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Restricted-Oriented Convex Sets

Gregory J.E. Rawlins
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Data Structuring Group
Research Report CS-89-01

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Restricted-Oriented Convex Sets *

Gregory J. E. Rawlins † Derick Wood ‡

January 13, 1989

Abstract

A *restricted-oriented convex set* is a set of points whose intersection with any line, in a given set of orientations, is either empty or connected. This notion generalizes both orthogonal convexity and normal convexity.

The aim of this paper is to establish a mathematical foundation for the theory of restricted-oriented convex sets. To this end, we prove the restricted-oriented analogs of some basic properties of convex sets and also present a decomposition theorem for them.

Keywords: convex sets, convex hulls, restricted-orientation convexity, computational geometry.

Introduction

. the fifteen years or so of its existence the field of computational geom-ry has bifurcated quite markedly into the study of algorithms for either thogonal or arbitrarily oriented objects. Possibly the main reason for .is is that the major application areas of computational geometry, namely LSI design, computer aided design, digital picture processing, computer sion, and computer graphics, have traditionally placed heavy emphasis on thogonally-oriented objects. This in turn is due to technical limitations; r example, most input/output devices and layout schemes have been or-.ogonal. Recent technical advances in VLSI design however now allow)jects to have more than the usual two orientations and, as a result, de-;ners are now concerned with objects with horizontal, vertical and lines

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of 45° and 135° [23]. Some companies also offer the capability of any finite number of orientations.

Another reason for the special study of orthogonal polygons is that algorithms for orthogonal polygons are simpler and, often, more efficient [18]. It is natural to speculate whether we can increase the number of allowed orientations and still have fast and simple algorithms.

Convex sets are a comparatively recent but very fruitful concept in geometry having applications in optimization, statistics, geometric number theory, functional analysis and combinatorics [9,12] and this is one of the reasons for the inordinate interest in convex sets in computational geometry. But their study is also practically motivated since the convex hull of an object typically has much less complexity than the object itself and so is much used in testing for intersections among objects [12,21]. The same reason suffices to explain the great popularity of the "bounding box" of an object in computer graphics and computer vision. Finally, the convex hull was one of the first concepts studied in computational geometry [19] and so deserves especial attention.

In [14] we defined and gave optimal algorithms to construct various new versions of the convex hull of a *finitely-oriented* polygon (meaning a polygon whose edge orientations belong to only a fixed *finite* set of orientations). The new notion of convexity introduced in that paper was a natural generalization of the well-known concept of *orthogonal* convexity (see [11], for example) and the new convex hulls we introduced generalized orthogonally-convex hulls. As it turned out, this was an advantageous generalization since the new notion completely encompassed the old and there was no additional complexity. In fact, the convex hull algorithms were simplified; the reason being that the generalization allowed the identification of inessential details that were specific only to orthogonal polygons.

In this paper we investigate the more general concept of *restricted-orientation convexity* and apply it to *arbitrary* sets of points, thereby generalizing our previous results and also, at the same time, verifying some otherwise unsupported observations in the literature.

The aim of the present paper is, apart from introducing restricted-orientation convexity, to establish the following analogs of the basic properties of convex sets [5]. In the following $\mathbf{P}$ is a planar convex set:

Simple Connectedness. $\mathbf{P}$ is simply connected.

Line Intersection. The intersection of $\mathbf{P}$ and any line is either empty or a connected set.

Intersection. $\mathbf{P}$ is the intersection of all convex sets which contain it.

Separation. If $p \notin \mathbf{P}$, then there exists a line separating p and $\mathbf{P}$.

[**alfplane Intersection.** $\mathbf{P}$ is the intersection of all halfplanes which contain it.

'**isibility.** If p, $q \in \mathbf{P}$, then the line segment joining p and q is in $\mathbf{P}$.

xcept for simple connectedness, these properties are defining characteristics f convex sets.

In Section 2 we define sets $\mathcal{O}$ of orientations, the associated $\mathcal{O}$-convex :ts, and prove some of their more elementary properties. In Section 3 we stablish the intersection and simple connectedness properties, while in Sec-.on 4 we prove the Separation Theorem. Stairlines, the restricted-oriented nalog of lines, are introduced in Section 5 and used to prove the stair-alfplane intersection property. We then in Section 6, prove the visibility roperty and, finally, in Section 7 provide a decomposition theorem. We onclude, in Section 8, with a summary of what we have accomplished and discussion of further work.

? Definitions

Ve assume the reader's familiarity with such elementary topological con-epts as (path-)connectedness, closure, simplicity, separability, support, in-erior and boundary of planar figures. We denote subsets of $\Re^2$ by bold face ppercase letters (for example, $\mathbf{P}$ and $\mathbf{Q}$) and elements of such sets by lower ase italic letters (for example, p and q). We treat a subset of $\Re^2$ as a set of iterior points together with its boundary (if it has one).

The *orientation of a directed line* is the counterclockwise angle made with he horizontal in a directed plane (in the goniometric sense). The *orientation f an undirected line* is the smaller of the two possible orientations. We only iscuss undirected lines in this paper. We use the symbol $\mathcal{O}$, with or without ubscripts, to refer to *a set* (possibly empty) *of orientations.*

A collection of lines, segments and rays is said to be $\mathcal{O}$-*oriented* if the et of orientations of the elements of the collection is a subset of $\mathcal{O}$. Thus, 'e speak of $\mathcal{O}$-*lines*, $\mathcal{O}$-*segments*, and $\mathcal{O}$-*rays* to mean $\mathcal{O}$-oriented lines, egments and rays. By extension, we call a polygon an $\mathcal{O}$-*polygon* if its dges are $\mathcal{O}$-segments.

Because we wish to preserve symmetry of direction in this paper we as-ume that the set $\mathcal{O}$ is symmetric about the horizontal, that is, if it contains n orientation $\theta < 180°$ it also contains an orientation $\theta + 180°$ and similarly or $\theta > 180°$. Hence, we specify a set of orientations only by the set of rientations less than 180°, it being understood that all the complementary rientations are present. So, for example, if we say that the set of orienta-ions $\mathcal{O}$ has two orientations we mean that it has four orientations two of 'hich are complementary to the other two.

