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Although the optimality problems of § 2 were posed for unstructured meshes, the solution technique shows that the optimal meshes can be obtained with structured meshes. This raises a speculation as to whether unstructured meshes are fundamentally more effective with respect to error control, as suggested at the start of § 3.2, as opposed to , say, simply being algorithmically more convenient. Does the presence of a finite, possibly geometrically complex, domain boundary in the x coordinate plane play a role in this question? The application of the first stage of this technique involves mapping a Riemannian space to a Euclidean coordinate system in the presence of a boundary, which is a familiar research topic in the literature on transformation methods for structured meshes. In fact, it is possible, at least theoretically, to specify an arbitrary convex shape for the image domain for general given domains of the x plane. This claim is based on the theory of harmonic maps which is surveyed by Dvinsky, [4], Chapter 8, and by Liao in [4], Chapter 9. Topics related to generation of such transformations for the purposes of generating structured meshes are discussed in other chapters of this reference. It may be that questions of optimal error control for general domains may be reduceable to questions of geometric optimization of triangulations in the square by this route.

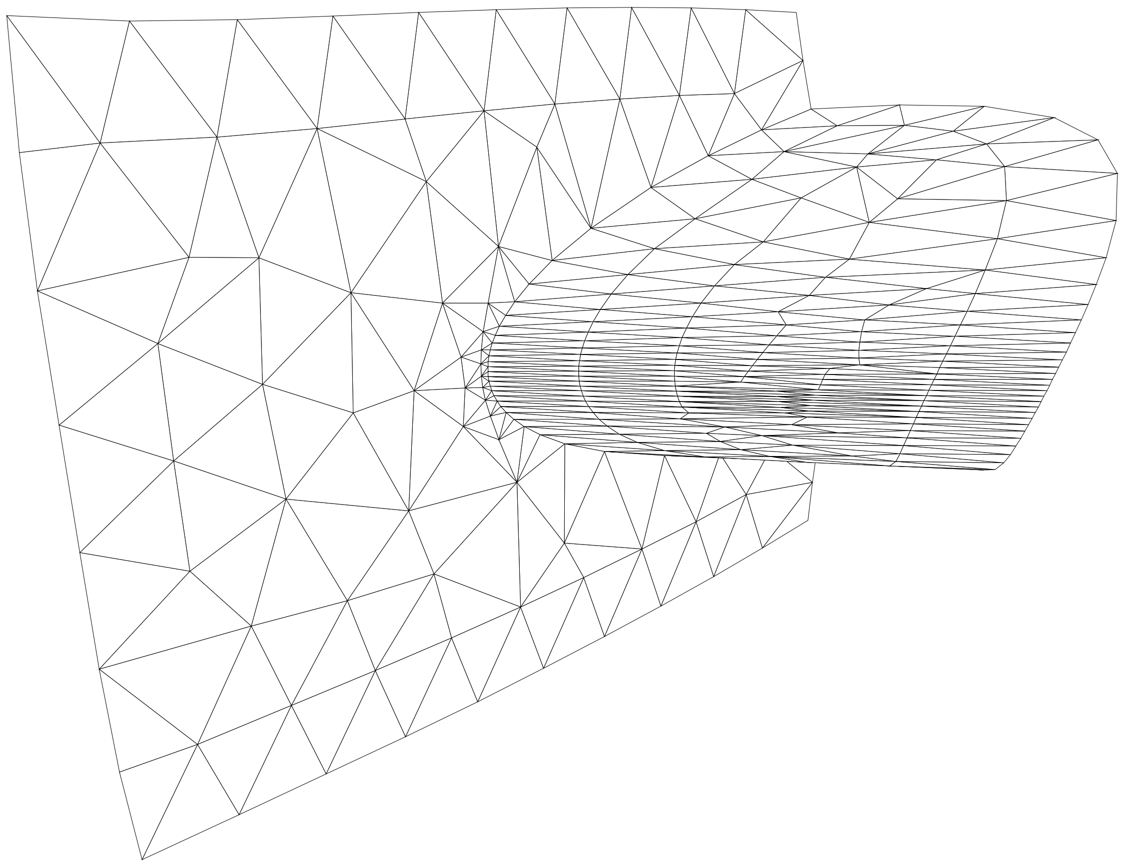

Figure 3.2 - Triangulation of airfoil - fuselage surface

courtesy S Farestam, CERFACS

4 Summary and Speculation

To summarize the techniques reviewed here, the task of generating a general unstructured mesh for a domain D with prescribed, possibly anisotropic , error control is approached in two stages. In the first, the prescribed error behaviour is identified with a Riemannian metric for the coordinate system in which the mesh is sought, and a transformation to a Euclidean coordinate system for this domain is sought, the anisotropic mesh transformation. If this stage is successful, the desired error control becomes a purely geometric requirement on a mesh to be generated in a Euclidean space. This technique has been successfully applied to solve several mathematically posed model optimal mesh problems; the maximum efficiency and minimal error mesh problems for the entire plane, discussed for constant error metric in § 2 and with variable metrics in § 3.1. Prescribing some form of optimal error control will generally lead to a geometric optimality requirement of the Euclidean domain mesh. An extensive review of properties of triangulations in the Euclidean plane (and in 3-space) has been provided by Bern and Eppstein, [3] , including many optimality considerations.

of nodes is positioned to minimize the error and is used as an intermediate process between major mesh adaptions to reduce the frequency of these more expensive modifications. Moving mesh techniques are often described by analogy to physical configurations in which the mesh edges are identified as springs connecting the vertices which are free to respond to the springs in the interior of the domain, but are fixed, or restricted to one degree of freedom on the boundary of D. Using this analogy, the technique of Bank and Smith can be described as involving anisotropic springs, with spring constants determined by a posteriori estimates of the directional error in the solution gradient on a control mesh. The use of the gradient error, as compared with the solution error, is reported to reduce the tendency of moving mesh methods to mesh degeneracy and entanglement, and performs better for nonconvex boundaries, interfaces in D.

