Description of Generalized Continued Fractions by Finite Automata

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Abstract.

A generalized continued fraction algorithm associates with every real number x a sequence of integers; x is rational iff the sequence is finite. For a fixed algorithm, call a sequence of integers *valid* if it is the result of that algorithm on some input x_0. We show that, if the algorithm is sufficiently well-behaved, then the set of all valid sequences is accepted by a finite automaton.

I. Introduction.

It is well known that every real number x has a unique expansion as a *simple continued fraction* in the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$
$$= [a_0, a_1, a_2, ...]$$

where $a_i \in \mathbb{Z}$ for $i \geq 0$, $a_j \geq 1$ for $j \geq 1$, and if the expansion terminates with a_n, then $a_n \geq 2$.

Given x, we may find its simple continued fraction expansion with the following algorithm:

Algorithm SCF(x); outputs $(a_0, a_1, \ldots)$:

SCF1. Set $x_0 \leftarrow x$; set $i \leftarrow 0$.

SCF2. Set $a_i \leftarrow \lfloor x_i \rfloor$.

SCF3. If $a_i = x_i$ then stop. Otherwise set $x_{i+1} \leftarrow 1/(x_i - a_i)$; set $i \leftarrow i + 1$ and go to step SCF2.

For example, SCF(52/43) $= (1, 4, 1, 3, 2)$.

In fact, the rules "$a_j \geq 1$ for $j \geq 1$" and "if the expansion terminates with a_n, then $a_n \geq 2$" exist precisely so the set of valid expansions *coincide* with the possible outputs of the continued fraction algorithm.

There exist other versions of the continued fraction algorithm. For example, the so-called *nearest integer continued fraction* (NICF) satisfies the following rules: $a_j \leq -2$ or $a_j \geq 2$ for $j \geq 1$; if $a_j = -2$ then $a_{j+1} \leq -2$; if $a_j = 2$ then $a_{j+1} \geq 2$; and if the expansion terminates with a_n, then $a_n \neq 2$. The NICF is generated by algorithm SCF above with step SCF2 replaced by

SCF2′. Set $a_i \leftarrow \lfloor x_i + \frac{1}{2} \rfloor$.

For example, NICF(52/43) $= (1, 5, -4, -2)$.

(Actually, the NICF is usually described slightly differently in the literature, but our formulation is essentially the same. See [Hur2].)

The concept of "rules" that describe the set of possible outputs of a continued fraction expansion also appears in a paper of Hurwitz [Hur1] which describes the nearest integer continued fraction algorithm in $\mathbb{Z}[i]$.

In this paper, we are concerned with the following questions:

(1) Which functions f are suitable replacements for the floor function in Algorithm SCF (i. e. yield generalized continued fraction algorithms)?

(2) Which of these functions correspond to generalized continued fraction algorithms which have "easily describable" outputs (i. e. accepted by a finite automaton)?

In this paper, we will answer question (1) by *fiat*, and then examine the consequences for question (2).

II. Real Integer Functions and Finite Automata.

Let us introduce some notation. By $[a_0, a_1, a_2, ..., a_n]$ we will mean the *value* of the expression

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n}}},$$

and *not* necessarily the result of the algorithm SCF.

Let $f : \mathbf{R} \to \mathbb{Z}$. We say f is a *real integer function* if

(i) $|f(x) - x| < 1$ for all $x \in \mathbf{R}$;

(ii) $f(x + j) = f(x) + j$ for all $x \in \mathbf{R}$, $j \in \mathbb{Z}$.

Examples are $f(x) = \lfloor x \rfloor$, $f(x) = \lceil x \rceil$, $f(x) = \lfloor x + \frac{1}{2} \rfloor$.

Real integer functions induce generalized continued algorithms by imitating algorithm SCF above:

Algorithm $\mathrm{CF}_f(x)$;outputs $(a_0, a_1, \ldots)$:

CF1. Set $x_0 \leftarrow x$; set $i \leftarrow 0$.

CF2. Set $a_i \leftarrow f(x_i)$.

CF3. If $a_i = x_i$ then stop. Otherwise set $x_{i+1} \leftarrow 1/(x_i - a_i)$, $i \leftarrow i + 1$ and go to step C2.

We leave it to the reader to verify that (i) The algorithm CF_f terminates iff x is rational and (ii) if $\mathrm{CF}_f(x)$ terminates, with $(a_0, a_1, \ldots a_n)$ as output, then $x = [a_0, a_1, \ldots, a_n]$.