The notion of $\mathcal{O}$-orientation has been previously defined, but only for finite $\mathcal{O}$, in [6,14,23,24] and, in a slightly related form, in [3]. As mentioned in the Introduction there is a vast literature concerning the special case of $\mathcal{O} = \{0°, 90°\}$ or more exactly, $\mathcal{O} = \{0°, 90°, 180°, 270°\}$. $\{0°, 90°\}$-objects are more usually called *orthogonal* (also; rectilinear, isothetic, iso-oriented, x-y or aligned) objects; see [12,25] for further references.

Throughout the paper, we assume that $\mathcal{O}$ is representable as a set of disjoint *closed* ranges[1], where some (or all) of the ranges may collapse to single orientations. Not only is the number of orientations allowed to be infinite, but also the number of closed ranges can be infinite too. For example, the set $\{\theta_1, [\theta_2, \theta_3], [\theta_4, \theta_5], \theta_6, \theta_7\}$ (all $\theta_i < 180°$). There is a natural ordering amongst the closed ranges in $\mathcal{O}$, so we can speak of the next range in $\mathcal{O}$ (the successor of the last range is the first range). In the example, $\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 < \theta_6 < \theta_7$ and $[\theta_2, \theta_3]$ is the next range after the range $\theta_1 (= [\theta_1, \theta_1])$.

We say the open range (θ_1, θ_2) is *$\mathcal{O}$-free* if there are no orientations in $\mathcal{O}$ in the range (θ_1, θ_2) and it is *maximal* if $\theta_1, \theta_2 \in \mathcal{O}$. If $\mathcal{O}$ is specified by n ranges, then $\mathcal{O}$ divides $[0°, 360°)$, into at most $2n$ maximal $\mathcal{O}$-free ranges. In the example the maximal $\mathcal{O}$-free ranges are $(\theta_1, \theta_2), (\theta_3, \theta_4)$, (θ_5, θ_6), (θ_6, θ_7), $(\theta_7, \theta_1 + 180°)$, together with the five complementary ranges.

The line passing through the points p and q is denoted by $L(p, q)$ and, similarly, $LS(p, q)$ denotes the line segment with endpoints p and q. The orientation of L is denoted by $\Theta(L)$, where L is a line, segment, or ray. If L is a line, segment, or ray and $\Theta(L) \notin \mathcal{O}$, then by *the* maximal $\mathcal{O}$-free range of L we mean the unique maximal $\mathcal{O}$-free range in which $\Theta(L)$ lies.

Any collection of lines, segments and rays having (one, two or) three orientations in the plane can be mapped onto another collection having the same incidence structure as the first but with (one, two or) three completely different orientations [14]. For this reason we frequently, for ease of exposition, assume that $(0°, 90°)$ is L's maximal $\mathcal{O}$-free range, for a particular L and $\mathcal{O}$, where $\Theta(L) \notin \mathcal{O}$ and $\mathcal{O}$ has two or more orientations.

The line intersection property of convex sets can be taken as a defining characteristic of convex sets. In other words, a set is *convex* if its intersection with *any* line is either empty or connected. We use this approach to define *restricted-orientation convexity.* The phrasing is somewhat unfortunate since it implies that it is a restriction of normal convexity when, in fact, the opposite is the case: restricted-orientation convexity includes (normal) convexity as a special case.

Assume that we have some fixed set $\mathcal{O}$ of orientations; none of our results depend on the particular set.

---

[1]Note, however, the the one exception to this, namely, any range that ends at 180°

Definition 2.1 *We say that* $\mathbf{P}$ *is* $\mathcal{O}$-convex *if the intersection of* $\mathbf{P}$ *and any* $\mathcal{O}$*-line is either empty or connected.*

This is a natural generalization of orthogonal convexity and normal convexity.

Figure 1 contains some example figures which are $\mathcal{O}$-convex for various $\mathcal{O}$. Figure 1 (a) is not $\mathcal{O}$-convex for any non-empty $\mathcal{O}$, but is $\mathcal{O}$-convex if $\mathcal{O} = \emptyset$, as are all the other figures. Figures 1 (b) and (c) are convex with respect to any horizontal line, as are (d), (e) and (f), so they are all $\{0°\}$-convex besides being $\emptyset$-convex. Note that (b) and (c) are not convex in any other direction. Figures 1 (d), (e) and (f) are convex with respect to any vertical line as well and so they are also $\{0°, 90°\}$-convex. Note that (d) is not convex in any other direction. Figures 1 (e) and (f) are convex with respect to any line with orientation in the range $\{[90°, 180°)\}$ and so they are also $\{[90°, 180°)\}$-convex. Note that (e) is not convex in any other direction. Figure 1 (f) is $\mathcal{O}$-convex for any $\mathcal{O}$.

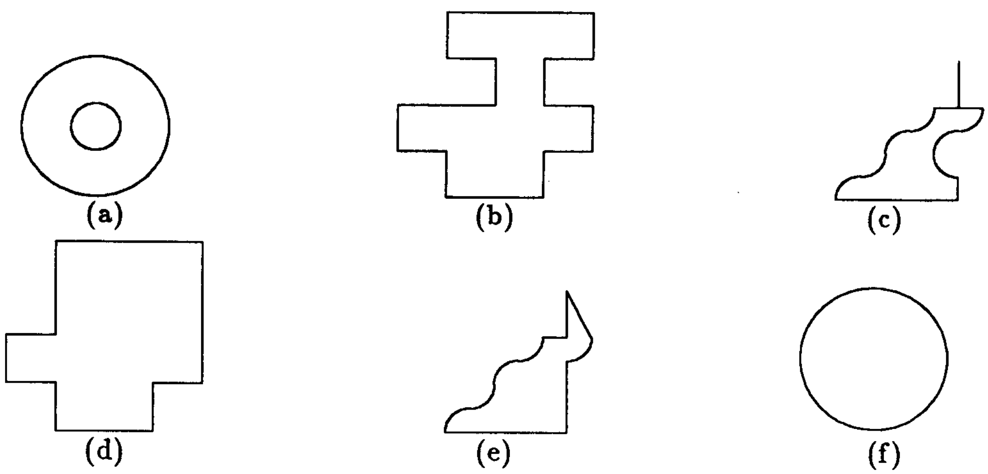

Figure 1: $\mathcal{O}$-convex figures.

