ell = 1, \ldots, m-1) \\ &= C_{m-1}(B)[\bar{i}, J_k]. \end{aligned}$$

Hence from (4.2) and (4.3), we have

$$\begin{aligned} &[(\wedge^{m-1}X_1) \wedge (\wedge^{m-1}X_2) \wedge \cdots \wedge (\wedge^{m-1}X_m), \\ &\qquad (\wedge^{m-1}U_1) \wedge (\wedge^{m-1}U_2) \wedge \cdots \wedge (\wedge^{m-1}U_m)] \\ &= \det\left[\ [X_i, U_j]\ \right] \quad (\text{for } i, j = 1, \ldots, m) \\ &= \det C_{m-1}(B)[:, \{J_1, J_2, \ldots, J_m\}] \ \neq 0. \end{aligned}$$

□

Proofs of Theorems 1.1 and 1.2: The proof of Theorem 1.1 follows immediately from Theorem 2.1 and the calculations of Section 2. The necessity of the $(n-k)$-intersection property for the linear independence of the elementary vectors of the null space of A, $y(I_1), y(I_2), \ldots, y(I_t)$, is an immediate consequence of the calculations of Section 2.

An argument is needed to derive the converse of Theorem 1.2 from Theorem 2.1, since the assumption that the matrix A is generic does not imply that the matrix B (defined in Section 2), whose row space is the null space of A, is generic. But we shall overcome this by first choosing a generic B and then defining A.

Assume first only that A is a nondegenerate matrix and the sets $\{\overline{I}_1, \overline{I}_2, \ldots, \overline{I}_t\}$ satisfy the $(n-k)$-intersection property. Since the entries of each elementary vector are polynomials in the entries of A, it follows that the elementary vectors in the null space of A, $y(I_1), y(I_2), \ldots, y(I_t)$, are linearly dependent if and only if the determinantal polynomial vanishes identically for every submatrix of order t of the t by n matrix Y formed by these elementary vectors. The theorem follows if we can show that there exists at least one nondegenerate k by n matrix A of rank k for which $y(I_1), y(I_2), \ldots, y(I_t)$ are linearly independent, for then at least one of these determinantal polynomials does not vanish identically. Let B be an $n-k$ by n generic matrix. Let $x(I_1), x(I_2), \ldots, x(I_t)$ be elementary vectors of the row space of B with supports $I_1, I_2, \ldots, I_t$, respectively. Choose A to be any k by n matrix of rank k such that $AB^T = O$. Since $BA^T = O$ the arguments in Section 2 show that A is nondegenerate. Since the $(n-k)$-intersection property holds, we conclude from Theorem 1.1 that $x(I_1), x(I_2), \ldots, x(I_t)$ are linearly independent elementary vectors in the row space of B. We now take the vectors $x(I_1), x(I_2), \ldots, x(I_t)$ as the elementary vectors $y(I_1), y(I_2), \ldots, y(I_t)$ in the null space of A. This completes the proof. □

6. Proof of the Main Theorem. In this section we give the proof of Theorem 4.1. The following two elementary lemmas, used in our proof, concern vector spaces generated by certain operations on subspaces of a vector space, and we review these operations now. If V_1 and V_2 are subspaces of a finite dimensional vector space W, then their *union*

$$V_1 \cup V_2 = \{v : v \in V_1\} \cup \{v : v \in V_2\}$$

is in general not a vector space, since it is not necessarily closed under vector addition. The *sum*

$$V_1 + V_2 = \{v_1 + v_2 : v_1 \in V_1, v_2 \in V_2\},$$

and *intersection*

$$V_1 \cap V_2 = \{v : v \in V_1 \cap V_2\},$$

are vector spaces, and it is easy to verify that the sum is the smallest vector space that contains the vectors in $V_1 \cup V_2$.

LEMMA 6.1. *Let k be a positive integer and let $V, V_1, \ldots, V_k$ be subspaces of a finite dimensional vector space W over $\mathbf{R}$. Then $V \subseteq V_1 \cup \cdots \cup V_k$ if and only if $V \subseteq V_i$ for some i.*

Proof. Let $V_i' = V_i \cap V$ for $i = 1, \ldots, k$. Then each V_i' is a subspace of V. If each V_i' is a proper subspace of V, then $V \setminus \cup_{i=1}^k V_i = V \setminus \cup_{i=1}^k V_i'$ is a set of positive Lebesgue measure of dimension $\dim V$. □

