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by

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Abstract

We consider joining two, tensor-product, Bézier patches, $\mathbf{F}(u,t)$ and $\mathbf{G}(r,t)$, so that they agree with G^1 continuity along a common boundary. The continuity condition may be given as

$$\alpha(t)D_u\mathbf{F}(1,t) + \beta(t)D_r\mathbf{G}(0,t) + \gamma(t)D_t\mathbf{G}(0,t) = 0 \ ,$$

where t has been used as the boundary parameter. Using cubic patches for the exposition, it is usual to argue from the orders of $D_u\mathbf{F}(1,t)$, $D_r\mathbf{G}(0,t)$, and $D_t\mathbf{G}(0,t)$ that $\alpha(t)$ and $\beta(t)$ must be constant and $\gamma(t)$ must be linear. This makes the geometric association of the control vertices for the patches at one end of the common boundary depend in a restrictive way upon the control vertices at the other end of the boundary. In order to avoid this limitation, it has been thought necessary to work with patches of higher order so as to obtain flexibility in constructing surfaces for computer-aided geometric design. We demonstrate that higher-order patches are not always required.

By assuming α, β, and γ to be polynomials of higher order than the degrees of $D_u\mathbf{F}(1,t)$, $D_r\mathbf{G}(0,t)$, and $D_t\mathbf{G}(0,t)$ might suggest and by expressing the result in Bernstein-Bézier form, it is possible to produce constraints on the control vertices of patches $\mathbf{F}$ and $\mathbf{G}$ in terms of the coefficients of the polynomials α, β, and γ that imply the continuity condition above.

It will be shown how these constraints can be expressed in terms of matrices derivable from the Bernstein polynomials. As long as the control vertices reside in a space associated with these matrices, the patches will meet with G^1 continuity.

§1. General Setting

Consider two bi-variate polynomial patches, $\mathbf{F}(u,v)$ and $\mathbf{G}(r,s)$, that intersect along a curve,

$$\mathbf{C}(t) = \mathbf{F}(u(t),v(t)) = \mathbf{G}(r(t),s(t)) \ .$$

The two patches are said to agree with G^d continuity if they admit a common approximating surface of at least degree d that varies "smoothly" along the intersection curve. (The degree of smoothness that is reasonable for the variation of the approximating surface is the issue that lies at the heart of the observations that we make in this paper.)

Two recent surveys of the concepts involved are to be found in [1,2]. We will be restricting our attention to G^1 continuity, that is, in which there is a common linear approximation (tangent plane) at each point of the intersection curve. Reflecting the material in [1], an analytic requirement that should be imposed upon the patches is that, at each such point $\mathbf{C}_0 = \mathbf{C}(t_0)$, $\mathbf{F}$ and $\mathbf{G}$ admit a common reparameterization with variables p and q,

$$u = u(p,q)\ , \quad v = v(p,q)\ , \quad r = r(p,q)\ , \quad \text{and} \quad s = s(p,q)\ ,$$

providing Taylor agreement to first order around the parameter values (p_0, q_0), which correspond to $t = t_0$:

$$\mathbf{F}(p_0,q_0) + (p - p_0)D_p\mathbf{F}(p_0,q_0) + (q - q_0)D_q\mathbf{F}(p_0,q_0)$$

$$= \mathbf{G}(p_0,q_0) + (p - p_0)D_p\mathbf{G}(p_0,q_0) + (q - q_0)D_q\mathbf{G}(p_0,q_0)\ .$$

This should hold along the intersection curve, $p = p(t)$, $q = q(t)$, in a neighborhood of t_0. From the independence of the functions $(p - p_0)$ and $(q - q_0)$, this is equivalent to asking for $D_p\mathbf{F}(p_0,q_0) = D_p\mathbf{G}(p_0,q_0)$ and $D_q\mathbf{F}(p_0,q_0) = D_q\mathbf{G}(p_0,q_0)$ to hold. Using the chain rule to reformulated these equations in terms of u and v, we expect

$$\left(\frac{\partial u}{\partial p}\right) D_u\mathbf{F} + \left(\frac{\partial v}{\partial p}\right) D_v\mathbf{F} = \left(\frac{\partial r}{\partial p}\right) D_r\mathbf{G} + \left(\frac{\partial s}{\partial p}\right) D_s\mathbf{G} \tag{1}$$

to hold as a function of t. An equivalent condition is obtained by considering $\frac{\partial}{\partial q}$. This condition reinforces what we knew from general principles of linear algebra: we can't have a common approximating tangent plane to to $\mathbf{F}$ and $\mathbf{G}$ along $\mathbf{C}$ unless the vectors $D_u\mathbf{F}$ and $D_v\mathbf{F}$ (respectively $D_r\mathbf{G}$ and $D_s\mathbf{G}$) lie in the plane defined by $D_r\mathbf{G}$ and $D_s\mathbf{G}$ (respectively $D_u\mathbf{F}$ and $D_v\mathbf{F}$).