The main result of this paper is that the outputs of CF_f are easily describable in most of the interesting cases, including all the examples mentioned above. Let us define more rigorously what we mean by "easily describable".

Call a (finite or infinite) sequence of integers *valid* if it is the result of $\mathrm{CF}_f(x)$ for some x. We envision a finite automaton which reads a purported finite expansion $a = (a_0, a_1, \ldots a_n)$ and reaches a final state on the last input iff a is valid. Also, given a valid infinite sequence $(a_0, a_1, \ldots)$, the automaton should never "crash" (i. e. attempt to make a transition for which the resulting state is undefined), though it may fail to "crash" on invalid infinite expansions.

We emphasize again that our description must in some sense cover *all* valid outputs of the algorithm, and is *not* concerned with, for example, the periodicity for specific inputs.

One minor problem with the model described above is that the a_i belong to $\mathbb{Z}$, but in defining finite automata we usually insist that our alphabet Σ is finite. We can get around this in one of two ways: first, we could expand the definition of finite automata so that there can be infinitely many transitions (but still only finitely many states). Second, we could redefine our strings as numbers encoded in a particular base. (Even if a state has infinitely many transitions associated with it, they are all of a certain form that is easily describable by a regular set.) It turns out that either approach is satisfactory, but for simplicity we choose the first.

Definition.

A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where Q is a finite set of states, Σ is a (not necessarily finite) input alphabet, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states, and δ is the transition function mapping $Q \times \Sigma$ to Q. δ may be incomplete; i. e. $\delta(q, a)$ may be undefined for some pairs q, a.

We extend δ to a function which maps $Q \times \Sigma^*$ to Q in the obvious fashion.

The reader to whom these definitions are unfamiliar should consult [HU].

Notation.

If A is a set, then by A^{-1} we mean the set $\{x \in \mathbf{R} : \mathbf{x^{-1}} \in \mathbf{A}\}$. If f is a function, then by $f^{-1}[a]$ we mean, as usual, the set $\{x \in \mathbf{R} : \mathbf{f(x)} = \mathbf{a}\}$. If A is a set, then by $A - a$ we mean the set $\{x : x + a \in A\}$. We will say x is *quadratic* if x is the real root of a quadratic equation with integer coefficients.

Definition.

Let f be a real integer function. Then we say that the finite automaton $A = (Q, \mathbb{Z}, \delta, q_0, F)$ accepts the outputs of CF_f if

(i) $\delta(q_0, a_0 a_1 a_2 \cdots a_n) \in F$ iff there exists $q \in \mathbb{Q}$ such that $\mathrm{CF}_f(q) = (a_0, a_1, \ldots, a_n)$.

(ii) If x is irrational and $\mathrm{CF}_f(x) = (a_0, a_1, \ldots)$, then $\delta(q_0, a_0 a_1 \cdots a_n)$ is defined for all $n \geq 0$.

The object of this paper is to prove the following theorem:

Theorem 1.

Let $f^{-1}[0]$ be the finite union of intervals. Then there exists a finite automaton accepting the outputs of CF_f iff all the endpoints of the intervals are rational or quadratic.

In section III below, we will prove one direction of this theorem; in section IV, we prove the other.

Comment.

No simple characterization seems to exist in the case where f is not the finite union of intervals. In section IV below, we will give an example of an f that is accepted by a finite automaton, but $f^{-1}[0]$ is not the finite union of intervals.

III. One direction of the theorem.

Let $f^{-1}[0]$ be the finite union of intervals. We will create a finite automaton as follows: states will correspond to certain subsets of $f^{-1}[0]$, and transitions will correspond to partial quotients a_i. We will define $\delta(q_0, a_0) = f^{-1}[0]$ for all $a_0 \in \mathbb{Z}$ and inductively define

$$\delta(q_i, a) = q_j$$

where $q_j = (q_i^{-1} \cap f^{-1}[a]) - a$. We say $q_i \in F$ if $0 \in q_i$.

To verify that this construction works, we need to show that (i) the automaton accepts CF_f and (ii) this process generates only a finite number of distinct states.

Let us agree to the following unpleasant notation. When we write

$$\mathrm{CF}_f(x) = (a_0, a_1, a_2, \ldots, a_n, \ldots)$$

we will mean that the first $n+1$ outputs of the algorithm CF_f on x are given by a_0 through a_n; there may be more outputs or not.