In the preceding applications, a important factor leading to a need for mesh anisotropy can be identified as the convection of a transport phenomenon. We comment now on an application in which the same techniques apply, but for which the requirement for anisotropy in the mesh is quite different; i.e. generating a suitable triangulation for a surface in three space. Consider the coordinates (x_1, x_2) as the parameters of a parametric surface description

$$Z_i(x_1, x_2), \quad i = 1, 2, 3 \quad \text{for } x \in D \tag{12}$$

, for which we wish to construct a piecewise linear approximation. This construction can be done by triangulating D and using (12) to project the triangles into three space. For parts of the surface which are rapidly curving, small triangle edges are necessary for surface resolution; and such curving may be different in two different directions in the plane. Analytically, this curving information is captured in the parameter space domain, D, by the second fundamental form of the surface, which is a tensor computable from (12) (see e.g. Dubrovin et al [9] or Ding and Davies, [8]). This tensor has the form

$$Q - \lambda G \tag{13}$$

where G is the surface metric tensor (positive definite). The two generalized eigenvalues of (13) are the curvatures of the surface and determine the two length scales to be used in the parameter space. [3] In the work of Farestam, [11], this tensor , i.e. these length scales, and their orientations, is used as an error metric tensor to direct the anisotropy of the triangulation of the parameter space domain, D. In Figure 3.2, we show a triangulation of a portion of an airfoil where it meets the fuselage of an aircraft which is based on this technique. The surface has a single parameterization, but the second fundamental form for this surface is in fact discontinuous, and hence piecewise defined. The leading edge of the airfoil is shown on the right half of the figure, with short triangle sides in the high curvature vertical direction of the wing and long sides along the airfoil , where there is little curvature. The low curvatures on the fuselage and the surface of the wing results in large relatively equilateral triangles there.

[3] These two length scales are G-orthogonal.

ellipse configuration , for free stream Mach number $\simeq 8$, and Reynold's number $\simeq 10^7$, an anisotropic structured mesh is used to represent the boundary layer (which requires about 70% of the vertices). The advancing front technique of the thesis is used to generate an anisotropic unstructured grid for the rest of the flow field. The error metric tensor is based on estimating solution Hessian matrices in a local L_2 sense. The mesh adaption typically requires 4 remeshing steps, 10^4 triangles, and stretching ratios typically in the range 6 to 25, with extremes in excess of 100.

In [21], and [23], computations involving the use of anisotropic unstructured meshes by the group of CFD researchers associated with INRIA, Rocquencourt, and Dassault Aviation are presented. This aspect of the research is comprehensively summarized by M-G Vallet in her thesis, [22] in which Figure 1.1 appears. In particular, this thesis provides details of experimental computations on heuristics for determining an error metric tensor from flow characteristics and representation and smoothing of the data for this tensor on the control mesh. As an example, we comment on the computation of compressible Navier-Stokes flow past a wedge at Reynold's number $\simeq 10^3$ in which the error metric tensor is based on a local Mach number, modified to sense the boundary layer as well as shock waves. After two remeshings, a mesh of about 1.6×10^4 triangles was accepted in which the average stretching ratio was $\simeq 10$ and the extreme was $\simeq 150$.

In [14], Mavriplis describes the generation of anisotropic meshes for supporting compressible Navier-Stokes flows based on vertex insertion techniques, combined with an aft. As indicated above, the basic step of the technique involves using an anisotropic mesh transformation to map the local mesh configuration to an isotropic coordinate system , u. The u-coordinate image of the vertex to be inserted, is connected to the local mesh by the isotropic Delaunay insertion, and these edges identified as the connections for the anisotropic mesh in the x coordinate plane. This technique for selecting incidences was shown to be optimal for the model problem of § 2 with a fixed vertex set by D'Azevedo and Simpson in [6], and has been used as a heuristic for data fitting by Dyn et al in [10]. Calculations for the compressible Navier-Stokes flow around a two-element airfoil are reported in [14] for free stream Mach number of $\simeq 0.5$ and Reynold's number $\simeq 5 \times 10^3$. The adapted mesh covers recirculation zones, boundary layers and the wake region with $\simeq 2 \times 10^5$ triangles and stretching rations up to $\simeq 100$ in the boundary layers and $\simeq 1000$ in the wake.

3.3 Some other applications

The CFD applications just reviewed have the character of the maximum efficiency model meshing problem of § 2.1. Vertices are generated in an efficient, anisotropic distribution to achieve acceptable solution resolution. Solution resolution is identified with error metric tensors that estimate second derivative behaviour of some solution feature. The requirements of anisotropic meshes for supporting semiconductor device modeling has many similarities to CFD; the model is based on a transport phenomenon which can show high convection and the solutions can show sharp transition zones, through orders of magnitude much larger that common in other fluid flows. Adaptive mesh techniques have been common in both fields.