LEMMA 6.2. *Let $k \geq 2$ be an integer and let $V_1, \ldots, V_k$ be subspaces of a finite dimensional inner product space W over $\mathbf{R}$. Then*

$$\cap_{i=1}^k V_i = (V_1^\perp + \cdots + V_k^\perp)^\perp.$$

Proof. First suppose that $k=2$. Then the proof follows by choosing an orthonormal basis B_{12} of $V_1 \cap V_2$, extending to orthonormal bases $B_{12} \cup B_1$ of V_1 and $B_{12} \cup B_2$ of V_2, and then extending to an orthonormal basis $B_{12} \cup B_1 \cup B_2 \cup B$ of W. Then $B \cup B_1 \cup B_2$ is an orthonormal basis of $V_1^\perp + V_2^\perp$, and it follows that B_{12} is an orthonormal basis of $(V_1^\perp + V_2^\perp)^\perp$. We now assume that $k>2$ and use induction on k. Using the inductive assumption twice, we obtain

$$\cap_{i=1}^k V_i = (\cap_{i=1}^{k-1} V_i) \cap V_k = ((\cap_{i=1}^{k-1} V_i)^\perp + V_k^\perp)^\perp = ((V_1^\perp + \cdots V_{k-1}^\perp) + V_k^\perp)^\perp.$$

□

Proof of Theorem 4.1: Let $U_1, U_2, \ldots, U_m$ be m subspaces of W of dimension $m-1$. Then $\mathcal{W}_m(U_1, U_2, \ldots, U_m)$ is clearly a subvariety of $\mathcal{W}_m$. Thus the theorem is only concerned with whether or not it equals $\mathcal{W}_m$.

This proof is technically the most demanding part of the paper, and hence we provide a sketch of our proof technique before we embark on proving the theorem. The necessity of the dimension intersection property is the easier part of the proof. We use dimension counting arguments to show that certain subspaces occurring in the determinantal representation (4.3) of the inner product (4.2) have nontrivial intersection, leading to a large zero submatrix that makes the determinant zero. Sufficiency is harder, and is proved by induction on m, by showing that when the dimension intersection property is satisfied there exists a subspace X of dimension m, constructed using $U_1^\perp$, …, $U_m^\perp$, such that (4.2) does not hold.

First assume that the dimension m-intersection property (4.1) does not hold. Without loss of generality, assume that $V \equiv \cap_{i=1}^p U_i$ satisfies

$$\dim V = m - p + 1 \tag{6.1}$$

where p is an integer with $2 \le p \le m$. Let X be any subspace of W of dimension m. We have $\dim X^\perp = n - m$, and by (6.1), $\dim V^\perp = n - m + p - 1$; hence there exists a subspace F, contained in both $V^\perp$ and X, of dimension $p-1$. Choose a set $\{x_1, x_2, \ldots, x_{p-1}\}$ of $p-1$ linearly independent vectors spanning F. For $j = 1, 2, \ldots, p$, by Lemma 6.2

$$U_j^\perp \subseteq (U_1^\perp + \cdots + U_p^\perp) = V^\perp.$$

Since F and $U_j^\perp$ are subspaces of $V^\perp$, and

$$\dim F + \dim U_j^\perp = (p-1) + (n-m+1) = n - m + p > \dim V^\perp = n - m + p - 1,$$

we have $F \cap U_j^\perp \neq \{0\}$ for each $j = 1, 2, \ldots, p$. We extend $x_1, x_2, \ldots, x_{p-1}$ to a basis $x_1, x_2, \ldots, x_m$ of X and let X_i be the subspace of X with basis $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$ $(i = 1, 2, \ldots, m)$. By the above it follows that

$$X_i \cap U_j^\perp \neq \{0\} \quad (i = p, p+1, \ldots m; j = 1, 2, \ldots, p),$$

since such subspaces X_i contain F and $F \cap U_j^\perp \neq \{0\}$. Hence by Lemma 3.1

$$[X_i, U_j] = 0 \quad (i = p, p+1, \ldots, m; j = 1, 2, \ldots, p).$$

Therefore the matrix in (4.3) whose (i,j)-entry equals $[X_i, U_j]$ $(i, j = 1, 2, \ldots, m)$ has an $m - p + 1$ by p zero submatrix with $(m - p + 1) + p = m + 1$, and it follows from the Frobenius-König theorem that its determinant equals zero. This implies that (4.2) holds for every subspace X of W of dimension m and hence $\mathcal{W}_m(U_1, U_2, \ldots, U_m) = \mathcal{W}_m$.