§2. Smoothness of Variation

For a plane to be tangent to $\mathbf{F}$ at a point given by $(u(t), v(t))$ and tangent to $\mathbf{G}$ at a point given by $(r(t), s(t))$, we need the unit normals to $\mathbf{F}$ and $\mathbf{G}$ to be well-defined and be equal along $\mathbf{C}(t)$:

$$\frac{D_u\mathbf{F}(u(t),v(t)) \times D_v\mathbf{F}(u(t),v(t))}{\|D_u\mathbf{F}(u(t),v(t)) \times D_v\mathbf{F}(u(t),v(t))\|} \tag{2}$$

$$= \frac{D_r\mathbf{G}(r(t),s(t)) \times D_s\mathbf{G}(r(t),s(t))}{\|D_r\mathbf{G}(r(t),s(t)) \times D_s\mathbf{G}(r(t),s(t))\|}\ .$$

The degree of complexity possible in the variation of the common approximating tangent plane is indicated by this. For two intersecting bi-cubics, for example, $C(t)$ is, in general, a rational function whose numerator and denominator have degrees much higher than cubic [3]. Each of $D_u\mathbf{F}$ and $D_v\mathbf{F}$ can therefore have a like complexity, and the cross product of these two functions can be more complicated still.

In general, (2) suggests that we needn't expect a common linear approximation to two bi-cubics to vary with only low-degree smoothness, much less the approximation of degree d to two more general bi-variate patches. This suggests further that (1) might involve coefficient functions $\frac{\partial u}{\partial p}$ for $D_u\mathbf{F}$, $\frac{\partial v}{\partial p}$ for $D_v\mathbf{F}$, and so on, that reflect the complexity of the partial derivatives of $\mathbf{F}$ and $\mathbf{G}$ along $\mathbf{C}(t)$. Our investigations in this paper start with this motivation and explore whether one can assume a more complicated variation than cubic in the approximating tangent plane.

§3. Tensor Product

We will restrict the discussion to two tensor-product Bézier patches that abut along one of their boundary curves. This permits us to associate, say, the variable v of $\mathbf{F}$ and the variable s of $\mathbf{G}$ with the variable t of the intersection curve (the common boundary). Likewise, the derivatives $D_v\mathbf{F}$ and $D_s\mathbf{G}$ will be co-linear, and they can be represented by, say, $D_t\mathbf{G}$. Combining terms in (1) and rewriting the terms involving partial derivatives of p and q as functions α, β, and γ yields the requirement that

$$\alpha(t)D_u\mathbf{F}(1,t) + \beta(t)D_r\mathbf{G}(0,t) + \gamma(t)D_t\mathbf{G}(0,t) = 0 \tag{3}$$

along the common boundary. This, or some variant of it, is the typical condition to be found in the literature for G^1 continuity between Bézier patches [1,2,4,5,6]. This form is to be preferred, because it avoids the rational coefficients implicit in some of the variants.

We will assume cubics for this presentation, though higher orders will evidently follow suit. A schematic view of the control vertices for $\mathbf{F}$ and $\mathbf{G}$ in this case is given by

$$\begin{array}{ccccccc} \mathbf{N}_0 & \mathbf{O}_0 & \mathbf{P}_0 & \mathbf{Q}_0 & \mathbf{R}_0 & \mathbf{S}_0 & \mathbf{T}_0 \\ \mathbf{N}_1 & \mathbf{O}_1 & \mathbf{P}_1 & \mathbf{Q}_1 & \mathbf{R}_1 & \mathbf{S}_1 & \mathbf{T}_1 \\ \mathbf{N}_2 & \mathbf{O}_2 & \mathbf{P}_2 & \mathbf{Q}_2 & \mathbf{R}_2 & \mathbf{S}_2 & \mathbf{T}_2 \\ \mathbf{N}_3 & \mathbf{O}_3 & \mathbf{P}_3 & \mathbf{Q}_3 & \mathbf{R}_3 & \mathbf{S}_3 & \mathbf{T}_3 \end{array}$$

The patch $\mathbf{F}$ is defined by vertices N through Q and the patch $\mathbf{G}$ is defined by vertices $\mathbf{Q}$ through $\mathbf{T}$. The common boundary curve is defined by the control vertices $\mathbf{Q}$; the variable u runs from 0 to 1 along the direction $\mathbf{N}$ through $\mathbf{Q}$; the variable r runs from 0 to 1 along the direction $\mathbf{Q}$ through $\mathbf{T}$, and the variable t runs from 0 at the point $\mathbf{Q}_0$ to 1 at the point $\mathbf{Q}_3$.

