Printing Requisition/Graphic Services

18589

TITLE OR DESCRIPTION
Relief from the Pain of Overlap - Generalized Schwarz Splittings - CS-89-04

DATE REQUISITIONED April 25, 1990

DATE REQUIRED May 10th, 1990

ACCOUNT NO. 1 2 6 6 2 8 4 4 1

REQUISITIONER- PRINT Colleen Bernard

PHONE 2192

SIGNING AUTHORITY (signature)

MAILING INFO -
NAME Colleen Bernard
DEPT. CS
BLDG. & ROOM NO. DC-2326
[X] DELIVER
[ ] PICK-UP

NUMBER OF PAGES 29
NUMBER OF COPIES 15

TYPE OF PAPER STOCK
[X] BOND [ ] NCR ___ PT. [ ] COVER [ ] BRISTOL [ ] SUPPLIED [ ] ___

PAPER SIZE
[X] 8½ x 11 [ ] 8½ x 14 [ ] 11 x 17 [ ] ___

PAPER COLOUR [X] WHITE [ ] ___
INK [X] BLACK [ ] ___

PRINTING [X] 1 SIDE ___ PGS. [X] 2 SIDES ___ PGS.
NUMBERING FROM ___ TO ___

BINDING/FINISHING 3 down left side
[X] COLLATING [X] STAPLING [ ] ___ HOLE PUNCHED [ ] PLASTIC RING

FOLDING/ PADDING
CUTTING SIZE

Special Instructions

Math fronts and backs enclosed

NEGATIVES		QUANTITY	OPER. NO.	TIME	LABOUR CODE
F L M					C 0 1
F L M					C 0 1
F L M					C 0 1
F L M					C 0 1
F L M					C 0 1

PMT					
P M T					C 0 1
P M T					C 0 1
P M T					C 0 1

PLATES					
P L T					P 0 1
P L T					P 0 1
P L T					P 0 1

STOCK					
					0 0 1
					0 0 1
					0 0 1
					0 0 1

COPY CENTRE
OPER. NO. ___ BLDG. ___ MACH. NO. ___

DESIGN & PASTE-UP
OPER. NO. ___ TIME ___ LABOUR CODE
D 0 1
D 0 1
D 0 1

TYPESETTING QUANTITY
P A P 0 0 0 0 0 | T 0 1
P A P 0 0 0 0 0 | T 0 1
P A P 0 0 0 0 0 | T 0 1

BINDERY					
R N G					B 0 1
R N G					B 0 1
R N G					B 0 1
M I S 0 0 0 0 0					B 0 1

OUTSIDE SERVICES

PROOF
P R F
P R F
P R F

$ ___ COST

January 19, 1989

Sue!

80 please for Wei-Pai Tang using
the Math covers.

Thanks — go ahead and sign the req.
126-6284-41

Printing Requisition/Graphic Services

26013

1. Please complete unshaded areas on form as applicable.
2. Distribute copies as follows: White and Yellow to Graphic Services. Retain Pink Copies for your records.
3. On completion of order the Yellow copy will be returned with the printed material.
4. Please direct enquiries, quoting requisition number and account number, to extension 3451.

TITLE OR DESCRIPTION

Relief from the Pain of Overlap - Generalized Schwarz Splittings   CS-89-04

DATE REQUISITIONED Jan. 20/89

DATE REQUIRED ASAP

ACCOUNT NO. 1 2 6 6 2 8 4 4 1

REQUISITIONER- PRINT W.P. Tang

PHONE 4468

SIGNING AUTHORITY _S. DeAngelis / W.P. Tang_

MAILING INFO – **NAME** Sue DeAngelis   **DEPT.** C.S.   **BLDG. & ROOM NO.** DC 2314   ☒ DELIVER ☐ PICK-UP

Copyright: I hereby agree to assume all responsibility and liability for any infringement of copyrights and/or patent rights which may arise from the processing of, and reproduction of, any of the materials herein requested. I further agree to indemnify and hold blameless the University of Waterloo from any liability which may arise from said processing or reproducing. I also acknowledge that materials processed as a result of this requisition are for educational use only.

NUMBER OF PAGES 29

NUMBER OF COPIES 80

TYPE OF PAPER STOCK
☒ BOND ☐ NCR _____ PT. ☒ COVER ☐ BRISTOL ☒ SUPPLIED ☐ _____

PAPER SIZE
☒ $8\frac{1}{2}$ x 11 ☐ $8\frac{1}{2}$ x 14 ☐ 11 x 17 ☐ _____

PAPER COLOUR ☒ WHITE ☐ _____
INK ☒ BLACK ☐ _____

PRINTING ☐ 1 SIDE _____ PGS. ☒ 2 SIDES _____ PGS.
NUMBERING FROM _____ TO _____

BINDING/FINISHING 3 down left side
☒ COLLATING ☒ STAPLING ☐ _____ PUNCHED ☐ PLASTIC RING

FOLDING/PADDING
CUTTING SIZE

Special Instructions

Math fronts and backs enclosed.

NEGATIVES	QUANTITY	OPER. NO.	TIME	LABOUR CODE
FILM				C 0 1
FILM				C 0 1
FILM				C 0 1
FILM				C 0 1
FILM				C 0 1
PMT				
PMT				C 0 1
PMT				C 0 1
PMT				C 0 1
PLATES				
PLT				P 0 1
PLT				P 0 1
PLT				P 0 1
STOCK				
				0 0 1
				0 0 1
				0 0 1
				0 0 1
BINDERY				
RNG				B 0 1
RNG				B 0 1
RNG				B 0 1
MIS 0 0 0 0 0				B 0 1
OUTSIDE SERVICES				

COPY CENTRE	OPER. NO.	BLDG.	MACH. NO.

DESIGN & PASTE-UP	OPER. NO.	TIME	LABOUR CODE
			D 0 1
			D 0 1
			D 0 1

TYPESETTING	QUANTITY			CODE
PAP 0 0 0 0 0				T 0 1
PAP 0 0 0 0 0				T 0 1
PAP 0 0 0 0 0				T 0 1

PROOF				
PRF				
PRF				
PRF				

$ _____ COST

TAXES – PROVINCIAL ☐ FEDERAL ☐   GRAPHIC SERV. OCT. 85 482-2

Notes

Jan 12

Ane,

Can you prepare another
printing req. as the title
for the paper was changed.
I made a new front page
to save you the bother. He
would like 20 copies this
time instead of 50. All else
is the same - thanks ✓.

Printing Requisition/Graphic Services

12147

TITLE OR DESCRIPTION
CS-89-04

DATE REQUISITIONED	DATE REQUIRED	ACCOUNT NO.
Jan. 15/90	ASAP	1 2 6 6 2 8 4 4 1

REQUISITIONER – PRINT	PHONE	SIGNING AUTHORITY
W.P. TANG	2192	S. DeAngelis / W.P. Tang

MAILING INFO – NAME	DEPT.	BLDG. & ROOM NO.	
S. DeANGELIS	C.S.	DC 2314	☑ DELIVER ☐ PICK-UP

NUMBER OF PAGES **25**   NUMBER OF COPIES **20**

TYPE OF PAPER STOCK
☑ BOND ☐ NCR ____ PT. ☑ COVER ☐ BRISTOL ☑ SUPPLIED ☐ ____

PAPER SIZE
☑ 8½ x 11 ☐ 8½ x 14 ☐ 11 x 17 ☐ ____

PAPER COLOUR
☑ WHITE ☐ ____   INK ☑ BLACK ☐ ____

PRINTING
☐ 1 SIDE ____ PGS. ☑ 2 SIDES ____ PGS.   NUMBERING FROM ____ TO ____

BINDING/FINISHING
☑ COLLATING ☑ STAPLING 3 down left side ☐ HOLE PUNCHED ☐ PLASTIC RING

FOLDING/PADDING   CUTTING SIZE

Special Instructions

Math fronts & backs enclosed.

NEGATIVES	QUANTITY	OPER. NO.	TIME	LABOUR	CODE
F L M					C 0 1
F L M					C 0 1
F L M					C 0 1
F L M					C 0 1
F L M					C 0 1

PMT					
P M T					C 0 1
P M T					C 0 1
P M T					C 0 1

PLATES					
P L T					P 0 1
P L T					P 0 1
P L T					P 0 1

STOCK					
					0 0 1
					0 0 1
					0 0 1
					0 0 1

COPY CENTRE	OPER. NO.	BLDG.	MACH. NO.

DESIGN & PASTE-UP	OPER. NO.	TIME	LABOUR CODE
			D 0 1
			D 0 1
			D 0 1

TYPESETTING	QUANTITY			
P A P 0 0 0 0 0 0				T 0 1
P A P 0 0 0 0 0 0				T 0 1
P A P 0 0 0 0 0 0				T 0 1

BINDERY					
R N G					B 0 1
R N G					B 0 1
R N G					B 0 1
M I S 0 0 0 0 0 0					B 0 1

OUTSIDE SERVICES

PROOF				
P R F				
P R F				
P R F				

$ _____ COST

TAXES – PROVINCIAL ☐ FEDERAL ☐   GRAPHIC SERV. OCT. 85 482-2

Generalized Schwarz Splittings

by

Wei Pai Tang

Research Report CS-89-04
January, 1989

Faculty

of

Mathematics

University of Waterloo
Waterloo, Ontario, Canada

N2L   3G1

Generalized Schwarz Splittings

by

Wei Pai Tang

GENERALIZED SCHWARZ SPLITTINGS*

WEI PAI TANG[†]

Abstract.

A classic mathematical technique, Schwarz Alternating Method (*SAM*), has recently attracted much attention from the research community of parallel computations as well as theoreticians [2], [14], [11], [12], [5], [17], [4], [10], [3]. Its potential in parallelism, wide applicability and great flexibility in implementation make *SAM* a competitive choice in parallel computations. However, the computational performance of the classical *SAM* and its modern extensions strongly depends on the amount of overlap between the neighbouring subregions. Introducing a large overlap has changed the image of *SAM* from an impractical theoretical technique to a rather competitive numerical approach. Meanwhile, concerns about the duplicated work on these overlapped regions are increasing. Reducing the amount of overlap without affecting the speed of convergence has become an important performance issue.

