Pm Numbers, Ambiguity, and Regularity *

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Abstract

We introduce the pseudo-m-ary (Pm) number system in which numbers are represented by sums of the form $\sum_{i\geq 0} a_i(m^{i+1}-1)$. We characterize the Pm representations that are produced by the greedy algorithm and show that they form a regular set. In addition, we show that the set of Pm representations that are the sole representations for their corresponding numbers is also a regular set.

1 Introduction

Many number systems can be viewed as ways of representing integers based on finite or infinite integer sequences $1 = u_0 < u_1 < u_2 < \cdots$. A common method of finding a representation of an integer in any such number system is the greedy algorithm; see Fraenkel [Fra85]. To find the greedy representation of an integer N, we find the largest u_i that is no larger than N and then repeatedly we set $a_i \longleftarrow \lfloor N/u_i \rfloor$, $N \longleftarrow N - a_i u_i$, and $i \longleftarrow i-1$, until $i = 0$. In some number systems, some integers may have representations other than the one obtained via the greedy algorithm. (A number that has more than one representation in the given number system is said to be *ambiguous*; otherwise, it is *unambiguous*.)

There appears to be a close relationship between the properties of number systems and the properties of formal languages; see Shallit [Sha91], for example. Two intriguing problems about this relationship are:

Problem 1.1 *For which number systems are the sets of greedy representations regular?*

Problem 1.2 *For which number systems are the sets of unambiguous numbers regular?*

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$Bin(0)$, the empty tree
Size: 0

$Bin(h+1)$, for $h \geq 0$
Size: $2^{h+1} - 1$

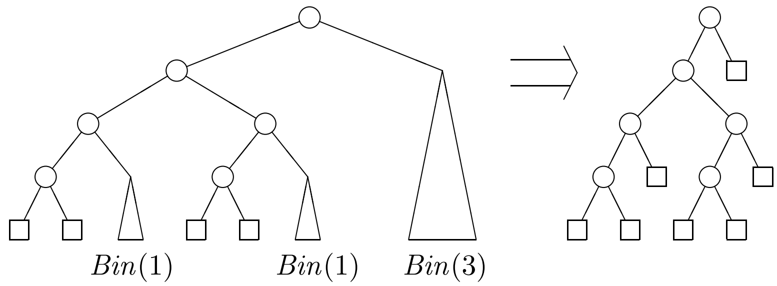
Figure 1: A recursive definition of the perfect binary tree of height h ($Bin(h)$).

We introduce the *pseudo-m-ary (Pm) number system* and show that the set of greedy representations and the set of representations of unambiguous numbers in the Pm number system are regular sets. For any fixed integer $m > 1$, the Pm number system is based on the sequence $m^1 - 1, m^2 - 1, m^3 - 1, \ldots$. As we will see, when $m = 2$ (in the P2 number system), every integer is representable; however, when $m > 2$, only multiples of $m - 1$ are representable.

The P2 number system has been studied previously. Allouche, Betrema, and Shallit [ABS89] characterized the set of integers that can be represented by P2 representations using only the digits 0 and 1. Their interest in the P2 number system arose from a study of the sequence of parentheses occurring in the recursive definition of the integers.

We have used the characterization of the greedy representations in the P2 number system in Cameron [Cam91] and Cameron and Wood [CW91] to establish an upper bound result for a class of binary trees. Every binary tree can be viewed as a perfect binary tree (a binary tree whose leaves all appear on one level; see Figure 1) with some perfect binary subtrees removed. Each node of a perfect binary tree has two perfect binary subtrees, so each remaining node has 0, 1, or 2 perfect binary children removed by the pruning; see Figure 2. A perfect binary subtree contains $2^h - 1$ nodes, where h is the height of the tree (the distance of the leaves from the root of the tree). Thus, we became interested in numbers of the form $\sum_{i \geq 0} a_i(2^{i+1} - 1)$, where $a_i = 0$, 1, or 2, because they give the total size of the subtrees we have removed by pruning. These numbers are exactly the P2 representations.

Similarly, each node of a perfect m-ary tree has m perfect m-ary subtrees. Pruning such a tree removes $0, 1, 2, \ldots,$ or m perfect m-ary subtrees from each remaining node. Again, because a perfect m-ary tree of height h contains $(m^h - 1)/(m - 1)$ nodes, we have a relationship between the



Prune a $Bin(3)$ and two $Bin(1)$ subtrees from a $Bin(4)$ tree.

The resulting binary tree.