With cubics it is usually observed that $D_u\mathbf{F}(1,t)$ and $D_r\mathbf{G}(0,t)$ are cubic in t, while $D_t\mathbf{G}(0,t)$ is quadratic. This has led to the typical assumption, e.g.

[5], justified by counting degrees, that $\alpha(t)$ is constant, $\beta(t)$ is constant, and $\gamma(t)$ is linear. This implies that the "natural" degree of variation of any tangent plane to $\mathbf{F}$ and $\mathbf{G}$ along their common boundary curve would be limited by the cubic character of (3).

Substituting the (trivial) Bézier representations $\alpha(t) = \alpha_0$, $\beta(t) = \beta_0$, and $\gamma(t) = \gamma_0(1-t) + \gamma_1 t$ into (3), using the Bézier representations for $D_u\mathbf{F}(1,t)$, $D_r\mathbf{G}(0,t)$, and $D_t\mathbf{G}(0,t)$, and expressing the result in Bernstein-Bézier form yields

$$A_0 B_0^3(t) + A_1 B_1^3(t) + A_2 B_2^3(t) + A_3 B_3^3(t) \ , \qquad (4)$$

where $B_i^d(t)$ is the i-th Bernstein polynomial of degree d and where

$$A_0 = \alpha_0(3\mathbf{Q}_0 - 3\mathbf{P}_0) + \beta_0(3\mathbf{R}_0 - 3\mathbf{Q}_0) + \gamma_0(3\mathbf{Q}_1 - 3\mathbf{Q}_0) \ ,$$

$$A_1 = \alpha_0(3\mathbf{Q}_1 - 3\mathbf{P}_1) + \beta_0(3\mathbf{R}_1 - 3\mathbf{Q}_1) + \gamma_0(2\mathbf{Q}_2 - 2\mathbf{Q}_1) + \gamma_1(\mathbf{Q}_1 - \mathbf{Q}_0) \ ,$$

$$A_2 = \alpha_0(3\mathbf{Q}_2 - 3\mathbf{P}_2) + \beta_0(3\mathbf{R}_2 - 3\mathbf{Q}_2) + \gamma_0(\mathbf{Q}_3 - \mathbf{Q}_2) + \gamma_1(2\mathbf{Q}_2 - 2\mathbf{Q}_1) \ ,$$

$$A_3 = \alpha_0(3\mathbf{Q}_3 - 3\mathbf{P}_3) + \beta_0(3\mathbf{R}_3 - 3\mathbf{Q}_3) + \gamma_1(3\mathbf{Q}_3 - 3\mathbf{Q}_2) \ .$$

In order to have this expression equal to zero for $0 \le t \le 1$, the coefficients A_i must all be zero. These conditions, provide conditions on the control vertices $\mathbf{P}$, $\mathbf{Q}$, and $\mathbf{R}$ necessary to ensure G^1 continuity with a tangent plane having variation limited by the cubic nature of (3).

The conditions produced in this manner are too restrictive for practical applications. Along the shared boundary between $\mathbf{F}$ and $\mathbf{G}$, for example, they require that

$$\alpha_0(\mathbf{Q}_0 - \mathbf{P}_0) + \beta_0(\mathbf{R}_0 - \mathbf{Q}_0) + \gamma_0(\mathbf{Q}_1 - \mathbf{Q}_0) = 0 \qquad (5)$$

and

$$\alpha_0(\mathbf{Q}_3 - \mathbf{P}_3) + \beta_0(\mathbf{R}_3 - \mathbf{Q}_3) + \gamma_1(\mathbf{Q}_3 - \mathbf{Q}_2) = 0 \qquad (6)$$

both hold. Since α_0 and β_0 appear in each of these expressions, this would appear to require changes at the $t = 0$ (equation (5)) end of the common boundary to be reflected in necessary changes at the $t = 1$ (equation (6)) end. If the two patches under consideration are but two in a line of patches, this would suggest that changes at one point of the line would propagate throughout the entire line. Such non-locality is undesirable.

