Optimal Data Placement on CLV Optical Discs

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Research Report CS-89-22

December, 1989

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ABSTRACT

Optimal data placement on the surface of a disc has as an objective the minimization of the expected access cost of random data retrievals from the disc when the probabilities of access of data items are different. The problem of optimal data placement for optical discs is both more important and more difficult than the corresponding problem on magnetic discs. A good data placement on optical discs is more important, because data on optical discs such as WORM and CD ROM cannot be modified or moved once they are placed on the disc. Even rewritable optical discs currently are best suited for applications which are archival in nature. The problem of optimal data placement on CLV format optical discs is more difficult mainly because the useful storage space is not uniformly distributed across the disc surface (along a radius). This leads to a positional performance trade-off not present for magnetic discs.

We present a model which encompasses all the important aspects of the placement problem on CLV format optical discs. The model takes into account the non-uniform distribution of useful storage, the dependency of the rotational delay on disc position, a parameterized seek cost function for optical discs, and the varying access probabilities of data items. We show that the optimal placement satisfies a unimodality property. Based on this observation, we solve the optimal placement problem. We also study the impact of the relative weights of the problem parameters and show that the optimal data placement may be very different than the optimal data placement on magnetic discs.

1. Introduction

One important goal of physical database design is to obtain the best retrieval performance possible from the storage system or device on which a database resides. An accurate measure of retrieval performance is the expected time delay required to access the records qualifying in a query. For both magnetic and optical discs this delay is dominated by the time needed to physically reposition the device's access mechanism during the retrieval process. Typical seek times for magnetic and optical discs are 30 and 400 milliseconds respectively. For some optical discs such as CD ROM, the seek time can be as much as one second. It is obviously of critical importance to minimize the expected delays that result from using such slow devices.

A technique frequently employed in physical database design to improve retrieval performance is data clustering. Clustering improves performance by storing those records which are likely to be retrieved frequently, in locations on the storage device that are physically near each other, such as in the same or adjacent tracks. This physical grouping reduces both the expected number of seeks that the access mechanism will execute, and the expected distance it will travel. The idea of positioning data to improve performance can be extended to encompass the entire arrangement of sectors on the disc and leads directly to the problem of finding a total disc sector arrangement that minimizes the expected cost of a single disc access.

This optimal sector placement problem is an important one for optical discs. Their large storage capacities and low cost make them ideal for large database systems, their only real drawback is their slower seek performance. Any technique that can mitigate the impact on retrieval performance of the slow retrieval times. This is particularly true for CLV optical discs. First because they typically have slower access times than CAV optical discs. Second, because data on CD ROM's and WORM's are never modified or placed in a different position (unlike magnetic disks where data is modified frequently, and also data placement is frequently variable and transparent to users). Even rewritable CLV optical discs may be used very frequently an an archival medium. Since CD ROM's are media used for distribution of information, and other optical discs are typically media for archiving information, the benefit of finding an optimal or good sector arrangement for the data will be reaped over the many, usually thousands, of copies of the discs, or over may years of use.

The problem of record or sector arrangement to improve retrieval performance has been investigated previously for magnetic discs. The results of those investigations, however, have very limited applicability to the same problem for optical discs. The differences in the physical characteristics of magnetic and optical discs are significant enough to invalidate many important underlying assumptions used in determining the solutions for magnetic discs. For instance, all previous investigations of the optimal sector placement problem for magnetic discs, implicitly (and quite naturally, since virtually no other formats were available) assumed that the disc employed the CAV storage format. This assumption implies that each track on the disc has the same uniform storage capacity (same number of sectors). This is simply not valid for optical discs that use the CLV storage format. With this format, the distribution of storage space varies across the disc surface; the tracks nearest the centre of the disc have fewer sectors than those nearest the outer edge. For CD ROM discs, which use the CLV format, the ratio between the capacity of the outer and inner tracks can be as great as three to one.

The asymmetric distribution of storage capacity changes the optimal sector arrangement problem significantly for CLV format optical discs. The higher capacity tracks at the disc's outer edge allow for more data clustering there than at the disc's inner edge. To add a further difficulty to the problem, the rotational latency also varies as a function of position. These characteristics of the recording formats lead to a three-way positional performance trade-off, the balance point of which is determined by the parameters of the problem. The presence of the trade-off makes the placement problem much more complex and difficult than the CAV format placement problem.

In the next section, we describe previous investigations of this problem for (CAV format) magnetic discs. In the sections following that, we give an overview of the placement problem and discuss the positional performance trade-off in more detail. We then develop a model for our analysis which encompasses virtually all aspects of the placement problem, including the distribution of storage across the disc, the access probability distribution, the seek performance of the drive, and the complication of a rotational latency function which varies with the position of the access mechanism. Next, we develop an analysis of the problem and then analyze the performance trade-off and the roles played by the model parameters in determining the optimal solution.

2. Previous Research

The problem of finding an optimal arrangement of probabilities to minimize the expected distance function between regularly spaced probability points in linear space, was first investigated by [Hardy 34]. They found that an arrangement called the *Organ-Pipe Permutation* produced the optimal solution. This permutation places the largest probability in the centre of the space and then positions the rest of the probabilities, in decreasing order, on alternating sides of the centre. This results in two optimal solutions, one the mirror of the other.

This work was extended by [Bergmans 72] who developed the notion of the *Pairwise Majorizing Property* or *PMP* as a necessary condition for the optimality of probability arrangements. The basic notion behind the condition is that all probabilities on one side of a point, line, or plane, depending on the dimensions of the space, must be greater than or equal to their respective probabilities on the opposite side. Bergmans showed that the PMP was a sufficient condition for optimality in linear and circular spaces (i.e., the probabilities are placed around the circumference of a circle) and result in the Organ-Pipe permutation (an analogue of the organ-pipe is produced for circular spaces). The PMP condition, however, is not a sufficient condition for general spaces, but it does imply that all optimal solutions under these assumptions must be unimodal.

These results were later applied to the problem of optimal track arrangements on magnetic discs. In the models used to approach the problem, the tracks or cylinders of the disc were viewed as forming a linear space. [Grossman 73] considered the case of assigning a single data element with a known access probability per position (track/cylinder) and showed that the Organ-Pipe permutation was optimal for the physical characteristics of a magnetic disc.

The problem of optimal arrangement when each position can be assigned more than one data element (i.e., a sector or record), and each position contains exactly the same number of elements, was studied by [Yue 73]. They showed that with an Organ-Pipe permutation of the tracks, based on the sum of the access probabilities of the sectors/records assigned to the track, that the "ranking partition scheme" or "greedy partition scheme" [Wong 80] was optimal. This partition scheme results in each track being assigned a continuous subsequence of the total ordering of the data elements (sectors/records) according to their access probabilities; this implies, that the access probabilities of the elements in each track are consecutive in value with regard to a total ordering of all element access probabilities. Thus, the sectors in the centre track, which has the highest probability sum, will have the sectors with the highest access probabilities.

3. Problem Overview

The optimal placement problem for optical discs which employ the CLV format is considerably different and more difficult than the same problem for discs which use the CAV format. The reason for this increased difficulty is that a positional performance trade-off is present for discs which use the CLV format. This trade-off is not present for discs which use the CAV format and is primarily a consequence of the constant recording density of CLV discs which skews the distribution of storage capacity towards the outer edge of the disc (where disc tracks are longer and they can contain more data).

This skewed distribution produces variations in the amount of clustering possible at different positions on the disc and in the rotational latency encountered at different positions. These variations can be exploited to improve access performance. For example, improved access performance might be obtained by clustering more frequently accessed sectors together in the higher capacity tracks (near the outer disc edge) or by reducing their rotational latency (by placing them closer to the first recording track near the centre of the disc). It might also be obtained by placing frequently accessed sectors near the middle of the set of tracks on the whole disk in order to reduce the average distance the access mechanism must travel between them and other parts of the disc. The cost function that we derive reflects all of the above factors. The optimal solution finds the position which best balances these improvements.

It is interesting to note how the uniform storage capacity distribution of a CAV format disc eliminates the possibility of trading off positional performance improvements. This is because all tracks on such discs have the same capacity and rotational latency. This uniformity makes the distance between disc sectors the only factor in determining the expected retrieval cost. The organ-pipe arrangement, which minimizes this distance in order of the frequency of sector accesses, is the resulting optimal solution. In that sense, the problem of optimal placement on CAV discs is much simpler.

The balance point in the performance trade-off for CLV discs is determined by the relative significance of each of the performance improvements possible. These in turn are a direct consequence of the parameters of the placement problem and their own relative significance. These parameters are: 1) the distribution of storage capacity across the disc surface; 2) the distribution of the relative or absolute sector access probabilities; 3) the seek cost function, and 4) the rotational latency function.

The distribution of storage space affects the trade-off by determining the variability in both the clustering of disc sectors and the rotational latency function. The significance of sector clustering is determined by the sector access probability distribution. For example, if the storage distribution becomes more skewed, the performance improvement from the increased clustering of frequently accessed sectors becomes more significant, the opposite is true when the distribution becomes more uniform. On the other hand, increased skewness will increase the rotational delays of the larger tracks.

Given the previous parameters, the seek cost and rotational latency functions determine the actual balance point between the performance improvements that can result from increased clustering, reduced average travel distance or reduced rotational delay. If the seek cost function is the more significant of the two, and it is relatively independent of distance (e.g., some constant), then increased clustering will provide the best performance improvement (zero cost for accesses in the same track). If the opposite is true and seek cost function is relatively more dependent upon distance then the best improvement will come from reducing the average travel distance. If the rotational latency is the more significant of the two (the disc turns very slowly compared to the seek time), then the best improvement will come from reducing the rotational latency.

The complication of the positional performance trade-off in the optimal placement problem for CLV format discs, necessitates a different model and analysis than the one that has been used in previous investigations for CAV format discs. In the next section, we present a continuous model which retains the essence of the placement problem while allowing it to be analyzed to reveal the basic intuition and trade-offs. The continuous model transforms the problem from one of placing discrete disc sectors on a discrete disc, to one of placing infinitesimal probability masses on a continuous disc.

In our model, we also adopt a simplification which restricts the sector access probabilities to two values, a high probability subset and a low probability subset. This restriction retains the essence of the placement problem while avoiding the extra complication of determining the exact optimal placement of a general probability distribution. It is also directly applicable when the exact probability distribution of access is not known. For example, it may often be the case that the only information known is that indexes are accessed more frequently than all other data (and what the relative ratio between the frequency of access of the two is). In the later part of this report we present extensions to more general probability density functions.

4. The Placement Model

To avoid some of the inherent complications of a discrete model, we develop a continuous model for our analysis. In moving from the discrete to the continuous domain, the problem changes from one of placing sectors on a discrete disc to one of positioning probability masses on a continuous disc. We now describe the parameters of the continuous model below.

Capacity Distribution

Since the track capacity on discs which employ the CLV formats increases linearly as the track's position moves away from the centre of the disc platter (circumference is a linear function of the radius of a circle), the distribution of storage can be modeled exactly by a straight line function.

The storage distribution function is $C(x)$ and is specified by its relative slope k and intercept j. The capacity of a position is measured relative to capacity of the middle track on the disc which has capacity 1. The slope is also specified relative to the capacity of the middle position, since capacity is a linear function of position and j cannot be less than zero, this restricts the value of the slope to the range 0 to 2. To see that this is so, consider that the greatest value for the slope corresponds to $j=0$, since the capacity of the middle position is always 1, then the value for the slope for that case will be $(1-0)/(\frac{1}{2}-0) = 2$. From this we also have $j = 1 - \frac{1}{2}k$.

The relative storage capacity is given by:

$$C(x) = k\,x + j,\ \ 0 \le x \le 1,\ \ 0 \le k \le 2$$

A position on the disc is specified as a value between 0 and 1, where 0 represents the innermost position (track) on the disc and 1, the outermost. A disc model is illustrated in Figure 1.

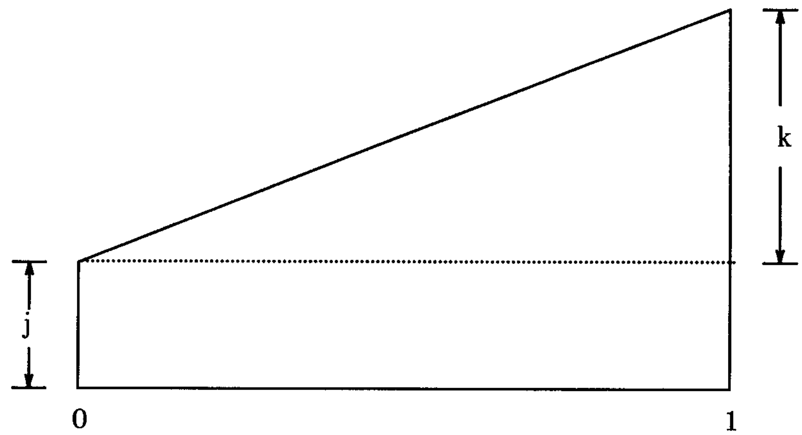

Figure 1. Model of Distribution of Storage Capacity

Random Access Probability Distribution

Our probability model assumes that requests are independent of each other. This is consistent with the models used in the analysis of the problem of optimal data placement on magnetic discs.

In addition,, to simplify our analysis we restrict the access probabilities of the point masses to two relative values P_1 and P_2 ($P_1 > P_2$). In a later section, we will describe how this model can be generalized to provide good data allocations for arbitrary probability distributions.

The proportion of point masses with relative access probability P_1 is specified by r; the proportion with value P_2 is $1-r$. Figure 2. illustrates the two value relative probability distribution.

The absolute access probability value of a probability mass with relative value P_1 is $\frac{P_1}{P_1 r + P_2(1-r)} \frac{1}{\mu}$ and for P_2 is $\frac{P_2}{P_1 r + P_2(1-r)} \frac{1}{\mu}$, where μ is the number of probability points (proportional to the area occupied by these points), which tends to infinity in the continuous model. We will let $W = P_1 r + P_2(1-r)$ be the normalizing factor.