A recently developed moving mesh strategy of R E Bank and R K Smith for transient simulations is based on sensing directionality in the gradient error of the solutions , [17], [2]. It has the character of the minimum error model meshing problem of § 2.4, i.e. a fixed set

3.2 Anisotropic meshes in CFD

In computational fluid dynamics, two major alternative types of mesh are used :

- structured curvilinear coordinate meshes (body fitted coordinate meshes) typically based on a transformation from an Euclidean coordinate system.
- unstructured triangular meshes; as discussed in this article

We do not review the trade-offs between these alternatives, but simply quote two common arguments in support of unstructured meshes:

- facility for accommodating complicated flow region boundaries
- flexibility to distribute the vertices of the mesh efficiently

The goal referred to in this latter argument is reflected in the optimality principles of our model problems of § 2.

In the data fields of computational fluid dynamics, anisotropies arise from flow features such as rapid variations, or discontinuities, in pressure (shock waves), density (contact surfaces) , velocity components (boundary layers , shear layers). The mesh anisotropy information for efficient meshes for these flows must be determined adaptively, typically by computing an approximate flow on a perhaps relatively crude and inefficient mesh, and then producing a new, more efficient mesh by modifying the old one, or regenerating the new one entirely, referred to as remeshing. The information for controlling the improvements in the new mesh is summarized in a tensor, which we will continue to call the error metric tensor, which is stored as part of the data for the old mesh, which we will refer to as the control mesh (see [12] § 13.7 for further details). Typically, the tensor data is stored as the size of one of the length scales for the triangles, δ, the orientation of this length scale , θ, and a stretching ratio, s , giving the ratio of the longer to the shorter length scale.

The generation of complete meshes is typically done using an advancing front technique (aft) while mesh refinements may be done by triangle bisections or vertex insertion techniques. The basic local operations of the first and last of these techniques involves locating a new vertex in the unmeshed region, and/or connecting a vertex to the existing triangulation by appropriate new edges. The desired anisotropy in the mesh is typically obtained by using a local anisotropic mesh transformation based on the error metric tensor , as per (9) of § 2, to transform the local mesh configuration of the original x coordinates to new isotropic coordinates u. The local operation is then carried out isotropically in the u coordinate plane, and the results transformed back to the x coordinate description of the mesh.

Meshes generated this way have been used for explicit integration of the dynamic equations for compressible Euler and Navier-Stokes flows by adaptive remeshing. An early discussion was presented by Peraire et al, [16], for Euler flows in which the error metric tensor used was determined by sensing second derivatives of the solution flow variables in a local L_2 norm. Relatively small test examples are reported with stretching ratios of up to 6.

In [19], and in his thesis, [20], Tilch provides the details of efficient implementation of an advancing front technique anisotropic mesh generator and demonstrates the meshes produced for computations of compressible Navier-Stokes flows. In a flow past a double existence is that the Riemann-Christoffel tensor of the error metric tensor vanish identically. The construction of the anisotropic mesh transformation is then accomplished by integrating a system of ordinary differential equations. Once that transformation has been constructed, which can be done numerically with an initial value problem solver, the optimal error control mesh is the image under the reverse transformation, $x(u)$, of a mesh of equilateral triangles in the u coordinate plane. In Figure 3.1, we show a sample mesh [2] generated by this technique which is taken from [7]. Actually, this mesh is the image in the x plane of a uniform mesh of right angle triangles formed by splitting squares in the u plane, and hence has a slightly higher mesh density than optimal. The data function for this example is

$$f(x_1, x_2) = ((x_1 - .5)^2 + (x_2 + .2)^2)^{-1/2}$$

For this mesh of about 4000 triangles, the maximum interpolation error is 10^{-3} (excluding triangles that touch the boundary,); if this rather unremarkable looking mesh has a claim to fame, it would be that no mesh of a significantly lower triangle density can meet the error tolerance,

$$E(M) \leq 10^{-3} \quad .$$

In principle, then, this construction provides an avenue for forming the anisotropic mesh transformation for any anisotropy described by a tensor, H, which has vanishing R-C tensor. However, there does not appear to be any intrinsic reason why the tensors for controlling anisotropic vertex distributions in meshes should obey this condition , i.e. have vanishing R-C tensors. In particular, for general data functions, $f(x)$, the Hessians do not satisfy this condition, although large classes of functions do, e.g. solutions of any partial differential equations of the form

$$a\partial^2 f/\partial x_1^2 + b\partial^2 f/\partial x_1 \partial x_2 + c\partial^2 f/\partial x_2^2 = 0$$

for any constants a, b, c.

In [7], E F D'Azevedo and this author extended the approach of [5] reviewed in the preceding subsection, to generate a mesh for optimal control of the maximum gradient error in piecewise linear interpolation.

The error metric tensor for the maximum gradient error, H_g , is the square of the error metric tensor for the interpolation error itself, i.e. $H_g = H^2$, where H is the Hessian of the data function. The remarkable feature of this is that, as shown in [7], the Riemann-Christoffel tensor for H_g always vanishes identically. Hence, for any data function and for this error measure i.e. this error metric tensor, the classical differential geometry construction of a global anisotropic mesh transformation can be constructed.

In [15], E Nadler studied the model maximum efficiency mesh problem of §2 but for the piecewise linear least squares error approximation of a convex quadratic data function rather than the maximum error. The conclusions are essentially the same; i.e. the optimal mesh is the image of an equilateral triangular mesh in the u plane, under the anisotropic mesh transformation (9).