The usual correction to this, e.g. [5], is found in taking $\mathbf{F}$ and $\mathbf{G}$ to be of higher degree than cubic. This justifies higher degrees for α, β, and γ, and the resulting conditions for G^1 continuity are more flexible, allowing for local surface manipulation. The considerations of Section 2, however, provide us with a motivation for seeing whether there is more flexibility for lower-degree patches than a tangent plane consistent with a cubic version of (3) might suggest. Indeed, flexibility becomes evident when one retains the given degree of the patches but raises the degrees of α, β, and γ.

§4. Degree Raising

Let a, b, and c stand for the degrees of α, β, and γ respectively. Then the degree of (3) will be given by the maximum of $(3+a)$, $(3+b)$, and $(2+c)$. It is useless to choose a, b, or c so that one of the these expressions is larger than the other two, since we want the (3) to be identically zero, and we hope to achieve this by cancelling one of the polynomial terms in (3) off against the other two. Still, there is wide latitude in the choice. For the sake of presentation, we will be working with $a = 3$, $b = 3$, $c = 4$, which results in a degree 6 expression for (3), but we have explored all the reasonable combinations in the ranges $0 \leq a \leq 3$, $0 \leq b \leq 3$, and $0 \leq c \leq 4$, and the observations we outline below is representative of all combinations.

The model we followed above in the case $a = 0$, $b = 0$, $c = 1$ to produce (4) is the one to be followed in general. α, β, and γ are expressed in Bernstein-Bézier form, as are the derivatives of $\mathbf{F}$, $\mathbf{G}$. The products of terms in (3) are brought to Bernstein-Bézier form,

$$\alpha(t)D_u\mathbf{F}(1,t) + \beta(t)D_r\mathbf{G}(0,t) + \gamma(t)D_t\mathbf{G}(0,t)$$
$$= A_0B_0^d(t) + A_1B_1^d(t) + \cdots \ ,$$

and the coefficients A_i are equated to zero. The result yields conditions on the control vertices involving the (unknown) coefficients of α, β, and γ. In detail for the chosen degrees a, b, and c:

$$\alpha(t) = \alpha_0B_0^3(t) + \alpha_1B_1^3(t) + \alpha_2B_2^3(t) + \alpha_3B_3^3(t)$$
$$\beta(t) = \beta_0B_0^3(t) + \beta_1B_1^3(t) + \beta_2B_2^3(t) + \beta_3B_3^3(t)$$
$$\gamma(t) = \gamma_0B_0^4(t) + \gamma_1B_1^4(t) + \gamma_2B_2^4(t) + \gamma_3B_3^4(t) + \gamma_4B_4^4(t)$$
$$D_u\mathbf{F}(1,t) = 3(\mathbf{Q}_0 - \mathbf{P}_0)B_0^3(t) + 3(\mathbf{Q}_1 - \mathbf{P}_1)B_1^3(t)$$
$$+3(\mathbf{Q}_2 - \mathbf{P}_2)B_2^3(t) + 3(\mathbf{Q}_3 - \mathbf{P}_3)B_3^3(t)$$
$$D_r\mathbf{G}(0,t) = 3(\mathbf{R}_0 - \mathbf{Q}_0)B_0^3(t) + 3(\mathbf{R}_1 - \mathbf{Q}_1)B_1^3(t)$$
$$+3(\mathbf{R}_2 - \mathbf{Q}_2)B_2^3(t) + 3(\mathbf{R}_3 - \mathbf{Q}_3)B_3^3(t)$$

and

$$D_t\mathbf{G}(0,t) = 3(\mathbf{Q}_1 - \mathbf{Q}_0)B_0^2(t) + 3(\mathbf{Q}_2 - \mathbf{Q}_1)B_1^2(t)$$
$$+3(\mathbf{Q}_3 - \mathbf{Q}_2)B_2^2(t) \ .$$

The chief exercise in carrying out the algebra involves replacing products of Bernstein polynomials by an appropriate Bernstein polynomial of higher order [6, page 57] $B_i^m(t)B_j^n(t) = \frac{\binom{m}{i}\binom{n}{j}}{\binom{m+n}{i+j}}B_{i+j}^{m+n}(t)$. The following condition, taken from the coefficient of $B_2^6(t)$, is typical

$$\alpha_0(\frac{3}{5}\mathbf{Q}_2 - \frac{3}{5}\mathbf{P}_2) + \beta_0(\frac{3}{5}\mathbf{R}_2 - \frac{3}{5}\mathbf{Q}_2)$$

$$+\gamma_0(\frac{1}{5}\mathbf{Q}_3 - \frac{1}{5}\mathbf{Q}_2)$$

$$+\alpha_1(\frac{9}{5}\mathbf{Q}_1 - \frac{9}{5}\mathbf{P}_1) + \beta_1(\frac{9}{5}\mathbf{R}_1 - \frac{9}{5}\mathbf{Q}_1)$$