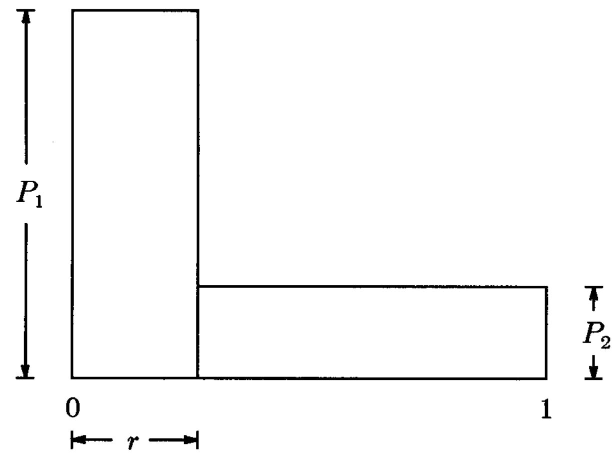

Figure 2. Two Value Relative Access Probability Distribution

Seek Cost Function

We model seek performance with the function $Sc(t)$, which is illustrated in Figure 3.

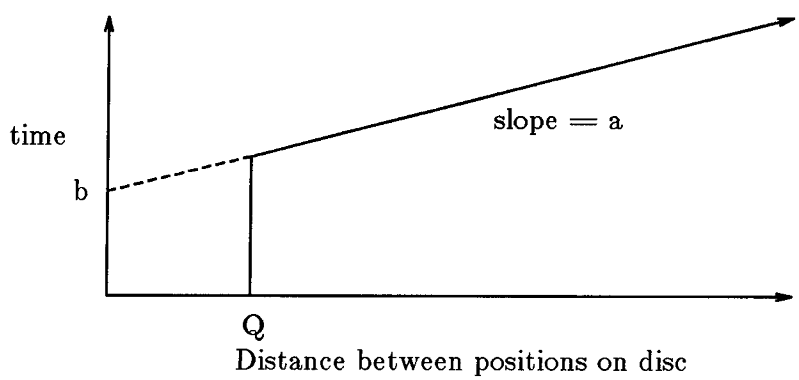

Figure 3. Seek Cost Function

The parameter a is the slope of the line and represents the time delay incurred when moving the access mechanism a given distance over the disc surface. The parameter b is the intercept of the line with the vertical axis. The parameter Q is the span size and represents the portion of the disc on one side of the anchor position of the access mechanism that can be accessed without incurred the seek cost penalty; Q, and distances t, are both specified in terms of fractions of disc units. The parameter Q represents the ability of some disc drives to access more than one track on a disc from a single position of the access mechanism. This is usually accomplished by tilting a mirror in the viewing mechanism of the drive which deflects the laser used to read the disc. If possible, the time to tilt the mirror is on the order of 5-10 milliseconds, very much smaller than the time to reposition the access mechanism (seek). A typical non-zero value for Q is 10 to 40 tracks.

The definition of the function $Sc(t)$ is given below.

$$Sc(t) = \begin{cases} a\,t + b & \text{if } t > Q \\ 0 & \textit{otherwise} \end{cases}$$

Rotational Latency Function

Rotational latency on CLV format discs varies as a function of position. This is because the speed at which the disc platter rotates is adjusted by the disc drive to match the position of the access mechanism; the platter rotates more slowly when accesses are near the outer edge than when accesses are near its centre. This adjustment is necessary for the drive to read the disc because it ensures that the data recordings, whose density is constant across the disc surface, pass beneath the access mechanism at a constant rate. Thus, the rotation rate of the disc platter is a direct linear function of the capacity of the position (track) being accessed.

If h is the time required for the entire middle position to be read, then the expected rotational latency of accesses from a position x is:

$$Rd(x) = \frac{1}{2} h\, C(x)$$

Summaries of the variables and functions defined in the model are given in Tables 1. and 2. below.

Variables		
k	-	Slope of storage distribution
j	-	Intercept of storage distribution
a	-	Slope of seek cost function
b	-	Intercept of seek cost function
T	-	Number of tracks on disc
Q	-	Span size (proportion of disc)
$\overline{Q}$	-	Span size in tracks ($\overline{Q} = Q \cdot T$)
P_1	-	Relative access probability of most frequently accessed data
P_2	-	Relative access probability of least frequently accessed data
W	-	Probability normalizing factor
r	-	Proportion of probability mass that has value P_1
h	-	Time to read middle position of disc (milliseconds)
m	-	Centre position of P_1 group
x,y	-	A position on the disc

Table 1. Summary of Placement Model Variables

Functions		
$Pm(x)$	-	Probability mass assigned to position x
$C(x)$	-	Storage capacity of position x
$Sc(t)$	-	Seek cost function (milliseconds)
$Rd(y)$	-	Expected rotational delay at position y

Table 2. Summary of Placement Model Functions

The Expected Random Access Retrieval Cost

The objective of our analysis of the placement problem is to determine an arrangement of probability masses that minimizes the expected random access retrieval cost (seek and rotational delay). The overall expected retrieval cost is computed by summing the cost in time units of successive accesses to each pair of positions on the disc (in both directions since the rotational latency will vary), weighted by the product of the probabilities of accesses to each of the two positions. The cost of moving from one position to another is the sum of the value of the seek cost function for the distance between the two positions and the value of the rotational delay function at the destination position. We first develop a discrete expression for the expected random access retrieval cost and then extend it to the continuous domain.

The cost function in our model is:

$$Cost = \sum_{i=1}^{T} \sum_{j=1}^{T} Pm(i)Pm(j)(Sc(|(i-j)/T|) + Rd(j))$$

In the expression above, i is a start position on the disc, and j is the destination position, $Pm(i)$ is the probability mass assigned to a position i. $Sc(|(i-j)/T|)$ is the seek cost between the positions i and j and $Rd(j)$ is the expected rotational delay at position j.

$$Cost = \sum_{i=1}^{T} \sum_{j=1}^{T} Pm(i)Pm(j)Sc(|(i-j)/T|) + \sum_{i=1}^{T} \sum_{j=1}^{T} Pm(i)Pm(j)Rd(j)$$

We expand the function $Sc(|(i-j)/T|)$ which changes the limits on the summations. In the discrete case, we use $\overline{Q}$ to represent the number of tracks, to the left or right of the current track, which are in the current span. Thus, the span has a total of $2\overline{Q}+1$ tracks.

$$= \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i+\overline{Q}+1}^{T} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b)$$

$$+ \sum_{i=\overline{Q}+2}^{T} \sum_{j=1}^{i-\overline{Q}-1} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b)$$

$$+ \sum_{i=1}^{T} Pm(i) \sum_{j=1}^{T} Pm(j)Rd(j)$$

To simplify the expression, we add the following terms which sum to zero,

$$\sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b) - \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b)$$

$$\sum_{i=\overline{Q}+2}^{T} \sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b) - \sum_{i=\overline{Q}+2}^{T} \sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b)$$

and use the fact that the sum of the probability masses equals one ($\sum_{i=1}^{T} Pm(i) = 1$) to simplify the rotational delay term and obtain:

$$Cost = \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{T} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b) - \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b)$$

$$+ \sum_{i=\overline{Q}+2}^{T} \sum_{j=1}^{i} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b) - \sum_{i=\overline{Q}+2}^{T} \sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b)$$

$$+ \sum_{j=1}^{T} Pm(j)Rd(j)$$

The expected retrieval cost function can be divided into three components, one representing the delay due strictly to the *distance* between successively accessed positions, one representing the cost resulting from successive accesses which do not fall within the same span which we will call the *clustering delay*, and one representing the *rotational delay*.

Using the simplification $\sum_{i=1}^{T} Pm(i) \sum_{j=1}^{T} Pm(j)\, b = b$, and adding further zero sum terms, we have:

$$Cost = \sum_{i=1}^{T} \sum_{j=i}^{T} Pm(i)Pm(j)\frac{a}{T}(j-i) + \sum_{i=1}^{T} \sum_{j=1}^{i} Pm(i)Pm(j)\frac{a}{T}(i-j)$$

$$+\, b + \sum_{i=1}^{T} Pm(i)Pm(i)\, b$$

$$- \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b) - \sum_{i=T-\overline{Q}}^{T} \sum_{j=i}^{T} Pm(i)Pm(j)(\frac{a}{T}(j-i)+b)$$

$$- \sum_{i=\overline{Q}+2}^{T} \sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b) - \sum_{i=1}^{\overline{Q}+1} \sum_{j=1}^{i} Pm(i)Pm(j)(\frac{a}{T}(i-j)+b)$$

$$+ \sum_{j=1}^{T} Pm(j)Rd(j)$$

Expected Cost for Small Span Sizes

We can obtain a closed form solution for the case where the span size is small in comparison to the number of tracks of the disc ($\frac{Q}{T} \rightarrow 0,\ T \rightarrow 0$). Our expression for the expected random access retrieval cost simplifies to the following:

$$Cost = \sum_{i=1}^{T} \sum_{j=i}^{T} Pm(i)Pm(j)\frac{a}{T}(j-i) + \sum_{i=1}^{T} \sum_{j=1}^{i} Pm(i)Pm(j)\frac{a}{T}(i-j)$$

$$+\, b$$

$$- \sum_{i=1}^{T} Pm(i)\,Pm(i)\,b$$

$$+ \sum_{j=1}^{T} Pm(j)\,Rd(j)$$

For $T \rightarrow 0$, we obtain:

$$\lim_{t\rightarrow\infty} \sum_{i=1}^{T} \sum_{j=i}^{T} Pm(i)\,Pm(j)\,\frac{a}{T}(j-i) = \int_{0}^{1} \int_{x}^{1} Pm(x)\,Pm(y)\,a(y-x)\,dy\,dx$$

$$\lim_{t\rightarrow\infty} \sum_{i=1}^{T} \sum_{j=1}^{i} Pm(i)\,Pm(j)\,\frac{a}{T}(j-i) = \int_{0}^{1} \int_{0}^{x} Pm(x)\,Pm(y)\,a\,(x-y)\,dy\,dx$$

When we move to continuous domain the clustering term vanishes because:

$$\lim_{t\rightarrow\infty} \sum_{i=1}^{T} Pm(i)\,Pm(i)\,b \le \lim_{t\rightarrow\infty} \sum_{i=1}^{T} (P_{\max} C(1) \frac{1}{T})^2\, b$$

$$\le \lim_{t\rightarrow\infty} \frac{b}{T^2} \sum_{i=1}^{T} P_{\max} C(1)$$

$$= \lim_{t\rightarrow\infty} \frac{b}{T^2}\, T\, P_{\max} C(1) = \lim_{t\rightarrow\infty} \frac{b}{T}\, P_{\max} C(1) = 0$$

Also,

$$\lim_{t\rightarrow\infty} \sum_{j=1}^{T} Pm(j)\,Rd(j) = \int_{0}^{1} Pm(x)\,Rd(x)\,dx$$

Thus, for $\frac{Q}{T} \rightarrow 0$, $T \rightarrow 0$, the cost function becomes:

$$Cost = \int_{0}^{1} \int_{x}^{1} Pm(x)\,Pm(y)\,a(y-x)\,dy\,dx + \int_{0}^{1} \int_{0}^{x} Pm(x)\,Pm(y)\,a\,(x-y)\,dy\,dx$$

$$+ b + \int_0^1 Pm(x)Rd(x)dx$$

5. Proof of Consecutivity and Unimodality

Theorem 1: {Consecutivity} In an optimal arrangement, there cannot exist two different positions on the disc, x and y, such that both a P_1 and P_2 mass element is assigned to position x and both a P_1 and P_2 mass element is assigned to position y.

Proof: Assume there there are two such positions. Consider the change in the expected retrieval cost if we exchange the P_1 mass at position x with the P_2 mass at position y. If the expected retrieval cost increases then the opposite operation of exchanging the P_1 mass element at position y with the P_2 mass element at position x will decrease the expected retrieval cost and would violate our assumption that the arrangement was optimal.

Corollary: The above theorem for discrete discs implies that in an optimal arrangement at most one track will have sectors with probabilities P_1 and P_2. All other tracks will have sectors with either just P_1 or just P_2.

Due to the assumption of infinitely small elements and disc space continuity, as well as due to the consecutivity theorem, we need only to consider placements where all the elements in a column defined by two points x_i and x_j are occupied by elements of the same probability value.

It is also clear that when the elements of a column defined by x_i and x_j (which correspond to the area of the trapezoid defined by x_i and x_j) are moved to another location on the disc they will occupy all the area of the trapezoid between two points x_i' and x_j'. This is true since given an area A smaller than the area of the disc, and a point x_i we can always find another point x_j such that the area between x_i and x_j is A. The above observations allow us to describe the problem without worrying about columns that are not occupied completely by elements of a given kind.

We will show that the optimal placement of such columns is unimodal. This means that given two columns with probability elements of value P_1, there cannot exist a column between them with probability elements of value P_2.

Theorem 2: {Unimodality} The optimal arrangement of two probability masses is unimodal.

Proof: The intuition behind the unimodality of the optimal arrangement is simple. If the frequently accessed elements are placed together, the distance that the access mechanism will travel for most accesses will be reduced. The best position for the group of frequently accessed sectors, within the limit of being placed too far from the bulk of the probability mass on the disc, will depend upon the relative merits of placing the group in a position with a low rotational delay or in a position which allows more elements to be placed closer together.

We prove unimodality by showing that if an arrangement is not unimodal, then we can always make a small change in it which reduces the expected retrieval cost. The basic intuition behind the proof given below is to show that if we alter the arrangement by moving the position of some of the masses in one direction, and that move increases the expected retrieval cost, then moving some other masses in the opposite direction must reduce the cost.