Figure 2: Pruning a complete binary tree.

number of nodes pruned from a perfect m-ary tree and sums of the form $\sum_{i\geq 0} a_i(m^{i+1}-1)$, where $a_i = 0, 1, 2, \ldots,$ or m; that is, between m-ary trees and Pm representations.

In the following sections, all numbers discussed are assumed to be non-negative integers, and we assume that m is some fixed integer greater than 1.

2 The Pm Number System and the Greedy Algorithm

In this section, we define the Pm number system and introduce the Pm representations obtained via the greedy algorithm.

The base m number system, for some integer $m > 1$, is based on the integer sequence $1 = m^0 < m^1 < m^2 \cdots$. If we wish to represent an integer in base m, then we use the digits $0, \ldots, m-1$, and the i^{th} digit of a base m representation is the coefficient of m^i. (The least significant digit corresponds to index 0, and we count up from there.) We consider the pseudo-m-ary (Pm) number system, which is based on the integer sequence $1 \leq m^1 - 1 < m^2 - 1 < m^3 - 1 < \cdots$. It uses the digits $0, \ldots, m$, and the i^{th} digit of a Pm representation is the coefficient of $m^{i+1} - 1$.

Thus, a Pm representation is either ϵ or a sequence of integers of the form $a_n \cdots a_0$, where $n \geq 0$, $1 \leq a_n \leq m$, and $0 \leq a_i \leq m$, for all i, $0 \leq i < n$. The value of the Pm representation ϵ is 0. The value of any other Pm representation $a_n \cdots a_0$ is denoted by value$(a_n \cdots a_0)$ and is defined to be $\sum_{i=0}^{n} a_i(m^{i+1} - 1)$. If we consider all non-zero Pm representations with exactly $n+1$ digits, for some $n \geq 0$, the Pm representation consisting of a 1 digit followed by n zero digits (that is, the Pm representation 10^n, using the formal language notation 0^n to mean a string of n zeros) has the smallest value among all non-zero Pm representations with exactly $n+1$ digits. Similarly, the Pm representation consisting of $n+1$ digits equal to m (the Pm representation m^{n+1}) has the largest value among all non-zero Pm representations with exactly $n+1$ digits. Thus, the value of the non-zero Pm representation $a_n \cdots a_0$ is bounded by

$$m^{n+1} - 1 \leq \text{value}(a_n \cdots a_0) \leq m \left(\frac{m^{n+2} - 1}{m - 1} - n - 2 \right).$$

For $m = 2$, we will show that every nonnegative integer has at least one Pm representation. But, if $m > 2$, then only some of the nonnegative integers have Pm representations. For example, when $m = 3$, the integer 5 has no representation in the Pm number system. We will show that an integer has a Pm representation if and only if the integer is a nonnegative-integer multiple of $m - 1$. (Since $m - 1 = 1$ when $m = 2$, we will have shown that each nonnegative integer has at least one representation in the P2 number system.)

It is well-known that $m - 1$ divides $m^k - 1$, for all $k > 0$; thus, since $\text{value}(a_n \cdots a_0) = \sum_{i=0}^{n} a_i(m^{i+1} - 1)$, $\text{value}(a_n \cdots a_0)$ is divisible by $m - 1$. Now, we show that each nonnegative-integer multiple of $m - 1$ is representable in the Pm number system. We will use the greedy algorithm and a result of Fraenkel [Fra85].

The greedy algorithm produces a representation $a_n \cdots a_0$ (if one is possible) in a number system $1 \leq u_0 < u_1 < u_2 < \cdots$ for a positive integer N as follows:

> `Find the largest index` n `such that` $u_n \leq N$.
> $i \longleftarrow n$
> `Repeat`
> $a_i \longleftarrow \lfloor N/u_i \rfloor$
> $N \longleftarrow N - a_i u_i$
> $i \longleftarrow i - 1$
> `Until` $i = 0$.

Note that

$$\sum_{i=0}^{k} a_i u_i < u_{k+1}, \text{ for all } k,\ 0 \leq k \leq n,$$

because n is the largest index such that $u_n \leq N$ and because we remove as many multiples of u_i as possible from what remains of N before considering lower-order digits in the greedy representation, for all i, $0 \leq i \leq n$. The following result of Fraenkel [Fra85] implies that, in certain number systems, the greedy representation is the only representation to satisfy

$$\sum_{i=0}^{k} a_i u_i < u_{k+1}, \text{ for all } k,\ 0 \leq k \leq n,$$

and that every nonnegative integer has such a representation in these number systems.