[2] modified to fit the boundaries of the square in which it is displayed

The viewpoint of this extension is a standard viewpoint of differential geometry, i.e. the localized model of §2 can be regarded as holding at the differential level. In this view, the anisotropic error growth formula, (7), is interpreted as inducing a Riemannian metric on the x plane

$$ds^2 = H_{1,1}dx_1^2 + 2H_{1,2}dx_1dx_2 + H_{2,2}dx_2^2 \qquad (11)$$

The metric tensor of (11) will be referred to as the error metric tensor. It is defined by the two principal length scales $\sqrt{|\lambda_1|}$, $\sqrt{|\lambda_2|}$, and the orientation θ where $\lambda_1(x)$, $\lambda_2(x)$, and $\theta(x)$ are define by the Hessian of the data function, $f(x)$, as in §2.

For an anisotropic mesh transformation, we look for a transformation to a new coordinate system, $u(x)$, for the Riemannian plane currently described by the x coordinates so that distance is measured by the Euclidean metric, i.e.

$$ds^2 = du_1^2 + du_2^2$$

analogously to (10).

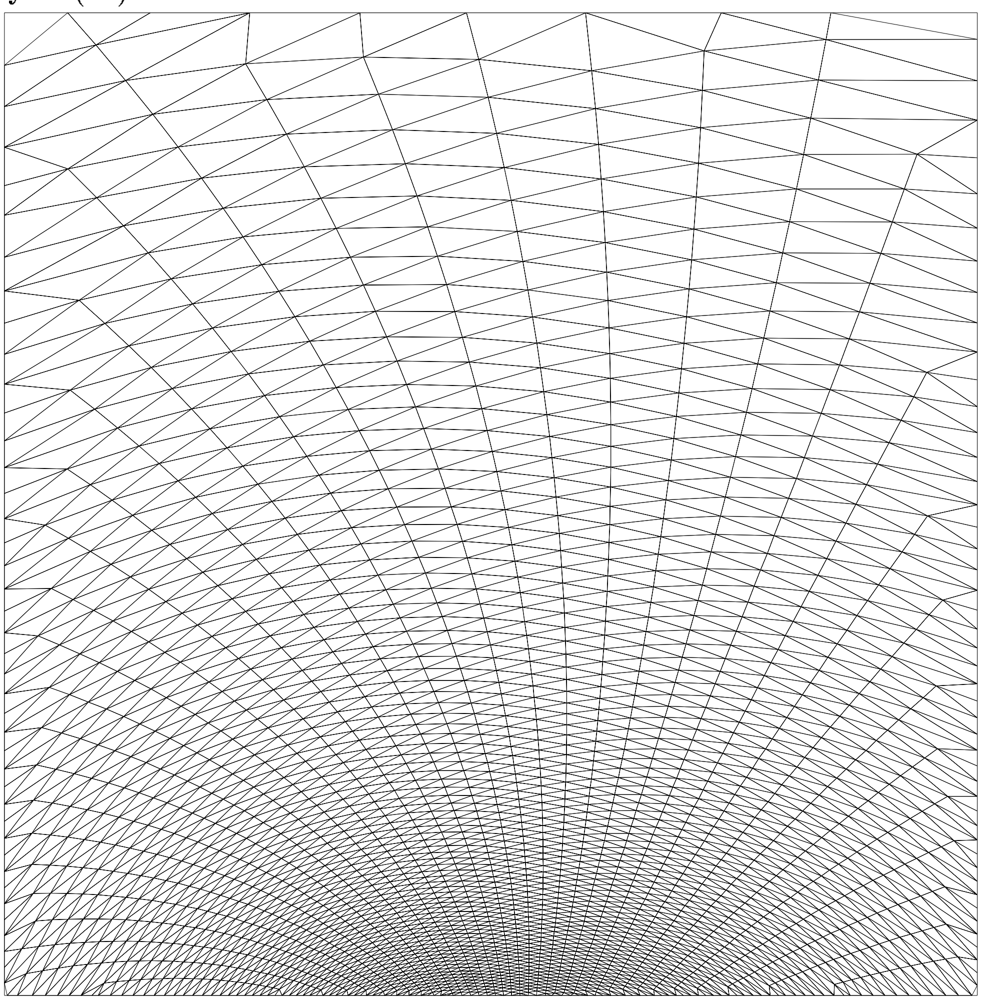

Figure 3.1 - Optimal Efficiency Mesh Example

Such a transformation would be a (typically nonlinear) generalization of (9). The existence and construction of such a coordinate transformation are textbook topics in traditional differential geometry,(see e.g. Dubrovin et al , [9], or Sokolnikoff, [18].) The basic condition for it would be plausible to expect monotone dependence of some measures of error to depend monotonically on the single length scale of each mesh triangle. In this hypothesized scenario, it would reasonable to expect equidistributing of this error measure to be sufficient to identify the optimal mesh over the class of isotropic meshes. However, as we have seen, for optimality over the class of general unstructured meshes, the full anisotropy of the triangles is permitted; the error over each triangle typically depends on several parameters and equidistributing is not a sufficient condition for optimality.

3 Generalizations and Implications

The elementary model based discussion of the previous section formulated two related optimality problems, demonstrated a solution technique, and established a connection between anisotropy and optimal error control. in this section, we use this model discussion as a basis for reviewing two convergent lines of research. One of them, undertaken primarily by E F D'Azevedo and the author, has focussed on extensions of the technique to more general optimal error control. This line of research was not particularly oriented to anisotropy in the resulting meshes although the mechanics for including it are all present, and the example computations in this research were made with data functions which lead to optimal, variable , but essentially isotropic meshes.