Assume that we have an optimal arrangement which is not unimodal, then there must exist two points X_1 and X_4 such that P_1 probability masses are assigned to the immediate left (call the group q_l) and P_2 probability masses are assigned to the immediate right (call the group u_l) of X_1; and for X_4, P_1 probability masses (q_r) are assigned to the immediate right and P_2 probability masses (u_r) to the immediate left. This situation is illustrated in Figure 4.

We select the points X_0, X_2, so that the regions X_0 to X_1 and X_1 to X_2 have the same area A_{left}. We do the same on the right so that the regions X_3 to X_4 and X_4 to X_5 have the area A_{right}.

Since the arrangement is assumed to be optimal, any change in it cannot result in a decrease in the value of the expected retrieval cost. Thus, if we exchange the positions of q_l and u_l on the left, or the positions of q_r and u_l on the right, the expected retrieval cost must either not change, or it must increase in value.

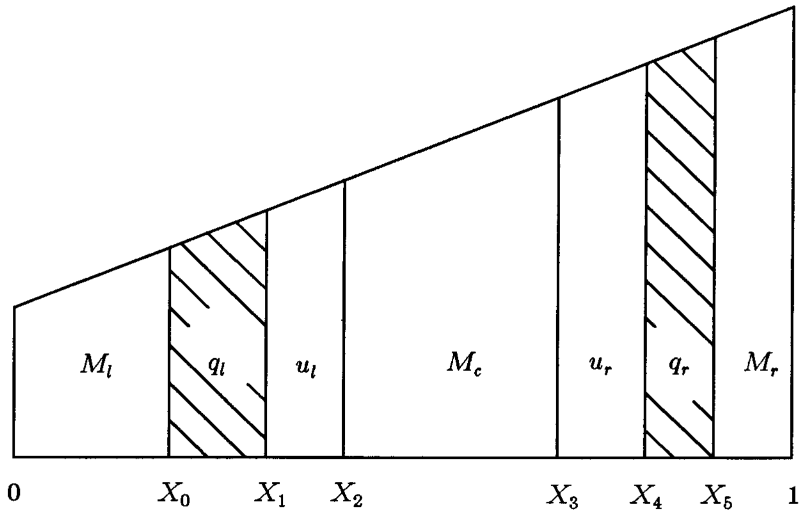

Figure 4. Non-Unimodal Arrangement

Below, we develop and express the change resulting from an exchange in terms of the rotational delay and the distance components.

Rotational Delay

The difference between rotational delay components of the cost function before and after the exchange is given below. The terms representing the rotational delay outside of the region X_0 to X_2 do not change and so cancel in the subtraction.

$$After - Before = \int_{X_0}^{X_1} Pm(x)_{After}\, Rd(x)\, dx - \int_{X_0}^{X_1} Pm(x)_{Before}\, Rd(x)\, dx$$

$$+ \int_{X_1}^{X_2} Pm(x)_{After}\, Rd(x)\, dx - \int_{X_1}^{X_2} Pm(x)_{Before}\, Rd(x)\, dx$$

$$= \tfrac{1}{2} h \int_{X_0}^{X_1} (Pm(x)_{After} - Pm(x)_{Before})\, C(x)\, dx$$

$$+ \tfrac{1}{2} h \int_{X_1}^{X_2} (Pm(x)_{After} - Pm(x)_{Before}\, C(x)\, dx$$

$$= \frac{\frac{1}{2}h\,(P_2 - P_1)}{W} \int_{X_0}^{X_1} C(x)^2\,dx + \frac{\frac{1}{2}h\,(P_1 - P_2)}{W} \int_{X_1}^{X_2} C(x)^2\,dx$$

$$= \frac{\frac{1}{2}h\,(P_2 - P_1)}{W} \left(\int_{X_0}^{X_1} C(x)^2\,dx - \int_{X_1}^{X_2} C(x)^2\,dx \right)$$

And for the exchange on the right,

$$After - Before = \frac{\frac{1}{2}h\,(P_1 - P_2)}{W} \left(\int_{X_3}^{X_4} C(x)^2\,dx - \int_{X_4}^{X_5} C(x)^2\,dx \right)$$

If you consider then only the rotational delay, the expected retrieval cost would be reduced if you were to transfer the region X_4 to X_5 to the left, however, this may not reduce the overall average distance between masses because of the values of M_l and M_r.

Distance

We expand the expression for the distance component of the cost function to separate out the terms which will not change during the exchange, from those which will.

For the exchange on the left we have:

$$\int_0^1 Pm(x) \int_0^1 Pm(y)\,(a\,|x-y|)\,dy\,dx$$

$$= \int_0^{X_0} Pm(x) \int_0^1 Pm(y)\,(a\,|x-y|)\,dy\,dx + \int_{X_0}^{X_2} Pm(x) \int_0^1 Pm(y)\,(a\,|x-y|)\,dy\,dx$$

$$+ \int_{X_2}^1 Pm(x) \int_0^1 Pm(y)\,(a\,|x-y|)\,dy\,dx$$

$$= \int_0^{X_0} Pm(x) \left(\int_0^{X_0} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_0}^{X_2} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_2}^1 Pm(y)\,(a\,|x-y|)\,dy \right) dx$$

$$+ \int_{X_0}^{X_2} Pm(x) \left(\int_0^{X_0} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_0}^{X_2} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_2}^{1} Pm(y)\,(a\,|x-y|)\,dy \right) dx$$

$$+ \int_{X_2}^{1} Pm(x) \left(\int_0^{X_0} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_0}^{X_2} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_2}^{1} Pm(y)\,(a\,|x-y|)\,dy \right) dx$$

We now subtract the distance component of the cost before the exchange from the distance component after the exchange.

$$After - Before = \int_0^{X_0} Pm(x) \left(\int_{X_0}^{X_2} Pm(y)_{After}\,(a\,|x-y|)\,dy - \int_{X_0}^{X_2} Pm(y)_{Before}\,(a\,|x-y|)\,dy \right) dx$$

$$+ \int_{X_0}^{X_2} Pm(x)_{After} \left(\int_0^{X_0} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_0}^{X_2} Pm(y)_{After}\,(a\,|x-y|)\,dy \right.$$

$$\left. + \int_{X_2}^{1} Pm(y)\,(a\,|x-y|)\,dy \right) dx$$

$$- \int_{X_0}^{X_2} Pm(x)_{Before} \left(\int_0^{X_0} Pm(y)\,(a\,|x-y|)\,dy + \int_{X_0}^{X_2} Pm(y)_{Before}\,(a\,|x-y|)\,dy \right.$$

$$\left. + \int_{X_2}^{1} Pm(y)\,(a\,|x-y|)\,dy \right) dx$$

$$+ \int_{X_2}^{1} Pm(x) \left(\int_{X_0}^{X_2} Pm(y)_{After}\,(a\,|x-y|)\,dy - \int_{X_0}^{X_2} Pm(y)_{Before}\,(a\,|x-y|)\,dy \right) dx$$

$$= \int_0^{X_0} Pm(x) \left(\int_{X_0}^{X_1} Pm(y)_{After}\,(a\,|x-y|)\,dy - \int_{X_0}^{X_1} Pm(y)_{Before}\,(a\,|x-y|)\,dy \right) dx$$

$$+ \int_0^{X_0} Pm(x) \left(\int_{X_1}^{X_2} Pm(y)_{After}\,(a\,|x-y|)\,dy - \int_{X_1}^{X_2} Pm(y)_{Before}\,(a\,|x-y|)\,dy \right) dx$$

$$+ \int_{X_0}^{X_1} Pm(x)_{After} \left(\int_0^{X_0} Pm(y)(a\,|x-y\,|)\,dy + \int_{X_2}^{1} Pm(y)(a\,|x-y\,|)\,dy\right) dx$$

$$+ \int_{X_1}^{X_2} Pm(x)_{After} \left(\int_0^{X_0} Pm(y)(a\,|x-y\,|)\,dy + \int_{X_2}^{1} Pm(y)(a\,|x-y\,|)\,dy\right) dx$$

$$+ \int_{X_0}^{X_2} Pm(x)_{After} \int_{X_0}^{X_2} Pm_{After}(y)(a\,|x-y\,|)\,dy\,dx$$

$$- \int_{X_0}^{X_1} Pm(x)_{Before} \left(\int_0^{X_0} Pm(y)(a\,|x-y\,|)\,dy + \int_{X_2}^{1} D(y)(a\,|x-y\,|)\,dy\right) dx$$

$$- \int_{X_1}^{X_2} Pm(x)_{Before} \left(\int_0^{X_0} Pm(y)(a\,|x-y\,|)\,dy + \int_{X_2}^{1} Pm(y)(a\,|x-y\,|)\,dy\right) dx$$

$$- \int_{X_0}^{X_2} Pm(x)_{Before} \int_{X_0}^{X_2} Pm(y)_{Before}(a\,|x-y\,|)\,dy\,dx$$

$$+ \int_{X_2}^{1} Pm(x) \left(\int_{X_0}^{X_1} Pm(y)_{After}(a\,|x-y\,|)\,dy - \int_{X_1}^{X_2} Pm(y)_{Before}(a\,|x-y\,|)\,dy\right) dx$$

$$+ \int_{X_2}^{1} Pm(x) \left(\int_{X_1}^{X_2} Pm(y)_{After}(a\,|x-y\,|)\,dy - \int_{X_0}^{X_1} Pm(y)_{Before}(a\,|x-y\,|)\,dy\right) dx$$

Simplifying

$$= 2a \int_0^{X_0} Pm(x) \left(\int_{X_0}^{X_1} Pm(y)_{After}\ |x-y\,|\,dy - \int_{X_0}^{X_1} Pm(y)_{Before}\ |x-y\,|\,dy\right) dx$$

$$+ 2a \int_0^{X_0} Pm(x) \left(\int_{X_1}^{X_2} Pm(y)_{After}\ |x-y\,|\,dy - \int_{X_1}^{X_2} Pm(y)_{Before}\ |x-y\,|\,dy\right) dx$$

$$+ 2a \int_{X_2}^{1} Pm(x) \left(\int_{X_0}^{X_1} Pm(y)_{After}\ |x-y\,|\,dy - \int_{X_0}^{X_1} Pm(y)_{Before}\ |x-y\,|\,dy\right) dx$$

$$+ 2a \int_{X_0}^{1} Pm(x) \left(\int_{X_1}^{X_2} Pm(y)_{After} \; |x-y| \, dy - \int_{X_1}^{X_2} Pm(y)_{Before} \; |x-y| \, dy \right) dx$$

$$+ \int_{X_0}^{X_1} Pm(x)_{After} \int_{X_0}^{X_1} Pm(y)_{After} \, (a \, |x-y|) \, dy \, dx$$

$$- \int_{X_0}^{X_1} Pm(x)_{Before} \int_{X_0}^{X_1} Pm(y)_{Before} \, (a \, |x-y|) \, dy \, dx$$

$$+ \int_{X_0}^{X_1} Pm(x)_{After} \int_{X_1}^{X_2} Pm(y)_{After} \, (a \, |x-y|) \, dy \, dx$$

$$- \int_{X_0}^{X_1} Pm(x)_{Before} \int_{X_1}^{X_2} Pm(y)_{Before} \, (a \, |x-y|) \, dy \, dx$$

$$+ \int_{X_1}^{X_2} Pm(x)_{After} \int_{X_0}^{X_1} Pm(y)_{After} \, (a \, |x-y|) \, dy \, dx$$

$$- \int_{X_1}^{X_2} Pm(x)_{Before} \int_{X_0}^{X_1} Pm(y)_{Before} \, (a \, |x-y|) \, dy \, dx$$

$$+ \int_{X_1}^{X_2} Pm(x)_{After} \int_{X_1}^{X_2} Pm(y)_{After} \, (a \, |x-y|) \, dy \, dx$$

$$- \int_{X_1}^{X_2} Pm(x)_{Before} \int_{X_1}^{X_2} Pm(y)_{Before} \, (a \, |x-y|) \, dy \, dx$$

The first four terms above calculate the change in the expected retrieval cost due to the relative change in the positions of the q_l and u_l masses with respect to the masses which remain stationary during the exchange. The remaining terms compute the change resulting from the q_l and u_l masses being in new positions with respect to each other. Below, we simplify the first four terms of the above expression and then, later, the remaining terms.

The first four terms become:

$$2\,a\,\frac{(P_2-P_1)}{W}\int_0^{X_0} Pm(x)\,(\int_{X_0}^{X_1} C(y)\,(y-x)\,dy - \int_{X_1}^{X_2} C(z)\,(z-x)\,dz)\,dx$$

$$2\,a\,\frac{(P_2-P_1)}{W}\int_{X_2}^{1} Pm(x)\,(\int_{X_0}^{X_1} C(y)\,(x-y)\,dy - \int_{X_1}^{X_2} C(z)\,(x-z)\,dz)\,dx$$

Let $A_{left} = \int_{X_0}^{X_1} C(y)\,dy = \int_{X_1}^{X_2} C(y)\,dy$ be the area of the two regions being exchanged on the left. If we multiply the equations by A_{left}/A_{left}, and modify the two inner integrations, we obtain:

$$2\,a\,\frac{(P_2-P_1)}{W}A_{left}\int_0^{X_0} Pm(x)\,(\int_0^{\gamma} \frac{1}{A_{left}}C(X_1-y)\,((X_1-x)-y)\,dy$$

$$-\int_0^{\epsilon} \frac{1}{A_{left}}C(X_1+z)\,((X_1-x)+z)\,dz)\,dx$$

$$2\,a\,\frac{(P_2-P_1)}{W}A_{left}\int_{X_2}^{1} Pm(x)\,(\int_0^{\gamma} \frac{1}{A_{left}}C(X_1-y)\,((x-X_1)+y)\,dy$$

$$-\int_0^{\epsilon} \frac{1}{A_{left}}C(X_1+z)\,((x-X_1)-z)\,dz)\,dx$$

Where, $\gamma = X_1 - X_0$ and $\epsilon = X_2 - X_1$. In general, for any two adjacent trapezoids of the same area A, γ and ϵ can be easily computed from expressions for the area of a trapezoid (Appendix 1). For a given position x, which is the boundary of two adjacent trapezoids of area A, we have:

$$\gamma = \frac{C(x) - \sqrt{C(x)^2 - 2kA}}{k} \quad \text{and} \quad \epsilon = \frac{-C(x) + \sqrt{C(x)^2 + 2kA}}{k}$$

Simplifying further:

$$2\,a\,\frac{(P_2-P_1)}{W}A_{left}\int_0^{X_0} Pm(x)\,(\int_0^{\gamma} \frac{1}{A_{left}}C(X_1-y)\,(-y)\,dy - \int_0^{\epsilon} \frac{1}{A_{left}}C(X_1+z)\,z\,dz)\,dx$$

$$2\,a\,\frac{(P_2 - P_1)}{W} A_{left} \int_{X_2}^{1} Pm(x)\,\Big(\int_{0}^{\gamma} \frac{1}{A_{left}} C(X_1 - y)\; y\; dy \;-\; \int_{0}^{\epsilon} \frac{1}{A_{left}} C(X_1 + z)\;(-z)\, dz\Big)\, dx$$

The inner integrals simply compute the distance between the positions of the centres of mass of q_l and u_l. Call this distance $\Delta x(X_1, A_{left})$.