Lemma 2.

$$\delta(q_0, a_0a_1 \cdots a_n) = \{x : \mathrm{CF}_f([a_0, a_1, a_2, \cdots, a_{n-1}, a_n + x]) = (a_0, a_1, \ldots, a_n, \ldots)\}.$$

Proof.

The lemma is proved by induction. It is easy to verify that

$$\delta(q_0, a_0) = f^{-1}[0] = \{x : \mathrm{CF}_f([a_0 + x]) = (a_0, \ldots)\}.$$

Assume true for k. Then

$$\delta(q_0, a_0 a_1 \cdots a_k) = \{x : \mathrm{CF}_f([a_0, a_1, \cdots, a_{k-1}, a_k + x]) = (a_0, a_1, \ldots a_k, \ldots)\}.$$

Thus

$$\delta(q_0, a_0 a_1 \cdots a_k)^{-1} = \{x^{-1} : \mathrm{CF}_f([a_0, a_1, \cdots, a_{k-1}, a_k + x]) = (a_0, a_1, \ldots a_k, \ldots)\}$$

$$\Rightarrow \delta(q_0, a_0 a_1 \cdots a_k)^{-1} = \{x : \mathrm{CF}_f([a_0, a_1, \cdots, a_{k-1}, a_k, x]) = (a_0, a_1, \ldots a_k, \ldots)\}$$

$$\Rightarrow \delta(q_0, a_0 a_1 \cdots a_k)^{-1} \cap f^{-1}[a_{k+1}] = \{x : \mathrm{CF}_f([a_0, a_1, \cdots, a_{k-1}, a_k, x]) = (a_0, a_1, \ldots a_k, a_{k+1}, \ldots)\}$$

$$\Rightarrow \delta(q_0, a_0 a_1 \cdots a_{k+1}) = (\delta(q_0, a_0 a_1 \cdots a_k)^{-1} \cap f^{-1}[a_{k+1}]) - a_{k+1}$$

$$= \{x : \mathrm{CF}_f([a_0, a_1, \cdots, a_k, a_{k+1} + x]) = (a_0, a_1, \ldots a_k, a_{k+1}, \ldots)\}.$$

which completes the proof. ■

Corollary.

$\delta(q_0, a_0 a_1 \cdots a_n) \in F$ iff there exists $q \in \mathbb{Q}$ such that $\mathrm{CF}_f(q) = (a_0, a_1, \ldots, a_n)$.

Proof.

Assume $\delta(q_0, a_0 a_1 \cdots a_n) \in F$. Then by the definition of the set of final states F, we must have $0 \in \delta(q_0, a_0 a_1 \cdots a_n)$. But by the lemma, then the first $n+1$ outputs of the algorithm CF_f on input $[a_0, a_1, \ldots, a_n]$ are precisely $(a_0, a_1, \ldots, a_n)$. Hence we may take $q = [a_0, a_1, \ldots, a_n]$.

Now assume that there exists $q \in \mathbb{Q}$ such that $\mathrm{CF}_f(q) = (a_0, a_1, \ldots, a_n)$. Then from the definition of CF_f, we see that $x_n = a_n$; hence

$$0 = x_n - a_n \in \delta(q_0, a_0 a_1 \cdots a_n)$$

which shows that $\delta(q_0, a_0 a_1 \cdots a_n)$ is a final state.

We leave it to the reader to verify that if x is irrational and $\mathrm{CF}_f(x) = (a_0, a_1, \ldots)$, then the automaton never crashes on any prefix of the output. ■

It remains to show that this construction generates a finite number of states. By the inductive definition of states as certain subsets of $f^{-1}[0]$, we see that if $e \in [-1, 1]$ is an endpoint of an interval in q_j, then either e is an endpoint of an interval of $f^{-1}[0]$, or

$e = (1/d) - f(1/d)$, where d is an endpoint of an interval q_i, where there exists a transition $\delta(q_i, a) = q_j$. Since for any particular x we have $f(x) = \lfloor x \rfloor$ or $f(x) = \lceil x \rceil$, it suffices to prove the following:

Lemma 3.

Define $s_1 : x \to (1/x) - \lfloor 1/x \rfloor$ and $s_2 : x \to (1/x) - \lceil 1/x \rceil$. Consider the semigroup u formed by s_1 and s_2 under composition. Let $u(x)$ be the *orbit* of x under elements of u.

Then $u(x)$ is finite iff x is rational or quadratic.

Proof.