The other line of research reviewed here arises from incorporating anisotropic mesh transformations in applications. The primary applications area has been computational fluid dynamics, but research in meshes for surface triangulation and for semiconductor device modeling is also reviewed. A positive definite symmetric tensor in the plane, i.e. two orthogonal length scales plus their orientation, is a natural device for specifying planar anisotropy. Hence mesh requirements which lead to anisotropic meshes typically specify such a tensor either explicitly or implicitly. In the requirements for a model maximum efficiency mesh in the previous section the data , τ and $f(x)$, implicitly determine the tensor $\tau^{-1}H$ to which an optimal mesh conforms in the sense described in § 2.3 The model maximum efficiency mesh problem reflects the practical objective of unstructured triangular mesh generation techniques to distribute the vertices so that the mesh does not contain significantly more vertices than necessary. In this section, we survey some research in which error control and the associated tensors play an important role in producing anisotropy in meshes for serving efficiency goals.

3.1 Variable H

The discussion of §2 was particularly elementary because the anisotropy requirement was constant throughout the plane, i.e. $\tau^{-1}H$ was a constant. In [5], E F D'Azevedo studied an extension of these techniques in which the data function, $f(x)$, is not necessarily quadratic, but is assumed to have a nonsingular Hessian at every point of the plane. This could be viewed as a model of the general problem of computing maximum efficiency meshes which is not so localized as that of §2, i.e. the boundaries of a general domain have still been pushed out to infinity, but we are now looking at the problem on a scale where we can see the Hessian , H , of $f(x)$ varying.

Hence, we conclude that any mesh of equilateral triangles of side length $\sqrt{3}\tau$ in the u plane, such as shown in Figure 2.2, transforms under the anisotropic mesh transformation, (9), to a maximal efficiency mesh in the x plane, such as is shown in Figure 2.3 . Clearly, then, a model maximum efficiency mesh will be anisotropic in general, with orthogonal length scales proportional to $\lambda_1^{-1/2}$ and $\lambda_2^{-1/2}$ with the orientation of the shorter length scale making an angle θ to the x_1 axis.

2.4 the model minimal error mesh problem

At the outset of this section §2, we formulated the maximum efficiency mesh problem for a general domain, D. However, there is an alternative optimality concept that we could have studied, i.e. the minimal error mesh problem, in which we fix the number of vertices, and hence number of triangles, to be used to mesh the domain, and ask for a mesh which minimizes the error, $E(M)$. If we introduced the same localizations of this general problem as was done at the outset of §2.1, we can pose a model minimum error mesh problem by fixing the average density, $W(M)$ of triangular meshes generated by a single triangle of arbitrary shape, and asking for the shape that minimizes the maximum error in piecewise linear interpolation of a quadratic data function, $f(x)$, i.e. that minimizes $E(M) = E(T)$ of (1).

If we consider the u plane introduced by the anisotropic mesh transformation (9) again, the specifications for the minimum error mesh problem are transformed as follows. Fixing the average triangle density amounts to fixing the area of the admissible generating triangle. So the goal of choosing the triangle shape, given a fixed area, to minimizing the error becomes the question of finding the shape that minimizes the radius of the circumscribing circle for a fixed area. Clearly, the optimal meshes for this model problem are also images of equilateral triangle meshes in the u coordinate plane.

Equidistributing is not sufficient

We can see that for both of these model problems in optimal error control, the solution meshes are structured, uniform meshes formed by the reflections, translations, of a single triangle of the optimal shape. Clearly, these meshes equidistribute the maximum interpolation error per triangle. Equidistributing as a sufficient criterion for optimality is familiar from one dimensional discretization problems, and its connection with one dimensional anisotropic mesh transformations is also well known. e.g. White [24], and Babuska and Rheinboldt, [1]. It is also sometimes used as a criterion for accepting meshes in the plane, so it is perhaps worth commenting on the fact that it is not in general a sufficient condition for optimality over the general class of unstructured meshes. For example, for the maximum efficiency mesh problem, any triangle in the x plane which has the same circumellipse as the optimally shaped triangle will generate a mesh which will equidistribute the prescribed error tolerance, τ, ; however, unless the triangle is the image of an equilateral triangle under the reverse anisotropic transformation,(9), it will generate a mesh with triangle density higher than that of the maximum efficiency mesh.

The role of equidistributing as a sufficient condition seems intimately connected to error behaviour which is monotone in a single parameter. Hence, for strictly isotropic meshes,

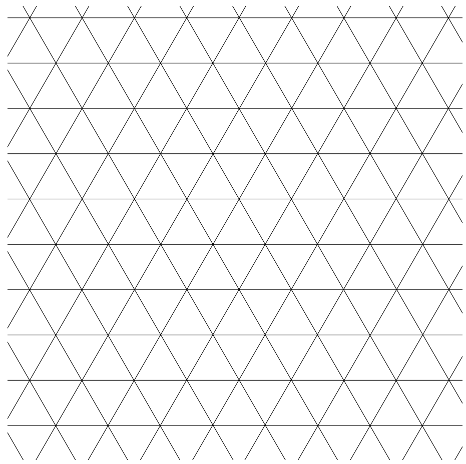

Figure 2.2
Equilateral mesh in u plane

i.e. the error growth is isotropic in the u plane. The zero error ellipse of the x plane transforms to a zero error circle of the u plane, and the question of a triangle shape of maximal area inscribed in this error ellipse of the x plane is transformed to the question of the triangle of maximal area in the zero error circle of the u plane. If we designate this latter triangle as $\overline{T}_{opt}$, it is obvious that the center of the inscribing circle of $\overline{T}_{opt}$ must lie inside this triangle (which excludes case (4) above from optimality) and that $\overline{T}_{opt}$ is an equilateral triangle of side length $\sqrt{3}\tau$.