Schwarz Splitting (*SS*)[17] has been proposed as one of *SAM*'s modern extensions in numerical algebra, and a generalized *SS* is presented in this paper. The new approach allows us to utilize the flexibility of the splitting to further improve convergence speed and complexity. A fast convergence is obtained by choosing a good splitting instead of increasing the overlap. The best performance from our generalized *SS* is much better than that from previously recommended *SS*, in which a large overlap is used. Both convergence analysis and numerical results are presented here.

Key Words. Schwarz Alternating Method(*SAM*), Schwarz Splitting(*SS*), generalized Schwarz splitting, domain decomposition, parallel computation, overlap.

1. Introduction. Experience with the new generation of parallel computers has promoted efforts to search for *truly* parallel algorithms rather than parallelizing the existing sequential algorithms. For coarse grain parallelism, domain decomposition has become an increasingly important focus of research on numerical solution of partial differential equations.

A classic mathematical approach–Schwarz alternating method (1869)– [16] appears to offer promise for the parallel solution of the very large systems of linear or nonlinear algebraic equations that arise when elliptic problems in elasticity, fluid dynamics, or other important areas are discretized by finite elements or finite differences. With this approach, a large problem is decomposed into several coupled subproblems. With proper ordering, these subproblems can be solved independently. Given an initial guess, and repeating the solutions of each subregion and exchanging the new information, this process will converge to the solution on the entire region. Flexibility in mapping these subproblems into different parallel computer topologies and the lower ratio between communication and computation make SAM a competitive choice in parallel processing. It is also crucial for some complex fluid flow calculations that different modelings or grids be applied to different subdomains of the flow. For example, in many applications we need to merge Euler's equation, the Navier-Stokes equations, potential flow and other models in suitable subregions for a single large problem. There are also applications where composite meshes in regions with complicated boundaries are needed. SAM can provide a natural framework within which all these requirements are met.

The recognition of SAM's potential in numerical computations was a rather recent event [2], [14], [11], [12], [5], [17], [4], [10], [3]. This delay may have been caused by some

disappointing experiments with earlier naive implementations of SAM [15]. During the past several years, understanding of SAM's computational behavior has become clearer. If the area of the overlap regions is a constant fraction of the subregions, several people have shown that the convergence of *SAM* is independent of the mesh [12], [3].

Several *modern extensions* have been proposed. For example, a generalization of SAM in linear algebra– Schwarz splittings– was introduced in [14], [17], and an additive version of SAM is being investigated [4], [3]. It is known now that the convergence of the classical SAM and most of its modern extensions strongly depends on the amount of overlap between subregions. Introducing a larger overlap does considerably improve the performance of SAM [12], [8]. Combining with other acceleration techniques, such as multilevel techniques, preconditioning or SOR accelerations, *SAM* has proven to be a competitive method in large scale scientific computations. In spite of of quick convergence, the duplicated work on the overlapped regions at each iteration is not desired. Reducing the amount of overlap without affecting speed of convergence has become an important performance issue.

Schwarz originally proposed a coupling between subregions which requires only the continuity of the unknown. Kantorovich and Krylov ([9], pp. 617-626) presented a rather general convergence result of *SAM* for a second order partial differential equation of the form:

$$L(u) = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}) = 0,$$

with Dirichlet boundary condition. They also used Dirichlet boundary condition on the artificial boundaries. It is not difficult to observe that we can replace the continuity of the unknown on these artificial boundaries by other couplings, for example, the continuity of the unknown's derivative. Several attempts to use different boundary conditions on these overlapped regions were not very successful [17], [13], leading us to conjecture two years ago that "For different problems the best choice of the type of boundary conditions may vary. It is a very interesting open problem for future research." For the model problem, it was shown that Dirichlet boundary condition has a better convergence rate than Neumann boundary condition [17].

In this paper, a successful coupling between the overlap subregions is identified. First, let us introduce a generalized version of *SAM*: consider the Dirichlet problem for a second order elliptic operator L

$$(1) \qquad \begin{cases} L(u(\mathbf{x})) &= 0, & \mathbf{x} \in \Omega, \\ u(\mathbf{x})\,|_{\Gamma_\Omega} &= \psi(\mathbf{x}), & \mathbf{x} \in \Gamma_\Omega \end{cases}$$

where Ω is a bounded region in k-dimensional space, Γ_Ω is the boundary of Ω, $\mathbf{x} = \{x_1, x_2, \cdots, x_k\}$ is the independent variable (Fig. 1 shows an example of two dimensional case). To simplify the discussion, we consider a case for two subregions, although direct generalization may be made to more subregions. We also assume that the solution to this problem exists and is unique.

Split the solution domain Ω into two overlapping subdomains Ω_1 and Ω_2 (see Fig. 1), provided $\Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset$. Denote Γ_{Ω_1}, Γ_{Ω_2}, $\Gamma_{\Omega_{12}}$ the boundaries of Ω_1, Ω_2 and Ω_{12} respectively. Let

$$\begin{aligned} \Gamma_{\Omega_1} &= \Gamma_1 \cup \Gamma'_1, \\ \Gamma_{\Omega_2} &= \Gamma_2 \cup \Gamma'_2. \end{aligned}$$
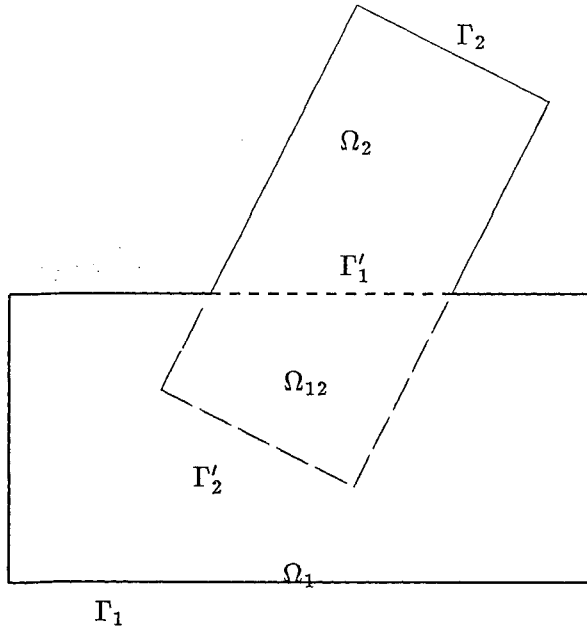
2

FIG. 1. *Two overlapping subregions*

where

$$\Gamma_1 = \Gamma_\Omega \cap \Gamma_{\Omega_1}, \quad \Gamma_1' = \Gamma_{\Omega_1} \cap \Gamma_{\Omega_{12}},$$
$$\Gamma_2 = \Gamma_\Omega \cap \Gamma_{\Omega_2}, \quad \Gamma_2' = \Gamma_{\Omega_2} \cap \Gamma_{\Omega_{12}}.$$

Denote u_1 and u_2 the solutions on subdomain Ω_1 and Ω_2, respectively. Then, the following couplings

$$(2) \qquad\qquad g_1(u_1)\,\big|_{\Gamma_1'} = g_1(u_2)$$

$$(3) \qquad\qquad g_2(u_2)\,\big|_{\Gamma_2'} = g_2(u_1)$$

are true on the artificial boundaries Γ_1' and Γ_2', where

$$(4) \qquad\qquad g_i(u) = \omega_i u + (1 - \omega_i)\frac{\partial u}{\partial n}, \qquad i = 1, 2.$$

With these new couplings we can formulate two coupled subproblems

$$(5) \qquad \begin{cases} L\big(u_1(\mathbf{x})\big) &= 0, & \mathbf{x} \in \Omega_1, \\ u_1(\mathbf{x})\,\big|_{\Gamma_1} &= \psi(\mathbf{x}), & \mathbf{x} \in \Gamma_1, \\ g_1\big(u_1(\mathbf{x})\big)\,\big|_{\Gamma_1'} &= g_1\big(u_2(\mathbf{x})\big), & \mathbf{x} \in \Gamma_1', \end{cases}$$

$$(6) \qquad \begin{cases} L\big(u_2(\mathbf{x})\big) &= 0, & \mathbf{x} \in \Omega_2, \\ u_2(\mathbf{x})\,\big|_{\Gamma_2} &= \psi(\mathbf{x}), & \mathbf{x} \in \Gamma_2, \\ g_2\big(u_2(\mathbf{x})\big)\,\big|_{\Gamma_2'} &= g_2\big(u_1(\mathbf{x})\big), & \mathbf{x} \in \Gamma_2'. \end{cases}$$

We have the following result:

3

THEOREM 1. *If the boundary value problem*

$$(7) \quad \begin{cases} L(w(\mathbf{x})) & = 0, & \mathbf{x} \in \Omega_{12}, \\ g_1(w(\mathbf{x}))\,|_{\Gamma_1'} & = 0, & \mathbf{x} \in \Gamma_1', \\ g_2(w(\mathbf{x}))\,|_{\Gamma_2'} & = 0, & \mathbf{x} \in \Gamma_2'. \end{cases}$$

has only trivial solution and the solution u_1, u_2 of (5) and (6) exist, then

1. $u_1(\mathbf{x}) = u_2(\mathbf{x})$, $\mathbf{x} \in \Omega_{12}$.
2. $u(\mathbf{x}) = u_1(\mathbf{x})$, $\mathbf{x} \in \Omega_1$ *and* $u(\mathbf{x}) = u_2(\mathbf{x})$, $\mathbf{x} \in \Omega_2$,

where u, u_1, u_2 be the solution of (1) and the solution of (5) and (6), respectively.

The proof of this theorem is straightforward and a direct generalization of this result to a finite number of overlapping subregions can be made. We define that the problem (1) is *equivalent* to (5) and (6). A version of this result in linear algebra will be shown in the next section. From Theorem 1, we can replace the problem (1) by (5) and (6).