Proposition 2.1 (Fraenkel) *Let $1 = u_0 < u_1 < u_2 < \cdots$ be any finite or infinite sequence of integers. Any nonnegative integer N has precisely one representation in the system $S = \{u_0, u_1, u_2, \ldots\}$ of the form $N = \sum_{i=0}^{n} a_i u_i$, where the a_i are nonnegative integers that satisfy*

$$a_k u_k + a_{k-1} u_{k-1} + \cdots + a_0 u_0 < u_{k+1} \ (k \geq 0).$$

Consider the number system S_{m-1} based on the integer sequence

$$1 = \frac{m-1}{m-1} < \frac{m^2-1}{m-1} < \frac{m^3-1}{m-1} < \cdots \ .$$

This system is simply the Pm number system divided by $m-1$. By Proposition 2.1, we see that, in S_{m-1}, the greedy representation $a_n \cdots a_0$ for a nonnegative integer p is the only representation for p that satisfies

$$a_k \geq 0$$

and

$$\sum_{i=0}^{k} a_i \frac{m^{i+1}-1}{m-1} < \frac{m^{k+2}-1}{m-1},$$

for all k, $0 \leq k \leq n$ and that there is such a representation for every nonnegative integer. Therefore, in the Pm number system, the corresponding greedy representation $a_n \cdots a_0$ for the nonnegative integer $p(m-1)$ is the only representation for $p(m-1)$ that satisfies

$$a_k \geq 0$$

and

$$\sum_{i=0}^{k} a_i (m^{i+1}-1) < m^{k+2}-1,$$

for all k, $0 \leq k \leq n$, and there is such a representation for every nonnegative-integer multiple of $(m-1)$. Since only nonnegative-integer multiples of $m-1$ have representations in the Pm number system, a nonnegative integer has a representation in the Pm number system if and only if it is a multiple of $m-1$.

3 The Regularity of Greedy Representations

We will show that the following regular language captures exactly the Pm numbers that are produced by the greedy algorithm.

Definition 3.1 *Let L_G be the regular language*

$$L_G = \{1,\ldots,m-1\}\{0,\ldots,m-1\}^* + \{1,\ldots,m-1\}\{0,\ldots,m-1\}^* m0^* + m0^* + \epsilon.$$

The regular language $\{1,\ldots,m-1\}\{0,\ldots,m-1\}^* + \epsilon$ is the set of Pm representations that do not have any digit equal to m. The regular language $\{1,\ldots,m-1\}\{0,\ldots,m-1\}^* m0^* + m0^*$ is the set of Pm representations that have exactly one digit equal to m and all lower-order digits are zero. Note that if $a_n \cdots a_0$ is in L_G, then $a_k \cdots a_0$, where $a_k > 0$, for some $0 \leq k < n$, is also in L_G. Also, if we consider the Pm representations with exactly $n+1$ digits in L_G, then we see that the Pm representation that consists of the digit m followed by n zero digits has the largest value among them all; that is, the value of the Pm representation $a_n \cdots a_0$ in L_G is bounded from above by

$$\text{value}(a_n \cdots a_0) \leq m(m^{n+1} - 1).$$

Now, we will show that L_G is the set of all Pm representations produced by the greedy algorithm.

Theorem 3.1 *The regular language L_G consists of exactly the Pm representations produced by the greedy algorithm for nonnegative-integer multiples of $m-1$. Hence, the set of Pm representations produced by the greedy algorithm for nonnegative-integer multiples of $m-1$ is regular.*

Proof: We first show that the set of greedy representatives is a subset of L_G. In other words, the Pm representations produced by the greedy algorithm for the number $p(m-1)$ is in L_G, for all $p \geq 0$. We use induction on p.

Basis: For $p = 0$, the greedy representative of 0 is ϵ, which is in L_G. For $p = 1, 2, \ldots, m$, the Pm representation produced by the greedy algorithm for the number $p(m-1)$ is p and p is in L_G.

Induction hypothesis: Assume that the Pm representation produced by the greedy algorithm for the number $p'(m-1)$ is in L_G, for all $0 \leq p' < p$, for some $p > m$.