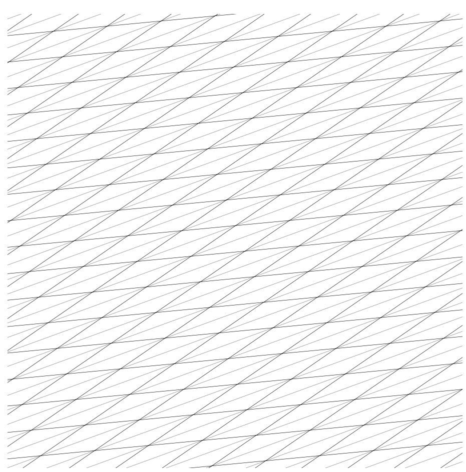

Figure 2.3
Anisotropic mesh in x plane

circumscribing error ellipse, but not its location in the plane, or orientation under reflections in a side, since these transformation simply results in translaions of the same error ellipse. Hence for a mesh, M, generated by translations and reflections from a single triangle,T , $E(M) = E(T)$.

The shapes of triangles that meet the error tolerance exactly can be determined from the shape of the family of error contour ellipses, as we now discuss. For case (3), we can set the contour value at the center of the family to $-\tau$, which then determines the zero level error ellipse. Any triangle circumscribed by this ellipse and containing the center in the triangle's interior will result in a maximum interpolation error of exactly τ . Similarly, for the case (4), if we take two ellipses of the error contour family whose values differ by τ, then any triangle which is circumscribed by the outer one, which lies between the two ellipses, and which has a side tangent to the inner one, will result in a maximum interpolation error of exactly τ. Clearly, then, a maximum efficiency mesh will be constructed by triangles of maximal area from this family of permissible shapes.

While one could now approach this maximal area question directly, it is instructive to consider a transformation of the problem, which is the basic anisotropic mesh transformation. The error of (2) can be written

$$e_T(x) = E_c + [x - x_c, H(x - x_c)] \tag{5}$$

where we use "[...]" to designate the usual Euclidean inner product and E_c is the error at $x_c = E(T)$ in case (3) . If we let Q be the matrix of the diagonalizing rotation of H [1] then

$$Q^T HQ = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ for } Q = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix}$$

and we make the coordinate transformation

$$z = Qx \tag{6}$$

we get

$$e_T(z) = E_c + \lambda_1(z_1 - z_{1,c})^2 + \lambda_2(z_2 - z_{2,c})^2 \tag{7}$$

where $\lambda_1 \geq \lambda_2$ are the eigenvalues of H, and θ is the angle that the eigenvector corresponding to λ_1 makes with the x_1 axis in the original coordinate plane.

We can further introduce rescaled coordinates

$$u_1 = \sqrt{\lambda_1}\, z_1 \ , \ u_2 = \sqrt{\lambda_2}\, z_2 \tag{8}$$

then, combining 6 with 8, we get

$$u = \Lambda Q x \tag{9}$$

where Λ is the diagonal matrix apparent from (8). Now the triangle, T, in the x plane transforms to a triangle, $\overline{T}$ in the u plane for which the interpolation error of the transformed interpolation problem is described by

$$e_{\overline{T}}(u) = E_c + ((u_1 - u_{1,c})^2 + (u_2 - u_{2,c})^2) \tag{10}$$

[1] e.g. See [13],pages 410, and 201

2.2 The geometry of optimal linear interpolation on a triangle

Let T be any triangle, and let $p(x)$ be the linear interpolant of convex quadratic $f(x)$ at the vertices of T. The error in this interpolation has a simple geometric description. We note that this error,

$$e_T(x) = f(x) - p(x) \tag{2}$$

is also a quadratic polynomial, with the same Hessian, H, as f. Hence , the contour lines of constant error will be a family of concentric ellipses. We will denote the center of this family by x_c. The geometric significance of interpolation at the vertices of T is that the zero error contour is the circumellipse of T from this family. For identifying the maximum error that occurs in T. i.e. $E(T) = \max_{x \in T} |e_T(x)|$, there are two cases. Either the center, x_c, lies in T, as shown in Figure 2.1 in which case

$$E(T) = |e_T(x_c)| \tag{3}$$

or x_c lies outside T, in which case

$$E(T) = |e_T(x_{mid})| \tag{4}$$

where x_{mid} is the midpoint of the side of T closest to x_c. While this latter case is perhaps less geometrically obvious, it is established, along with the algebraic formulae that correspond to this geometry, in D'Azevedo and Simpson, [6].

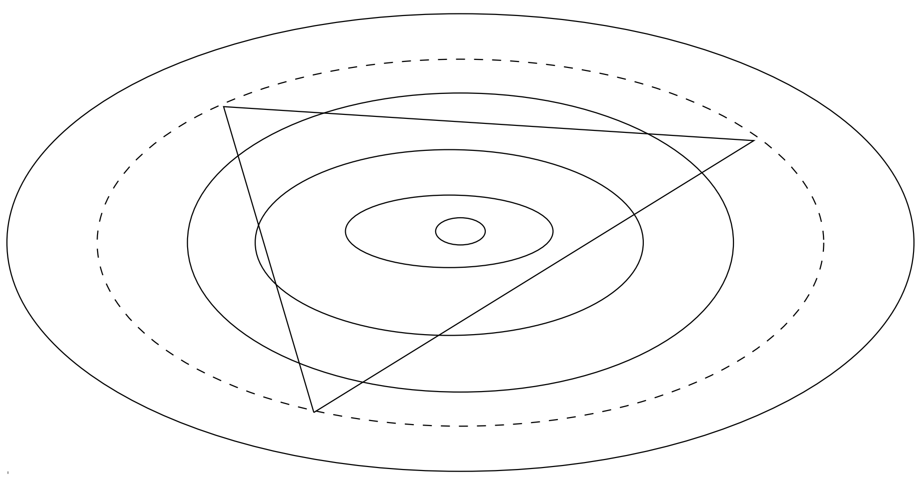

Figure 2.1
Error contours for interpolation on a triangle

2.3 Solving the model maximum efficiency mesh problem

In the model maximum efficiency mesh problem, the maximum error is specified not to exceed τ, and we seek an appropriate triangle shape with which to mesh the plane. We note from the preceding subsection that the maximum error for a given triangle depends on its by using interpolation as the approximations technique, instead of the finite element method, and two levels of geometric localization.