One direction is easy. Assume $u(x)$ is finite. Then in particular the set

$$x, s_1(x), s_1^{(2)}(x), \ldots$$

is finite. Hence we have $s_1^{(j)}(x) = s_1^{(k)}(x)$ for some $j \neq k$. But it is easily proved by induction that

$$x = [0, a_0, a_1, \ldots, a_{n-1} + s_1^{(n)}(x)]$$

for some sequence of integers $a_0, a_1, \ldots$; hence there exist integers such that

$$x = \frac{a_j + b_j s_1^{(j)}(x)}{c_j + d_j s_1^{(j)}(x)},$$

and similarly

$$x = \frac{a_k + b_k s_1^{(k)}(x)}{c_k + d_k s_1^{(k)}(x)}.$$

Thus we see that $s_1^{(j)}(x)$ is the root of a quadratic equation, and so is either quadratic or rational. Thus x itself is either quadratic or rational.

Now let us prove the other direction. The assertion is trivial for x rational, $x = p/q$, for then $s_1(p/q) = (q \bmod p)/p$ and $s_2(p/q) = -((-q) \bmod p)/p$. Thus an application of s_1 or s_2 decreases the absolute value of the numerator, while retaining the relationship $|x| < 1$. Thus iterated applications of s_1 and s_2 reduce p/q to 0.

Now let us consider the case where x is the root of a quadratic equation with integer coefficients. We use the classical theorem that the simple continued fraction for x is ultimately periodic iff x is quadratic. If x is quadratic, let $r(x)$ denote the length of the repeating portion (*period*) of the simple continued fraction for x, and let $q(x)$ denote the length of the leading portion of the continued fraction. (Example: if $x = \sqrt{7}$, then $\mathrm{SCF}(x) = (2, 1, 1, 1, 4, 1, 1, 1, 4, \ldots)$; hence $r(x) = 4$ and $q(x) = 1$.)

Let $S_1 : x \to 1/(x - \lfloor x \rfloor)$ and $S_2 : x \to 1/(x - \lceil x \rceil)$. Since $S_1(x) = s_1(x^{-1})^{-1}$ and $S_2(x) = s_2(x^{-1})^{-1}$, it suffices to prove the theorem for the semigroup U formed by S_1 and S_2 under composition.

Let x be quadratic. We will show that $U(x)$ is finite by showing that repeated application of the maps S_1 and S_2 can result in at most a finite number of distinct simple continued fraction expansions. More precisely, we show that every element in $U(x)$ has a simple continued fraction whose period is identical to or is a cyclic shift of the period for x; that there exists a uniform upper bound for $q(y)$ for $y \in U(x)$, and that the partial quotients of the continued fraction for each $y \in U(x)$ are also bounded.

Let the simple continued fraction expansion of x be given by $(a_0, a_1, a_2, a_3, \ldots)$. Then

$$\mathrm{SCF}(S_1(x)) = (a_1, a_2, a_3, \ldots). \tag{1}$$

The description of $S_2(x)$ is slightly more complicated:

$$\mathrm{SCF}(S_2(x)) = \begin{cases} (-(a_2+2), a_4+1, a_5, a_6, \ldots) & \text{if } a_1 = 1, a_3 = 1; \\ (-(a_2+2), 1, a_3-1, a_4, a_5, \ldots) & \text{if } a_1 = 1, a_3 \geq 2; \\ (-2, a_2+1, a_3, a_4, \ldots\ ) & \text{if } a_1 = 2; \\ (-2, 1, a_1-2, a_2, a_3, \ldots\ ) & \text{if } a_1 \geq 3. \end{cases} \tag{2}$$

For example, see [Knu, pp. 358, 600].

From equations (1) and (2), it is clear that

$$r(S_i(x)) = r(x)$$

for $i = 1, 2$. An application of S_i does not change the period, although by "sliding" elements off the left end of the continued fraction, it may shift the period cyclically.

Now define $t(x) = \max(q(x), r(x), 3)$. I claim that

$$t(S_i(S_j(x))) \leq t(x),$$

for $1 \leq i, j \leq 2$. This is a tedious verification of cases, and is left to the reader. Since t is bounded, it follows that q is also bounded.

It remains to show that the partial quotients of elements of $U(x)$ are bounded. Let $a(x; i)$ denote the ith partial quotient of the simple continued fraction for x. Let $x^{(k)}$ denote the kth iterate of x under one of the two maps S_1 and S_2. Let $M = \max_{i \geq 0} |a(x;i)|$. Clearly M is finite since the simple continued fraction for x is ultimately periodic.