$$+\gamma_1(\frac{8}{5}\mathbf{Q}_2 - \frac{8}{5}\mathbf{Q}_1)$$

$$+\alpha_2(\frac{3}{5}\mathbf{Q}_0 - \frac{3}{5}\mathbf{P}_0) + \beta_2(\frac{3}{5}\mathbf{R}_0 - \frac{3}{5}\mathbf{Q}_0)$$

$$+\gamma_2(\frac{6}{5}\mathbf{Q}_1 - \frac{6}{5}\mathbf{Q}_0) = 0$$

This can be written as a matrix expression,

$$CM_2\mathbf{V}$$

where

$$C = [\alpha_0, \beta_0, \gamma_0, \alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3, \gamma_4] \qquad (7)$$

$$\mathbf{V} = [\mathbf{P}_0, \mathbf{Q}_0, \mathbf{R}_0, \mathbf{P}_1, \mathbf{Q}_1, \mathbf{R}_1, \mathbf{P}_2, \mathbf{Q}_2, \mathbf{R}_2, \mathbf{P}_3, \mathbf{Q}_3, \mathbf{R}_3]^T$$

and the matrix M_2 is given by

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -3/5 & 3/5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -3/5 & 3/5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/5 & 0 & 0 & 1/5 & 0 \\
0 & 0 & 0 & -9/5 & 9/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -9/5 & 9/5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -8/5 & 0 & 0 & 8/5 & 0 & 0 & 0 & 0 \\
-3/5 & 3/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3/5 & 3/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -6/5 & 0 & 0 & 6/5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$CM_2\mathbf{V}$ must be interpreted as three products in parallel, since each entry of $\mathbf{V}$ is a 3-component vector, e.g. $\mathbf{P}_0 = [P_0^x, P_0^y, P_0^z]$. For the chosen degrees, $a = 3$, $b = 3$, $c = 4$, yielding a sixth-degree version of (3), we have the products $CM_0\mathbf{V}, \ldots, CM_6\mathbf{V}$. This represents 21 vector-matrix-vector products, for which all being zero will imply that the patches $\mathbf{F}$ and $\mathbf{G}$ are G^1 with a tangent plane that can vary in accord with the sixth-degree character of (3).

§5. Pragmatics

To take stock of where we are, having chosen degrees a, b, and c for $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ respectively, we have produced matrix conditions of the form $CM_i\mathbf{V}$. The number of these conditions depend on the degree of the patches and the degree to which (3) has been raised by the choice of the degrees a, b, and c. The 7 conditions being used for the discussion derive from cubic patches, cubic α and β, and quartic γ. The column dimension of the matrices M_i derives solely from the degree of the patches. Cubic patches produce 12 columns for these matrices. The row dimension of the matrices M_i derives solely from the composite of degrees chosen for the coefficients α, β, and γ. For the presentation, the matrices have 13 rows. The matrix entries are determined purely from the degrees and the properties of the Bernstein polynomials. The matrices M_i are not dependent upon the specific patches, that is to say upon $\mathbf{V}$, nor upon the polynomials α, β, and γ. A necessary and sufficient condition for equation (3) to hold for all t is that all these vector-matrix-vector products should be zero for some choice of the coefficients α_f, β_g, and γ_h of the polynomials $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ respectively.

If (3) does not hold, patches $\mathbf{F}$ and $\mathbf{G}$ may admit a tangent plane of a higher-degree variation than is implicit in the degrees of the terms in (3), for example. (A tangent plane that varies in a manner consistent with a lower-degree version of (3) will be subsumed by (3).) Moreover, (3) does not rule out situations for which $\alpha(t) = \beta(t) = \gamma(t) = 0$ at some t, or for which $D_u\mathbf{F}(1,t)$, $D_r\mathbf{G}(0,t)$, and $D_t\mathbf{G}(0,t)$ no longer span a 2-dimensional subspace for some t, that is,

$$\text{rank}\,[D_u\mathbf{F}(1,t)|D_r\mathbf{G}(0,t)|D_t\mathbf{G}(0,t)] < 2\ . \tag{8}$$

We will assume no singularities and concentrate on an approach for adjusting $\mathbf{F}$ and $\mathbf{G}$ so that (3) holds.