Note that, if $\mathcal{O}$ is empty, then, vacuously, all sets are $\mathcal{O}$-convex. It is straightforward to prove the following lemma.

Lemma 2.1 *1. All planar convex sets are* $\mathcal{O}$*-convex.*

2. A planar set is convex if and only if it is $\{[0°, 180°)\}$*-convex.*

It is easy to construct examples to show that the second statement of the lemma holds for no smaller set of orientations. For example if we delete just *one* orientation (say θ_1), then any set consisting of just two distinct points on a $\{\theta_1\}$-line is $\{\theta\}$-convex for all $\theta \neq \theta_1$ but is, of course, not convex. Moreover, examples like these establish that the statement "for all $\mathbf{P}$, $\mathbf{P}$ is connected if $\mathbf{P}$ is $\mathcal{O}$-convex" holds if and only if $\mathcal{O} = \{[0°, 180°)\}$.

Note that the following sets are convex, and hence $\mathcal{O}$-convex for any $\mathcal{O}$: .e empty set, $\Re^2$, and, any point, line, segment, ray or halfplane in $\Re^2$.

Intersection and Simple Connectedness

. [16] the notion of a convexity space is used as a tool to analyze vari- ıs notions of convexity that have appeared in the literature. Convexity ›aces are not new (see [2,7,10,20], for example), but their application in ›mputational geometry is. As far as this paper is concerned our interest in .em is twofold. First, $\mathcal{O}$-convex sets form a convexity space and, second, .is implies that $\mathcal{O}$-convex sets are closed under intersection. This result stated as Lemma 3.1. Results already available for convexity spaces es- .blish immediately the intersection property for $\mathcal{O}$-convex sets; see Lemma 2.

Convex sets are simply connected; however, this does not hold for $\mathcal{O}$- ›nvex sets in general. If $\mathbf{P}$ is connected, its $\mathcal{O}$-hull is simply connected, ıt otherwise it consists of a set of connected components, as we prove in heorems 3.3 and 3.4.

emma 3.1 *Let $C_{\mathcal{O}}$ be the collection of all $\mathcal{O}$-convex sets. Then, $(\Re^2, C_{\mathcal{O}})$ a convexity space; that is, $\emptyset$ and $\Re^2$ are $\mathcal{O}$-convex and the intersection of ›ery subcollection of $\mathcal{O}$-convex sets is itself $\mathcal{O}$-convex.*

roof: See [16]. □

efinition 3.1 *The intersection of all $\mathcal{O}$-convex sets containing $\mathbf{P}$ is called ›e* $\mathcal{O}$-hull *of $\mathbf{P}$; it is denoted by $\mathcal{O}$-hull$(\mathbf{P})$.*

bserve that, $\forall\mathcal{O}$ and $\forall\mathbf{P}$, $\mathbf{P} \subseteq \mathcal{O}\text{-}hull(\mathbf{P})$ even when $\mathcal{O} = \emptyset$ or $\mathbf{P} = \emptyset$ (or ›th). Since $(\Re^2, C_{\mathcal{O}})$ is a convexity space; then $\forall\mathbf{P}$, $\mathcal{O}\text{-}hull(\mathbf{P})$ exists, is ıique and is the smallest $\mathcal{O}$-convex set which contains $\mathbf{P}$.

If $\mathcal{O} = \emptyset$, then $\mathcal{O}\text{-}hull(\mathbf{P}) = \mathbf{P}$, for all $\mathbf{P}$, since $\mathbf{P}$ is the smallest set ›ntaining $\mathbf{P}$ which is not required to be convex in any direction. Similarly, $\mathbf{P} = \emptyset$, then $\mathcal{O}\text{-}hull(\mathbf{P}) = \mathbf{P}$, for all $\mathcal{O}$, since the intersection of every -line and $\mathbf{P}$ is empty. When $\mathcal{O} = \{\theta\}$ and $\mathbf{P}$ is a polygon, then the $\mathcal{O}$-*hull* f $\mathbf{P}$ has been called the "θ-visibility hull" of $\mathbf{P}$ [18,22].

Note that in Figure 1, (f) is the $\mathcal{O}$-*hull* of (a), for any non-empty $\mathcal{O}$, and 1) and (e) are the $\{90°\}$-*hulls* of (b) and (c), respectively.

Directly from known results for convexity spaces (see [8], for example), e obtain the following properties of $\mathcal{O}$-convex sets.

emma 3.2 *1.* $\forall\mathcal{O}, \mathbf{P}$; $\mathbf{P}$ is $\mathcal{O}$-convex if and only if $\mathcal{O}\text{-}hull(\mathbf{P}) = \mathbf{P}$.

2. $\forall \mathcal{O}, \mathbf{P}$; $\mathcal{O}\text{-}hull(\mathcal{O}\text{-}hull(\mathbf{P})) = \mathcal{O}\text{-}hull(\mathbf{P})$.
3. $\forall \mathcal{O}, \mathbf{P}, \mathbf{Q}$; $\mathbf{P} \subseteq \mathbf{Q} \Longrightarrow \mathcal{O}\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}\text{-}hull(\mathbf{Q})$.

Part (1) of this lemma establishes the intersection property for $\mathcal{O}$-convex sets. When $\mathcal{O}$ is the set of all orientations, this is equivalent to the intersection property for convex sets.

Theorem 3.3 *If $\mathcal{O}$ is non-empty and $\mathbf{P}$ is connected; then, $\mathcal{O}$-hull($\mathbf{P}$) is simply connected. In other words, if $\mathcal{O}$ is non-empty and $\mathbf{P}$ is connected and $\mathcal{O}$-convex, then $\mathbf{P}$ is simply connected.*

Proof: If $\mathbf{P}$ is empty we have nothing to prove, so suppose $\mathbf{P}$ is non-empty.