This gives us:

$$2\,a\,\frac{(P_2 - P_1)}{W} A_{left}\,(-\Delta x(X_1, A_{left}))\,M_l$$

$$2\,a\,\frac{(P_2 - P_1)}{W} A_{left}\,\Delta x(X_1, A_{left})\,(M_c + q_r + u_r + M_r)$$

Or,

$$-2\,a\,\frac{(P_2 - P_1)}{W} A_{left}\,(M_l - M_r)\,\Delta x(X_1, A_{left})$$

$$2\,a\,\frac{(P_2 - P_1)}{W} A_{left}(M_c + u_r + q_r)\,\Delta x(X_1, A_{left})$$

The first term corresponds to the change in the cost resulting from moving the q_l mass away from (and the u_l mass towards) the mass (M_l) to the left of the point X_0, and towards the mass (M_r) to the the right of X_5. Similarly, the second term corresponds to the change resulting from moving the q_l mass towards (and the u_l mass away from) the remaining mass ($M_c + q_r + u_r$) to the right of the point X_2.

The remaining terms in the change in the expected retrieval cost due to the change in the distance component are easier to simplify:

$$+\,a\,\Big(\int_{X_0}^{X_1} Pm(x)_{After} \int_{X_0}^{X_1} Pm(y)_{After}\,|x-y|\,dy\,dx$$

$$-\int_{X_1}^{X_2} Pm(x)_{Before} \int_{X_1}^{X_2} Pm(y)_{Before}\,|x-y|\,dy\,dx\Big)$$

$$+ a \left(\int_{X_1}^{X_2} Pm(x)_{After} \int_{X_1}^{X_2} Pm(y)_{After} \, |x-y| \, dy \, dx \right.$$

$$\left. - \int_{X_0}^{X_1} Pm(x)_{Before} \int_{X_0}^{X_1} Pm(y)_{Before} \, |x-y| \, dy \, dx \right)$$

The terms above represent the change resulting from the changes in the distances between points internal to the q_l mass and internal to the u_l mass. The q_l mass moves to an area which is not as wide so its mass moves closer together (reducing the cost); the u_l mass moves to a wider area so its mass becomes spread farther apart (increasing the cost).

The size of the area also changes linearly with the distance moved. Therefore, the distance of each element from its centre of mass changes linearly with the distance.

When the exchange is made on the left, between q_l and u_l, the sum of the terms represents a reduction in the cost since the net result of the exchange is to move the larger q_l mass closer together the same amount that the smaller u_l mass moves apart. The opposite however, is true when the exchange is made on the right between q_r and u_r. The net result in that case, because the larger q_r mass moves apart the same amount that the smaller u_l mass moves together, is an increase in the cost.

Since we are assuming that any change will increase the expected retrieval cost we can ignore these terms for the left exchange. We must however account for the terms which result from the exchange on the right.

The net change in the expected retrieval cost due to the exchange on the left is:

$$\Delta Cost_{left(distance)} = -2\,a\,\frac{(P_2 - P_1)}{W} A_{left}\,(M_l - M_r)\,\Delta x(X_1, A_{left})$$

$$+ 2\,a\,\frac{(P_2 - P_1)}{W} A_{left}(M_c + u_r + q_r)\,\Delta x(X_1, A_{left})$$

Similarly, the total change in the distance component resulting from the exchange on the right is:

$$\Delta Cost_{right(distance)} = -2\,a\,\frac{(P_2 - P_1)}{W} A_{right}\ (M_r\ -\ M_l)\ \Delta x(X_4, A_{right})$$

$$+\ 2\,a\,\frac{(P_2 - P_1)}{W} A_{right}(M_c\ +\ u_l\ +\ q_l)\ \Delta x(X_4, A_{right})$$

$$+\ a\,(\int_{X_3}^{X_4} Pm(x)_{After} \int_{X_3}^{X_4} Pm(y)_{After}\,|x-y\,|\,dy\ dx$$

$$-\int_{X_4}^{X_5} Pm(x)_{Before} \int_{X_4}^{X_5} Pm(y)_{Before}\,|x-y\,|\,dy\ dx)$$

$$+\ a\,(\int_{X_4}^{X_5} Pm(x)_{After} \int_{X_4}^{X_5} Pm(y)_{After}\,|x-y\,|\,dy\ dx$$

$$-\int_{X_3}^{X_4} Pm(x)_{Before} \int_{X_3}^{X_4} Pm(y)_{Before}\,|x-y\,|\,dy\ dx)$$

Again, the first two terms represent the change due to the relative movements of the q_r and u_r masses towards and away from the masses outside of the region X_3 to X_5. The terms represent the cost change due to the change in the shape of the areas occupied by the two groups. As noted above, the net sum of the last two terms for the exchange on the right is positive.

For the exchange on the right, we observed that the last two terms have a positive sum. We can however make this sum as small as we like by reducing the area A_{right} (and hence q_r and u_r) involved in the exchange. If we make it smaller in absolute value than the second term (which is negative), we can also ignore these three terms for the exchange on the right.

The intuition behind this result is that while the exchange of positions on the right makes the q_r mass spread apart, which in a sense means it is moving away from itself, the exchange also moves it closer towards the other exchange pair q_l and u_l (and any mass M_c between the exchange pairs). If the mass of the other pair is large enough compared to the mass of the pair being exchanged (and we can always make the pair on the right as small as we like), we will be moving towards more than we are moving away, and this reduces the expected cost.

Consider the sum of the following two terms from the expression for change due to the right exchange:

$$a \int_{X_4}^{X_5} Pm(x)_{After} \int_{X_4}^{X_5} Pm(y)_{After} \, |x-y| \, dy \, dx \; - \; a \int_{X_4}^{X_5} Pm(x)_{Before} \int_{X_4}^{X_5} Pm(y)_{Before} \, |x-y| \, dy \, dx$$

The result will be negative since the smaller u_r mass will be in the X_4 to X_5 region after the exchange, and the larger q_r mass, before. As such, we can eliminate them from consideration, as we can the term

$$- \, a \int_{X_3}^{X_4} Pm(x)_{Before} \int_{X_3}^{X_4} Pm(y)_{Before} \, |x-y| \, dy \, dx$$

which is negative.

This leaves the only positive term $a \cdot \int_{X_3}^{X_4} Pm(x)_{After} \int_{X_3}^{X_4} Pm(y)_{After} \, |x-y| \, dy \, dx$. If we make it larger by substituting $(X_5 - X_3)$ for $|x-y|$, we can simplify the term to $a \cdot q_r \cdot q_r \, (X_5 - X_3)$.

By making the area A_{right} small enough, we can make this positive term smaller in absolute value than the term $2\,a \, \frac{(P_2 - P_1)}{W} A_{right} \, (M_c + u_l + q_l) \, \Delta x(X_4, A_{right})$, which is negative.

Thus, we need to satisfy the following inequality:

$$2\,a \, \frac{(P_2 - P_1)}{W} A_{right} (M_c + u_l + q_l) \, \Delta x(X_4, A_{right}) + a \cdot q_r \cdot q_r \, (X_5 - X_3) < 0$$

$$\frac{P_2}{P_1} - 1 + \frac{q_r \, (X_5 - X_3)}{2 \, (M_c + u_l + q_l) \, \Delta x(X_4, A_{right})} \cdot \frac{q_r \, W}{P_1 \, A_{right}} < 0$$

$$\frac{q_r}{(M_c + u_l + q_l)} \cdot \frac{(X_5 - X_3)}{2 \, \Delta x(X_4, A_{right})} < 1 - \frac{P_2}{P_1}$$

We can always satisfy the inequality, because as we make the area A_{right} smaller, the ratio $\frac{(X_5 - X_3)}{2\,\Delta x(X_4, A_{right})}$, approaches unity, and the ratio $\frac{q_r}{(M_c + u_l + q_l)}$ approaches zero.

This leaves us with simple expressions for the change in the expected retrieval cost due to the distance component.

On the left:

$$\Delta Cost_{left(distance)} = 2\,a\,\frac{(P_1 - P_2)}{W} A_{left}\,(M_l - M_r)\,\Delta x(X_1, A_{left})$$

And on the right

$$\Delta Cost_{right(distance)} = 2\,a\,\frac{(P_1 - P_2)}{W} A_{right}(M_r - M_l)\,\Delta x(X_4, A_{right})$$

Contradiction Development

We use the expressions we have developed above to derive a contradiction. First we express the changes in the the expected retrieval cost resulting from the two exchanges by combining the changes due to the two components, rotational delay and distance.

The change in the expected retrieval cost due to the exchange on the left is:

$$\Delta Cost_{left} = 2a\,(M_l - M_r)\frac{(P_1 - P_2)}{W} A_{left}\,\Delta x(X_1, A_{left})$$

$$+ \frac{½\,h\,(P_2 - P_1)}{W}\left(\int_{X_0}^{X_1} C(x)^2\,dx - \int_{X_1}^{X_2} C(y)^2\,dy\right) \geq 0$$

The change due to the exchange on the right is:

$$\Delta Cost_{right} = 2a\,(M_r - M_l)\frac{(P_1 - P_2)}{W} A_{right}\,\Delta x(X_4, A_{right})$$

$$+ \frac{½\,h\,(P_1 - P_2)}{W}\left(\int_{X_3}^{X_4} C(x)^2\,dx - \int_{X_4}^{X_5} C(y)^2\,dy\right) \geq 0$$

Through straightforward, but tedious manipulations (Appendix 2), it is possible to show that

$$\frac{\left(\int_{X_0}^{X_1} C(x)^2 dx - \int_{X_1}^{X_2} C(y)^2 dy\right)}{A_{left}\,\Delta x(X_1, A_{left})} = \frac{\left(\int_{X_3}^{X_4} C(x)^2 dx - \int_{X_4}^{X_5} C(y)^2 dy\right)}{A_{right}\,\Delta x(X_4, A_{right})} = -k$$

Thus, we have:

$$\Delta Cost_{left} = 2a\,(M_l - M_r)\frac{(P_1 - P_2)}{W} - k\,\frac{½\,h\,(P_2 - P_1)}{W} \geq 0$$

$$\Delta Cost_{right} = -2a\,(M_l - M_r)\frac{(P_1 - P_2)}{W} + k\,\frac{½\,h\,(P_2 - P_1)}{W} \geq 0$$

Or, $\Delta Cost_{left} = -\Delta Cost_{right}$. Since both changes in the expected cost must be positive, by assumption, we have a contradiction. Therefore, our assumption that the non-unimodal arrangement was an optimal arrangement must be false. Therefore, optimal probability arrangements must be unimodal.

6. Analysis

We showed in the previous section that our optimal arrangement of probability masses must be unimodal. (e.g., all of the P_1 mass elements will be as close to each other as is possible). The form of a solution in our model is a specification of the position of the P_1 probability mass on the disc. We shall specify this position as the location on the disc of the mid-point between the left and right boundaries of the P_1 group. We denote this point as m $(m = \frac{Xl + Xr}{2})$. The relevant points are illustrated in Figure 5. The goal of our analysis is then to develop an expression for m which can be computed from the parameters of the model.

Before tackling the main problem, we first derive some preliminary expressions. We must compute the two points Xl and Xr which denote the left and right boundaries of the P_1 group on the disc.