Now we prove sufficiency of the dimension intersection property. Assume that $U_1, U_2, \ldots, U_m$ are subspaces of W of dimension $m-1$ satisfying the dimension m-intersection property (4.1); in particular, no two of $U_1, U_2, \ldots, U_m$ are equal. We prove by induction on m that there exists a subspace X of W of dimension m for which (4.2) does not hold.

First we consider the base case $m = 2$. Then U_1 and U_2 are distinct subspaces of W of dimension 1 and we choose X to be the subspace of dimension 2 spanned by u_1 and u_2 where u_1 is a basis for U_1 and u_2 is a basis for U_2. Then $u_1 \wedge u_2 \in (\wedge^2(\wedge^1 X)) \cap ((\wedge^1 U_1) \wedge (\wedge^1 U_2))$ and $(u_1 \wedge u_2, u_1 \wedge u_2) \neq 0$. Hence (4.2) does not hold.

Now suppose that $m > 2$. If U is a subspace of W of dimension $m-1$, then we define a subvariety $\mathcal{F}(U)$ of $\mathcal{W}_m$ by

$$\mathcal{F}(U) = \{X : X \in \mathcal{W}_m, \dim X \cap U^\perp \geq 2\}.$$

Let X be a subspace in $\mathcal{W}_m \setminus \mathcal{F}(U)$. Since $\dim X = m$ and $\dim U^\perp = n - m + 1$, by the choice of X we have $\dim X \cap U^\perp = 1$. Thus the subspace $X^* = X \cap (X \cap U^\perp)^\perp$ is in $\mathcal{W}_{m-1}$. The map

$$\phi_U : \mathcal{W}_m \setminus \mathcal{F}(U) \to \mathcal{W}_{m-1}, \text{ where } \phi_U(X) = X^*,$$

is a rational map.

We proceed to construct a subspace X of dimension m, $m-1$ subspaces of X of dimension $m-2$, and $m-1$ subspaces U_i' of dimension $m-2$ to set up the inductive step in the proof.

Since $U_1, U_2, \ldots, U_m$ are distinct subspaces of the same dimension $m-1$, it follows from Lemma 6.1 that

$$U_m^\perp \not\subseteq U_1^\perp \cup U_2^\perp \cup \cdots \cup U_{m-1}^\perp.$$

Let x_m be any vector satisfying

$$x_m \in U_m^\perp \setminus (U_1^\perp \cup \cdots \cup U_{m-1}^\perp). \tag{6.2}$$

Let X be a subspace in $\mathcal{W}_m \setminus \mathcal{F}(U_m)$ containing x_m and let $\{x_1, \ldots, x_{m-1}, x_m\}$ be an orthogonal basis of X containing x_m. Let X_i be the subspace of X that is spanned by $\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\}$, $(i = 1, 2, \ldots, m)$. We have $X_m = \phi_{U_m}(X)$. Since the vector x_m belongs to $U_m^\perp$ and X_1, …, X_{m-1}, by Lemma 3.1 we have

$$[X_i, U_m] = 0 \quad (i = 1, 2, \ldots, m-1). \tag{6.3}$$

Further, since $\dim X \cap U_m^\perp = 1$,

$$[X_m, U_m] \neq 0. \tag{6.4}$$

By (4.3), (4.2) is not zero if and only if

$$\det \Big[\, [X_i, U_j] \, \Big] \quad (\text{for } i, j = 1, \ldots, m)$$

is nonzero, and hence by (6.3) and (6.4), if and only if

$$\det \Big[\, [X_i, U_j] \, \Big] \quad (\text{for } i, j = 1, \ldots, m-1)$$

is nonzero. Thus (4.2) does not hold if and only if

$$\begin{aligned} &[(\wedge^{m-1} X_1) \wedge (\wedge^{m-1} X_2) \wedge \cdots \wedge (\wedge^{m-1} X_{m-1}), \\ &\quad (\wedge^{m-1} U_1) \wedge (\wedge^{m-1} U_2) \wedge \cdots \wedge (\wedge^{m-1} U_{m-1})] \neq 0. \end{aligned} \tag{6.5}$$

We now reduce the dimensions of $U_1, \ldots, U_{m-1}$ by one in order to apply the inductive assumption.