Suppose that $\mathcal{O}$-*hull*($\mathbf{P}$) is not connected. Since $\mathbf{P}$ is connected it can only belong to one of the connected components of $\mathcal{O}$-*hull*($\mathbf{P}$) (it must belong to at least one otherwise $\mathcal{O}$-*hull*($\mathbf{P}$) does not contain $\mathbf{P}$). This component must be $\mathcal{O}$-convex, otherwise the entire hull is not $\mathcal{O}$-convex. Hence, we may discard all of the other components of $\mathcal{O}$-*hull*($\mathbf{P}$) and have a smaller $\mathcal{O}$-convex set which contains $\mathbf{P}$. But, $\mathcal{O}$-*hull*($\mathbf{P}$) is the smallest such set. Therefore $\mathcal{O}$-*hull*($\mathbf{P}$) must be connected if $\mathbf{P}$ is connected.

Suppose that $\mathcal{O}$-*hull*($\mathbf{P}$) is connected but contains a hole. Since $\mathcal{O}$ is non-empty there must exist at least one $\mathcal{O}$-line which cuts this hole. Hence, there is an $\mathcal{O}$-line whose intersection with $\mathcal{O}$-*hull*($\mathbf{P}$) is neither empty nor connected. But, this implies that $\mathcal{O}$-*hull*($\mathbf{P}$) is not $\mathcal{O}$-convex; hence, $\mathcal{O}$-*hull*($\mathbf{P}$) must be simply connected. □

Theorem 3.4 *A set is $\mathcal{O}$-convex if and only if it is the union of disjoint connected components such that each component is $\mathcal{O}$-convex and no $\mathcal{O}$-line intersects any pair of components.*

Proof: Let $\mathbf{P}$ be the union of disjoint connected components such that each component is a connected $\mathcal{O}$-convex set and no $\mathcal{O}$-line intersects any pair of components. Since no $\mathcal{O}$-line can intersect any two of them simultaneously and each component is separately $\mathcal{O}$-convex, the entire collection is $\mathcal{O}$-convex.

Conversely, let $\mathbf{P}$ be $\mathcal{O}$-convex, but not connected. If one of its components is not $\mathcal{O}$-convex, then $\mathbf{P}$ cannot be $\mathcal{O}$-convex. Similarly, if there exists an $\mathcal{O}$-line which intersects any two components, then $\mathbf{P}$ cannot be $\mathcal{O}$-convex. □

Observe that if $\mathcal{O}$ is the set of all orientations, then for each pair of connected components there exists at least one $\mathcal{O}$-line which intersects them. Hence, all $\{[0°, 180°)\}$-convex sets are connected.

4 Separation

A crucial property of convex sets is separability; that is, the separation property. We prove, in this section, a separability result for any set of orientations and in the next section, after introducing stairlines, we prove the separation property. We also present a separation theorem for $\mathcal{O}$-convex sets that are not connected.

Lemma 4.1 *If* $\mathbf{P}$ *is connected and* $p \in \mathcal{O}$*-hull*$(\mathbf{P})$*; then, each* $\mathcal{O}$*-line through* p *intersects* $\mathbf{P}$.

Proof: If either $\mathcal{O}$ or $\mathbf{P}$ is empty, then the lemma is vacuously true since then $\mathcal{O}$-*hull*$(\mathbf{P}) = \mathbf{P}$. Further, if $p \in \mathbf{P}$ we have nothing to prove. So suppose that both $\mathcal{O}$ and $\mathbf{P}$ are non-empty and that $p \notin \mathbf{P}$.

Suppose that there exists a $\theta \in \mathcal{O}$ such that the $\{\theta\}$-line through p does not intersect $\mathbf{P}$. Then, by the continuity of $\Re^2$ and the fact that $\mathbf{P}$ is connected, there exists a convex set (and hence an $\mathcal{O}$-convex set) which contains $\mathbf{P}$ and does not contain p, namely, any halfplane bounded by a $\{\theta\}$-line separating p and $\mathbf{P}$. Hence p cannot be in the intersection of all $\mathcal{O}$-convex sets which contain $\mathbf{P}$ and so cannot be in the hull. □

Theorem 4.2 (The Separation Theorem) *Let* $\mathbf{P}$ *be connected and* $p \notin \mathbf{P}$*. Then,* $p \in \mathcal{O}$*-hull*$(\mathbf{P})$ *if and only if there exists a* $\theta \in \mathcal{O}$ *such that the* $\{\theta\}$*-line through* p *intersects* $\mathbf{P}$ *in, at least, two points on either side of* p.

Proof: If either $\mathcal{O}$ or $\mathbf{P}$ is empty, then the lemma is vacuously true since then $\mathcal{O}$-*hull*$(\mathbf{P}) = \mathbf{P}$. So suppose that both $\mathcal{O}$ and $\mathbf{P}$ are non-empty.

If $p \notin \mathbf{P}$ and there exists an $\mathcal{O}$-line which intersects $\mathbf{P}$ at two points which bracket p, then p must be in the $\mathcal{O}$-*hull* of $\mathbf{P}$ (else the $\mathcal{O}$-*hull* would not be $\mathcal{O}$-convex).

Conversely, if $\mathbf{P}$ is connected and $p \in \mathcal{O}$-*hull*$(\mathbf{P}) \setminus \mathbf{P}$ then all $\mathcal{O}$-lines through p must intersect $\mathbf{P}$ (Lemma 4.1).

We shall prove the claim for the three cases in which we have either exactly one orientation in $\mathcal{O}$, two or more with at least one $\mathcal{O}$-free range and finally if $\mathcal{O}$ is all orientations (that is, there are no $\mathcal{O}$-free ranges).

