Abstract

A numeration system based on a strictly increasing sequence of positive integers $u_0 = 1, u_1, u_2, \ldots$ expresses a non-negative integer n as a sum $n = \sum_{j=0}^{i} a_j u_j$. In this case we say the string $a_i a_{i-1} \cdots a_1 a_0$ is a *representation* for n. If $\gcd(u_0, u_1, \ldots) = g$, then every sufficiently large multiple of g has some representation.

If the lexicographic ordering on the representations is the same as the usual ordering of the integers, we say the numeration system is *order-preserving*. In particular, if $u_0 = 1$, then the *greedy representation*, obtained via the greedy algorithm, is order-preserving. We prove that, subject to some technical assumptions, if the set of all representations in an order-preserving numeration system is regular, then the sequence $u = (u_j)_{j \geq 0}$ satisfies a linear recurrence. The converse, however, is not true.

The proof uses two lemmas about regular sets that may be of independent interest. The first shows that if L is regular, then the set of lexicographically greatest strings of every length in L is also regular. The second shows that the number of strings of length n in a regular language L is bounded by a constant (independent of n) iff L is the finite union of sets of the form xy^*z.