In the global version of this problem, we would specify a bounded polygonal domain, D, and an error tolerance, τ, and a data function $f(x)$ for $x = (x_1, x_2)$. For any mesh, M, on D, we will denote the piecewise linear interpolant by $p(x)$, the pointwise interpolation error as

$$e_M(x) \;=\; f(x) \;-\; p(x)$$

and the maximum error by

$$E(M) \;=\; \max_{x \in D} |e_M(x)| \;.$$

A maximal efficiency mesh, then, is a triangulation, M_{eff}, that meets the error tolerance

$$E(M_{eff}) \;\leq\; \tau \tag{1}$$

and such that there is no mesh with fewer triangles in it which also meets this tolerance. While it is clear that such meshes exist, the problem of determining them is currently unsolved in the sense that there is neither an algorithm for their construction, nor criteria for recognizing one.

2.1 An elementary model problem

To get a tractable optimal mesh problem, we introduce two levels of localization. First, we push the boundary of D out to infinity; i.e. we mesh the entire plane. Secondly, we assume that $f(x)$ is smooth and look at it on such a small scale that it appears quadratic. I.e. we assume that the Hessian matrix of second derivatives of f is a constant, positive definite matrix.

$$\begin{pmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1 \partial x_2 \\ \partial^2 f/\partial x_1 \partial x_2 & \partial^2 f/\partial x_2^2 \end{pmatrix} \;=\; H$$

Since we will now require infinitely many triangles, we will measure the efficiency of meshes by their average densities, $W(M)$

$$W(M) = \lim_{r \to \infty} \mathrm{Num}(r)/4r^2$$

where $\mathrm{Num}(r)$ is the number of triangles in a square of side length $2r$ centered on $x = (0, 0)$. A maximal efficiency mesh for this model problem then is defined as one which meets the error tolerance as in (1) and such that there is no mesh of lower density which also meets this tolerance. Now, since we are measuring efficiency by average triangle density,the problem is somewhat degenerate as posed for arbitrary meshes, Clearly, we could replace any fixed number of triangles in mesh M by some other fixed number of different triangles to get a new mesh $\overline{M}$ and still have $W(M) = W(\overline{M})$. However, by assuming a constant H for the data function, we are in effect assuming that the requirements for resolving the data are the same everywhere. So it is consistent with this model to restrict the admissible meshes of our optimization to those generated by reflections and translations of a single triangle shape.

direction than in the second, i.e. if the mesh is anisotropic. In the next section of this paper, we extend this observation to demonstrate the connection between anisotropic meshes and theoretical analysis of optimal error control, using simple model problems. A review of further optimal error control research is then presented in §3. The primary application area of anisotropic meshes, to date, has been in computational fluid dynamics, but they have also been used in surface triangulation, and in modeling semiconductor devices; we also review this research in §3.

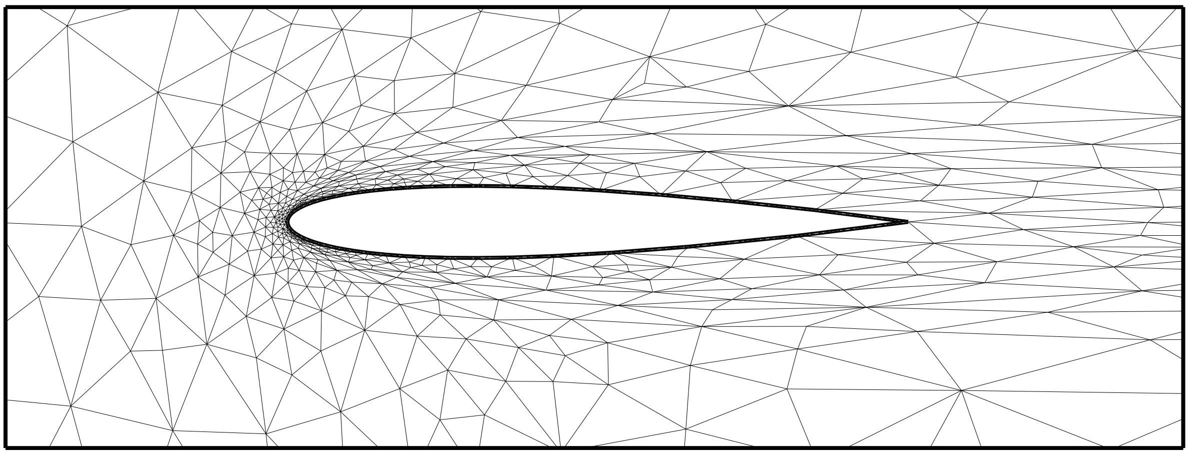

Figure 1.1 - Anisotropic mesh for flow past an aerofoil

courtesy M-G Vallet, [23]

In all these contexts, the desired distribution of triangle shapes, or equivalently of vertices, is specified by a symmetric tensor, and the anisotropic mesh is constructed as the transform of an isotropic mesh by a coordinate transformation based on this tensor. Hence one can view anisotropic mesh generation as an extension of, rather than an alternative to, isotropic mesh generation.