The scale chosen for C and $\mathbf{V}$ will be immaterial as far as the conditions $CM_i\mathbf{V} = 0$ are concerned. Theoretically we could normalize these vectors to suit our convenience. Since we want C and $\mathbf{V}$ to be non-trivial, we could ensure this by insisting that C have unit norm, for example, as well as the x, y, and z component vectors in $\mathbf{V}$. In practice, however, we are limited from normalizing $\mathbf{V}$, since it will arise from some application, will be non-trivial by the practicalities of that application, and will have a scale imposed by the nature of the setting. C, on the other hand, is simply the implicit byproduct of the existence of a tangent plane of a certain degree of variability. Its specific coefficients are of no interest and nothing is lost by normalizing them so that $\|C\| = 1$.

We begin with the optimistic situation: suppose that $\mathbf{F}$ and $\mathbf{G}$ are given and are G^1 consistent with (3) for the chosen degrees a, b, and c. Then some unit coefficient vector C exists that defines $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ in Bernstein-Bézier form such that (3) holds for all t. Finding α, β, and γ is equivalent to asking for a unit vector C for which $CM = 0$, where the matrix M is composed of all products of M_i with the x, y, and z component vectors represented by

V

$$M = [M_0 V^x | M_0 V^y | M_0 V^z | M_1 V^x \cdots] \ . \tag{9}$$

Using the degrees chosen for illustration, $a = 3$, $b = 3$, and $c = 4$, for example, M has 13 rows and 21 columns.

A C satisfying $CM = 0$ will exist if and only if M has a left nullspace. A basis of unit vectors for this nullspace, if it exists, is obtainable from the SVD decomposition [7,8] of M,

$$M = L\Sigma R^T \ . \tag{10}$$

In this decomposition, Σ is a matrix of the same row and column dimensions as M with zeros in all except the "diagonal" (j, j) positions, and L and R are square, orthogonal matrices of the appropriate sizes. The diagonal entries of Σ are the *singular values* of M. They are the square roots of the eigenvalues of the smaller of the two matrices $M^T M$ and MM^T. The number of zero diagonal entries in Σ gives the dimension of the nullspace of M, and the columns of L that are multiplied by these zeros in forming the product (10) will constitute a basis for the nullspace. C can be chosen as any one of those columns, or any linear combination (if more than one such column exists) with unit norm.

Algorithms for computing the SVD decomposition of any matrix have been known for several decades. They are readily available [8,9,10], fast, accurate, and numerically stable.

Thus, if (3) can be made to hold for all t for chosen degrees a, b, and c, then we have a mechanism for determining this and explicitly finding all possible $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ that will satisfy (3). This, alone, does not provide an indication of singular situations of the sort mentioned previously, but the explicit representation for the patch derivatives and the coefficients in (3) are now available for analysis to determine whether singularities exist.

The pessimistic situation comes next: suppose $\mathbf{V}$ is given, the matrix M is formed, and the SVD decomposition has no zero singular values. Then either the patches are not G^1, or they are G^1, but this fact cannot be detected because the chosen degrees a, b, and c are too low. In this case we consider finding C together with a change $\Delta\mathbf{V}$ to $\mathbf{V}$ so that (3) is satisfied. A reasonable problem to solve to achieve this is:

$$\begin{gathered} \text{minimize } \|\Delta\mathbf{V}\| \\ \text{such that } \|C\| = 1 \\ \text{and } CM_i(\mathbf{V} + \Delta\mathbf{V}) = 0 \ \ \forall\, i \ . \end{gathered} \tag{11}$$

Without the normalization condition on C this would be a simple, linearly constrained, least-squares problem for which economical solution algorithms exist. Without the normalization, however, the solution is the trivial one: $C = 0$ and $\Delta\mathbf{V} = 0$. With the normalization included, a non-linear optimization is required, and algorithms to accomplish this are costly and relatively difficult to apply.

We have investigated a method to achieve a C and a change to $\mathbf{V}$ that is suggested by a two-phase, suboptimal approximation to the solution of (11). In the first phase we find a C of unit norm that achieves the closest possible values to zero for the products $CM_i\mathbf{V}$. In the second phase we find a $\Delta\mathbf{V}$ of smallest possible norm that satisfies

$$CM_i(\mathbf{V} + \Delta\mathbf{V}) = 0 \ \ \forall\, i$$

with the vector C found in the first phase. This works well in practice, as we indicate in a closing example.

The precise formulation we use for the first-phase problem is

$$\text{minimize } \|CM\|$$

$$\text{such that } \|C\| = 1 \ .$$

Equivalently, we can minimize CMM^TC^T over the unit vectors C. The optimum C is provided by the eigenvector (or eigenvectors) associated with the smallest eigenvalue of MM^T. These are given, equivalently, by the columns of L in (10) associated with a singular value of smallest magnitude. In the usual situation, there is only one smallest singular value, corresponding to the final column of L, and the transpose of that column is taken to be C.