Induction step: Let $a_n \cdots a_0$ be the Pm representation produced by the greedy algorithm for the number $p(m-1)$. There are two cases to consider: either a_n is the only non-zero digit in the Pm representation $a_n \cdots a_0$ or there is more than one non-zero digit in the Pm representation $a_n \cdots a_0$.

a_n **is the only non-zero digit.** Then, $a_n \cdots a_0 = a_n 0^n$. Since the Pm representation is produced by the greedy algorithm, $\text{value}(a_n \cdots a_0) = a_n(m^{n+1} - 1) < m^{n+2} - 1$. Thus, $a_n \leq m$. But, $\{1, 2, \ldots, m\}0^* \subset L_G$; therefore, if a_n is the only non-zero digit, then the Pm representation $a_n \cdots a_0$ is in L_G.

a_n **is not the only non-zero digit.** Let k be the second largest index of a non-zero digit in the Pm representation $a_n \cdots a_0$; that is, let $a_n \cdots a_0 = a_n 0^{n-k-1} a_k \cdots a_0$, where $a_k > 0$. Because $a_n \cdots a_0$ is a greedy representation, $a_i \geq 0$ and $\sum_{j=0}^{i} a_j(m^{j+1} - 1) < m^{i+2} - 1$, for all $i \geq 0$. Therefore, $a_k \cdots a_0$ satisfies these two conditions as well. The greedy representation of $R_k = \text{value}(a_k \cdots a_0)$ must satisfy these two conditions and, as we argued above, there is only one representation for R_k that satisfies these two conditions. Therefore, $a_k \cdots a_0$ is the greedy representative for R_k. Note that $R_k = \text{value}(a_n \cdots a_0) - a_n(m^{n+1} - 1)$ is a smaller multiple of $m - 1$ than $p(m - 1)$, since $\text{value}(a_n \cdots a_0) = p(m-1)$ and $m^{n+1} - 1$ is a positive multiple of $m - 1$. By the induction hypothesis, since $R_k < p(m - 1)$, the Pm representation $a_k \cdots a_0$ is in L_G. If we can show that $a_n < m$, then $a_n \cdots a_0 = a_n 0^{n-k-1} a_k \cdots a_0$ is in L_G, too. (If $a_n = m$, then, for the Pm representation $a_n \cdots a_0$ to be in L_G, we would need $a_i = 0$, for $0 \leq i < n$. Since $a_k > 0$, we must have $a_n < m$.) Since $a_n \cdots a_0$ is produced by the greedy algorithm, $\text{value}(a_n \cdots a_0) < m^{n+2} - 1$. Since a_n and a_k may not be the only non-zero digits in the Pm representation $a_n \cdots a_0$, we have $a_n(m^{n+1} - 1) + a_k(m^{k+1} - 1) \leq \text{value}(a_n \cdots a_0)$. If $a_n \geq m$, then, since $a_k > 0$ and $k \geq 0$, we have $a_n(m^{n+1} - 1) + a_k(m^{k+1} - 1) \geq m(m^{n+1} - 1) + 1(m^{k+1} - 1) \geq m^{n+2} - 1$, a contradiction. Therefore, $a_n < m$. Thus, if a_n is not the only non-zero digit, the Pm representation $a_n \cdots a_0$ is in L_G.

Therefore, a Pm representation produced via the greedy algorithm is in L_G.

Now, we show that any Pm representation in L_G is produced by the greedy algorithm for the corresponding number. We now show that every Pm representation $a_n \cdots a_0$ in L_G satisfies $a_i \geq 0$ and $\sum_{j=0}^{i} a_j(m^{j+1} - 1) < m^{i+2} - 1$, for all $i \geq 0$, thus proving, by Proposition 2.1, that every element of L_G is a greedy representation. Let $a_n \cdots a_0$ be in L_G. Clearly, $a_k \geq 0$, for all k, $0 \leq k \leq n$. Also, for any k, $0 \leq k \leq n$, the Pm representation $a_k \cdots a_0$ (ignoring leading zeros) is in L_G. Now, the value of the Pm representation $a_k \cdots a_0$ in L_G is bounded from above by $\text{value}(a_k \cdots a_0) \leq m(m^{k+1} - 1)$. But $m(m^{k+1} - 1) < m^{k+2} - 1$, so $\text{value}(a_k \cdots a_0) < m^{k+2} - 1$, as required. □

4 The Pm Representations of the Unambiguous Numbers

There are many Pm representations that are not in L_G; namely, all those that have a digit equal to m and some other lower-order non-zero digit. Since the value of a Pm representation that is not in L_G is also the value of some Pm representation that is in L_G, the numbers corresponding to Pm representations that are not in L_G are ambiguous. For example, the Pm number $m0^{n-2}1$, which is not in L_G, has value $m(m^n-1)+(m-1) = m^{n+1}-1$ and so does the Pm number 10^n, which is in L_G. Thus, the number $m^{n+1}-1$ is ambiguous in the Pm number system. We prove that the set of numbers that are unambiguous in the Pm number system is a regular set.