Case 1: $\mathcal{O} = \{\theta\}$

The $\{\theta\}$-line through p must cut $\mathbf{P}$. Suppose that it only cuts it on one side of p (say to the right of p). Then we may delete p and all other points in $\{\theta\}$-*hull*$(\mathbf{P})$ on the left θ-ray from p and so obtain a smaller $\{\theta\}$-convex set which contains $\mathbf{P}$. But $\{\theta\}$-*hull*$(\mathbf{P})$ is the smallest such set. Hence p cannot be in $\{\theta\}$-*hull*$(\mathbf{P})$. Hence the $\{\theta\}$-line through p must cut $\mathbf{P}$ on both sides of p.

Case 2: $\mathcal{O}$ contains two or more orientations but not all.

Every $\mathcal{O}$-line through p must cut $\mathbf{P}$. Suppose that none of them cut $\mathbf{P}$ both to the left and to the right of p. Since $\mathbf{P}$ is connected this means that there exists at least one $\mathcal{O}$-convex halfplane containing $\mathbf{P}$ and not p. The simplest such halfplane is bounded by the first $\{\theta\}$-line through p which does not cut $\mathbf{P}$ to the left of p and the first $\{\theta\}$-line through p which does not cut $\mathbf{P}$ to the right of p (see Figure 2 for a simple example with $\mathcal{O} = \{0°, 90°, 135°\}$). This halfplane must be $\mathcal{O}$-convex as no $\mathcal{O}$-line can intersect both of the boundary $\mathcal{O}$-lines since the entire range is $\mathcal{O}$-free.

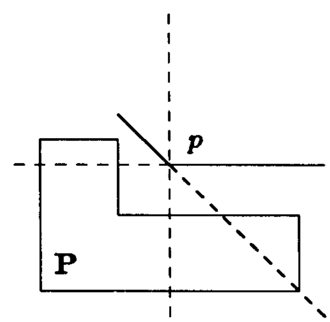

Figure 2: A halfplane containing $\mathbf{P}$ and not p.

Hence p cannot be in $\mathcal{O}$-*hull*$(\mathbf{P})$ for it would not be contained in the intersection of all $\mathcal{O}$-convex sets which contain $\mathbf{P}$. Hence at least one of the $\mathcal{O}$-lines through p must cut $\mathbf{P}$ to the left and to the right of p.

Case 3: $\mathcal{O} = \{[0°, 180°)\}$.

Here $\mathcal{O}$-*hull*$(\mathbf{P})$ is the convex hull of $\mathbf{P}$. The result then follows from the well-known separation theorem of Fenchel ([4]). □

This theorem is false if $\mathbf{P}$ is not connected as the following example shows. In figure 3, $\mathbf{P}$ is the set of points indicated by the bullets. The point p is not in $\mathbf{P}$ yet it is in $\mathcal{O}$-*hull*$(\mathbf{P})$ whenever $\{0°, 90°\} \subseteq \mathcal{O}$. However, there does not exist an $\mathcal{O}$-line through p which cuts $\mathbf{P}$ on both sides of p.

The final result of this section is a separation result for $\mathcal{O}$-convex sets that are not connected.

Theorem 4.3 *A set is $\mathcal{O}$-convex if and only if it is the union of disjoint $\mathcal{O}$-convex sets such that for each pair of connected components there is a point through which every $\mathcal{O}$-line separates the two components.*

Proof: This is a straightforward modification of Theorem 3.5. □

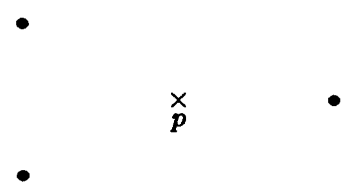

Figure 3: Disconnected Point Sets Do Not Support Separation

5 Halfplane Intersection

n order to characterize $\mathcal{O}$-convex sets in terms of halfplane intersections, ve need a new definition of line more appropriate to $\mathcal{O}$-convex sets. These ;eneralized lines are called *stairlines*; we define and investigate them in his section before establishing the new halfplane intersection property as Corollary 5.5. We then go on to characterize the boundary of $\mathcal{O}$-convex sets, vhen they have one, in Theorem 5.7.

First we need the concept of the *span* of a continuous curve in the plane.

Definition 5.1 *We say that the continuous plane curve S has* span $[\theta_1, \theta_2]$, *where $\theta_1 \leq \theta_2$, if for any two distinct points $p, q \in S$, $\Theta(L(p,q)) \in [\theta_1, \theta_2]$.*

(Of course, $\theta_1 = \theta_2$ if and only if the curve is a line, segment or ray with orientation θ_1.)

As an illustration: if S is a continuous curve with span $[0°, 90°]$ and $(x_1, y_1), (x_2, y_2)$ are any two distinct points on S, then either ($x_1 \leq x_2$ and $y_1 \leq y_2$) or ($x_1 \geq x_2$ and $y_1 \geq y_2$).

Definition 5.2 *We say that a continuous curve in the plane with span $[\theta_1, \theta_2]$ is an $\mathcal{O}$-*stairline *if (θ_1, θ_2) is $\mathcal{O}$-free.*

(Note that if $\theta_1 = \theta_2$, then (θ_1, θ_2) is vacuously $\mathcal{O}$-free since there are *no* orientations in the range and so any line, segment, or ray is an $\mathcal{O}$-stairline.)

We have chosen the name "stairline" as a combination of (orthogonal) *staircase* [25] and (straight) *line*. By analogy with lines, segments, and rays we also use the terms *$\mathcal{O}$-stairsegment* and *$\mathcal{O}$-stairray* with the obvious meanings. Note that a line, segment, or ray of *any* orientation is an $\mathcal{O}$-stairline, $\mathcal{O}$-stairsegment, or $\mathcal{O}$-stairray, for any $\mathcal{O}$.

Remark: To avoid excessive terminology, we shall assume for the rest of this section that $\mathcal{O}$ is understood and we shall just refer to stairlines (stairsegments, and stairrays). Also, if a result is stated for stairlines we do not dd the cumbersome qualifications that it also holds for stairsegments and tairrays.