Then we will show that, for all $k \geq 0$,

(a) $1 \leq a(x^{(k)}; j) \leq M$ for all $j \geq 2$.

(b) $1 \leq a(x^{(k)}; 1) \leq M + 1$.

(c) $|a(x^{(k)}; 0)| \leq M + 2$.

Assume not. Then there exists a minimal superscript m such that one of the conditions above fails for $x^{(m)}$.

Write $\mathrm{SCF}(x^{(m-1)}) = (a_0, a_1, a_2, \ldots)$. Then using the lemma above, we have

$$\mathrm{SCF}(x^{(m)}) = \begin{cases} (a_1, a_2, a_3, \ldots) & \text{(i)} \\ (-(a_2+2), a_4+1, a_5, a_6, \ldots) & \text{(ii)} \\ (-(a_2+2), 1, a_3-1, a_4, a_5, \ldots) & \text{(iii)} \\ (-2, a_2+1, a_3, a_4, \ldots) & \text{(iv)} \\ (-2, 1, a_1-2, a_2, a_3, \ldots) & \text{(v)} \end{cases}$$

Assume (a) fails for $k = m$. Then $a(x^{(m)}; j) > M$ for some j. But this is clearly false for $j \geq 3$. For $j = 2$, it is clearly false for cases (i), (ii), and (iv). For case (iii), $a_3 - 1 > M \Rightarrow a_3 > M + 1$, which is impossible by minimality of m. For case (v), $a_1 - 2 > M \Rightarrow a_1 > M + 2$, impossible by minimality of m.

Now assume (b) fails for $k = m$. Then $a(x^{(m)}; 1) > M + 1$. But this is clearly false for cases (i), (iii), (v). For case (ii), $a_4 + 1 > M + 1 \Rightarrow a_4 > M$, which is a contradiction. For case (iv), $a_2 + 1 > M + 1 \Rightarrow a_2 > M$, a contradiction.

Now assume (c) fails for $k = m$. Then $|a(x^{(m)}; 0| > M + 2$. But this is clearly false for cases (i), (iv), and (v). For cases (ii) and (iii), $|-(a_2+2)| > M + 2 \Rightarrow a_2 > M$, which is a contradiction.

This completes the proof of the Lemma 3. ■

Combining Lemmas 2 and 3 completes the proof of one direction of Theorem 1.

We now give an example of the construction of the finite automaton. Let us obtain the description of the outputs for CF_f for $f(x) = \lfloor x + \frac{\sqrt{2}}{2} \rfloor$. We find $q_0 = f^{-1}[0] = [-\frac{\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2})$; $q_1 = [-\frac{\sqrt{2}}{2}, 1 - \sqrt{2}]$; $q_2 = (\sqrt{2} - 2, \frac{2-\sqrt{2}}{2})$; $q_3 = [1 - \sqrt{2}, \frac{2-\sqrt{2}}{2})$. The transitions $\delta(q_i, a)$ are given by the following table:

Insert table here

IV. Completing the proof of Theorem 1.

We now wish to show that if $f^{-1}[0]$ consists of the finite union of intervals, but one of those intervals has an endpoint that is not rational or quadratic, then no finite automaton can accept CF_f.

Assume that such an automaton A exists. Then we may assume that each state is in fact reachable from q_0; otherwise this state may be discarded without affecting A. For each state q_j, construct an input sequence $a_0a_1\cdots a_i$ such that $\delta(q_0, a_0a_1\cdots a_i) = q_j$. Let us label each state q_j with a subset of $\mathbb{Q}$, $L(q_j)$, by the following rule: If $\delta(q_0, a_0a_1\cdots a_i) = q_j$, then

$$L(q_j) = \{x \in \mathbb{Q} : \mathrm{CF}_f([a_0, a_1, \ldots, a_{i-1}, a_i + x]) = (a_0, a_1, \ldots, a_i, \ldots)\}.$$

We need to show that this map is indeed well-defined, in the sense that different paths from q_0 to q_j give the same labels $L(q_j)$. Assume that

$$\delta(q_0, a_0a_1\cdots a_i) = q_j$$

and

$$\delta(q_0, b_0b_1\cdots b_k) = q_j,$$

and there exists a rational number p such that

$$p \in S_1 = \{x \in \mathbb{Q} : \mathrm{CF}_f([a_0, a_1, \ldots, a_{i-1}, a_i + x]) = (a_0, a_1, \ldots, a_i, \ldots)\} \qquad (3)$$

but

$$p \notin S_2 = \{x \in \mathbb{Q} : \mathrm{CF}_f([b_0, b_1, \ldots, b_{k-1}, b_k + x]) = (b_0, b_1, \ldots b_k, \ldots)\}. \qquad (4)$$