By (6.2), x_m does not belong to the subspaces $U_1^\perp$, …, $U_{m-1}^\perp$, and hence for each $i = 1, 2, \ldots, m-1$, there exists a basis $\{u_1^i, u_2^i, \ldots, u_{m-1}^i\}$ of U_i with

$$(x_m, u_j^i) = 0 \ (j = 1, 2, \ldots, m-2) \text{ and } (x_m, u_{m-1}^i) = 1. \tag{6.6}$$

Let subspaces of W be defined by

$$U_i' = U_i \cap \{x_m\}^\perp \quad (i = 1, 2, \ldots, m-1).$$

By (6.2) we have

$$\dim U_i' = m - 2 \quad (i = 1, 2, \ldots, m-1).$$

We now use the bases of X_i and U_j, Lemma 3.1, and the determinantal formula in (3.1) to compute $[X_i, U_j]$ for $i, j = 1, 2, \ldots, m-1$. Let X_i' be the subspace of X_i with basis $\{x_j : j = 1, \ldots, i-1, i+1, \ldots, m-1\}, (i = 1, 2, \ldots, m-1)$. Using the Laplace expansion of the determinant in (3.1) by the last row (which is the vector $(0, \ldots, 0, 1)$ by (6.6)), we see that each $[X_i, U_j] = m[X_i', U_j']$. Hence (6.5) equals

$$\begin{aligned}[(\wedge^{m-2}X_1') \wedge (\wedge^{m-2}X_2') \wedge \cdots \wedge (\wedge^{m-2}X_{m-1}'),\\ (\wedge^{m-2}U_1') \wedge (\wedge^{m-2}U_2') \wedge \cdots \wedge (\wedge^{m-2}U_{m-1}')].\end{aligned} \tag{6.7}$$

It now follows that (4.2) is not identically zero provided (6.7) is not zero. By the induction hypothesis (6.7) is not zero provided $U_1', U_2', \ldots, U_{m-1}'$ satisfy the dimension $(m-1)$-intersection property. Our proof will be complete if we show that these subspaces satisfy the required dimension intersection property for some choice of x_m.

Assume to the contrary that for any admissible choice of x_m in $U_m^\perp \setminus (U_1^\perp \cup \cdots \cup U_{m-1}^\perp$ there exists an integer k with $2 \le k \le m-1$ and a subset of $\{1, 2, \ldots, m-1\}$ (both depending on x_m) of cardinality k, say the subset $\{1, 2, \ldots, k\}$, such that

$$\dim \cap_{i=1}^k U_i' \ge (m-1) - k + 1 = m - k. \tag{6.8}$$

Since $\cap_{i=1}^k U_i' \subseteq \cap_{i=1}^k U_i$, we have

$$\dim \cap_{i=1}^k U_i \ge m - k,$$

and since $U_1, U_2, \ldots, U_m$ satisfy the dimension m-intersection property, we have

$$\dim \cap_{i=1}^k U_i = m - k. \tag{6.9}$$

Hence there exists a set $Z \subseteq U_m^\perp \setminus (U_1^\perp \cup \cdots \cup U_{m-1}^\perp)$ of positive Lebesgue measure in $U_m^\perp$ such that

$$\cap_{i=1}^k U_i = \cap_{i=1}^k U_i'. \tag{6.10}$$

We now show that (6.10) leads to a contradiction of the dimension m-intersection property (4.1).

If (6.10) holds for all $x_m \in Z$, we claim that

$$U_m^\perp \subseteq U_1^\perp + \cdots + U_k^\perp. \tag{6.11}$$

(Note that now we are considering the *sums* of the vector spaces, and not the *unions* considered in (6.2).) The proof of the claim is also by contradiction. If the claim were not true, then $(U_1^\perp + \cdots + U_k^\perp) \cap U_m^\perp$ is a proper subspace of $U_m^\perp$ and hence we may choose the vector $x_m \in Z$ in (6.2) so that x_m is in $U_m^\perp \setminus (U_1^\perp + \cdots + U_k^\perp)$. Let V be the subspace of W spanned by x_m. Then using the definitions of the subspaces U_i', we have

$$\cap_{i=1}^k U_i' = V^\perp \cap (\cap_{i=1}^k U_i) = (V + U_1^\perp + \cdots + U_k^\perp)^\perp \subset (U_1^\perp + \cdots + U_k^\perp)^\perp = \cap_{i=1}^k U_i,$$

where we have used Lemma 6.2 twice. The containment relation we have obtained contradicts (6.10). We conclude that (6.11) is true whenever (6.10) holds.