For the second-phase problem, let

$$N = \begin{bmatrix} CM_0 \\ CM_1 \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

and solve

$$\text{minimize } \|\Delta\mathbf{V}\| \tag{12}$$

$$\text{such that } N\Delta\mathbf{V} = -N\mathbf{V} \ .$$

This will always have a solution, since the right-hand side of the equality constraints lies in the column space of N. In fact, for reasonable choices of the degrees a, b, and c associated with the degrees of $\mathbf{F}$ and $\mathbf{G}$, the system of equalities in this problem will be underdetermined. With the presentational choice of cubic α and β and quartic γ, the result is a matrix N with 7 rows and 12 columns. In any event, the solution to (12) is easily obtained from the *SVD* of N.

Because N will be underdetermined with reasonable choices for a, b, and c, it is often possible to include a small number of additional linear constraints to (12). Obvious constraints to add, in order to suppress the propagation of a change at one locality of the patches to another locality, are simple equalities of the form $\Delta \mathbf{V}_j = 0$, so long as adding such rows to N, and corresponding elements to $-N\mathbf{V}$, does not result in a right-hand side that is out of the column space of the matrix. Since the *SVD* provides complete information about the row and column spaces of a matrix, the means for avoiding the addition of an improper constraint is available.

A practical setting in computer-aided surface design is one in which two G^1 patches are adjusted by the movement of a single control vertex, and one would like a system to adjust some small number of the other control vertices to return the surface to a G^1 state. The addition of constraints limiting changes resulting from the adjustment of a control vertex at one end of the patches, say around $\mathbf{Q}_0$ from requiring any movement of the control vertices at the other end, around $\mathbf{Q}_3$, has been possible in all of the examples we have tried.

§6. Numerical Example

The patches for this example were taken from the data for the surface of an automobile. The computations were done using the MATLAB system [11,10] on a DEC VAXStation-2000. The "%" introduces a MATLAB comment, ">>" is MATLAB's prompt for a command, and most commands shown relate to MATLAB script files that we have written to carry out the details of various computational algorithms we have mentioned. Common scale factors for the elements of arrays in MATLAB are printed separately at the head of the array. Thus, our example begins

```
-     % Read patches
-
-     >> Cvs
-
-     V =
-
-        1.0e+03 *
-
-         5.7209     0.7153     1.1160
-         5.7091     0.7290     1.1164
-         5.6840     0.7585     1.1172
-         5.7512     0.7291     1.0347
-         5.7394     0.7429     1.0351
-         5.7143     0.7723     1.0359
-         5.7305     0.7289     0.8935
-         5.7246     0.7431     0.8939
-         5.7045     0.7911     0.8949
-         5.7230     0.7268     0.8121
-         5.7171     0.7410     0.8124
-         5.6970     0.7890     0.8135
```

The arrangement of the array V, which contains **V**, is

$$\begin{matrix} P_0^x & P_0^y & P_0^z \\ Q_0^x & Q_0^y & Q_0^z \\ R_0^x & R_0^y & R_0^z \\ P_1^x & P_1^y & P_1^z \\ Q_1^x & Q_1^y & Q_1^z \\ R_1^x & R_1^y & R_1^z \\ P_2^x & P_2^y & P_2^z \\ Q_2^x & Q_2^y & Q_2^z \\ R_2^x & R_2^y & R_2^z \\ P_3^x & P_3^y & P_3^z \\ Q_3^x & Q_3^y & Q_3^z \\ R_3^x & R_3^y & R_3^z \end{matrix}$$

and the prefix "1.0e+03 *" indicates that the numbers in the first row are to be read as

```
-        5720.9    715.3    1116.0
```

The M_i are set up next using the degrees $a = 3$, $b = 3$, and $c = 4$, and the singular values of the matrix in (8) is computed at a randomly chosen point on the common patch boundary. The fact that all singular values are non-zero demonstrates that the vectors $D_u\mathbf{F}$, $D_r\mathbf{G}$, and $D_t\mathbf{G}$ are not co-planar at that point. Hence, the patches are not joined in a G^1 fashion.

```
-     % Set up M(i) matrices
-
-     >> a3b3c4
-
-     % Check whether patches are G1
-
-     >> G1verify
-
-     Dsvals =
-
-       275.3926
-       127.5683
-         0.0524
```