Since there are unknowns which are coupled in the boundary conditions of (5) and (6), we cannot solve the two problems independently. Given an initial guess $u\,|_{\Gamma_1'} = \psi_0$, we will then be able to construct a sequence $\{u_1^{(i)}, u_2^{(i)}\}$ as follows:

$$(8) \quad \begin{cases} L(u_1^{(0)}) & = 0, & \mathbf{x} \in \Omega_1, \\ u_1^{(0)}\,|_{\Gamma_1} & = \psi, & \mathbf{x} \in \Gamma_1, \\ g(u_1^{(0)})\,|_{\Gamma_1'} & = g(\psi_0), & \mathbf{x} \in \Gamma_1', \end{cases}$$

$$(9) \quad \begin{cases} L(u_2^{(i)}) & = 0, & \mathbf{x} \in \Omega_2, \\ u_2^{(i)}\,|_{\Gamma_2} & = \psi, & \mathbf{x} \in \Gamma_2, \\ h(u_2^{(i)})\,|_{\Gamma_2'} & = h(u_1^{(i-1)}), & \mathbf{x} \in \Gamma_2'. \end{cases}$$

$$(10) \quad \begin{cases} L(u_1^{(i)}) & = 0, & \mathbf{x} \in \Omega_1, \\ u_1^{(i)}\,|_{\Gamma_1} & = \psi, & \mathbf{x} \in \Gamma_1, \\ g(u_1^{(i)})\,|_{\Gamma_1'} & = g(u_2^{(i)}), & \mathbf{x} \in \Gamma_1', \end{cases}$$

$$i = 1, 2, \cdots.$$

A key question is to ask under what conditions the sequence $\{u_1^{(i)}, u_2^{(i)}\}$ will converge to the solutions $\{u_1, u_2\}$ of (5) and (6). If it converges, then from the solution of (5) and (6), the solution of (1) can be constructed. An analysis for the model problem will be given in Sec. 3.

From the description of generalized *SAM*, we can view this idea as a general framework. Many flexibilities within this framework can be used to improve the performance of a particular implementation. Namely, we can tailor this approach more efficiently to different problems or to different computer environments for the same problem. The following flexibilities are rather useful:

- Flexibility in the choice of the couplings $g_i(u)$. A perfect choice of g_i can yield substantial improvement in performance (see numerical results in Section 3).
- Flexibility in the geometrical shapes of the subdomains. This flexibility makes it possible to choose the geometry of most of the subregions to meet the requirements imposed by fast solvers or by grids. A fast biharmonic solver on irregular domains using generalized *SAM* is studied in [1].

4

- Flexibility in the solution techniques for each subproblem. We are able to use different solution techniques for different subproblems. It is also possible to use different ways to obtain the solution of the same subproblem in the different stages of the computation, allowing us to use an optimal approach at any particular moment and in any particular location. Hierarchical grid and inexact solution strategies are typical examples here [18], [7], .
- Flexibility in the numerical model for each subproblem. Special boundary shapes or local behavior of the solution may require different models in different subregions. The decoupled subproblems allow us to localize the special treatment to the place where needed. Composite grids are a good example of this case.

Proper use of these flexibilities can yield an efficient algorithm. A particularly important application of *SAM* is for parallel computations. In the previous description of generalized *SAM*, the parallelism is not very obvious. When the number of the subregions is greater than or equal to the number of processors, we can divide the subregions into the same number of groups as processors. Now each processor is assigned to solve one group of the subproblems, so these solution processes can be done independently. As we can see there are many issues which need to be considered in a real implementation such as load balancing, communication, synchronization and ordering of the solution of the subproblems. These are very important in terms of parallel efficiency, however, we shall not study them in depth here.

From our description, we can also observe that generalized *SAM* not only provides parallelism in the algorithm, it also has the advantages of a higher ratio between computation and communications, the local communication pattern, and the hiding of global information exchange. All these features make generalized *SAM* an attractive candidate as a parallel algorithm.

In the next section, a generalized Schwarz Splitting (generalized *SS*) and an equivalence theorem are presented. This generalization is an analogy of the generalization from *SAM* to Schwarz splitting. Then, an application of this generalized *SS* to the solution of elliptic equations is shown in Section 3. The convergence analysis of the strip case and our numerical results indicate that the performance of a proposed generalized *SS* depends mostly on a coupling parameter α. A fast convergence rate based on a proper choice of α can be obtained with very little overlap, thus the concern about too much duplicated computation in the traditional *SAM* can be alleviated.

2. Generalized Schwarz Splittings. In this section we present an extension of the generalized *SAM* to numerical linear algebra. For a matrix equation $Ax = f$, we first introduce a Schwarz enhanced equation $\widetilde{A}\widetilde{x} = \widetilde{f}$. The corresponding matrix $\widetilde{A}$ is called a Schwarz enhanced matrix. A necessary and sufficient condition for the equivalence of the original equation and Schwarz enhanced equation is shown. The analogy of applying generalized *SAM* to the matrix equation is equivalent to applying a particular block Gauss-Seidel scheme to the Schwarz enhanced matrix. The corresponding splitting of the Schwarz enhanced matrix is called generalized Schwarz splitting (generalized *SS*). With this extension, many classical results in numerical linear algebra can be applied to this problem.

2.1. Definitions. Here the generalized *SAM* is discussed in terms of matrix theory. This approach provides many new opportunities for generalizing and improving the original *SAM*.

Consider a matrix problem:

(11)
$$Ax = f,$$

where A is an $N \times N$ nonsingular matrix, f and x are N vectors. A partitioned form of the equation (11) will be used in the rest of this paper. A partition is defined by the integers $n_1, n_2, \cdots, n_{2k+1}$ such that

(12)
$$n_1 + n_2 + \cdots + n_{2k+1} = N.$$

(13)
$$n_{2i} > 0, \quad n_{2i+1} \geq 0, \quad i = 1, \cdots, k$$

Given a set $\{n_i\}_{i=1}^{2k+1}$ which satisfies (12) and (13) the $(2k+1) \times (2k+1)$ partitioned form of the matrix A is then given by

(14)
$$
A = \begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,2k+1} \\
A_{2,1} & A_{2,2} & \cdots & A_{2,2k+1} \\
\vdots & \vdots & \ddots & \vdots \\
A_{2k+1,1} & A_{2k+1,2} & \cdots & A_{2k+1,2k+1}
\end{bmatrix},
$$

where $A_{i,j}$ is an $n_i \times n_j$ submatrix. We always assume that the unknown vector x and the known vector f in the matrix equation $Ax = f$ are partitioned in a form consistent with A. Thus, if A is given by (14), then x is assumed to be partitioned as

(15)
$$x = [x_1, x_2, \cdots, x_{2k+1}]^T,$$

where x_i is an $n_i \times 1$ matrix (column vector). An *augmented vector* of x

(16)
$$\tilde{x} = [x_1, x_2, x_2, x_3, x_4, x_4, x_5, \cdots, x_{2k}, x_{2k}, x_{2k+1}]^T$$

is defined such that: all even subvectors x_{2i}, $i = 1, \cdots, k$ are duplicated once in their places, and all odd subvectors remain the same.

To make the formula more readable, we will present the cases for $N = 3$ and 5 here. The generalization to a large N is direct. A dense 3×3 partitioned matrix can be written as:

$$
A = \begin{bmatrix}
A_{1,1} & A_{1,2} & A_{1,3} \\
A_{2,1} & A_{2,2} & A_{2,3} \\
A_{3,1} & A_{3,2} & A_{3,3}
\end{bmatrix}
$$

If the operator $L(u)$ in equation (1) is a linear second order elliptic operator, then the discretized problem can be written as a matrix equation:

(17)
$$
Ax = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix} = f.
$$

The order of the unknowns is arranged so that $[x_1, x_2]$ corresponds to the unknowns in Ω_1, $[x_2, x_3]$ corresponds to the unknowns in Ω_2 and $[x_2]$ corresponds to the unknowns

in Ω_{12}, which is the overlapped part of the two. The numerical generalized SAM for the above problem solves the following subproblems alternately:

$$(18) \qquad \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & B_2 \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i+1/2)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} 0 & A_{13} \\ C_2 & A_{23} \end{bmatrix} \begin{bmatrix} x_2^{(i-1)} \\ x_3^{(i-1)} \end{bmatrix},$$

$$\begin{bmatrix} B_2' & A_{23} \\ A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_2^{(i)} \\ x_3^{(i)} \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} C_2' & A_{13} \\ 0 & A_{23} \end{bmatrix} \begin{bmatrix} x_1^{(i)} \\ x_2^{(i+1/2)} \end{bmatrix}.$$

where

$$(19) \qquad A_{22} = B_2 + C_2 = B_2' + C_2'.$$

The splittings in (19) correspond to the couplings in (2), (3). In the next section we will show that a good choice of the splitting of A_{22} can significantly affect the convergence of SAM. It is therefore a very interesting research problem for further improvement of SAM.

It is not difficult to observe that this procedure is equivalent to a 2×2 block Gauss–Seidel iteration for the following matrix equation:

$$(20) \qquad \tilde{A}\tilde{x} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{13} \\ A_{21} & B_2 & C_2 & A_{23} \\ A_{21} & C_2' & B_2' & A_{23} \\ A_{31} & 0 & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_2' \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \end{bmatrix} = \tilde{f}.$$

Under certain conditions we know that the procedure (18) will converge [14], [17], the solution of equation (20) satisfies $\tilde{x}_2 = \tilde{x}_2'$, and $[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]^T$ is a solution of equation (17). This is to say that the augmented vector of the solution of (17) is the solution of (20) and vice versa. Later we will prove that this conclusion can be true only when $(B_2 - C_2')^{-1}$ exists. For most approximations of an elliptic partial differential equation this restriction is not very difficult to satisfy. We shall call the equation (20) the generalized Schwarz enhanced equation of (17) and the corresponding matrix $\tilde{A}$ in (20) the generalized Schwarz enhanced matrix of the matrix A.