Definition 4.1 *Let L_U be the regular language*

$$L_U = \{1,\ldots,m-1\}^+[0m+\{1,\ldots,m-1\}\{0,\ldots,m\}+m0]+\{\epsilon,1,\ldots,m0\}.$$

Thus, L_U contains all Pm representations that fall, in lexicographic order, between (and including) ϵ and $m0$, and L_U contains all Pm representations $a_n\cdots a_0$, for $n \geq 2$, such that the last two digits a_1a_0 fall, in lexicographic order, between (and including) $0m$ and $m0$, and $0 < a_i < m$, for all i, $2 \leq i \leq n$.

Clearly, L_U is a subset of L_G; that is, a Pm representation in L_U has at most one digit equal to m, and, if it has a digit equal to m, then all lower-order digits are zero. Furthermore, if $a_n\cdots a_0$ is in L_U, for some $n \geq 2$, then $a_{n-1}\cdots a_0$ is in L_U, too.

We will show that the Pm representations in L_U are exactly the Pm representations of the unambiguous numbers. To do this, we first show that no two Pm representations in $\{0,1,\ldots,mm\}$ have the same value and then we bound the values of the Pm representations in L_U.

Lemma 4.1 *Let S_2 be the set of Pm representations with one or two digits; that is, let $S_2 = \{\epsilon,1,\ldots,mm\}$. If x and y are in S_2 and $x \neq y$, then* $\text{value}(x) \neq \text{value}(y)$.

Proof: For convenience, we treat all Pm numbers in S_2 as if they have two digits, by adding leading zeros if necessary. Let a_1a_0 and b_1b_0 be in S_2 and let $a_1a_0 \neq b_1b_0$. There are two cases to consider: either $a_1 \neq b_1$, or $a_1 = b_1$ and $a_0 \neq b_0$.

If $a_1 \neq b_1$, then assume, without loss of generality, that $a_1 < b_1$. Consider $\text{value}(a_1a_0) = a_1(m^2-1)+a_0(m-1)$. Since $a_1 < b_1$ and $a_0 \leq m$, we have $\text{value}(a_1a_0) \leq (b_1-1)(m^2-1)+m(m-1) = b_1(m^2-1)-(m-1)$. Since $b_0 \geq 0$, we have $\text{value}(a_1a_0) < b_1(m^2-1)+b_0(m-1) = \text{value}(b_1b_0)$.

If $a_1 = b_1$ and $a_0 \neq b_0$, assume, without loss of generality, that $a_0 < b_0$. Then, $\text{value}(a_1a_0) \leq b_1(m^2-1)+(b_0-1)(m-1)$. Since $0 \leq a_0 < b_0$ and $m > 1$, we have $\text{value}(a_1a_0) < b_1(m^2-1)+b_0(m-1) = \text{value}(b_1b_0)$.

In both cases, $\text{value}(a_1a_0) \neq \text{value}(b_1b_0)$. □

Note that this result does not establish the unambiguity of the numbers with representations in S_2 because it does not consider Pm representations with more than two digits. Indeed, the numbers corresponding to some two digit Pm representations are ambiguous. For example, the number $m(m^2-1)+m-1$ is represented in the Pm number system by $m1$ and 100.

Lemma 4.2 *Let $a_n \cdots a_0$ be in L_U. If $n < 2$, then*

$$0 \leq \text{value}(a_n \cdots a_0) \leq m(m^2-1).$$

Otherwise,

$$\frac{m^{n+2}-1}{m-1} - 2m - n \leq \text{value}(a_n \cdots a_0) \leq m^{n+2} - 1 - n(m-1).$$

Proof: The set of Pm representations with zero, one, or two digits is $L_U(2) = \{\epsilon, 1, \ldots, m0\}$ and this set consists of all Pm representations that fall, in lexicographic order, between (and including) ϵ and $m0$. By Lemma 4.1, no two of these representations have the same value. If we list the elements of $L_U(2)$ in lexicographic order, their values are strictly increasing. To see this, consider the Pm representation that comes after a_1a_0 (we add leading zeros as necessary to obtain two digits). If $a_0 < m$, then the next representation is $a_1(a_0+1)$ and

$$\begin{aligned} \text{value}(a_1a_0) &= a_1(m^2-1) + a_0(m-1) \\ &< a_1(m^2-1) + (a_0+1)(m-1) \\ &= \text{value}(a_1(a_0+1)). \end{aligned}$$

If $a_0 = m$, then the next number is $(a_1+1)0$ and

$$\begin{aligned} \text{value}(a_1a_0) &= a_1(m^2-1) + m(m-1) \\ &< a_1(m^2-1) + (m+1)(m-1) \\ &= (a_1+1)(m^2-1) \\ &= \text{value}((a_1+1)0). \end{aligned}$$