Write $\mathrm{CF}_f(p) = (0, a_{i+1}, \ldots, a_n)$; by our definition of what it means to accept the output of CF_f, we know that

$$\delta(q_j, a_{i+1}\cdots a_n) = q_r \in F,$$

a final state. Let $y = [b_0, b_1, \ldots, b_k, a_{i+1}, \ldots a_n]$. Then since the automaton is in state q_j upon reading inputs $b_0 b_1 \cdots b_k$, we have

$$\delta(q_0, b_0 b_1 \cdots b_k a_{i+1} \cdots a_n) = q_r.$$

Hence $\mathrm{CF}_f(y) = (b_0, b_1, \ldots, b_k, a_{i+1}, \ldots a_n)$. But then $y = [b_0, b_1, \ldots, b_k + p]$ which shows that indeed $p \in S_2$, a contradiction.

Thus we may assume that sets $L_i = L(q_i)$ are well-defined. Let $\bar{A}$ denote the *closure* of the set A in $\mathbf{R}$, and consider the sets $\bar{L_i}$. I claim that since $f^{-1}[0]$ consists of the finite union of intervals, so does each of the sets $\bar{L_i}$; this follows easily from the definition of CF_f. Suppose $\delta(q_i, a) = q_j$; then the endpoints e of intervals of $\bar{L}_j$ are those of $f^{-1}[0]$ or are related to the endpoints E of $\bar{L}_i$ by the equation

$$e = \frac{1}{E} - a.$$

Since $f^{-1}[0]$ contains an endpoint which is not rational or quadratic, so must $\bar{L}_0$. Hence there exists a transition $\delta(q_0, a) = q_i$ such that $\bar{L}_i$ contains an endpoint which is not rational or quadratic. Continuing in this fashion, and remembering that there are only a finite number of states, we eventually return to a state previously visited, which gives one of the two equations

$$e = [0, a_1, \ldots, a_k]$$

or

$$e = [0, a_1, \ldots, a_k + e]$$

which shows that e is rational or quadratic, contrary to assumption.

This completes the proof of Theorem 1. ■

Now let us give an example of an f such that $f^{-1}[0]$ is not the finite union of intervals, but nevertheless there is a finite automaton accepting CF_f.

Let $f(x)$ be defined by

$$f(x) = \begin{cases} \lfloor x \rfloor, & \text{if } x \text{ is rational;} \\ \lceil x \rceil, & \text{if } x \text{ is irrational.} \end{cases}$$

Then

$$f^{-1}[0] = \{x : x \text{ rational }, 0 \le x \le 1\} \cup \{x : x \text{ irrational }, -1 \le x < 0\}.$$

Clearly $f^{-1}[0]$ cannot be written as the finite union of intervals. Then it is easily verified that the procedure of section III generates a finite automaton with four states that accepts CF_f.

It may be of interest to remark that the automata accepting the result of CF_f may be arbitrarily complex. For example, it can be easily shown that the automaton corresponding to

$$f^{-1}[0] = [-\frac{F_{n-1}}{F_n}, \frac{F_{n-2}}{F_n})$$

has $n+1$ states. (Here F_n denotes the nth Fibonacci number.)

V. Epilogue.

Several other writers have noted connections between finite automata and continued fractions. One of the best known papers is that of Raney, who showed how to obtain the simple continued fraction for

$$\beta = \frac{a\alpha + b}{c\alpha + d}$$

in terms of the continued fraction for α. See [Ran], [Bey].

Istrail considered the language consisting of all prefixes of the continued fraction for x, and observed that this language is context-free and non-regular iff x is a quadratic irrational [Ist].

Allouche discusses several applications of finite automata to number theory, including continued fractions [All].

In this paper, we have been concerned with a different approach; namely, describing the "set of rules" associated with a generalized continued fraction algorithm. One immediately wonders if similar theorems may be obtained for continued fraction algorithms in $\mathbb{Z}[i]$, such as those discussed by Hurwitz [Hur1] and McDonnell [McD].

In [Sha], the author proved that the McDonnell's continued fraction algorithm can be described by a finite automaton with 25 states. The corresponding result for Hurwitz's algorithm is not known.

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