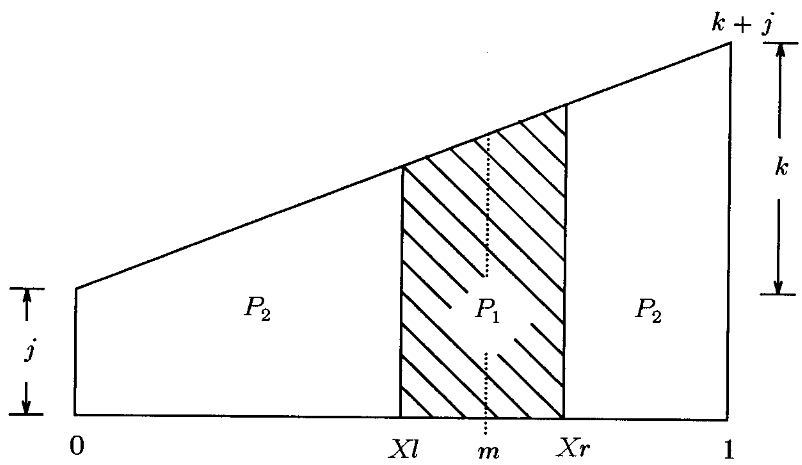

Figure 5. Optimal Placement

To compute Xl and Xr from m we need to know the "width" in disc units of the high probability group. We know that the proportion of the area occupied by the high probability group must be equal to r. If w is the width of the group then from the formula for the area of a trapezoid, we have:

$$r = \frac{(k\,m + j)\,w}{1}$$

Therefore:

$$w = \frac{r}{k\,m + j} = \frac{r}{C(m)}$$

From this we derive the simple expressions for Xl and Xr.

$$Xl(r,m) = m - \tfrac{1}{2}\,\frac{r}{C(m)} \qquad\qquad Xr(r,m) = m + \tfrac{1}{2}\,\frac{r}{C(m)}$$

Given these bounds, we can specify our probability assignment function:

$$Pm(x,r,m) = \begin{cases} \dfrac{P_1}{P_1 r + P_2(1-r)} C(x), & \text{if } Xl \le x \le Xr \\[2ex] \dfrac{P_2}{P_1 r + P_2(1-r)} C(x), & \textit{otherwise} \end{cases}$$

From the model our expected retrieval cost is:

$$Cost = b + 2\int_0^1 \int_x^1 Pm(x)Pm(y)\, a\,(y-x)\, dy\, dx \qquad \text{(Distance)}$$

$$+ \int_0^1 Pm(x)Rd(x)\, dx \qquad \text{(Rotational delay)}$$

The distance component is:

$$= b + \frac{2P_2^2}{W^2} \int_0^{Xl(r,m)} \int_x^{Xl(r,m)} C(x)C(y)\, a\,(y-x)\, dy\, dx$$

$$+ \frac{2P_2P_1}{W^2} \int_0^{Xl(r,m)} \int_{Xl(r,m)}^{Xr(r,m)} C(x)C(y)\, a\,(y-x)\, dy\, dx$$

$$+ \frac{2P_2^2}{W^2} \int_0^{Xl(r,m)} \int_{Xr(r,m)}^{1} C(x)C(y)\, a\,(y-x)\, dy\, dx$$

$$+ \frac{2P_1^2}{W^2} \int_{Xl(r,m)}^{Xr(r,m)} \int_x^{Xr(r,m)} C(x)C(y)\, a\,(y-x)\, dy\, dx$$

$$+ \frac{2P_1P_2}{W^2} \int_{Xl(r,m)}^{Xr(r,m)} \int_{Xr(r,m)}^{1} C(x)C(y)\, a\,(y-x)\, dy\, dx$$

$$+ \frac{2P_2^2}{W^2} \int_{Xr(r,m)}^{1} \int_x^{1} C(x)C(y)\, a\,(y-x)\, dy\, dx$$

Where $W = \mu(P_1 r + P_2(1-r))$. The rotational delay component is:

$$\tfrac{1}{2}h \int_0^1 Pm(x,r,m)\,C(x)\,dx$$

$$= \tfrac{1}{2}h \int_0^{Xl(r,m)} Pm(x,r,m)\,C(x)\,dx$$

$$+ \tfrac{1}{2}h \int_{Xl(r,m)}^{Xr(r,m)} Pm(x,r,m)\,C(x)\,dx$$

$$+ \tfrac{1}{2}h \int_{Xr(r,m)}^{1} Pm(x,r,m)\,C(x)\,dx$$

$$= \frac{\tfrac{1}{2}h}{W} P_2 \int_0^{Xl(r,m)} C(x)^2\,dx + \frac{\tfrac{1}{2}h}{W} P_1 \int_{Xl(r,m)}^{Xr(r,m)} C(x)^2\,dx + \frac{\tfrac{1}{2}h}{W} P_2 \int_{Xr(r,m)}^{1} C(x)^2\,dx$$

The compete expression for the expected random access retrieval cast has been evaluated using the Maple symbolic mathematics package [Char 85]. The complete evaluation is shown in Appendix 3, below is a small fragment (first and last terms) of the resulting expression.

$$Cost(k,j,a,b,P_1,P_2,r,h,m) = \frac{(60P_2^2 j^7 h + \cdots - 60P_2^2 j^7 hr)}{120\,(km+j)^5\,(P_1 r + P_2(1-r))^2}$$

To derive the optimal value for m we take the derivative, again using the Maple symbolic mathematics package, of the cost function with respect to m. The complete result is in Appendix 4, again below is a small fragment (first and last terms) of the resulting expression.

$$\frac{\partial\, Cost(k,j,a,b,P_1,P_2,r,h,m)}{\partial\, m} = \frac{r\,(48aP_2^2 j^7 + \cdots + 48k^6 m^6 jaP_2^2)}{(km+j)^6 (P_1 r + P_2 - P_2 r)^2}$$

Using Maple, we determine the roots of the derivative and finally arrive at the desired expression (with $P_2 = 1$) for the optimal location of m (shown in its entirety).

$$m_{optimal} = -\frac{j}{k} + \frac{\left(\sqrt{3}\,C_1 + \sqrt{-24j^2 + 48j^4a^2C_2k + C_3k^2 + C_4k^3 + C_5k^4}\ \right)^{1/2}}{2\sqrt{2}\,3^{1/4}k\sqrt{a}}$$

$$C_1 = a\,(4j^2 + 2k^2 + 4jk) + hk^2(-P_1r + r - 1)$$

$$C_2 = 96j^3a^2$$

$$C_3 = 32P_1r^2a^2 + aP_1rh + 24j^2arh - 24j^2ah + 96j^2a^2 - 48r^2a^2$$

$$C_4 = -24jah - 24jaP_1rh + 24jarh + 48ja^2$$

$$C_5 = 12a^2 + 3P_1^2r^2h^2 - 6P_1r^2h^2 + 6P_1rh^2 - 12aP_1rh$$

$$-12ah + 3h^2 + 12arh + 3r^2h^2 - 6rh^2$$

7. Impact of the Model Parameters on the Optimal Data Placement

An expression for the optimal position of the group of the most frequently accessed sectors was derived previously. In the discussion below, we examine and explain the variation in the optimal solution as the values of the parameters of the placement problem are changed. We examine m, the centre of the optimal position for the highest probability mass, for a range of placement problem parameters, plotting m as a function of each parameter in turn.

We examine the relative impact of the rotational latency and seek costs first, and then the storage and access probability distribution.

To determine a typical value for P_1, the relative access frequency of the most frequently accessed disc sectors, we use the rule-of-thumb that states that 80% of all disc accesses will be to 20% of the disc sectors. This means that our value for r is 0.20 and that the area in the P_1 group in the distribution must be 80% of the total distribution area.

$$\frac{r \cdot P_1}{r \cdot P_1 + (1 - r) \cdot P_2} = 0.8$$

With $r = 0.2$ and $P_2 = 1$, we have:

$$\frac{0.2 \cdot P_1}{0.2 \cdot P_1 + (1 - 0.2)} = 0.8$$

$$\frac{0.2}{0.8} P_1 = 0.2 \cdot P_1 + 0.8$$

$$0.2 \cdot P_1 - 0.16 \cdot P_1 = 0.64$$

$$0.04 \cdot P_1 = 0.64$$

$$P_1 = \frac{0.64}{0.04} = 16$$

Impact of Rotational Latency

The graph in Figure 5. plots the optimal position m as a function of the rotational latency. The solid line in the graph plots m while the top dashed line plots the position of Xr and the bottom dashed line plots the position of Xl.

The graph shows that the rotation latency plays a significant role in determining the optimal position. As the retrieval delays due to rotational latency become more significant in relation to other delays, the optimal position shifts towards the lower capacity tracks which have a lower rotational latency. Typical values for the time to read a sector which determines h (the time to read the middle position) range from 2 milliseconds for fast drives (e.g., Alcatel-Thomson), to 13.3 milliseconds for slower drives such as CD ROM drives. The sector transfer time for CD.lf 2 ROM drives is fixed by international standard (1/75 seconds per sector).

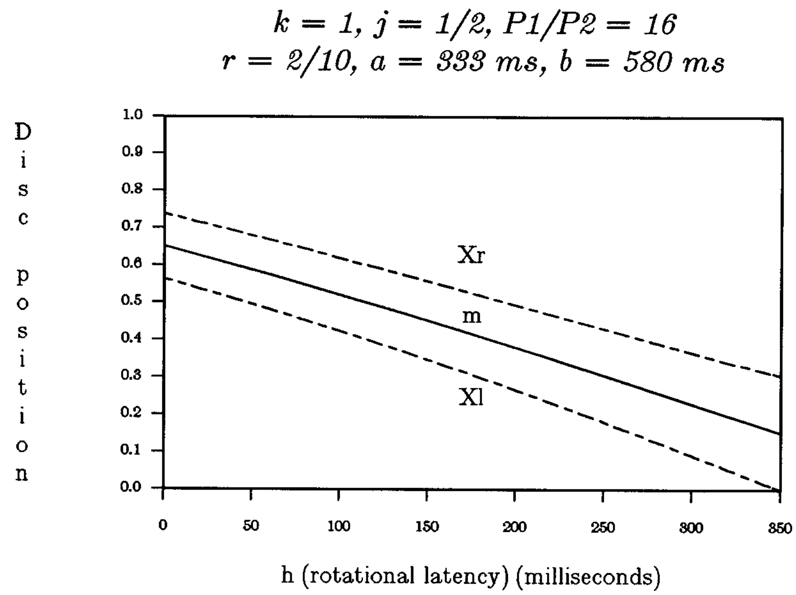

Figure 6. Location of centre as function of rotational latency

Impact of Seek Cost

The graph in Figure 7. plots the optimal position as a function of the seek cost parameter a (slope of seek cost). It shows that as the value of the seek cost function becomes more dependent upon distance (greater value for the slope a), the optimal position moves away from the inner edge of the disc. A similar behaviour would be observed, if for a constant slope a, the rotational latency would decrease (so that the seek cost would become the most important cost). The limiting position depends upon the exact distribution of storage across the disc, and is essentially the location of the centre of gravity of the "probability mass".

The slope of the cost function can be determined either by experimentation, or by manufacturer specifications. Typical values for the slope range from 100 to 350 milliseconds per disc. The Alcatel-Thomson GD 1001 disc drive has a seek cost slope of 160 milliseconds per disc and the Hitachi CDR-1503S has a slope of 333 milliseconds per disc.

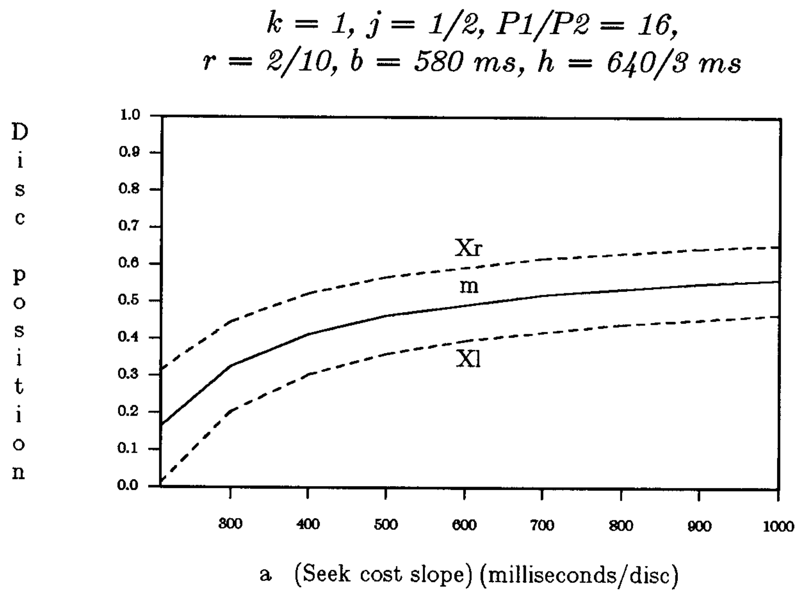

Figure 7. Location of centre as function of seek cost slope

The Impact of the Storage Distribution

The next parameters we examine are those which determine the distribution of storage capacity on the disc. The graph in Figure 8. plots the optimal position as a function of k, the relative slope of the capacity distribution function, for two different values for the rotational delay parameter h.

When h is significant with respect to the slope of the seek cost function ($h = 640/3$), the optimal position shifts towards the inner edge of the disc to take advantage of the reduced rotational delay at that position. When h is insignificant with respect to the seek cost slope ($h = 10$), the optimal position shifts towards the outer edge of the disc (it is moving with the moment of the probability mass).

Note that for $k = 0$ (uniform distribution) the optimal position for both values of h are at the centre of the disc as we would expect since that is the optimal position for CAV format discs.

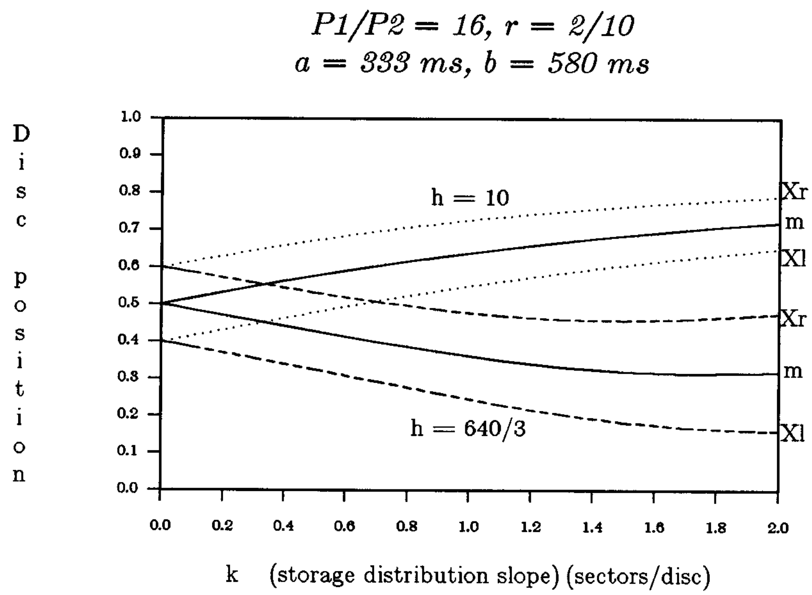

Figure 8. Location of centre as function of capacity slope

The Impact of the Data Access Probabilities

The access probability distribution also plays a role in determining the optimal solution. As the distribution becomes more skewed (P_1/P_2 increases, keeping r constant) the impact of the rotational delay component for the high probability masses (P_1) becomes the more significant component of the cost. As a result, the optimal location of the high probability mass moves towards the inner tracks to take advantage of the reduced rotational delay. The plot in Figure 9. shows how the optimal solution moves towards the lower capacity tracks as the P_1/P_2 ratio is increased.