We begin our study of stairlines by proving that they are $\mathcal{O}$-convex, just s lines are convex.

emma 5.1 *If S is a stairline, then S is $\mathcal{O}$-convex.*

roof: Suppose S is a stairline with span $[\theta_1, \theta_2]$. If $\theta_1 = \theta_2$, then S is a traight line and, hence, is $\mathcal{O}$-convex. Suppose, then that $\theta_1 \neq \theta_2$ and there xists an $\mathcal{O}$-line L which cuts S at two distinct points p and q. Since S has pan $[\theta_1, \theta_2]$, $\Theta(L) = \Theta(L(p,q)) \in [\theta_1, \theta_2]$ and since (θ_1, θ_2) is $\mathcal{O}$-free, $\Theta(L)$ an only be θ_1 or θ_2.

Suppose $\Theta(L) = \theta_1$ and $\theta_1 \in \mathcal{O}$. Without loss of generality assume that $\theta_1, \theta_2] = [0°, 90°]$ and that p is to the left of q (that is, p and q lie on a orizontal line). Consider any point r on S between p and q. r must be *on r above* the horizontal line segment $LS(p,q)$, otherwise $\Theta(L(p,r)) \notin [\theta_1, \theta_2]$. imilarly, r must be *on or below* the horizontal line segment $LS(p,q)$, otherwise $\Theta(L(q,r)) \notin [\theta_1, \theta_2]$. Hence, $r \in LS(p,q)$, for all r in S between p and . That is, between p and q, S is a line segment. Hence, even if θ_1 (or θ_2) s an orientation in $\mathcal{O}$, S is $\mathcal{O}$-convex. □

If S divides the plane into two halfspaces, we call them both *stair-alfplanes* for obvious reasons. Beware! It is easy to fall into the habit f thinking of stair-halfplanes as just halfplanes with wavy line boundaries. This is only true if $\mathcal{O} \neq \emptyset$.

orollary 5.2 *All stair-halfplanes are $\mathcal{O}$-convex.*

Ve can now prove the separation property for $\mathcal{O}$-convex sets.

orollary 5.3 *Let* $\mathbf{P}$ *be connected. If $p \notin \mathcal{O}$-hull$(\mathbf{P})$ then there exists an $\mathcal{O}$-stairline separating p and* $\mathbf{P}$.

roof: If $p \notin \mathcal{O}$-*hull*$(\mathbf{P})$, then from Theorem 4.2 there is no $\theta \in \mathcal{O}$ such that $L[\theta, p]$ intersects $\mathbf{P}$ on both sides of p. Thus we may construct an $\mathcal{O}$-stairline eparating p and $\mathbf{P}$ as in the proof of Theorem 4.2. □

Vhen $\mathcal{O} = [0°, 180°)$ all $\mathcal{O}$-stairlines are lines and this corollary becomes he separation property for convex sets.

We state without proof a converse of this result.

orollary 5.4 *If* $\mathbf{P}$ *is connected and there exists a stair-halfplane which ontains* $\mathbf{P}$ *and not the point p, then $p \notin \mathcal{O}$-hull$(\mathbf{P})$.*

s a final corollary we have the stair-halfplane intersection property for $\mathcal{O}$-convex sets.

Corollary 5.5 *If* **P** *is connected, then* **P** *is* $\mathcal{O}$*-convex if* **P** *is the intersection of all stair-halfplanes that contain it.*

The boundaries of closed convex sets are characterized in terms of line segments. We now provide a similar characterization theorem for closed $\mathcal{O}$-convex sets in terms of stairsegments. To this end we begin with the following definition.

Definition 5.3 *A stairline composed of a sequence of connected line segments is a* polygonal *stairline.*

It is easy to show that if a connected sequence of segments $l_1, l_2, \ldots, l_m$ forms a stairline, stairsegment, or stairray with span $[\theta_1, \theta_2]$, then

1. $\forall 1 \leq i \leq m$; $\Theta(l_i) \in [\theta_1, \theta_2]$.
2. $\forall 2 \leq i \leq m-1$; l_i meets l_{i-1} and l_{i+1} only at its endpoints.

Polygonal stairlines have been previously defined for the special case of orthogonal objects; see [25] for references. In this case they are known as *staircases.* See Figure 4 for examples of a stairsegment, a polygonal stairsegment, and an $\mathcal{O}$-oriented polygonal stairsegment for $\mathcal{O}$ any subset of $\{[90°, 180°)\}$.

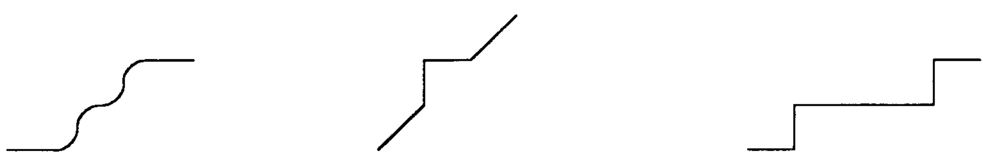

Figure 4: A variety of stairsegments.

Definition 5.4 *The set of all stairsegments joining p and q is called the* $\mathcal{O}$-region *of p and q and it is written as $\mathcal{O}$-region(p,q).*

Note that if $\Theta(LS(p,q)) \in \mathcal{O}$, then $\mathcal{O}$-*region*$(p,q) = LS(p,q)$. Of course, if $\mathcal{O}$ consists of all orientations, then, for all p and q, $\mathcal{O}$-*region*$(p,q) = LS(p,q)$. On the other hand, if $\mathcal{O}$ is empty, then every range is $\mathcal{O}$-free and so *any* continuous curve connecting p and q, for any p and q, is a stairsegment; hence, $\emptyset$-*region*$(p,q) = \Re^2$.

Definition 5.5 *If $\mathcal{O}$ has at least two orientations, then we say that* the parallelogram induced by p and q, $\|pq$, *is $LS(p,q)$ if $\Theta(LS(p,q)) \in \mathcal{O}$. Otherwise, it is the parallelogram with diagonal endpoints p and q and with sides of orientations θ_1 and θ_2, where (θ_1, θ_2) is $LS(p,q)$'s maximal $\mathcal{O}$-free range.*

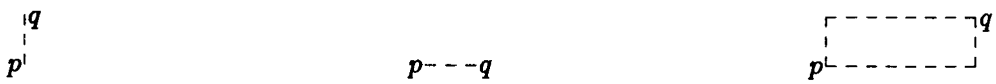

Figure 5: Orthogonal induced parallelograms.