Although we primarily limit our review to literature of the unstructured mesh school of grid generation, it will doubtless not escape the reader that this subject has much in common with the coordinate transform techniques of the structured mesh school of grid generation. We conclude §4 with some comments on this connection.

2 Elementary theory of optimal error control

To serve applications, unstructured triangular meshes must meet a combination of requirements . Typically, they must conform to specified boundaries, and provide adequate resolution of data. If the context is the finite element method, there may be some requirements imposed by the properties of the stiffness matrix ; limits on the condition number, monotonicity for physically conserved quantities.

In this section, we suppress such requirements in order to isolate the role of error control. We construct a model mesh generation problem and its solution to illustrate how optimal error control can require anisotropic meshes. The simplicity of the model problem is obtained have several independent origins, we make the case that they are convergent, or at least complementary, by demonstrating that meshes solving explicitly posed model problems of mesh optimality consist of triangles with independent length scales, and by observing that a common mechanism is present in these two lines of research, which we refer to as anisotropic mesh transformation. A third relevant line of research is the study of triangulation algorithms and geometric properties undertaken in the computational geometry developments of the past 15 years. We refer to the extensive review of Bern and Eppstein for this line of research, [3].

The intent of mesh generation methods to produce triangles of a single length scale has been reflected in concern about avoiding small angles in the generated triangles, about maintaining similarity, or near similarity in the generated triangles generated by local refinements, and in the use of the Delaunay triangulation; it has also been reflected in measures of mesh quality that tabulate statistics on the deviations from equilateral shape in the triangles of a generated mesh.

Mesh terminology is fairly consistent in the literature; in an 'unstructured' triangular mesh, there is no restriction placed on the number of triangles which can meet at any vertex of the mesh. If some pattern is specified for how many triangle can meet at the mesh vertices, the mesh is 'structured', e.g. the simplest pattern would be exactly 6 triangles at each internal mesh vertex, or one might specify that either 4 or 8 triangles are allowed to meet at an internal mesh vertex, etc. A 'uniform' mesh is one for which the length scales of the triangles are translation invariant. We will regard 'regular' and 'irregular' as synonymous with 'structured' and 'unstructured'. It has become common to refer to meshes as 'isotropic' if the two length scales of all the triangles are essentially the same, and meshes which include triangles in which the two length scales are clearly different as 'anisotropic'. Clearly, this distinction is a matter of degree, rather than a mathematical partitioning of the set of triangular meshes. Equally relevant for our discussion is the intention of a mesh generation method. As mentioned above, until recently most triangular mesh generation techniques intended to produce isotropic meshes, so it would be appropriate to call them 'isotropic' mesh generation methods. They can, of course, produce anisotropic meshes, but this would be regarded as evidence of inappropriate input data, or poor performance of the method. There is however a minor quandary about how to refer to methods in which the triangles intentionally have two independent length scales, since in their normal operation, they could produce meshes in which both length scales of each of the triangles were essentially equal, i.e. isotropic meshes, from appropriate data. We might chose to designate such methods by 'general' , but in fact the term 'anisotropic' for such methods has already gained some acceptance, so we will use it, with the caveat that anisotropic mesh generation methods can acceptably produce isotropic meshes for appropriate data. In Figure 1.1, we show an example of an unstructured, anisotropic mesh generated to support the computation of the flow around an airfoil, further details of the method used to construct this example are given in § 3.2.

A connection between mesh anisotropy and error behaviour of piecewise polynomial approximations based on the mesh is certainly to be expected. Consider, for example, a mesh to be used for piecewise linear approximation of a bivariate function which changes rapidly, i.e. has a large second derivative, in one direction but changes slowly, (a small second derivative) in the perpendicular direction. It is intuitively apparent that, for a fixed number of vertices, the approximation will be better if the vertices are spaced more closely in the first

Anisotropic Mesh Transformations and Optimal Error Control *

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Abstract

Recently, research originating in several different applications has appeared on unstructured triangular meshes in which the vertex distribution is not locally uniform, i.e. anisotropic unstructured meshes. The techniques used have the common features that the distribution of triangle shapes for the mesh is controlled by specifying a symmetric tensor, and that the anisotropic mesh is the transform of an isotropic mesh. We discuss how these mechanisms arise in the theory of optimal error control, using simple model mesh generation problems, and review the related research in applications to computational fluid dynamics, surface triangulation, and semiconductor simulation.

1 Introduction

In order to fully exploit the flexibility of triangular meshes in the plane, it would intuitively seem likely that it would be necessary to take advantage of the independence of the two length scales of the triangles, e.g. the length of the longest edge, and the perpendicular distance to the opposite vertex. Nevertheless, most triangular mesh generation techniques and research until the late 1980's concentrated on generating meshes in which the length scales of the triangles were essentially the same. Typically, these methods have been heuristic approaches to an implicitly posed optimality goal for the resulting mesh. I.e.. the methods have employed tactics which arguably served some optimality goal, but whether the goal itself was attainable, whether criteria existed for recognizing a mesh that attained the goal, or whether, under some circumstances, a tactic lead to a mesh that achieved the goal, have been unaddressed theoretical questions. In this paper, we review recent research in two areas related to these comments; research on methods for generating unstructured meshes in which the triangles are intended to have two independent length scales, and research in posing optimality concepts for unstructured mesh problems. Although these research lines

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