Notice that: the second equation in (17) becomes a pair of dual equations in (20):

$$A_{21}x_1 + B_2 x_2 + C_2 x_2' + A_{23}x_3 = f_2$$
$$A_{21}x_1 + C_2' x_2 + B_2' x_2' + A_{23}x_3 = f_2$$

They are almost identical, except the term $A_{22}x_2$ in (17) is split in two different ways:

$$A_{22}x_2 \implies B_2 x_2 + C_2 x_2',$$
$$A_{22}x_2 \implies C_2' x_2 + B_2' x_2'.$$

Here is another example of a 5×5 block matrix equation and its generalized Schwarz enhanced equation:

$$Ax = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = f.$$

$$\widetilde{A}\widetilde{x} = \begin{bmatrix} A_{11} & A_{12} & 0 & A_{13} & A_{14} & 0 & A_{15} \\ A_{21} & B_2 & C_2 & A_{23} & A_{24} & 0 & A_{25} \\ A_{21} & C_2' & B_2' & A_{23} & A_{24} & 0 & A_{25} \\ A_{31} & 0 & A_{32} & A_{33} & A_{34} & 0 & A_{35} \\ A_{41} & 0 & A_{42} & A_{43} & B_4 & C_4 & A_{45} \\ A_{41} & 0 & A_{42} & A_{43} & C_4' & B_4 & A_{45} \\ A_{51} & 0 & A_{52} & A_{53} & 0 & A_{54} & A_{55} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2' \\ x_3 \\ x_4 \\ x_4' \\ x_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_2 \\ f_3 \\ f_4 \\ f_4 \\ f_5 \end{bmatrix} = \widetilde{f}.$$

where

$$A_{22} = B_2 + C_2 = B_2' + C_2'$$
$$A_{44} = B_4 + C_4 = B_4' + C_4'$$

For a general partitioned matrix (14), the splittings of the submatrices $A_{2i,2i}$ are:

$$A_{2i,2i} = B_{2i} + C_{2i} = B_{2i}' + C_{2i}'$$

From these examples, we can summarize the following rules of constructing the Schwarz enhanced equation: The odd number equations in $Ax = f$ is changed to:

$$\sum_{j=1}^{i} A_{2i-1,2j-1} x_{2j-1} + \sum_{j=1}^{i-1} A_{2i-1,2j} x_{2j}' + \sum_{j=2i}^{2k+1} A_{2i-1,j} x_j = f_{2i-1},$$
$$i = 1, \cdots, k+1,$$

while the even number of equations becomes a pair of dual equations in the generalized Schwarz enhanced equation:

$$(21) \quad \begin{aligned} \sum_{j=1}^{i} A_{2i,2j-1} x_{2j-1} + \sum_{j=1}^{i-1} A_{2i,2j} x_{2j}' + B_{2i} x_{2i} + C_{2i} x_{2i}' + \sum_{j=2i+1}^{2k+1} A_{2i,j} x_j = f_{2i} \\ \sum_{j=1}^{i} A_{2i,2j-1} x_{2j-1} + \sum_{j=1}^{i-1} A_{2i,2j} x_{2j}' + C_{2i}' x_{2i} + B_{2i}' x_{2i}' + \sum_{j=2i+1}^{2k+1} A_{2i,j} x_j = f_{2i} \\ i = 1, \cdots, k \end{aligned}$$

Only two terms are different in the two dual equations. We will not describe the details of how to form the generalized Schwarz enhanced matrix in general cases, as it is similar to the Schwarz enhanced matrix described in [17]. From the construction of the generalized Schwarz enhanced equation, it is easy to see the following result:

LEMMA 1. *If the vector* $x = (x_1, x_2, \cdots, x_{2k+1})^T$ *is the solution of equation (11), then its augmented vector* $\widetilde{x}$ *is the solution of generalized Schwarz enhanced equation* $\widetilde{A}\widetilde{x} = \widetilde{f}$, *where* $\widetilde{f}$ *is the augmented vector of* f.

The matrices $A_{2i,2i}$, $i = 1, \cdots, k$ are also called overlapped blocks. Let two matrices $\widetilde{B}$ and $\widetilde{C}$ be the Schwartz enhanced matrices of the same matrix A and their overlapped blocks are $B_{2i,2i}$ and $C_{2i,2i}$, $i = 1, \cdots, k$, respectively. If $B_{2i,2i}$ and $C_{2i,2i}$ have a relationship such that each $B_{2i,2i}$ is a submatrix of the corresponding $C_{2i,2i}$, we then say $\widetilde{C}$ has *more overlap* than $\widetilde{B}$. This overlap is closely related to the overlap area of the solution regions for the subregions mentioned in the introduction. As we have shown in [17], for the continuous model problem, if the amount of overlap increases then the convergence rate will increase if a traditional SAM is applied. For the matrix model we have a similar result [12].

8

2.2. Equivalence Theorem. A necessary and sufficient condition for the equivalence of equation (11) and its Schwarz enhanced equation (20) is given in this section. Let A be the same partitioned matrix in (14) and $\widetilde{A}$ be its Schwarz enhanced matrix.

THEOREM 2. *Let $\lambda(A)$, $\lambda(\widetilde{A})$ and $\lambda(B_{2i} - C'_{2i})$, $i = 1, \cdots, k$ be the sets of eigenvalues of $A, \widetilde{A}$ and $(B_{2i} - C'_{2i})$, $i = 1, \cdots, k$, respectively. Then $\lambda(\widetilde{A}) \subset \lambda(A) \cup (\bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i}))$.*

Proof. Let λ be an eigenvalue of $\widetilde{A}$ and

$$\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}'_2, \cdots, \widetilde{x}_{2k+1})$$

be the corresponding eigenvector. Substituting $\widetilde{x}$ into the equation $2i$ and its dual equation, we have:

$$\sum_{j=1}^{i} A_{2i,2j-1}\widetilde{x}_{2j-1} + \sum_{j=1}^{i-1} A_{2i,2j}\widetilde{x}'_{2j} + B_{2i}\widetilde{x}_{2i} + C_{2i}\widetilde{x}'_{2i} + \sum_{j=2i+1}^{2k+1} A_{2i,j}\widetilde{x}_j = \lambda\widetilde{x}_{2i},$$

$$\sum_{j=1}^{i} A_{2i,2j-1}\widetilde{x}_{2j-1} + \sum_{j=1}^{i-1} A_{2i,2j}\widetilde{x}'_{2j} + C'_{2i}\widetilde{x}_{2i} + B'_{2i}\widetilde{x}'_{2i} + \sum_{j=2i+1}^{2k+1} A_{2i,j}\widetilde{x}_j = \lambda\widetilde{x}'_{2i},$$

As we mentioned in the last section, only two terms are different in the left hand sides of the two equations. Subtracting the first equation from the second, we have:

$$(B_{2i} - C'_{2i})(\widetilde{x}_{2i} - \widetilde{x}'_{2i}) = \lambda(\widetilde{x}_{2i} - \widetilde{x}'_{2i}), \qquad i = 1, \cdots, k.$$

If $\widetilde{x}_{2i} - \widetilde{x}'_{2i} \neq 0$ for some i, then we have $\lambda \in \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$. If $\lambda \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$, then $\widetilde{x}_{2i}$ has to be equal to $\widetilde{x}'_{2i}$ for $i = 1, \cdots, k$. Therefore, $\widetilde{x}$ is a augmented vector of $x = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \cdots, \widetilde{x}_{2k+1})^T$, which will satisfy equation

$$Ax = \lambda x.$$

Thus $\lambda \in \lambda(A)$, which concludes the proof.

Define the Schwarz enhanced equation (20) as *equivalent* to (11) if $\widetilde{A}^{-1}$ exists and the solution vector $\widetilde{x}$ is a augmented vector of the solution x of (11). Similarly, we say that the Schwarz enhanced matrix $\widetilde{A}$ is *equivalent* to matrix A if $\widetilde{A}^{-1}$ exists. With this definition and the result from Theorem 2 we have

THEOREM 3. *If a matrix $\widetilde{A}$ is a Schwarz enhanced matrix of the nonsingular matrix A, then the following conditions are equivalent:*

1. Matrix $\widetilde{A}$ is equivalent to matrix A.

2. $0 \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$.

Proof. If $0 \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$, then from Theorem 2 we know $\widetilde{A}^{-1}$ exists. Applying the same strategy used in the previous proof, we can show that the solution $\widetilde{x}$ of $\widetilde{A}\widetilde{x} = \widetilde{f}$ is a augmented vector of the solution x of $Ax = f$.

Now we show that $0 \notin \bigcup_{i=1}^{k} \lambda(B_{2i} - C'_{2i})$ is also a necessary condition. Suppose there is a j such that $0 \in \lambda(B_{2j} - C'_{2j})$. We know that $(B_{2j} - C'_{2j})$ is singular, hence

so is $(B'_{2j} - C_{2j})$, since $(B_{2j} - C'_{2j}) = (B'_{2j} - C_{2j})$. Now, if we subtract row $2j$ from $2j'$ in matrix $\tilde{A}$, we will have

$$(22) \qquad 0, \cdots, 0, (B_{2j} - C'_{2j}), -(B'_{2j} - C_{2j}), 0, \cdots, 0$$

This means that $\tilde{A}$ is singular. The proof is complete.

If a matrix is a positive definite matrix or an M-matrix[1], any principal minor of this matrix is also a positive definite matrix or an M-matrix respectively. Thus, if we choose $C_{2i} = 0$ and $C'_{2i} = 0$[2], we immediately have

COROLLARY 1. *Any Schwarz enhanced matrix of a positive definite matrix A is equivalent to A if $C_{2i} = 0$ and $C'_{2i} = 0, i = 1, \cdots, k$.*

COROLLARY 2. *Any Schwarz enhanced matrix of an M-matrix A is equivalent to A if $C_{2i} = 0$ and $C'_{2i} = 0, i = 1, \cdots, k$.*

3. A Parameterized Generalized Schwarz Splitting. The general framework of a generalized SS is given in the last section. Here, the convergence behavior of a particular generalized SS for the elliptic equation, namely parameterized generalized SS, is studied. In a traditional approach of SS, we choose $C_{2i} \equiv 0$. In this case, it is well understood that the amount of overlap is a key factor which affects the convergence rate. Even though a larger overlap means more duplicated work on these overlapping regions, the overall complexity is still better than a smaller overlap. However, a natural question is raised: is a larger overlap the ultimate choice? The generalized Schwarz splitting discussed in the previous section provides a way to explore possibilities of further improving the performance of SAM. In particular, we will examine the importance of splitting

$$A_{2i,2i} = B_{2i} + C_{2i} = B'_{2i} + C'_{2i}.$$

First, an application of generalized SS to a two–point boundary value problem is investigated. A similar approach can also be applied to two dimensional problems.