The proportion r of the total probability distribution that has value P_1 also affects the optimal position of the P_1 group. The graph in Figure 10. plots the position of m as a function of r. When r is small, most of the probability mass is located outside of the P_1 region making the distance cost component between P_1 region and the two P_2 regions, the more significant contributor to the total cost. As a result, the optimal placement of the P_1 region is in the higher locations of the disc where the average distance between the P_1 and P_2 masses is minimized.

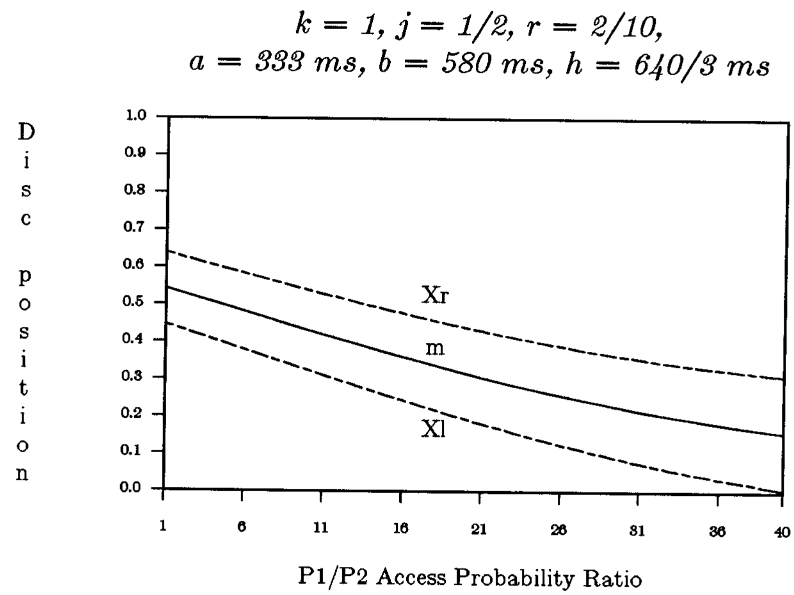

Figure 9. Location of centre as function of P1/P2 ratio

As r increases, less of the probability mass lies outside of the P_1 group so that the rotational delay component for the high probability masses becomes the more significant contributor to the total cost. As a result, the optimal location of P_1 moves towards the inside tracks of the disc to take advantage of the reduced rotational delay. As r increases still more, the P_1 group becomes the bulk of the probability mass which moves its centre back towards the centre of the disc.

For the chosen values of the parameters, the right edge of the P_1 group strikes the outer edge of the disc before r becomes 1. For different parameter values, the left and right edges of the P_1 group strike the edge(s) of the disc well before r approaches 1.

8. Larger Span Sizes

We can extend our analysis, for the case when $Q > 0$, and develop an approximate solution for the optimal placement problem. The difficulty with developing an exact solution using our approach is in obtaining a closed form expression for the expected retrieval cost. The main complication being in obtaining closed form expressions for the clustering terms which are present when $Q > 0$.

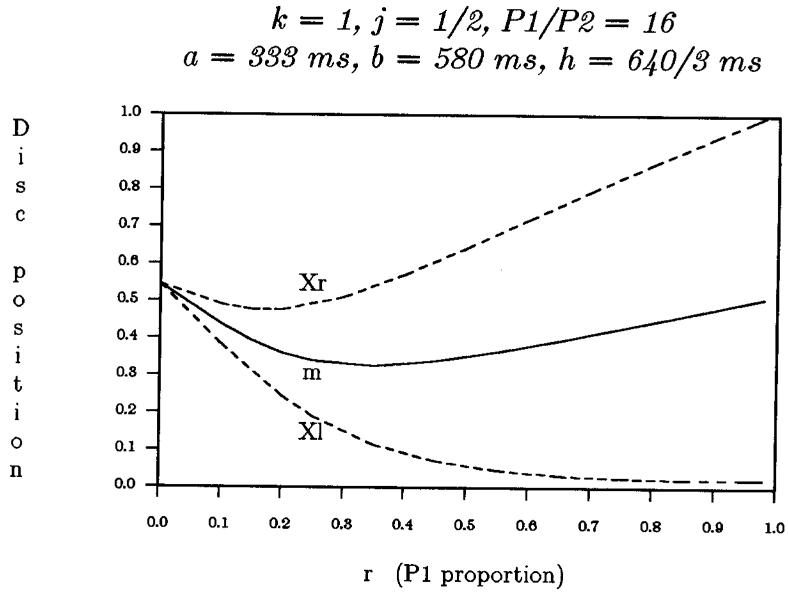

Figure 10. Location of centre as function of the size of P1 portion

The clustering terms essentially subtract from the distance component the cost of accessing the probability mass that falls within each possible position of a "span" ($2\overline{Q}+1$ tracks) on the disc (the distance component is computed assuming $\overline{Q}=0$). The problem is in computing the clustering terms for spans which overlap a boundary between the P_1 and a P_2 group. For example, consider spans which overlap both boundaries between the P_1 and P_2 groups; the values of their corresponding clustering terms cannot be expressed in terms of $P_1 C(x)\frac{1}{T}$ and $P_2 C(x)\frac{1}{T}$ (instead of the more general $Pm(x)$) without *min* and *max* functions to determine the limits of the summations.

From our model, the discrete clustering terms are:

$$-\sum_{i=1}^{T-\overline{Q}-1}\sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j)\left(\frac{a}{T}(j-i)+b\right) - \sum_{i=T-\overline{Q}}^{T}\sum_{j=i}^{T} Pm(i)Pm(j)\left(\frac{a}{T}(j-i)+b\right)$$

$$-\sum_{i=\overline{Q}+2}^{T}\sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j)\left(\frac{a}{T}(i-j)+b\right) - \sum_{i=1}^{\overline{Q}+1}\sum_{j=1}^{i} Pm(i)Pm(j)\left(\frac{a}{T}(i-j)+b\right)$$

In our approximate cost expression, we use the exact values for the distance and rotational delay components and approximations for the clustering terms above. In developing our approximation for the clustering terms, we first assume that Q is small enough for the given values of a and T so that $\frac{a}{T}(2\overline{Q}+1) \ll b$

This assumption can be used to simplify the clustering cost.

$$- b \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j) - b \sum_{i=T-\overline{Q}}^{T} \sum_{j=i}^{T} Pm(i)Pm(j)$$

$$- b \sum_{i=\overline{Q}+2}^{T} \sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j) - b \sum_{i=1}^{\overline{Q}+1} \sum_{j=1}^{i} Pm(i)Pm(j)$$

For small Q, is small we can ignore the terms representing spans which overlap the two ends of the disc:

$$- b \left(\sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i}^{i+\overline{Q}} Pm(i)Pm(j) + \sum_{i=\overline{Q}+2}^{T} \sum_{j=i-\overline{Q}}^{i} Pm(i)Pm(j) \right)$$

$$\approx -2b \left(\sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i+1}^{i+\overline{Q}} Pm(i)Pm(j) + \sum_{i=1}^{T} Pm(i)Pm(i) \right)$$

Adding in the distance and rotational delay components we have:

$$Cost \approx \sum_{i=1}^{T} \sum_{j=i}^{T} Pm(i)Pm(j) \frac{a}{T}(j-i) + \sum_{i=1}^{T} \sum_{j=1}^{i} Pm(i)Pm(j) \frac{a}{T}(i-j)$$

$$+ b + \sum_{i=1}^{T} Pm(i)Pm(i)b$$

$$- 2b \left(\sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i+1}^{i+\overline{Q}} Pm(i)Pm(j) + \sum_{i=1}^{T} Pm(i)Pm(i) \right)$$

$$+ \sum_{j=1}^{T} Pm(j)\, Rd(j)$$

Or,

$$Cost \approx \sum_{i=1}^{T} \sum_{j=i}^{T} Pm(i)\, Pm(j) \frac{a}{T}(j-i) + \sum_{i=1}^{T} \sum_{j=1}^{i} Pm(i)\, Pm(j) \frac{a}{T}(i-j)$$

$$+ b$$

$$- 2b \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i+1}^{i+\overline{Q}} Pm(i)\, Pm(j)$$

$$+ b \sum_{i=1}^{T} Pm(i)\, Pm(i)$$

$$+ \sum_{j=1}^{T} Pm(j)\, Rd(j)$$

Moving to the continuous model, we have:

$$\lim_{T \to \infty} - 2b \sum_{i=1}^{T-\overline{Q}-1} \sum_{j=i+1}^{i+\overline{Q}} Pm(i)\, Pm(j) = - 2b \int_{0}^{1-Q} \int_{x}^{x+Q} Pm(x)\, Pm(y)\, dy\, dx$$

And the approximate expected random access retrieval cost for the continuous model is:

$$Cost \approx \int_{0}^{1} \int_{x}^{1} Pm(x)\, Pm(y)\, a(y-x)\, dy\, dx$$

$$+ \int_{0}^{1} \int_{0}^{x} Pm(x)\, Pm(y)\, a\, (x-y)\, dy\, dx$$

$$+ b - 2b \int_{0}^{1-Q} \int_{x}^{x+Q} Pm(x)\, Pm(y)\, dy\, dx$$

$$+ \int_{0}^{1} Pm(x)\, Rd(x) dx$$

Because Q is small we ignore the overlap of spans with the boundaries between P_1 and P_2, and assume that the P_1 group never occupies a part of the left or rightmost portion of size Q on the disc. The clustering term then becomes:

$$- 2b \int_0^{1-Q} \int_x^{x+Q} Pm(x)Pm(y)\,dy\,dx = -2b\,\Big(\frac{P_2^2}{W^2} \int_0^{Xl(r,m)-Q} \int_x^{x+Q} C(x)C(y)\,dy\,dx$$

$$+ \frac{P_1^2}{W^2} \int_{Xl(r,m)}^{Xr(r,m)-Q} \int_x^{x+Q} C(x)C(y)\,dy\,dx$$

$$+ \frac{P_2^2}{W^2} \int_{Xr(r,m)}^{1-Q} \int_x^{x+Q} C(x)C(y)\,dy\,dx\,\Big)$$

We use the other terms from our previous analysis for $Q = 0$ to complete the expression. Unfortunately, the Maple symbolic mathematics system was unable to find a closed form for m from this cost expression. However, solutions can be found by numerical approximation.

The effect on the optimal position of the clustering is illustrated in the graph in Figure 11. which plots the optimal position as a function of the P_1/P_2 ratio (for this plot $a = 0$). The data points were approximated numerically.

We see in Figure 11. that as the window size increases and the significance of the clustering term increases, the optimal position moves away from the inner edge of the disc and towards the higher capacity tracks which increase the clustering possible (and reduce the expected cost). The size of Q in terms of discrete tracks varies between 0 and 1600 tracks which is well beyond that found on conventional disc drives.

9. Extensions to more general distributions

The approach taken in this paper can be extended to find solutions for more general access probability distributions. Given a general distribution we first approximate it with our two value probability model, as in Figure 12.

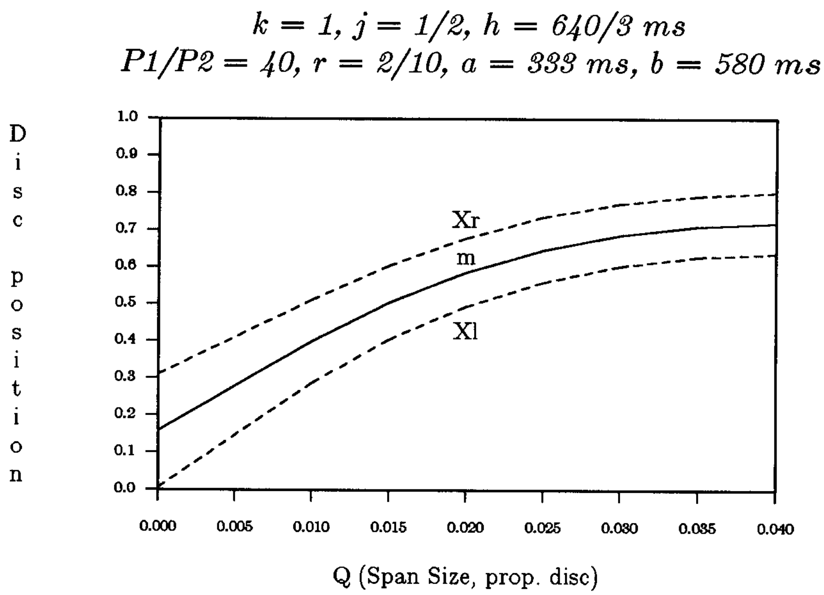

Figure 11. Location of centre as function of Q (span size)

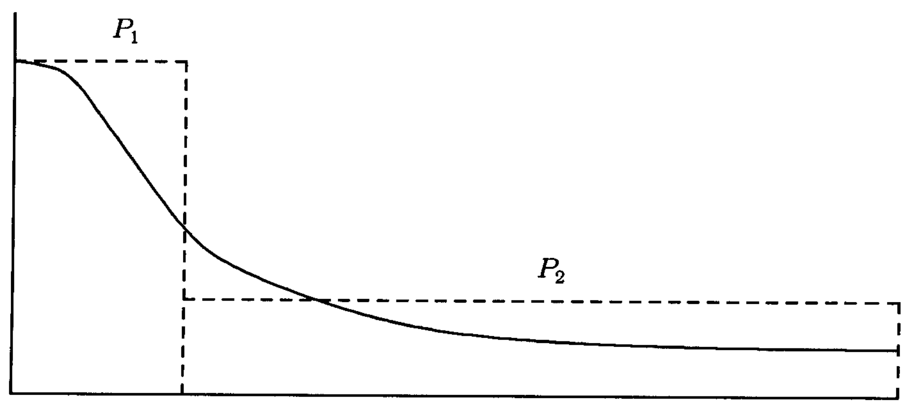

Figure 12. Two value approximation of general distribution

We determine an optimal solution using this approximation and place the disc sectors or probability masses. For those sectors corresponding to the P_2 access probability, the optimal two value probability problem that we provided previously will determine their final position. We then apply this procedure recursively to the part of the distribution represented by the P_1 values and the now smaller portion of the disc corresponding to the position of those P_1 values. We approximate the remainder of the distribution with two values and determine the optimal solution. This procedure is performed as often as necessary.