Note that if $\mathcal{O} = \{[0°, 180°)\}$, then, for all distinct p and q, $\|pq = LS(p,q)$. See Figure 5 for examples of $\|pq$ for $\mathcal{O} = \{0°, 90°\}$.

If $\Theta(LS(p,q)) \notin \mathcal{O}$, we call the two sets of segments connecting p and q the *arms* of $\|pq$.

Lemma 5.6 *If p and q are two points in the plane and $\mathcal{O}$ is a set of at least two orientations, then any stairsegment joining p and q must lie wholly in $\|pq$. Furthermore, all points in $\|pq$ lie on some stairsegment joining p and q.*

Proof: If $\Theta(LS(p,q)) \in \mathcal{O}$, then the lemma is true, so suppose otherwise. Without loss of generality, let $(0°, 90°)$ be $LS(p,q)$'s maximal $\mathcal{O}$-free range.

If any continuous path from p to q leaves the parallelogram $\|pq$, it can only be monotone in either the horizontal or vertical direction and, so, cannot be a stairsegment. Hence, when $\mathcal{O}$ contains two or more orientations, then all stairsegments must lie in $\|pq$.

If $r \in \|pq$, we can easily construct a stairsegment joining p and q passing through r; see Figure 6 for a simple example stairsegment for $\mathcal{O} = \{0°, 90°\}$. $\square$

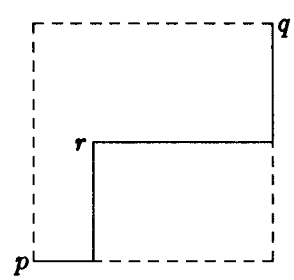

Figure 6: A stairsegment from p to q through r.

Hence, when $\mathcal{O}$ has two or more orientations, $\mathcal{O}$-*region*$(p,q) = \|pq$.

With respect to $\mathcal{O}$-convex sets, stairlines are the most natural analogs of straight lines with respect to convex sets, in that: there exists a stairsegment which realises the shortest distance between any two points; an $\mathcal{O}$-line meets a stairline in at most one point (unless collinear with some part of the stairline); and two stairlines with disjoint spans can only intersect in at most one point. However, the intersection of two stairlines with non-disjoint spans empty, connected, or not connected — unlike the simpler case of straight ıes. Furthermore, stairlines can be non-intersecting without being parallel n the conventional sense), and two *points* may define exactly one $\mathcal{O}$-line infinitely many stairlines (that is, all stairlines passing through their $\mathcal{O}$-*gion*). Perhaps a closer analogy would be to say that two stairlines with ›n-disjoint spans are *parallel* and that they are *collinear* if they intersect ıywhere. If two stairlines *have disjoint spans*, then they behave just like raight lines (that is, intersect exactly once, etc.).

With stairlines replacing lines we can generalize convexity in other ways ıan the one we investigate in this paper. For example, a set $\mathbf{P}$ is *strongly -convex* if, for every pair of points p and q in $\mathbf{P}$, *all* stairsegments with ıdpoints p and q lie in $\mathbf{P}$. It can be proved that this definition of convexity ways produces *convex* (in the normal sense) $\mathcal{O}$-*oriented* sets. Indeed, when $= \{0°, 90°\}$, the strong $\mathcal{O}$-convex hull of $\mathbf{P}$ is just the *bounding box* of $\mathbf{P}$. 'e have investigated strong $\mathcal{O}$-convexity in a previous paper [15] and we ıve shown in [16] that both $\mathcal{O}$-convexity and strong $\mathcal{O}$-convexity along with any other natural definitions of convexity are essentially the same.

We now characterize the boundary of a closed connected $\mathcal{O}$-convex set terms of stairsegments.

efinition 5.6 *A point p is an $\mathcal{O}$-*extremal point *of* $\mathbf{P}$ *if p is a point of ıpport of* $\mathbf{P}$ *with respect to an $\mathcal{O}$-line.*

efinition 5.7 *A portion of a continuous curve in the plane is a* maximal airsegment *in the curve if it is a stairsegment and it is not a proper subset f any other stairsegment in the curve.*

heorem 5.7 (The Boundary Characterization Theorem)
simply-connected closed set is $\mathcal{O}$-convex if and only if the portions of s boundary between every two consecutive $\mathcal{O}$-extremal points are maximal airsegments.

roof: If $\mathbf{P}$ is closed and simply connected and its boundary is made up nly of stairsegments meeting at $\mathcal{O}$-extremal points in $\mathbf{P}$, then the only way ı which $\mathbf{P}$ could fail to be $\mathcal{O}$-convex is if some $\mathcal{O}$-line intersects one of the airsegments more than once, since no $\mathcal{O}$-line can intersect such a set more ıan twice. But this is impossible, since any $\mathcal{O}$-line can only intersect a airline at most once (unless it is collinear with some part of the stairline). ence, such a set must be $\mathcal{O}$-convex.

Suppose now that $\mathbf{P}$ is a connected closed $\mathcal{O}$-convex set. Consider any air of distinct consecutive $\mathcal{O}$-extremal points p and q of $\mathbf{P}$. If $LS(p,q) \subseteq \mathbf{P}$, hen $\Theta(LS(p,q)) \in \mathcal{O}$ and, hence, $LS(p,q)$ is a stairsegment joining p and . So, let $S(p,q)$ be the portion of $\mathbf{P}$'s boundary connecting p and q, where

$S(p,q)$ is not a line segment. Since p and q are distinct consecutive extremal points of $\mathbf{P}$, $\Theta(LS(p,q)) \notin \mathcal{O}$. Without loss of generality, assume that $LS(p,q)$'s maximal $\mathcal{O}$-free range is $(0°, 90°)$ and that p is below and to the left of q (see Figure 7).