Writing (6.11) in the form

$$U_m^\perp \subseteq U_1^\perp + \cdots + U_k^\perp = (\cap_{i=1}^k U_i)^\perp,$$

we find

$$\cap_{i=1}^k U_i \subseteq U_m.$$

Therefore

$$U_m \cap (\cap_{i=1}^k U_i) = \cap_{i=1}^k U_i. \tag{6.12}$$

But now (6.9) and (6.12) contradict the dimension m-intersection property (4.1). This completes the inductive proof of sufficiency and the proof of the theorem. □

7. Coda. Theorem 4.1 implies a sufficient condition for a collection of vectors in the wedge product of a vector space to be linearly independent.

COROLLARY 7.1. *Let W be an inner product space over $\mathbf{R}$ of dimension n and let m be an integer with $2 \leq m \leq n$. If $U_1, U_2, \ldots, U_m$ are m subspaces of W of dimension $m-1$ which satisfy the dimension m-intersection property, then $\wedge^{m-1}U_1$, $\wedge^{m-1}U_2$, ..., $\wedge^{m-1}U_m$ as vectors in $\wedge^{m-1}W$ are linearly independent.*

Proof. Assume that $U_1, U_2, \ldots, U_m$ are subspaces of W of dimension $m-1$ satisfying the dimension m-intersection property. Recall that each $\wedge^{m-1}U_i$ is a subspace of $\wedge^{m-1}W$ of dimension 1 and thus can be regarded as a nonzero vector of $\wedge^{m-1}W$. It follows from Theorem 4.1 that there exists a choice of subspaces $X_1, X_2, \ldots, X_m$ of W of dimension $m-1$ such that

$$[\wedge^{m-1}X_1 \wedge \wedge^{m-1}X_2 \wedge \cdots \wedge \wedge^{m-1}X_m, \wedge^{m-1}U_1 \wedge \wedge^{m-1}U_2 \wedge \cdots \wedge \wedge^{m-1}U_m] \neq 0.$$

Let $\chi_i = \wedge^{m-1}X_i$ and $\omega_i = \wedge^{m-1}U_i$ for $i = 1, 2, \ldots, m$. It follows from (3.1) and elementary column operations that if $\omega_1, \omega_2, \cdots, \omega_m$ are linearly dependent, then $[\chi_1 \wedge \chi_2 \wedge \cdots \wedge \chi_m, \omega_1 \wedge \omega_2 \wedge \cdots \omega_m] = 0$. Hence $\omega_1, \omega_2, \cdots, \omega_m$ are linearly independent. □

We remark that the converse of Corollary 7.1 is not true in general. For example, let $n = 4$ and $m = 3$ and let e_1, e_2, e_3, e_4 be the standard basis of $W = \mathbf{R}^4$.

Also let U_1, U_2 and U_3 be the subspaces of W spanned by $\{e_1, e_4\}, \{e_2, e_4\}$ and $\{e_3, e_4\}$, respectively. Then U_1, U_2, U_3 do not satisfy the dimension 3-intersection property, since $U_1 \cap U_2 \cap U_3 \neq \{0\}$. Using the concrete realization of the wedge product, we see that $\wedge^2 U_1, \wedge^2 U_2$ and $\wedge^2\, U_3$ are spanned by $e_1 \wedge e_4 = (0,0,1,0,0,0), e_2 \wedge e_4 = (0,0,0,0,1,0)$ and $e_3 \wedge e_4 = (0,0,0,0,0,1)$, respectively, and hence are linearly independent.

However the converse of Corollary 7.1 is true if $m = n$.

COROLLARY 7.2. *Let $U_1, U_2, \ldots, U_t$ be subspaces of $\mathbf{R}^m$ of dimension $m-1$. Then $\wedge^{m-1}U_1, \wedge^{m-1}U_2, \ldots, \wedge^{m-1}U_t$ are linearly independent if and only if $U_1, U_2, \ldots, U_t$ satisfy the dimension m-intersection property.*

Proof. Let $J \subseteq \{1,2,\ldots,t\}$. Since $U_1, U_2, \ldots, U_t$ are $(m-1)$-dimensional subspaces of an m-dimensional space, the dimension of $\cap_{j\in J} U_j$ is at least $m - |J|$. By Lemma 6.2,

$$(\cap_{j\in J} U_j)^{\perp} = (\sum_{j\in J} U_j^{\perp})^{\perp}.$$

Hence $\dim(\cap_{j\in J} U_j) \leq m - |J|$ (and so equals $m - |J|$) if and only if the vectors $\wedge^{m-1}U_1, \wedge^{m-1}U_2, \ldots, \wedge^{m-1}U_t$ are linearly independent. □

The following lemma identifies the supports of the elementary vectors of an arbitrary subspace (taken as the row space of a matrix) of $\mathbf{R}^n$.