We note that errors in the approximation with respect to sectors represented by the P_2 value are not critical. If less than optimal positioning of some of these sectors results, their low access probabilities will minimize the impact on the total solution.

10. Impact of Data Placement on Expected Retrieval Cost

The optimal solutions presented in the previous section show that for CLV format discs the optimal solution can be very different from the optimal solution for CAV format discs (e.g., magnetic discs).

Consider the differences between the expected random access retrieval cost for an optimal positioning of the data and other data positionings on a CLV format disc with an outer track capacity of 48 sectors and an inner track capacity of 8 sectors ($k = 1.4$, $j = 0.3$), a rotational delay (h) of 640/3 milliseconds, a seek cost constant of 50 milliseconds and a seek cost slope of 200 milliseconds ($P_1/P_2 = 16$, $r = 0.2$), and very small span size ($Q = 0$). For this case, the optimal location of the frequently accessed group is at position 0.13, where the expected random access retrieval cost is 177 milliseconds. If the frequently accessed data is placed at the centre position of the disc ($m = 0.5$), the expected retrieval cost would be 192 milliseconds or about 8% greater. If it is placed at position at 0.9 the expected retrieval cost would be 237 milliseconds or 34% greater.

For a set of parameters which describe a model in which clustering is more important we obtain a different optimal solution, but a similar large difference between optimal and non-optimal solutions. For the same distribution of storage capacity, a rotational delay of 27 milliseconds, a seek cost constant of 580 milliseconds and slope of 333 milliseconds and a slightly higher probability ratio ($P_1/P_2 = 20$, $r = 0.2$), and a larger span size ($Q = 0.01$), the optimal location is at position 0.8 and the approximate expected cost is 587 milliseconds. At position 0.5 the expected cost is 601 milliseconds or 2% greater, and at position 0.1, the expected cost is 653 milliseconds or 11% greater.

11. Summary

We have presented a model for studying the problem of optimal placement of data with known access probabilities on the recording surface of CLV optical discs. The model takes into account the non-uniform distribution of storage capacity on the disc and the dependency of the rotational delay on the track location, as well as a parameterized seek cost function. We have shown that the optimal placement satisfies a unimodality property for the placement of high probabilities, and we have derived an analytic solution to the optimal data placement problem. We have shown that the optimal data placement may be drastically different than the optimal data placement on magnetic discs.

The data access probabilities were described by a parametrized, two valued, probability distribution. This problem formulation allowed us to derive optimal locations for the high probability data items. Since in many real life environments, precise knowledge about the access probabilities of data items may not be known, this problem formulation will be adequate for these environments. As a special case, indices are frequently considered to be high access frequency items (in comparison to data values). In this context our results suggest an optimal position for the indexes given the device characteristics.

In environments where more detailed knowledge of the access frequencies of the data items may be available, our method can be extended using a recursive approximation of the access probabilities. We have outlined such an algorithm.

The results of this paper complement the results presented in [Christodoulakis and Ford 88]. In that paper, the problem of the optimal order of file placement on CLV optical discs was studied. Files were considered to be independent of each other, and only one file was accessed at a given point in time. The cost function described the expected cost of a set of random and sequential requests from each file. Data items in each file were assumed to have the same access probability and files were allocated a consecutive set of addresses on the disc surface. In contrast, the results of this paper are applicable to environments where many users may exist, and/or data items of a single file may have different access probabilities. Data items are considered to be small (e.g., fit within a single sector) in contrast to files that typically occupy many tracks on the disc. Only random access of data items is considered, and the probabilities of access of data items are independent.

12. References

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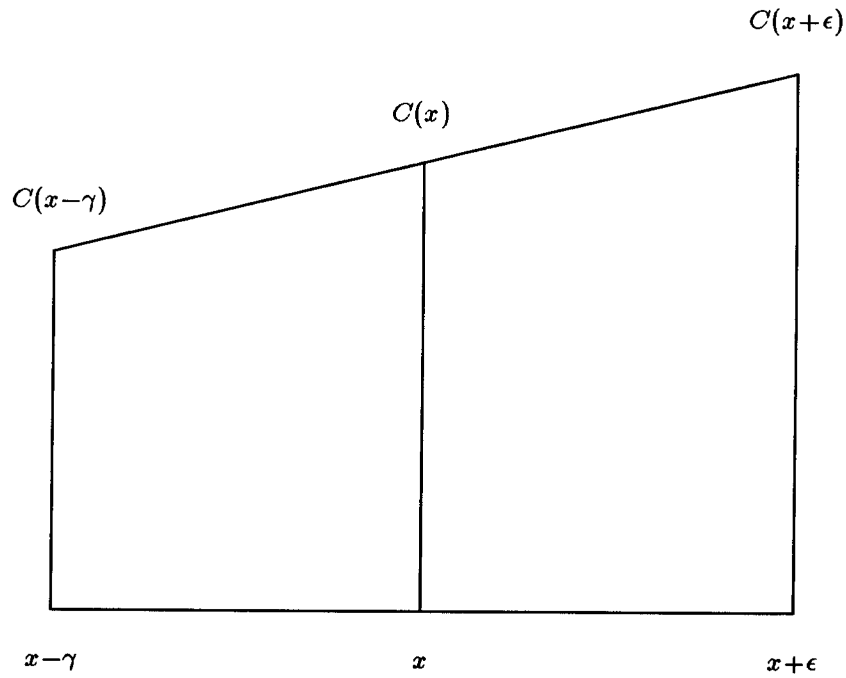
Appendix 1

Gamma and Epsilon

We compute the positions of X_0, X_2, X_3 and X_5 such that the areas of the four regions being exchanged are equal. Given a point x we wish to compute the two points, one to the left and one to the right, that give equal areas to the trapezoids on each side of the point x.



Let $x-\gamma$ be the point on the left, and $x+\epsilon$, the point on the right, and let A be the area, then we compute γ in the following manner:

$$\frac{C(x) + C(x-\gamma)}{2}\gamma = A$$

$$(C(x) + C(x) - k\gamma)\gamma = 2A$$

$$2C(x)\gamma - k\gamma^2 - 2A = 0$$

$$\gamma = \frac{-2\,C(x) +/- \sqrt{4C(x)^2 - 4(-k)(-2A)}}{-2k}$$

$$\gamma = \frac{C(x) - \sqrt{C(x)^2 - 2kA}}{k}$$

and ϵ:

$$\frac{C(x) + C(x+\epsilon)}{2}\epsilon = A$$

$$(C(x) + C(x) + k\epsilon)\epsilon = 2A$$

$$2C(x)\epsilon + k\epsilon^2 - 2A = 0$$

$$\epsilon = \frac{-2\,C(x) +/- \sqrt{4C(x)^2 - 4(k)(-2A)}}{2k}$$

$$\epsilon = \frac{-C(x) + \sqrt{C(x)^2 + 2kA}}{k}$$

Appendix 2

Ratio

For a given position x, we generalize

$$\frac{\left(\int_{X_0}^{X_1} C(x)^2\,dx - \int_{X_1}^{X_2} C(y)^2\,dy\right)}{A_{left}\,\Delta x(X_1, A_{left})} \quad \text{and} \quad \frac{\left(\int_{X_3}^{X_4} C(x)^2\,dx - \int_{X_4}^{X_5} C(y)^2\,dy\right)}{A_{right}\,\Delta x(X_4, A_{right})}$$

to the following, using the definitions for γ and ϵ given in appendix 2.

$$\frac{\int_0^{\gamma} C(x-y)^2\,dy - \int_0^{\epsilon} C(x+z)^2\,dz}{A\left(\int_0^{\gamma} \frac{1}{A} C(x-y)\,y\,dy + \int_0^{\epsilon} \frac{1}{A} C(x+z)\,z\,dz\right)}$$

If we expand each of the integrals we obtain:

$$\int_0^{\gamma} C(x-y)^2\,dy = \frac{1}{3k}((kx+j)^3 - ((kx+j)^2 - 2kA)^{3/2})$$

$$\int_0^{\epsilon} C(x+z)^2\,dz = \frac{1}{3k}(((kx+j)^2 + 2kA)^{3/2} - (kx+j)^3)$$

$$\int_0^{\gamma} C(x-y)\,y\,dy$$

$$= \frac{1}{6k^2}((kx+j) - \sqrt{(kx+j)^2 - 2kA})^2((kx+j) + 2(\sqrt{(kx+j)^2 - 2kA}))$$

$$= \frac{1}{6k^2}((kx+j)^3 + 2(kx+j)^2\sqrt{(kx+j)-2kA} - 2(kx+j)^2\sqrt{(kx+j)-2kA}$$

$$- 4(kx+j)((kx+j)-2kA) + (kx+j)((kx+j)-2kA)$$

$$+ ((kx+j)-2kA)\sqrt{(kx+j)-2kA})$$

$$= \frac{1}{6k^2}(-2(kx+j)^3 + 6kA(kx+j) + 2((kx+j)-2kA)^{3/2})$$

$$\int_0^{\epsilon} C(x+z)\,z\,dz = \frac{1}{6k^2}((kx+j) - \sqrt{(kx+j)^2 + 2kA})^2((kx+j) + 2(\sqrt{(kx+j)^2 + 2kA}))$$

$$= \frac{1}{6k^2}((kx+j)^3 + 2(kx+j)^2\sqrt{(kx+j)+2kA} - 2(kx+j)^2\sqrt{(kx+j)+2kA}$$

$$- 4(kx+j)((kx+j)+2kA) + (kx+j)((kx+j)+2kA)$$
$$+ ((kx+j)+2kA)\sqrt{(kx+j)+2kA})$$
$$= \frac{1}{6k^2}(-2(kx+j)^3 - 6kA(kx+j) + 2((kx+j)+2kA)^{3/2})$$

The numerator of our ratio becomes:

$$\int_0^\gamma C(x-y)^2\,dy - \int_0^\epsilon C(x+z)^2\,dz$$
$$= \frac{1}{3k}(2(kx+j)^3 - ((kx+j)^2-2kA)^{3/2} - ((kx+j)^2+2kA)^{3/2})$$

And our denominator

$$A\,(\int_0^\gamma \frac{1}{A}C(x-y)\,y\,dy + \int_0^\epsilon \frac{1}{A}C(x+z)\,z\,dz)$$
$$= \frac{1}{6k^2}(-4(kx+j)^3 + 2((kx+j)-2kA)^{3/2} + 2((kx+j)+2kA)^{3/2})$$
$$= \frac{1}{3k^2}(-2(kx+j)^3 + ((kx+j)-2kA)^{3/2} + ((kx+j)+2kA)^{3/2})$$

Simplifying the ratio we have:

$$\frac{\int_0^\gamma C(x-y)^2\,dy - \int_0^\epsilon C(x+z)^2\,dz}{A\,(\int_0^\gamma \frac{1}{A}C(x-y)\,y\,dy + \int_0^\epsilon \frac{1}{A}C(x+z)\,z\,dz)} = -k$$