Therefore, if $a_1a_0 \in L_U$, then

$$\text{value}(\epsilon) = 0 \leq \text{value}(a_1a_0) \leq m(m^2-1) = \text{value}(m0).$$

If $a_n \cdots a_0 \in L_U$ and $n \geq 2$, then, by similar arguments about the last two digits of this number,

$$\text{value}(a_n \cdots a_2 0m) \leq \text{value}(a_n \cdots a_0) \leq \text{value}(a_n \cdots a_2 m0).$$

If some $a_i > 1$, where $2 \leq i \leq n$, then we can subtract 1 from a_i to create a Pm representation in L_U with smaller value than value$(a_n \cdots a_0)$. Thus, value$(1^{n-1}0m) \leq$ value$(a_n \cdots a_0)$, where 1^{n-1} represents a string of

$n-1$ ones. Similarly, if some $a_i < m-1$, where $2 \le i \le n$, then we can add 1 to a_i to create another Pm representation in L_U with greater value than value$(a_n \cdots a_0)$. Thus, value$(a_n \cdots a_0) \le$ value$((m-1)^{n-1}m0)$, where $(m-1)^{n-1}$ represents a string of $m-1$'s of length $n-1$. □

Theorem 4.3 *A number is unambiguous in the Pm number system if and only if it has a representation in* L_U*. Hence, the set of Pm representations of unambiguous numbers is regular.*

Proof: We split the proof into two parts.
Claim 1: Each Pm representation in L_U is the only Pm representation with its value.

Clearly, ϵ is the only Pm representation for 0. Consider the Pm representations $a_n \cdots a_0$ in L_U with positive values. The proof is by induction on n.
Basis: The set $L_U(2) = \{1, \ldots, m0\}$ contains the only Pm representations in L_U, for $n = 0$ and $n = 1$. Any Pm representation with three or more digits has value at least value$(100) = m^3 - 1$. The values of the Pm representations $m1, m2, \ldots, mm$ (the only Pm representations with at most two digits that are not in $L_U(2)$) are at least value$(m1) = m^3 - 1$. By Lemma 4.2, the value of a Pm representation in $L_U(2)$ is at most $m(m^2-1) < m^3 - 1$, for all $m > 1$. By Lemma 4.1, no two of the Pm representations in $L_U(2)$ have the same value. Therefore, each representation in $L_U(2)$ is unambiguous.
Induction hypothesis: Assume that each Pm representation $a_k \cdots a_0$ in L_U is the only Pm representation for value$(a_k \cdots a_0)$, for all $k < n$, for some $n > 1$.
Induction step: Let $a_n \cdots a_0$ be a Pm representation in L_U. Assume that there exists some other Pm representation $b_k \cdots b_0$ (not necessarily in L_U or L_G) with the same value. There are three possibilities: either $k > n$, $k < n$, or $k = n$.

$k > n$. We show that value$(a_n \cdots a_0) <$ value$(b_k \cdots b_0)$; that is, we cannot have a Pm representation $b_k \cdots b_0$ with the same value as $a_n \cdots a_0$.

Clearly, we have value$(b_k \cdots b_0) \ge$ value$(10^k) = m^{k+1} - 1$. Since $a_n \cdots a_0 \in L_U$, by Lemma 4.2, value$(a_n \cdots a_0) \le m^{n+2} - 1 - n(m-1)$. But, $m^{n+2} - 1 - n(m-1) < m^{n+2} - 1$, since $m > 1$ and $n \ge 2$. Since $k > n$, value$(a_n \cdots a_0) < m^{n+2} - 1 \le m^{k+1} - 1 \le$ value$(b_k \cdots b_0)$, a contradiction.

$k < n$. We show that the difference value$(b_k \cdots b_0) -$ value$(a_{n-1} \cdots a_0)$ is different from value$(a_n \cdots a_0) -$ value$(a_{n-1} \cdots a_0) = a_n(m^{n+1} - 1)$. Thus, the Pm representations $a_n \cdots a_0$ and $b_k \cdots b_0$ cannot have the same value, a contradiction.