Consider a two–point boundary value problem

$$U''(x) + qU(x) = f(x), \qquad x \in (0,1),$$

$$U(0) = a_0; \qquad U(1) = a_1,$$

where $q \le 0$. After discretization using a centered finite difference, the resulting linear system is

$$(23) \qquad T_n(\beta)x = b,$$

where

$$T_n(\beta) = Tridiagonal\{-1, \quad \beta, \quad -1\}_{n \times n}$$

and $\beta \ge 2$. If there is no ambiguity, it will be abbreviated as T_n. Denote

$$T_n(x_1, x_2, x_3)$$

---

[1] Any $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \le 0$ for all $i \ne j$ is an M-matrix if A is nonsingular, and $A^{-1} \ge 0$.

[2] In the traditional approach of SAM, we always choose $C_{2i} = 0$ and $C'_{2i} = 0$.

10
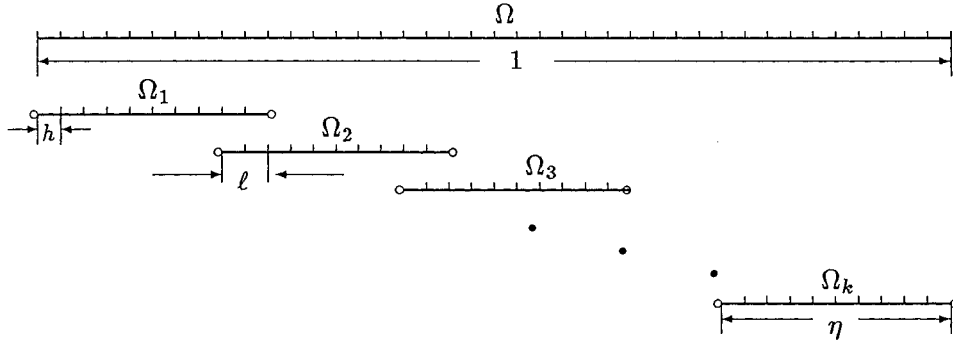
FIG. 2. *One-dimensional overlapping grid.*

as the same $n \times n$ tridiagonal matrix $T_n(x_2)$ except the first diagonal element is x_1 and the last is x_3.

The generalized *SAM* for solving this problem divides the region into k overlapping subregions Ω_i $i = 1, \cdots, k$ as shown in Figure 2 (To simplify the analysis we assume the overlap pattern is uniform.). Let h be the grid size, ℓ the length of the overlap and η the length of every subregion. Then let $n + 1 = \frac{1}{h}$, $l = \frac{\ell}{h}$ and $m + 1 = \frac{\eta}{h}$. Here we assume $l < m/2$, which means no three subregions have common overlap part. The open circle points in Figure 2 are the boundaries of the subregions.

To make the formula more readable, we will display the case of $k = 3$. The general case can be easily extended from this case. The partitioned form of (23) is now :

$$
T_n x =
\begin{bmatrix}
T_{m-l} & -F_1 & 0 & 0 & 0 \\
-E_2 & T_l & -F_2 & 0 & 0 \\
0 & -E_3 & T_{m-2l} & -F_3 & 0 \\
0 & 0 & -E_4 & T_l & -F_4 \\
0 & 0 & 0 & -E_5 & T_{m-l}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5
\end{bmatrix}
= b.
$$

Its corresponding Scwarz enhanced equation is:

$$
\widetilde{T}_n \widetilde{x} =
\begin{bmatrix}
T_{m-l} & -F_1 & 0 & 0 & 0 & 0 & 0 \\
-E_2 & B_2 & C_2 & -F_2 & 0 & 0 & 0 \\
-E_2 & C'_2 & B'_2 & -F_2 & 0 & 0 & 0 \\
0 & 0 & -E_3 & T_{m-2l} & -F_3 & 0 & 0 \\
0 & 0 & 0 & -E_4 & B_4 & C_4 & -F_4 \\
0 & 0 & 0 & -E_4 & C'_4 & B_4 & -F_4 \\
0 & 0 & 0 & 0 & 0 & -E_5 & T_{m-l}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x'_2 \\ x_3 \\ x_4 \\ x'_4 \\ x_5
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_2 \\ b_3 \\ b_4 \\ b_4 \\ b_5
\end{bmatrix}
= \widetilde{b}.
$$

The quantities above are defined as:

- F_1: an $(m - l) \times l$ matrix with zero elements everywhere except for a 1 in position $(m - l, 1)$.
- F_2: an $l \times (m - 2l)$ matrix with zero elements everywhere except for a 1 in position $(m - 2l, 1)$.
- F_3: an $(m - 2l) \times l$ matrix with zero elements everywhere except for a 1 in position $(m - 2l, 1)$.

11

- F_4: an $l \times (m - l)$ matrix with zero elements everywhere except for a 1 in position $(m - l, 1)$.
- E_2: an $l \times (m - l)$ matrix with zero elements everywhere except for a 1 in position $(1, m - l)$.
- E_3: an $(m - 2l) \times l$ matrix with zero elements everywhere except for a 1 in position $(1, l)$.
- E_4: an $l \times (m - 2l)$ matrix with zero elements everywhere except for a 1 in position $(1, m - 2l)$.
- E_5: an $(m - l) \times l$ matrix with zero elements everywhere except for a 1 in position $(1, l)$.

There are many ways to split the matrix T_l. We will introduce a parametrized generalized SS for this problem as follows: let

- F: an $l \times l$ matrix with zero elements everywhere except for a 1 in position (l, l) and

$$C = C_2 = C_4 = \alpha F, \qquad B = B_2 = B_4 = T_l - C.$$

- E: an $l \times l$ matrix with zero elements everywhere except for a 1 in position $(1, 1)$ and

$$C' = C'_2 = C'_4 = \alpha E, \qquad B' = B'_2 = B'_4 = T_l - C'.$$

where $0 \leq \alpha < 1$. It is not difficult to show that

$$\det(B - C') = \det(T_l(\beta - \alpha, \beta, \beta - \alpha)) \neq 0,$$

provided $\beta \geq 2$. The resulting Schwarz enhanced equation is equivalent to (23). Then the parametrized generalized SS of (23) is defined as:

$$
\begin{aligned}
T_n &= M(\alpha) - N(\alpha) \\
&= \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} - \begin{bmatrix} 0 & U_1 & 0 \\ L_2 & 0 & U_2 \\ 0 & L_3 & 0 \end{bmatrix}
\end{aligned}
$$

where

$$T_1 = T_m(\beta, \beta, \beta - \alpha), \quad T_2 = T_m(\beta - \alpha, \beta, \beta - \alpha), \quad T_3 = T_m(\beta - \alpha, \beta, \beta),$$

$$L_2 = \begin{bmatrix} E_2 & \alpha E \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad L_3 = \begin{bmatrix} E_4 & \alpha E \\ 0 & 0 \end{bmatrix},$$

$$U_1 = \begin{bmatrix} 0 & 0 \\ \alpha F & F_2 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \alpha F & F_4 \end{bmatrix}.$$

A simple calculation can show that the relationship between α and ω in (4) is

$$\omega = \frac{1 - \alpha}{1 - \alpha + h\alpha}$$

When $\alpha = 0$, we have $\omega = 1$. Thus, this parametrized generalized SS reduced to the traditional SS, namely a Dirichlet boundary condition, is used on these artificial

12

boundaries. If $\beta > 2$ and $\alpha = 1$, we have $\omega = 0$. It is equivalent to use a Neumann condition on the artificial boundaries (If $\beta = 2$ we can only use Neumann condition on one of the boundaries for interior subregions. Otherwise, the resulting Schwarz enhanced matrix is singular. For two dimensional problems, if a strip decomposition is employed, then a Neumann boundary condition can be used for both artificial boundaries). For $0 < \alpha < 1$, this generalized SS corresponds to

$$g_1(u) = g_2(u) = \omega u + (1 - \omega)\frac{\partial u}{\partial n}.$$

The convergence analysis of this parameterized generalized SS is therefore reduced to calculating the spectral radius of the block Jacobi matrix $J = M^{-1}N$. Notice that the matrix $N(\alpha)$ only has 8 nonzero elements. So the matrix $J = M^{-1}N$ has only 8 nonzero columns, provided $l < m/2$. They are only related to the elements in the last or first columns of the matrices T_1^{-1}, T_2^{-1} and T_3^{-1}. Let $t_{i,j}$ be the elements of the matrix $T_n^{-1}(\beta)$ and $D_i(\beta) = \det T_i(\beta)$. We have the following results (see [6]):

$$D_k(\beta) = \begin{cases} \sinh(n+1)\theta/\sinh\theta, & \beta > 2, \quad 2\cosh\theta = \beta, \\ n+1, & \beta = 2, \\ \sin(n+1)\theta/\sinh\theta, & \beta < 2, \quad 2\cos\theta = \beta, \end{cases}$$

$$t_{i,j} = \begin{cases} D_{j-1}(\beta)D_{n-i}(\beta)/D_n(\beta), & i \geq j, \\ D_{i-1}(\beta)D_{n-j}(\beta)/D_n(\beta), & i < j, \end{cases}$$

Based on this result, the elements of T_1^{-1} and T_2^{-1} can be easily derived from the Sherman-Morrison formula. We will not elaborate on the detailed derivation here.