Appendix 3

Cost Expression

$$Cost(k,j,a,b,P_1,P_2,r,h,m) = \frac{1}{120(km+j)^5(P_1r+P_2-P_2r)^2}$$

$$(60P_2^2j^7h+40aP_2^2j^7+120bj^5P_2^2-10P_2aP_1r^3k^4m^2+10k^2j^2r^3aP_2^2+400k^3m^3j^4aP_2^2$$

$$-800k^4m^3j^2raP_2^2-800k^3m^2j^3raP_2^2-400k^2j^4mraP_2^2+400k^2j^5m^2aP_2^2+2400k^2m^3j^4raP_2^2+1200k^4m^5j^2raP_2^2$$

$$+240k^5m^6rjaP_2^2+20k^3m^2r^3jaP_2^2+20j^3r^3kaP_2^2+20k^3jmr^3aP_2^2-80k^6m^5raP_2^2+200k^4m^4j^3aP_2^2$$

$$+40k^5m^5j^2aP_2^2-400k^5m^4jraP_2^2+240j^6mraP_2^2-600k^2m^2j^4raP_2^2+600j^2k^4m^4raP_2^2-3200k^3m^5j^3arP_2^2$$

$$-3k^2r^5aP_2^2-2800k^2m^4j^4arP_2^2+20r^3j^4aP_2^2-40k^4m^4r^3P_1aP_2+640k^5m^7jrP_1aP_2+2000k^4m^6j^2rP_1aP_2$$

$$+3200k^3m^5j^3rP_1aP_2+2800k^2m^4j^4rP_1aP_2+80k^6m^8rP_1aP_2-60r^3j^4P_1aP_2+240m^2j^6rP_1aP_2-260k^2m^2j^2r^3P_1aP_2$$

$$-120rj^6aP_2^2-160k^3m^3jr^3P_1aP_2+1280km^3j^5rP_1aP_2-200r^3kmj^3P_1aP_2-240m^2j^6arP_2^2-80k^6m^8raP_2^2$$

$$-640k^5m^7rajP_2^2+20k^2j^2r^3am^2P_2^2+40kj^3r^3amP_2^2-1280rj^5m^3akP_2^2+5k^2r^5P_1aP_2+120bk^5m^5P_1^2r^2$$

$$-2000k^4m^6j^2arP_2^2-10P_2aP_1r^3k^2j^2+60P_1krj^6hP_2+20P_2k^2hj^5P_1r-20P_2aP_1r^3kj^3+200kmj^6aP_2^2$$

$$+120k^6m^6raP_2^2-80kj^5raP_2^2+10k^4m^2r^3aP_2^2+1200j^5rkm^2aP_2^2+40k^2mj^2r^3aP_2^2-480kj^5mraP_2^2$$

$$+2400k^3m^4j^3raP_2^2+480jk^5m^5raP_2^2+240k^2j^2r^3am^2P_1^2+160kj^3r^3amP_1^2+160k^3m^3r^3ajP_1^2+40k^4m^4r^3aP_1^2$$

$$-2k^2r^5aP_1^2+40r^3j^4aP_1^2-20P_2aP_1r^3k^3mj-20P_2aP_1r^3k^3jm^2-40P_2aP_1r^3k^2j^2m-120P_2aP_1rm^6k^6$$

$$-480P_2aP_1rm^5k^5j-600P_2aP_1rm^4k^4j^2+80P_2aP_1rk^6m^5+400P_2aP_1rk^5m^4j+800P_2aP_1rk^4m^3j^2-240P_2aP_1rjk^5m^6$$

$$-1200P_2aP_1rj^2k^4m^5-2400P_2aP_1rj^3k^3m^4+800P_2aP_1rk^3m^2j^3-2400P_2aP_1rj^4k^2m^3+400P_2aP_1rk^2mj^4-1200P_2aP_1rj^5km^2$$

$$+600P_2aP_1rk^2m^2j^4+480P_2aP_1rkmj^5+80P_2aP_1rkj^5-240P_2aP_1rj^6m+120P_2aP_1rj^6+8aP_2^2k^7m^5$$

$$+40aP_2^2k^6jm^4+80aP_2^2k^5j^2m^3+80aP_2^2k^4m^2j^3+40aP_2^2k^3mj^4+8aP_2^2k^2j^5+40aP_2^2jk^6m^5$$

$$+200aP_2^2j^2k^5m^4+400aP_2^2j^3k^4m^3+400aP_2^2j^4k^3m^2+200aP_2^2j^5k^2m+40aP_2^2j^6k-600P_2^2k^4m^3j^3rh$$

$$-600P_2^2k^3m^2j^4rh-300P_2^2k^2mj^5rh-300P_2^2k^5m^4j^2rh-60P_2^2jk^6m^5rh-60P_2^2krj^6h+600P_1k^4m^3j^3rhP_2$$

$$+600P_1k^3m^2j^4rhP_2+300P_1k^2mj^5rhP_2+300P_1k^5m^4j^2rhP_2+60P_1jk^6m^5rhP_2+240bk^5m^5P_1rP_2-240bk^5m^5P_1r^2P_2$$

$$+120bk^5m^5P_2^2-240bk^5m^5P_2^2r+120bk^5m^5P_2^2r^2+600bjk^4m^4P_1^2r^2+1200bjk^4m^4P_1rP_2-1200bjk^4m^4P_1r^2P_2$$

$$+600bjk^4m^4P_2^2-1200bjk^4m^4P_2^2r+600bjk^4m^4P_2^2r^2+1200bj^2k^3m^3P_1^2r^2+2400bj^2k^3m^3P_1rP_2-2400bj^2k^3m^3P_1r^2P_2$$

$$+1200bj^2k^3m^3P_2^2-2400bj^2k^3m^3P_2^2r+1200bj^2k^3m^3P_2^2r^2+1200bk^2m^2j^3P_1^2r^2+2400bk^2m^2j^3P_1rP_2-2400bk^2m^2j^3P_1r^2P_2$$

$$+1200bk^2m^2j^3P_2^2-2400bk^2m^2j^3P_2^2r+1200bk^2m^2j^3P_2^2r^2+600bkmj^4P_1^2r^2+1200bkmj^4P_1rP_2-1200bkmj^4P_1r^2P_2$$

$$+600bkmj^4P_2^2-1200bkmj^4P_2^2r+600bkmj^4P_2^2r^2+120bj^5P_1^2r^2+240bj^5P_1rP_2-240bj^5P_1r^2P_2$$

$$-240bj^5P_2^2r+120bj^5P_2^2r^2+60P_1k^6m^6rhP_2+20P_2k^7hm^5P_1r+60P_1rj^6hP_2-2400P_2k^3m^3j^3r^2hP_1$$

$$-1200P_2^2k^3m^3j^3rh+1200P_2^2k^3m^3j^3r^2h-1800P_2k^2m^2j^4r^2hP_1-900P_2^2k^2m^2j^4rh+900P_2^2k^2m^2j^4r^2h-720P_2kmj^5r^2hP_1$$

$$-360P_2^2kmj^5rh+360P_2^2kmj^5r^2h-1800P_2k^4m^4j^2r^2hP_1-900P_2^2k^4m^4j^2rh+900P_2^2k^4m^4j^2r^2h-720P_2jk^5m^5r^2hP_1$$

$$-360P_2^2jk^5m^5rh+360P_2^2jk^5m^5r^2h-120P_2k^6m^6r^2hP_1-60P_2^2k^6m^6rh+60P_2^2k^6m^6r^2h-10P_2k^4r^4hm^2P_1$$
$$-5P_2^2k^4r^3hm^2+5P_2^2k^4r^4hm^2-20P_2k^3r^4hmjP_1-10P_2^2k^3r^3hmj+10P_2^2k^3r^4hmj-10P_2k^2r^4hj^2P_1$$
$$-5P_2^2k^2r^3hj^2+5P_2^2k^2r^4hj^2-120P_2r^2j^6hP_1-60P_2^2rj^6h+60P_2^2r^2j^6h+1200P_1^2k^3m^3j^3r^2h$$
$$+900P_1^2k^2m^2j^4r^2h+360P_1^2kmj^5r^2h+900P_1^2k^4m^4j^2r^2h+360P_1^2jk^5m^5r^2h+60P_1^2k^6m^6r^2h+5P_1^2k^4r^4hm^2$$
$$+10P_1^2k^3r^4hmj+5P_1^2k^2r^4hj^2+60P_1^2r^2j^6h+1200P_1k^3m^3j^3rhP_2+900P_1k^2m^2j^4rhP_2+360P_1kmj^5rhP_2$$
$$+900P_1k^4m^4j^2rhP_2+360P_1jk^5m^5rhP_2+5P_1k^4r^3hm^2P_2+10P_1k^3r^3hmjP_2+5P_1k^2r^3hj^2P_2+20P_2^2k^7hm^5$$
$$-20P_2^2k^7hm^5r+100P_2k^6hjm^4P_1r+100P_2^2k^6hjm^4-100P_2^2k^6hjm^4r+200P_2k^5hj^2m^3P_1r+200P_2^2k^5hj^2m^3$$
$$-200P_2^2k^5hj^2m^3r+200P_2k^4hm^2j^3P_1r+200P_2^2k^4hm^2j^3-200P_2^2k^4hm^2j^3r+100P_2k^3hmj^4P_1r+100P_2^2k^3hmj^4$$
$$-100P_2^2k^3hmj^4r+20P_2^2k^2hj^5-20P_2^2k^2hj^5r+60P_2^2k^6jhm^5+300P_2^2k^5j^2hm^4+600P_2^2k^4j^3hm^3$$
$$+600P_2^2k^3j^4hm^2+300P_2^2k^2j^5hm+60P_2^2kj^6h+60P_2j^2k^5hm^5P_1r+60P_2^2j^2k^5hm^5-60P_2^2j^2k^5hm^5r$$
$$+300P_2j^3k^4hm^4P_1r+300P_2^2j^3k^4hm^4-300P_2^2j^3k^4hm^4r+600P_2j^4k^3hm^3P_1r+600P_2^2j^4k^3hm^3-600P_2^2j^4k^3hm^3r$$
$$+600P_2j^5k^2hm^2P_1r+600P_2^2j^5k^2hm^2-600P_2^2j^5k^2hm^2r+300P_2j^6khmP_1r+300P_2^2j^6khm-300P_2^2j^6khmr$$
$$+60P_2j^7hP_1r-60P_2^2j^7hr)$$

Appendix 4

Derivative of Cost Expression

$$\frac{\partial \, Cost(k,j,a,b,P_1,P_2,r,h,m)}{\partial \, m} = \frac{r}{(km+j)^6(P_1r+P_{2-P_2}r)^2}$$

$$(48aP_2^2j^7+960k^3m^3j^4aP_2^2+720k^2j^5m^2aP_2^2+720k^4m^4j^3aP_2^2+288k^5m^5j^2aP_2^2-24P_1krj^6hP_2$$

$$+288kmj^6aP_2^2+144aP_2^2jk^6m^5+360aP_2^2j^2k^5m^4+480aP_2^2j^3k^4m^3+360aP_2^2j^4k^3m^2+144aP_2^2j^5k^2m$$

$$+24aP_2^2j^6k+240P_2^2k^4m^3j^3rh+180P_2^2k^3m^2j^4rh+72P_2^2k^2mj^5rh+180P_2^2k^5m^4j^2rh+72P_2^2jk^6m^5rh$$

$$+12P_2^2krj^6h-480P_1k^4m^3j^3rhP_2-360P_1k^3m^2j^4rhP_2-144P_1k^2mj^5rhP_2-360P_1k^5m^4j^2rhP_2-144P_1jk^6m^5rhP_2$$

$$-72P_2^2k^6jhm^5-180P_2^2k^5j^2hm^4-240P_2^2k^4j^3hm^3-180P_2^2k^3j^4hm^2-72P_2^2k^2j^5hm-12P_2^2kj^6h$$

$$-12k^2j^3r^2aP_2^2-3P_1k^3r^2hj^2P_2-3P_1^2k^3r^3hj^2+12P_1^2krj^6h+3P_2^2k^3r^2hj^2+2k^3r^4aP_1^2$$

$$-24P_2aP_1j^6k+6P_2aP_1r^2k^3j^2-6k^3j^2r^2aP_2^2-8r^2j^4aP_1^2k+12P_1kj^6hP_2+3k^3r^4aP_2^2$$

$$-48P_2aP_1j^7-5k^3r^4P_1aP_2-12r^2j^4aP_2^2k+20r^2j^4P_1aP_2k+12P_2aP_1r^2k^2j^3-3P_2^2k^3r^3hj^2$$

$$+6P_1k^3r^3hj^2P_2-6P_2^2k^4r^3hmj+6P_1k^5r^3hm^2P_2+12P_1k^4r^3hmjP_2-720P_2aP_1k^2m^2j^5+624P_2aP_1km^2j^6$$

$$-288P_2aP_1mj^6k+96P_2aP_1mj^7+12P_2^2k^7m^6rh-3P_2^2k^5r^3hm^2-144P_2aP_1k^2mj^5-144P_2aP_1m^5k^6j$$

$$-32k^2j^3r^2amP_1^2-32k^4m^3r^2ajP_1^2-8k^5m^4r^2aP_1^2-24P_2aP_1m^6k^7+24k^7m^6aP_2^2-48k^3j^2r^2am^2P_1^2$$

$$+12P_2aP_1r^2k^4mj-360P_2aP_1m^4k^5j^2-12k^4jmr^2aP_2^2+6P_2aP_1r^2k^5m^2-480P_2aP_1k^4m^3j^3+60P_2aP_1r^2m^2k^3j^2$$

$$+12P_2aP_1r^2k^4m^2j+24P_2aP_1r^2k^3mj^2+56P_2aP_1r^2mj^3k^2+48P_2aP_1k^7m^8+1728P_2aP_1k^2m^3j^5-24k^2mj^3r^2aP_2^2$$

$$-1296j^2k^5m^6aP_2^2-2640k^3m^4j^4aP_2^2-12k^3m^2j^2r^2aP_2^2-24P_1k^7m^6rhP_2-3P_1^2k^5r^3hm^2-360P_2aP_1k^3m^2j^4$$

$$-6k^5m^2r^2aP_2^2-24k^3mj^2r^2aP_2^2-2400j^3k^4m^5aP_2^2-384jk^6m^7aP_2^2-96j^7maP_2^2+32P_2aP_1r^2m^3k^4j$$

$$+8P_2aP_1r^2m^4k^5+384P_2aP_1k^6m^7j+240P_1k^4m^3j^3hP_2+180P_1k^3m^2j^4hP_2+6P_2^2k^4r^2hmj+3P_2^2k^5r^2hm^2$$

$$+240P_1^2k^4m^3j^3rh+180P_1^2k^3m^2j^4rh+72P_1^2k^2mj^5rh+12P_1^2k^7m^6rh-12P_2^2k^7m^6h-6P_1k^4r^2hmjP_2$$

$$-3P_1k^5r^2hm^2P_2+72P_1k^2mj^5hP_2+180P_1k^5m^4j^2hP_2+72P_1jk^6m^5hP_2+72P_1^2jk^6m^5rh-6P_1^2k^4r^3hmj$$

$$+180P_1^2k^5m^4j^2rh-48k^7m^8aP_2^2+12P_1k^7m^6hP_2+1296P_2aP_1k^5m^6j^2-48P_2aP_1k^6m^6j-288P_2aP_1k^5m^5j^2$$

$$+2400P_2aP_1k^4m^5j^3-720P_2aP_1k^4m^4j^3-960P_2aP_1k^3m^3j^4+2640P_2aP_1k^3m^4j^4-12k^4m^2jr^2aP_2^2-1728k^2m^3j^5aP_2^2$$

$$-624kj^6m^2aP_2^2+48k^6m^6jaP_2^2)$$