Consider the difference value$(b_k \cdots b_0) -$value$(a_{n-1} \cdots a_0)$. This difference should be $a_n(m^{n+1} - 1)$, since value$(b_k \cdots b_0) =$ value$(a_n \cdots a_0)$. Now,

$$\text{value}(b_k \cdots b_0) \leq m\left(\frac{m^{k+2}-1}{m-1} - k - 2\right) \leq m\left(\frac{m^{n+1}-1}{m-1} - n - 1\right).$$

Furthermore, since $a_{n-1} \cdots a_0 \in L_U$, by Lemma 4.2,

$$\frac{m^{n+1}-1}{m-1} - 2m - n + 1 \leq \text{value}(a_{n-1} \cdots a_0).$$

Therefore,

$$\begin{aligned} &\text{value}(b_k \cdots b_0) - \text{value}(a_{n-1} \cdots a_0) \\ \leq\ & m\left(\frac{m^{n+1}-1}{m-1} - n - 1\right) - \left(\frac{m^{n+1}-1}{m-1} - 2m - n + 1\right) \\ =\ & m^{n+1} - 1 - (m-1)(n-1) \\ <\ & m^{n+1} - 1, \end{aligned}$$

since $m > 1$ and $n \geq 2$. Thus, value$(b_k \cdots b_0) -$ value$(a_{n-1} \cdots a_0) \neq a_n(m^{n+1} - 1)$, a contradiction.

$k = n$. We know that $a_n \cdots a_0$ is in $L_U \subseteq L_G$; that is, $a_n \cdots a_0$ is produced by the greedy algorithm when it is given value$(a_n \cdots a_0)$. Therefore, $a_n = \lfloor \text{value}(a_n \cdots a_0)/(m^{n+1} - 1) \rfloor$. But this implies that b_n cannot be larger than a_n; that is, $b_n \leq a_n$.

Suppose $b_n = a_n$. Then, $b_{n-1} \cdots b_0$ is not equal to $a_{n-1} \cdots a_0$ and value$(b_{n-1} \cdots b_0) =$ value$(a_{n-1} \cdots a_0)$. Now, $a_{n-1} \cdots a_0$ is in L_U and, by the induction hypothesis, it is the only Pm representation for value$(a_{n-1} \cdots a_0)$. Therefore, we must have $b_{n-1} \cdots b_0 = a_{n-1} \cdots a_0$, a contradiction.

Now, if $b_n < a_n$, then the Pm representations $b_{n-1} \cdots b_0$ and $(a_n - b_n)a_{n-1} \cdots a_0$ are two different Pm representations with the same value. Since $a_n - b_n > 0$ and $a_n \cdots a_0$ is in L_U, the Pm representation $(a_n - b_n)a_{n-1} \cdots a_0$ is also in L_U. We have already shown above that we cannot have some Pm representation $(a_n - b_n)a_n \cdots a_0$ in L_U and some other Pm representation $b_{n-1} \cdots b_0$ such that value$(a_n \cdots a_0) =$ value$(b_k \cdots b_0)$. Thus, this case is not possible either.

Each possibility leads to a contradiction; therefore, our assumption that there exists some other Pm representation $b_k \cdots b_0$ that has the same value as $a_n \cdots a_0 \in L_U$ must be false. Thus, each Pm representation in L_U is the only Pm representation with the corresponding value.

Claim 2: Each number that does not have a representation in L_U is ambiguous.

Suppose the Pm representation $a_n \cdots a_0$ is not in L_U. We construct another Pm representation for value$(a_n \cdots a_0)$ to show that value$(a_n \cdots a_0)$ is ambiguous. There are two cases to consider: either $a_n \cdots a_0$ is in L_G or $a_n \cdots a_0$ is not in L_G.

If $a_n \cdots a_0$ is not in L_G, then, by Theorem 3.1, there exists some Pm representation $b_k \cdots b_0$ in L_G such that value$(b_k \cdots b_0)$ = value$(a_n \cdots a_0)$. Thus, value$(a_n \cdots a_0)$ is ambiguous.

Otherwise, $a_n \cdots a_0$ is in L_G and we use the equality

$$m^{k+1} - 1 = m(m^k - 1) + (m - 1)$$

to build another Pm representation with the same value as $a_n \cdots a_0$. There are two subcases to consider: either there are digits $a_{j-1} = 0$ and $a_j > 0$, for some j, $2 < j \leq n$, or there are not.

Two such digits, a_{j-1} and a_j, exist. If $a_0 = m$, then, by the definition of L_G, since a_0 is non-zero, a_1 cannot be m. Consider the Pm representation $b_n \cdots b_0$, where

$$\begin{aligned} b_j &= a_j - 1, \\ b_{j-1} &= a_{j-1} + m = m, \\ b_1 &= a_1 + 1, \\ b_0 &= 0, \text{ and} \\ b_i &= a_i, \text{ otherwise.} \end{aligned}$$

Since $j > 2$, we have not defined digit b_1 twice, so,

$$\begin{aligned} \text{value}(b_n \cdots b_0) &= \text{value}(a_n \cdots a_0) - (m^{j+1} - 1) \\ &\quad + m(m^j - 1) + m^2 - 1 - m(m - 1) \\ &= \text{value}(a_n \cdots a_0). \end{aligned}$$