LEMMA 7.3. *Let A be an m by n real matrix of rank m. Let I be a subset of $\{1,2,\ldots,n\}$. Then there exists an elementary vector of the row space of A with support I if and only if (i) the rank of $A[:,\overline{I}]$ equals $m-1$, and (ii) the rank of $A[:,\overline{I}\cup\{j\}]$ equals m for each $j \in I$.*

Proof. First assume that there is an elementary vector $x(I)$ with support I. If the rank of $A[:,\overline{I}]$ equals m, then any linear combination of the rows of A that vanishes on $\overline{I}$ is a trivial linear combination. If the rank of $A[:,\overline{I}\cup\{j\}]$ is less than m for some $j \in I$, then there is a nontrivial linear combination of the rows of A which vanishes on $\overline{I}\cup\{j\}$ and hence $x(I)$ is not an elementary vector. Assertions (i) and (ii) now follow.

Now assume that (i) and (ii) hold. Let K be any subset of $\overline{I}$ of cardinality $m-1$ such that the rank of $A[:,K]$ equals $m-1$. Then with $\overline{I}$ replaced by K, (2.1) defines an elementary vector $x(I)$ in the row space of A with support I. □

We now give a criterion for a set of elementary vectors of a subspace of $\mathbf{R}^n$ (again taken as the row space of a matrix) to be a basis.

THEOREM 7.4. *Let A be an m by n real matrix of rank m and let $x(I_1)$, $x(I_2)$, …, $x(I_m)$ be elementary vectors in the row space W of A. Let U_j be the subspace of $\mathbf{R}^m$ spanned by the columns of $A[:,\overline{I_j}]$ $(j = 1,2,\ldots,m)$. Then $\{x(I_1), x(I_2), \ldots, x(I_m)\}$ is a basis of W if and only if $U_1, U_2, \ldots, U_m$ satisfy the dimension m-intersection property.*

Proof. By Lemma 7.3 each of the subspaces U_j has dimension $m-1$. For each $j = 1,2,\ldots,m$ there exists a vector $y(I_j)$ such that $x(I_j) = y(I_j)A$. The vectors

$x(I_1), x(I_2), \ldots, x(I_m)$ are linearly independent if and only if $y(I_1), y(I_2), \ldots, y(I_m)$ are. By Lemma 7.3, there exists $K_j \subseteq \overline{I_j}$ such that $|K_j| = m-1$ and the rank of $A[:, K_j]$ equals $m-1$. We can identify the vector $y(I_j)$ with the vector $\wedge^{m-1} U_j$ (cf. (2.3)). If $y(I_1), y(I_2), \ldots, y(I_m)$ are linearly dependent then $\wedge^{m-1} U_1 \wedge \cdots \wedge \wedge^{m-1} U_m = 0$ and hence by Theorem 4.1, $U_1, U_2, \ldots, U_m$ do not satisfy the dimension m-intersection property.

Conversely, suppose that $U_1, U_2, \ldots, U_m$ do not satisfy the dimension m-intersection property. Then as remarked in the proof of Theorem 4.1 we have

$$[\wedge^{m-1} U_1 \wedge \cdots \wedge \wedge^{m-1} U_m, \wedge^{m-1} U_1 \wedge \cdots \wedge \wedge^{m-1} U_m] = 0$$

and hence $\wedge^{m-1} U_1, \cdots, \wedge^{m-1} U_m$ are linearly dependent. □

In the case that A is generic over $\mathbf{Q}$ we have shown that the subspaces $U_1, \ldots, U_m$ satisfy the dimension m-intersection property if and only if the sets $\overline{I_1}, \ldots, \overline{I_m}$ satisfy the m-intersection property. More generally we make the following conjecture.

Conjecture If A is an m by n matrix whose *nonzero* elements are algebraically independent over $\mathbf{Q}$, then the elementary vectors $x(I_1), x(I_2), \ldots, x(I_m)$ form a basis of the row space of A (that is, by Theorem 7.4, the subspaces $U_1, U_2, \ldots, U_m$ satisfy the dimension m-intersection property) if and only if

$$\text{rank } A[:, \cap_{i \in P} \overline{I_i}] \leq m - |P| \quad (\forall P \subseteq \{1, 2, \ldots, m\}, P \neq \emptyset).$$

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