Denote the last columns of the matrices T_1^{-1} and T_2^{-1} by $t^{(1)}$ and $t^{(2)}$ respectively:

$$t^{(1)} = (t_1^{(1)}, t_2^{(1)}, \cdots, t_m^{(1)})$$
$$t^{(2)} = (t_1^{(2)}, t_2^{(2)}, \cdots, t_m^{(2)}).$$

Note that elements $t_i^{(j)}$ are functions of α. Since matrix T_3^{-1} is a permuted matrix of T_1^{-1}, the first column of the inverse T_3^{-1} can be derived from the last column of T_1^{-1} by a simple permutation. Let P^T is a permutation matrix permute columns $m - l$, $m - l + 1$; $m + l$, $m + l + 1$; $2m - l$, $2m - l + 1$; $2m + l$, $2m + l + 1$ to $3m - k + 1$, $k = 8, 7, \cdots, 1$, respectively. J can be similarly transformed to $\tilde{J}$:

$$\tilde{J} = PJP^T = \begin{bmatrix} 0 & K \\ 0 & G \end{bmatrix},$$

where

$$G = \begin{pmatrix} 0 & 0 & -\alpha t_{m-l}^{(1)} & t_{m-l}^{(1)} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha t_{m-l+1}^{(1)} & t_{m-l+1}^{(1)} & 0 & 0 & 0 & 0 \\ t_{m-l+1}^{(2)} & -\alpha t_{m-l+1}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{l-1}^{(2)} & t_{l-1}^{(2)} \\ t_{m-l}^{(2)} & -\alpha t_{m-l}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{l}^{(2)} & t_{l}^{(2)} \\ t_{l}^{(2)} & -\alpha t_{l}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{m-l}^{(2)} & t_{m-l}^{(2)} \\ t_{l-1}^{(2)} & -\alpha t_{l-1}^{(2)} & 0 & 0 & 0 & 0 & -\alpha t_{m-l+1}^{(2)} & t_{m-l+1}^{(2)} \\ 0 & 0 & 0 & 0 & t_{m-l+1}^{(1)} & -\alpha t_{m-l+1}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{m-l}^{(1)} & -\alpha t_{m-l}^{(1)} & 0 & 0 \end{pmatrix}$$

13

Note that matrix G has only 4 independent columns. After a simple reduction, we know the following matrix includes 4 non-zero eigenvalues of G:

$$G' = \begin{bmatrix} 0 & t_{m-l}^{(1)} - \alpha t_{m-l+1}^{(1)} & 0 & 0 \\ t_{m-l}^{(2)} - \alpha t_{m-l+1}^{(2)} & 0 & 0 & t_l^{(2)} - \alpha t_{l-1}^{(2)} \\ t_l^{(2)} - \alpha t_{l-1}^{(2)} & 0 & 0 & t_{m-l}^{(2)} - \alpha t_{m-l+1}^{(2)} \\ 0 & 0 & t_{m-l}^{(1)} - \alpha t_{m-l+1}^{(1)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & g_1 & 0 & 0 \\ g_2 & 0 & 0 & g_3 \\ g_3 & 0 & 0 & g_2 \\ 0 & 0 & g_1 & 0 \end{bmatrix}.$$

Let

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

We have

$$HG'H = \begin{bmatrix} 0 & g_1 & 0 & 0 \\ g_2 + g_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_2 - g_3 \\ 0 & 0 & g_1 & 0 \end{bmatrix}.$$

Thus, the 4 eigenvalues of G' are:

$$\lambda_{1,2} = \pm\sqrt{g_1(g_2 + g_3)}$$
$$\lambda_{3,4} = \pm\sqrt{g_1(g_2 - g_3)}$$

We present two figures to show the relationship between the spectral radius and the parameter α in Fig 3. For both cases, the size of each subproblem is $m = 10$ and $\beta = 2$. When $\beta > 2$, generalized SS has a faster convergence rate. Figure 6–7 will show the results for the latter case. The x-axis is the parameter α while y-axis is the spectral radius of the Jacobi matrix for generalized SS. The top figure shows the case of one overlapping node while the lower one demonstrates the case of overlapping half the subregion. In Fig. 3, we can see that the traditional SS (when $\alpha = 0$) has a very poor convergence rate since the overlap is so small. The error reduction is 90% for each iteration. To reduce the residual by a factor of 10^6 requires more than 60 iterations. When the parameter α approaches 1, an amazing improvement of the convergence rate appears. For $\alpha = 0.9$, the convergence factor is less than 10^{-4}. That is to say only very few iterations are needed for any particular computation. From the second picture, we can observe when $\alpha = 0.85$ the convergence rate of the generalized SS approaches the optimum. But the optimal convergence rate in this case is even worse than having minimum overlap, and the only positive here is the sensitivity of the convergence rate with the parameter α.

A numerical test has verified this analysis. The problem we are testing is

$$y''(x) = 2e^x \cos x, \qquad x \in (0,1),$$
$$y(0) = 0, \qquad y(1) = e \sin(1).$$

14

which has a solution $y(x) = e^x \sin(x)$. The solution region is covered by three overlap subregions with m unknowns each. Two neighboring subregions have one overlapping grid node. A random initial guess is used when the iteration starts. For $\alpha = 0.948, m = 20$, and $l = 2$, the residual is reduced by a factor of 10^{14} after 3 iterations. The results are the same for different mesh sizes. For $\alpha = 0.9, m = 10$, and $l = 2$. the residual is reduced by a factor of 10^{15} after 3 iterations. By comparison, the traditional SS with the same overlap will take 60 iterations to reduce the residual by only a factor of 10^5.

We apply the traditional SS to the same problem with an overlap of half of the subregion. Numerical testing shows that 11 iterations are needed to achieve a reduction of the initial residual by a factor of 10^5. If an optimal α is used, 4 iterations are needed. This again verifies the analysis shown in Figure 3. The above results are summarized in the following table:

	minimum overlap		half overlap	
	Convergence factor	Number of iterations	Convergence factor	Number of iterations
Classical SAM	0.91	60	0.68	11
generalized SAM	10^{-4}	3	0.06	4

Similar to the traditional SS, the convergence rate will deteriorate when the number of subregions increases. For four overlapping subregions of size m, the non-zero eigenvalues for the Jacobi matrix of generalized SS are included in those of the following matrix:

$$G_4 = \begin{bmatrix} D_1 & L & 0 \\ U & D_2 & L \\ 0 & U & D_3 \end{bmatrix}$$

where

$$D_1 = \begin{bmatrix} 0 & g_1 \\ g_2 & 0 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & g_2 \\ g_2 & 0 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} 0 & g_2 \\ g_1 & 0 \end{bmatrix},$$

and

$$L = \begin{bmatrix} 0 & 0 \\ 0 & g_3 \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & g_3 \\ 0 & 0 \end{bmatrix}.$$

Using a similar reduction as in the case of $k = 3$, we can express the eigenvalues of matrix G_4 as the roots of the following two cubic equations:

$$\lambda^3 - \lambda^2 g_2 + g_1 g_2 \lambda + g_3^2 - g_1 g_2 = 0,$$
$$\lambda^3 - \lambda^2 g_2 - g_1 g_2 \lambda - g_3^2 + g_1 g_2 = 0.$$

The roots can be expressed as some complicated functions of $g_i, i = 1, 2, 3$ which we will not list here.

In general, for k overlapping subregions, the non-zero eigenvalues of the Jacobi

matrix are included in those of the following $(k-1) \times (k-1)$ block matrix:

$$
G_k = \begin{bmatrix}
D_1 & L & 0 & \cdot & \cdot & \cdot & 0 & 0 \\
U & D_2 & L & 0 & \cdot & \cdot & 0 & 0 \\
 & & \ddots & & \ddots & & & \\
 & & & \ddots & & \ddots & & \\
0 & 0 & \cdot & \cdot & \cdot & U & D_2 & L \\
0 & 0 & \cdot & \cdot & & 0 & U & D_3
\end{bmatrix}.
$$

There will not be a closed form for the eigenvalues of G_k when $k > 4$. In Figure 4 we present a numerical result for the spectral radius of J where 5 overlapping subregions exist. The first picture is for the minimum overlap case while the second is for the overlapping half of the subregions. Again, a significant improvement can be obtained by choosing a good parameter α.

For two dimensional problem

$$
\begin{aligned}
\triangle U(x,y) - qU(x,y) &= f(x,y), & (x,y) \in (0,1) \times (0,1), \\
U(x,y)\,|_\Gamma &= g(x,y)
\end{aligned}
$$

where $q < 0$, a matrix equation

$$
Ax = b
$$

can be derived using centered finite difference. Given grid size $h = 1/(n+1)$, A can be written as

$$
A = T_n(\beta) \otimes I_n + I_n \otimes T_n(2),
$$

where $\beta > 2$. Decompose the solution region into 3 overlapping subregions (in strip). The overlap pattern in x–direction is exactly the same as in one-dimensional case. The corresponding Schwarz enhanced matrix is

$$
\tilde{A} = \begin{bmatrix}
A_1 & -F_1 & 0 \\
-E_1 & A_2 & -F_1 \\
0 & -E_1 & A_3
\end{bmatrix}
$$

where

$$
\begin{aligned}
A_1 &= T_1 \otimes I_n + I_m \otimes T_n(2), \\
A_2 &= T_2 \otimes I_n + I_m \otimes T_n(2), \\
A_3 &= T_3 \otimes I_n + I_m \otimes T_n(2), \\
E_1 &= E_m \otimes I_n \\
F_1 &= F_m \otimes I_n
\end{aligned}
$$

and

- E_m: an $m \times m$ matrix with zero elements everywhere except for a 1 in position $(1, m-l)$ and α in position $(1, m-l+1)$.
- F_m: an $m \times m$ matrix with zero elements everywhere except for an α in position $(m, m-l)$ and 1 in position $(m, l-1)$.