If $a_0 < m$, consider the Pm representation $b_n \cdots b_0$, where

$$\begin{aligned} b_j &= a_j - 1, \\ b_{j-1} &= a_{j-1} + m = m, \\ b_0 &= a_0 + 1, \text{ and} \\ b_i &= a_i, \text{ otherwise.} \end{aligned}$$

We have

$$\begin{aligned} \text{value}(b_n \cdots b_0) &= \text{value}(a_n \cdots a_0) - (m^{j+1} - 1) \\ &\quad + m(m^j - 1) + m - 1 \\ &= \text{value}(a_n \cdots a_0). \end{aligned}$$

Thus, value($a_n \cdots a_0$) is ambiguous.

Two such digits, a_{j-1} and a_j, do not exist. Then, either $n \leq 2$, or $n > 2$ and $a_j > 0$, for all j, $2 \leq j \leq n$.

Let us first consider $n \leq 2$. (We add leading zeros as required to make all representations under consideration exactly three digits long.) Since $a_2a_1a_0$ is in L_G, if any digit is m, then all lower-order digits are zero. Since $a_2a_1a_0$ is not in L_U, either $a_2 = m$ or $0 < a_2 < m$ and a_1a_0 is in $\{00, 01, \ldots, 0(m-1), m1, m2, \ldots, mm\}$ or $a_2 = 0$ and a_1a_0 is in $\{m1, m2, \ldots, mm\}$. Combining these two restrictions, we see that if $n \leq 2$, then $a_2a_1a_0$ is in $\{m00\}+\{1, 2, \ldots, m-1\}\{00, 01, \ldots, 0(m-1)\}$. Since $a_2 > 0$, $a_1 = 0$, and $a_0 < m$ in each case, the representation $(a_2 - 1)m(a_0 + 1)$ is a valid Pm representation and

$$\begin{aligned} \text{value}((a_2-1)m(a_0+1)) &= \text{value}(a_2a_1a_0) - (m^3-1) \\ &\quad + m(m^2-1) + m - 1 \\ &= \text{value}(a_2a_1a_0). \end{aligned}$$

Now let us consider $n > 2$ and $a_j > 0$, for all j, $2 \leq j \leq n$. Since $a_n \cdots a_0$ is in L_G, if any digit is in m, then all lower-order digits are zero. Thus, since $a_j > 0$, for all $2 \leq j \leq n$, we have $a_j \neq m$, for all j, $2 < j \leq n$. Since $a_n \cdots a_0$ is not in L_U, either $a_2 = m$ (in which case $a_1a_0 = 00$, since $a_n \cdots a_0$ is in L_G), or $0 < a_2 < m$ and $a_1a_0 \notin [0m + \{1, \ldots, m-1\}\{0, \ldots, m\} + m0]$ (in which case $a_1a_0 \in \{00, 01, \ldots, 0(m-1)\}$, since $a_n \cdots a_0$ is in L_G). Since $a_2 > 0$, $a_1 = 0$, and $a_0 < m$ in each case, the representation $a_n \cdots a_3(a_2-1)m(a_1+1)$ is a valid Pm representation and

$$\begin{aligned} \text{value}(a_n \cdots a_3(a_2-1)m(a_1+1)) &= \text{value}(a_n \cdots a_0) - (m^3-1) \\ &\quad + m(m^2-1) + m - 1 \\ &= \text{value}(a_n \cdots a_0). \end{aligned}$$

Thus, once again value($a_n \cdots a_0$) is ambiguous.

Therefore, each number that does not have a Pm representation in L_U is ambiguous. □

5 Conclusion

We have characterized the set of Pm representations that are constructed by the greedy algorithm and the set of numbers that are unambiguous in the Pm number system and shown that these are regular sets.

One question that we have not answered is whether we need all the digits $0, 1, \ldots, m$. For instance, if we are not allowed to use the digit m, would some integer that had a Pm representation no longer have any Pm representation? We see that L_U uses all the digits from $\{0, 1, \ldots, m\}$ and each number with a representation in L_U has only one Pm representation. Thus, we need all the digits $0, 1, \ldots, m$, if all nonnegative integers of the form $p(m-1)$ are to be represented. This observation leaves an open problem: Characterize the integers that have Pm representations if the digit set is restricted to some subset of $\{0, 1, \ldots, m\}$.

As noted in the introduction, another more general problem that remains is: Characterize the number systems for which the set of greedy representations and the set of representations of unambiguous numbers are regular.

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