The matrix M is formed, and its singular values are reported.

```
-     % Get SVD of M(i)*V matrix
-
-     >> finda3b3c4
-
-     Msvals =
-
```

```
-        376.1164
-        331.2835
-        289.4835
-        237.9959
-        200.5062
-        177.5431
-        166.2289
-        131.9977
-         81.0544
-          9.3501
-          5.0842
-          2.6123
-          1.7793
```

The patches are not joined in a G^1 fashion, so the singular values of M are all non-zero. There is no C that will produce $CM\mathbf{V} = 0$. We take C to be the 13-th column of L, the column associated with the smallest singular value. The order of the entries in the array C is as indicated in (7).

```
-      % Set C to column of left orthogonal matrix in SVD
-
-      >> C=L(:,13)
-
-      C =
-
-         0.3403
-        -0.1594
-         0.0000
-         0.3288
-        -0.1518
-        -0.0001
-         0.4360
-        -0.1356
-        -0.0002
-         0.6914
-        -0.2029
-         0.0002
-         0.0000
```

Next the matrix N is formed and (12) is solved. Since the constraint equations, $N\Delta\mathbf{V} = -N\mathbf{V}$, form an underdetermined system, solving these equations via the *SVD* for N yields a vector $\Delta\mathbf{V}$ that minimizes $||\Delta\mathbf{V}||$ from among all possible solutions of the equation system.

```
-      % Repair V's without imposing constraints
-
-      >> rep
-
```

```
-     DELV =
-
-               0         0         0
-          7.2243   -2.6486   -0.1005
-         22.6509   -8.3072   -0.3189
-               0         0         0
-          7.3475   -3.2176   -0.0662
-         22.8764   -9.7669   -0.1445
-               0         0         0
-          2.0676   -4.7062    0.1862
-          9.7661  -22.3312    0.6939
-               0         0         0
-          0.0212   -0.0508    0.0032
-               0         0         0
```

The correction to **V**, which is on the order of 10^{-3} times the magnitude of the components of **V**, is added to **V** and the result is checked for the co-planarity of $D_u\mathbf{F}$, $D_r\mathbf{G}$, and $D_t\mathbf{G}$ at a random point.

```
-     % Check if new V's produce G1 patches
-
-     >> G1verify
-
-     Dsvals =
-
-       288.6854
-       140.6768
-         0.0000
```

The patches are now in a G^1 state. The point $\mathbf{P}_0$ is changed by a random amount.

```
-     % Change P0 (upper left twist point)
-
-     >> fudge=[500*(rand-0.5),500*(rand-0.5),500*(rand-0.5)]
-
-     fudge =
-
-        40.6570 -166.6412  -73.2498
-
-     >> V(1,:)=V(1,:)+fudge
-
-     V =
-
-        1.0e+03 *
-
-         5.7615    0.5486    1.0428
-         5.7163    0.7264    1.1163
```

```
-         5.7066    0.7502    1.1169
-         5.7512    0.7291    1.0347
-         5.7468    0.7397    1.0350
-         5.7372    0.7626    1.0358
-         5.7305    0.7289    0.8935
-         5.7267    0.7384    0.8940
-         5.7143    0.7688    0.8956
-         5.7230    0.7268    0.8121
-         5.7171    0.7409    0.8124
-         5.6970    0.7890    0.8135
```

The problem (12) is solved with the addition of 4 further equations that impose the conditions that $\mathbf{P}_0$, $\mathbf{P}_3$, $\mathbf{Q}_3$, and $\mathbf{R}_3$, must remain fixed. The adjustment to $\mathbf{V}$ largely moves $\mathbf{Q}_0$ and $\mathbf{R}_0$ to follow the move of $\mathbf{P}_0$ with minor adjustments to the interior control vertices.

```
-    % Repair V's with current C
-    % Use constraints so that P0 and P3,Q3,R3 remain fixed
-
-    >> repair
-
-    DELV =
-
-         0.0000    -0.0000     0.0000
-        40.5691  -166.2806   -73.0913
-        40.3791  -165.5020   -72.7490
-              0          0          0
-         0.0122    -0.0502    -0.0221
-         0.0652    -0.2673    -0.1175
-         0.0374    -0.1532    -0.0673
-        -0.0000     0.0000    -0.0000
-        -0.1274     0.5220     0.2294
-         0.0000     0.0000    -0.0000
-        -0.0000     0.0000     0.0000
-        -0.0000     0.0000     0.0000
```

Finally, the resulting change to $\mathbf{V}$ is checked with respect to a random point on the common boundary.

```
-    % Check if G1
-
-    >> G1verify
-
-    Dsvals =
-
-      382.5737
-       45.7186
-        0.0000
```

F and **G** now prove to be G^1.

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