16

The Jacobi iterative matrix for the generalized Schwarz splitting is

$$
J = \begin{bmatrix} A_1^{-1} & 0 & 0 \\ 0 & A_2^{-1} & 0 \\ 0 & 0 & A_3^{-1} \end{bmatrix} \begin{bmatrix} 0 & F_1 & 0 \\ E_1 & 0 & F_1 \\ 0 & E_1 & 0 \end{bmatrix}
$$

$$
= M^{-1}N.
$$

Let

$$
U = \begin{bmatrix} I_m \otimes X_n & 0 & 0 \\ 0 & I_m \otimes X_n & 0 \\ 0 & 0 & I_m \otimes X_n \end{bmatrix}
$$

where X_n is an orthogonal matrix. Each column in X_n corresponds to an eigenvector of matrix $T_n(2)$ and $X_n T_n(2) X_n = D_n = diag\{d_i\}$, $d_i = 2 + 2\cos\frac{i\pi}{n+1}$, $i = 1, \cdots, n$. Note U is orthogonal and $UNU = N$. So

$$
\begin{aligned}
J' &= UJU^T \\
&= (UMU^T)^{-1}N \\
&= \widetilde{M}N
\end{aligned}
$$

where

$$
\widetilde{M} = \begin{bmatrix} \tilde{A}_1 & 0 & 0 \\ 0 & \tilde{A}_2 & 0 \\ 0 & 0 & \tilde{A}_3 \end{bmatrix}
$$

$$
\begin{aligned}
\tilde{A}_i &= (I_m \otimes X_n)A_i(I_m \otimes X_n)^T \\
&= T_i \otimes I_n + I_m \otimes D_n
\end{aligned}
$$

Let P be the permutation matrix which permutes row $(k-1)n + i$ to $(i-1)3m + k$ $k = 1, \cdots, 3m$, $i = 1, \cdots, n$. Then

$$
PJ'P^T = \begin{bmatrix} J(d_1) & & & \\ & J(d_2) & & \\ & & \ddots & \\ & & & J(d_n) \end{bmatrix}
$$

Where each $J(d_i)$ is the Jacobi iterative matrix of the generalized SAM for matrix $T_n(d_i)$ in one-dimensional case. Similar to the traditional SAM, we found that the convergence of the lower frequency components are slower than that of higher frequencies. We present two pictures in Fig (5) to show how $\rho_{J(d_i)}$ changes when d_i changes. The first represents three subregions with minimum overlap while the second shows the same number of subregions with half overlap. Another two sets of figures present the relations between the spectral radius $\rho_{J(d_i)}$ and the parameter α. The first set is for $J(2.01)$ and the other is for $J(5.99)$, which represent the lowest and the highest frequencies of the eigenmodes respectively. Both sets have one figure for minimum overlapping (one grid line) and another for overlapping half of the subregion. We can see that the sensitivity of the convergence rate with the parameter α is better in

17

a two–dimensional problem. It is also noticeable that the convergence of the higher frequency mode is very fast for all α.

Numerical testing results for the model problem

$$
\begin{aligned}
\triangle U(x,y) &= -2x(1-x) - 2y(1-y), & (x,y) \in (0,1) \times (0,1), \\
U(x,y) \mid_\Gamma &= 0
\end{aligned}
$$

are given in Fig. (8). We present the relations between the number of iterations and the parameter α in these figures. Testing is carried out for three and five subregion cases, and for each decomposition, both minimum overlap and half overlap are tested. Initial guess is randomly generated. To make the programming easier, the grid size is slightly different for each case. h is between $1/40$ to $1/50$. The results plainly verify our analysis. The x-axis is the parameter α while y-axis is the number of iterations needed for reducing the initial error by a factor of 10^5.

4. Conclusion. From the above analysis, a generalization of the traditional *SAM* is presented and the improvement of its performance is significant. The results of this study suggest that there may be other interesting splittings or couplings with good or even better performance characteristics. So far, our analysis has been restricted to a simple case, namely the strip decomposition. In particular, the close form of the spectral radius did not provide us with an insight of how the convergence is related to the decomposition and the parameter α. We do not have the same intuitive understanding we had for the classical *SAM*. More interesting problems remain to be studied; a mixed coupling approach or multi-parameter generalized *SS* would be worth exploring. For variable coefficient problems, a local coupling approach should be an interesting topic, and how the geometry of the domain affects the optimal value of the parameter should be investigated. The effects of different couplings on the convergence of *SAM* have also been observed in a study for the fourth order equation[1].

We should also indicate that there is a severe sensitivity between the parameter α and the convergence rate of generalized *SS*. A better understanding of this sensitivity is needed to make this generalized *SS* a practical technique.

REFERENCES

[1] T. CHAN, D. GOOVAERTS, AND W. P. TANG, *A fast biharmonic solver on irregular domains using a generalized*, Tech. Rep. 37, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1, 1989.

[2] R. V. DINH, R. GLOWINSKI, AND J. PERIAUX, *Applications of domain decomposition techniques to the numerical solution of the navier–stokes equations*, tech. rep., INF–LAB, France, 1980.

[3] M. DRYJA, *An additive schwarz algorithm for two- and three–dimensional finite element elliptic problems*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Periaux, and O. Widlund, eds., Philadelphia, PA, 1989, SIAM, pp. 168–172.

[4] M. DRYJA AND O. WIDLUND, *An additive variant of schwarz alternating method for the case of many subregions*, Tech. Rep. 339, New York University, 251 Mercer St. New York, NY 10012, 1987.

[5] L. W. EHRLICH, *The numerical schwarz alternating procedure and sor*, SIAM J. on SCISC, 7 (1986), pp. 989–993.

[6] C. F. FISCHER AND R. A. USMANI, *Properties of some tridiagonal matrices and their application to boundary value problems*, SIAM J. Numer. Anal., 6 (1969), pp. 127–142.

[7] G. H. GOLUB AND M. L. OVERTON, *The convergence of inexact chebychev and richardson iterative methods for solving linear systems*, tech. rep., Stanford University, Stanford, CA 94305, 1987.

18

[8] A. GREENBAUM, C. LI, AND H. Z. CHAO, *Parallelizing preconditioned conjugate gradient algorithms*, tech. rep., New York University, 251 Mercer St. New York, NY 10012, 1988.

[9] L. V. KANTOROVICH AND V. I. KRYLOV, *Approximate methods of higher analysis*, P. Noordhoff LTD–Groningen–The Netherlands, 4th ed., 1958.

[10] P. L. LIONS, *On the schwarz alternating method. i*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, G. Glowinski, R.and Golub, G. Meurant, and J. Periaux, eds., Philadelphia, PA, 1988, SIAM, pp. 1–42.

[11] U. MEIER, *Two parallel sor variants of the schwarz alternating procedure*, tech. rep., Zentralinstitut für Angewandte Mathematik, 5170 Jülich –W. Germany, 1986.

[12] J. OLIGER, W. SKAMAROCK, AND W. TANG, *Convergence analysis and acceleration of the schwarz alternating method*, tech. rep., Stanford University, Computer Science Dept., 1986.

[13] G. RODRIGUE, *Personal communication*, 1987.

[14] G. RODRIGUE AND J. SIMON, *A generalization of the numerical schwarz algorithm*, in Computing Methods in Applied Sciences and Engineering VI, R. Glowinski and J. Lions, eds., North–Holland, Amsterdam–New York–Oxford, 1984, pp. 273–283.

[15] R. SCHREIBER, *Personal communication*, 1986.

[16] H. A. SCHWARZ, *Ueber einige abbildungsaufgaben*, Jour. f. die reine und angew. Math, 70 (1869), pp. 105–120.

[17] W. P. TANG, *Schwarz Splitting and Template Operators*, PhD thesis, Stanford University, Computer Science Dept. Stanford, CA 94305, 1987.

[18] H. YSERENTANT, *Hierarchical bases give conjugate type method a multigrid speed of convergence*, Appl. Math. and Comp., 19 (1986), pp. 147–158.

Three subregions with minimum overlap
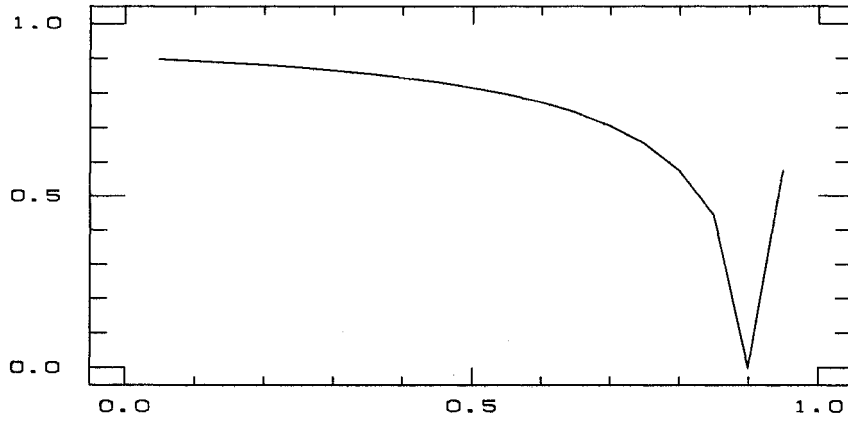
(One dimensional problem)

Three subregions with half overlap
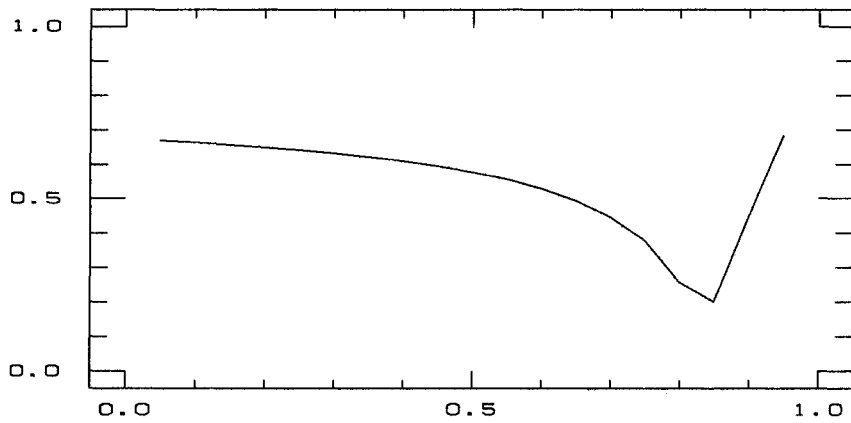
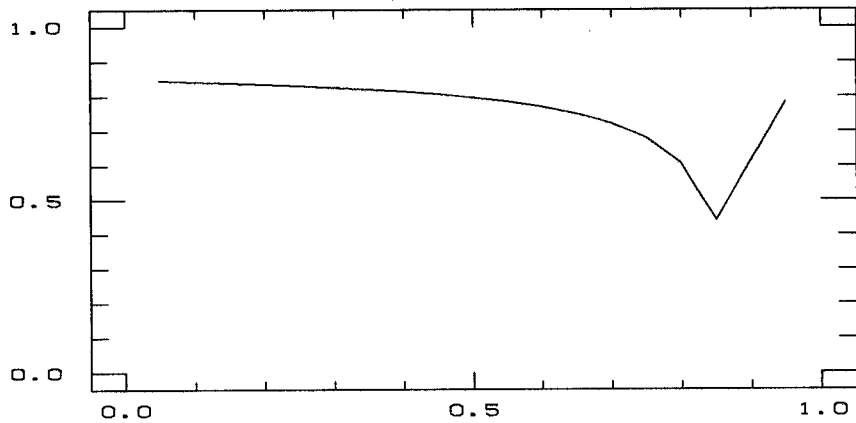Spectral radius versus alpha

FIG. 3.

Five subregions with minimum overlap
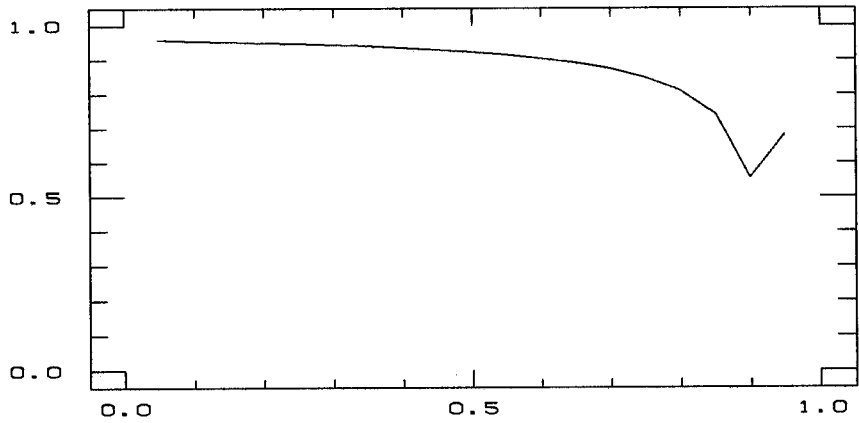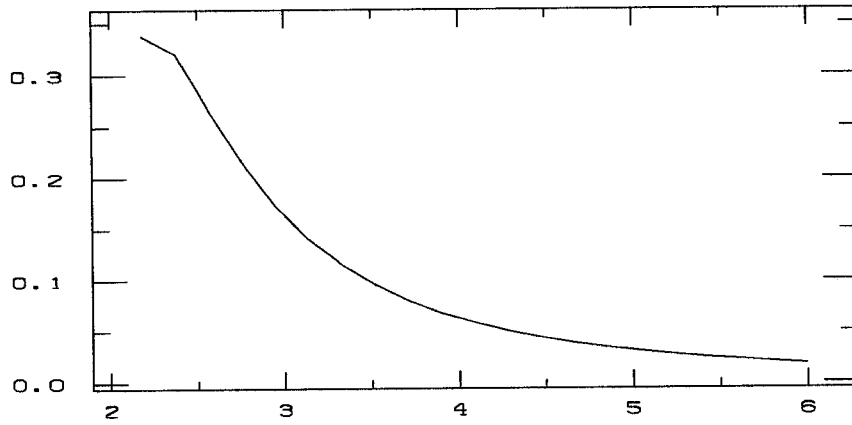
(One dimensional problem)

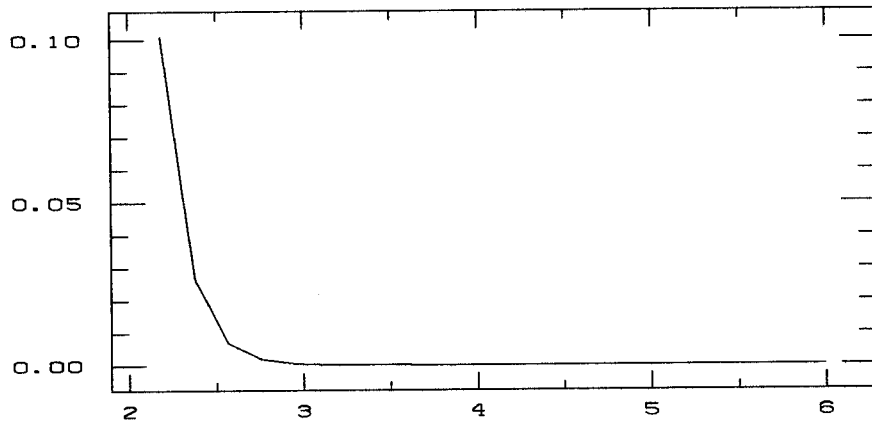Five subregions with half overlap

Spectral radius versus alpha
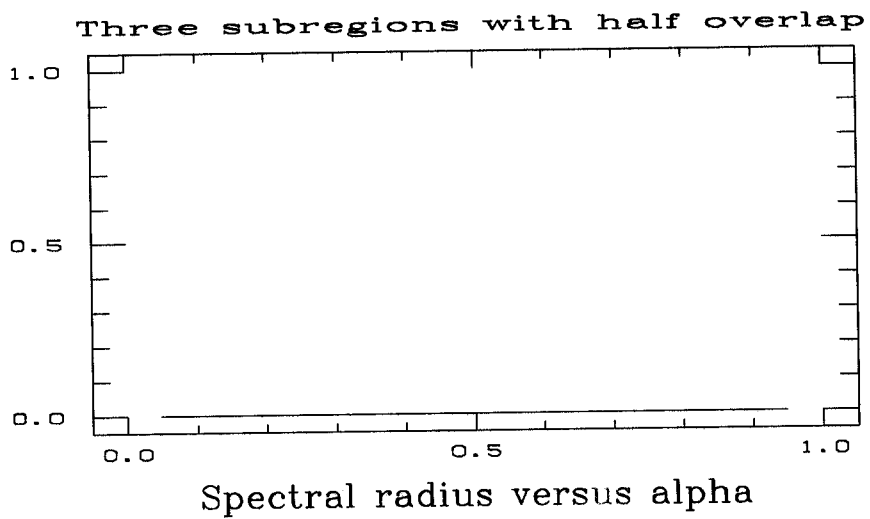
FIG. 4.

Three subregions with minimum overlap

0.3

0.2

0.1

0.0

2   3   4   5   6

(alpha = 0.87)

Three subregions with half overlap

0.10

0.05

0.00

2   3   4   5   6

Spectral radius versus eigenmode

FIG. 5.

Three subregions with minimum overlap

($J(d_1)$)

Three subregions with half overlap

Spectral radius versus alpha

FIG. 6.

Three subregions with minimum overlap

$(J(d_n))$

Three subregions with half overlap

Spectral radius versus alpha
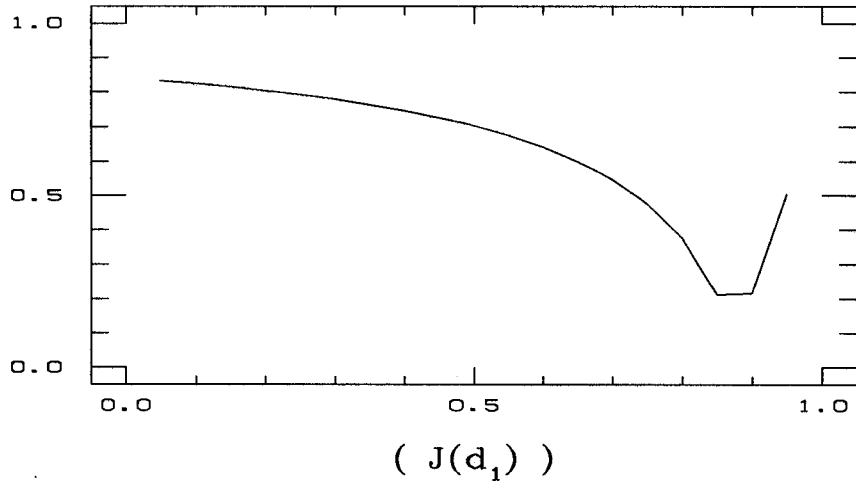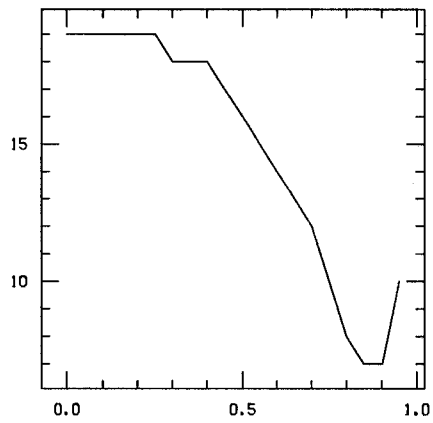
FIG. 7.

Three subregions with minimum overlap
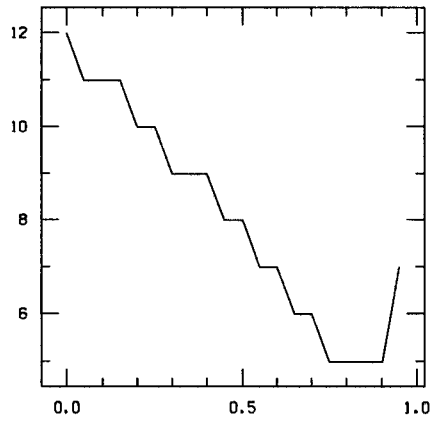
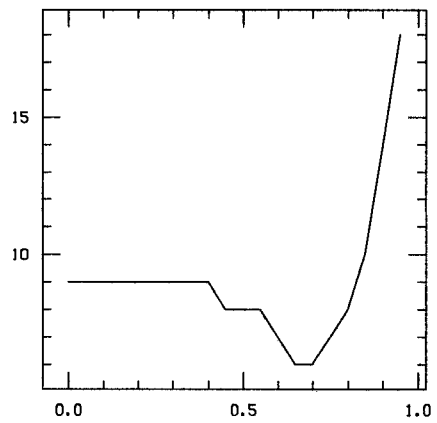Three subregions with half overlap
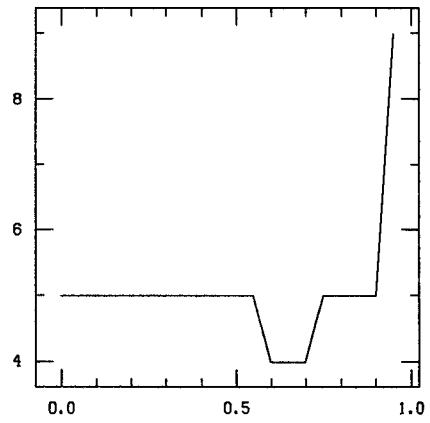
Two dimensional problem

Five subregions with minimum overlap

Five subregions with half overlap

Number of iterations versus alpha

FIG. 8.