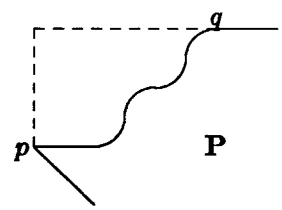

Figure 7: $S(p,q)$ is a maximal stairsegment.

Now sweep a horizontal line from q down to p. If at any time during the sweep this line intersects $S(p,q)$ more than once, then $\mathbf{P}$ cannot be $\mathcal{O}$-convex, and similarly for a vertical line sweeping from p to q. Hence $S(p,q)$ is a stairsegment connecting p and q. Trivially, it is maximal since it's end-points are $\mathcal{O}$-extremal in $\mathbf{P}$. □

Observe that in the normal convex hull (that is, $\mathcal{O} = \{[0°, 180°)\}$) *all* points are $\mathcal{O}$-extremal and so the maximal stairsegments in the boundary shrink to points.

Corollary 5.8 *A polygon is $\mathcal{O}$-convex if and only if its boundary consists of a sequence of polygonal stairsegments meeting at convex interior angles.*

Corollary 5.9 *An $\mathcal{O}$-polygon is $\mathcal{O}$-convex if and only if its boundary consists of a sequence of $\mathcal{O}$-oriented polygonal stairsegments meeting at convex interior angles.*

For the special case of finite $\mathcal{O}$, Corollary 5.8 has been stated without proof in [24] and it was proved in a different, more direct, way in [14]. Note that the characterization of the boundary of $\mathcal{O}$-convex polygons as a sequence of polygonal stairsegments is a direct generalization of the case for orthogonal polygons [25].

6 Visibility

In the theory of convex sets two points are said to be *visible* to each other in a set if the line segment joining them lies wholly in the set. Taking stairlines as the analogs of straight lines we are led to define $\mathcal{O}$-visibility as: two points in a set are $\mathcal{O}$-visible to each other if there exists *at least one $\mathcal{O}$-stairsegment* joining them that lies wholly in the set. This leads to a characterization of connected $\mathcal{O}$-convex sets in terms of $\mathcal{O}$-visibility — the final property of convex sets we consider.

Theorem 6.1 (The Visibility Theorem) *If* $\mathbf{P}$ *is connected, then* $\mathbf{P}$ *is* $\mathcal{O}$*-convex if and only if, for all* p *and* q *in* $\mathbf{P}$, p *and* q *are* $\mathcal{O}$*-visible to each other.*

Proof: Suppose that $\mathbf{P}$ is connected and, for all $p, q \in \mathbf{P}$, there is a stairsegment joining them that lies in $\mathbf{P}$. Consider any $\mathcal{O}$-line that intersects $\mathbf{P}$ in at least two points. Let p and q be two distinct points in $\mathbf{P}$ that lie on the $\mathcal{O}$-line. Then, $\Theta(LS(p,q)) \in \mathcal{O}$. This implies that the only one stairsegment joining p and q is $LS(p,q)$; therefore, by assumption, $LS(p,q) \subseteq \mathbf{P}$. But this implies that the intersection of $\mathbf{P}$ with each $\mathcal{O}$-line is either empty or connected; that is, $\mathbf{P}$ is $\mathcal{O}$-convex.

Conversely, suppose that $\mathbf{P}$ is connected and $\mathcal{O}$-convex. If $p, q \in \mathbf{P}$ and $\Theta(LS(p,q)) \in \mathcal{O}$, then there is a stairsegment that lies in $\mathbf{P}$ joining p and q — namely, $LS(p,q)$ (otherwise $\mathbf{P}$ is not $\mathcal{O}$-convex). Suppose, then that $\Theta(LS(p,q)) \notin \mathcal{O}$. Consider $\|pq$. If an arm of $\|pq$ lies in $\mathbf{P}$, there exists a stairsegment lying in $\mathbf{P}$ joining p and q since each arm of $\|pq$ is a stairsegment.

Assume then, that neither arm lies wholly in $\mathbf{P}$. Since the lower arm (say) consists of two $\mathcal{O}$-segments and it does not lie wholly in $\mathbf{P}$, then it must intersect the boundary of $\mathbf{P}$ exactly twice (otherwise $\mathbf{P}$ is not $\mathcal{O}$-convex). Both of these intersection points belong to one maximal stairsegment. For, if they belong to separate maximal stairsegments, there must be at least one $\mathcal{O}$-extremal point on $\mathbf{P}$'s boundary between the two intersection points. But, this implies that there is at least one $\mathcal{O}$-orientation in $LS(p,q)$'s $\mathcal{O}$-free range which is impossible.

We are now able to construct a stairsegment that lies in $\mathbf{P}$ and connects p and q as follows: starting at p, follow the lower arm until $\mathbf{P}$'s boundary is encountered; follow the boundary until the arm is met once more; and, finally, follow the arm to q. □

Note that in normal convexity this theorem collapses to the visibility property since all (normal) convex sets are connected.

7 Decomposition

Intuitively, we think of the action of forming the *$\mathcal{O}$-hull* of a set $\mathbf{P}$ as sweeping a line of each orientation in $\mathcal{O}$ across $\mathbf{P}$ and adding suitable line segments to the hull formed so far so that it is convex in each direction in $\mathcal{O}$. (Note that if $\mathcal{O}$ is empty, then we do not add anything to $\mathbf{P}$.) Thinking of it this way it does not seem sensible that the hull we eventually produce is changed if we decide to change the order of orientations in which we sweep. As we state in Theorem 7.2 this is, in fact, the case but only for *connected* sets. For sets that are not connected Lemma 7.1 is the strongest possible result. Observe that if $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $\mathcal{O}_1$-*hull*$(\mathbf{P}) \subseteq \mathcal{O}_2$-*hull*$(\mathbf{P})$, for all $\mathbf{P}$, since $\mathcal{O}_2$-*hull*$(\mathbf{P})$ contains $\mathbf{P}$ and is $\mathcal{O}_1$-convex. In some sense as a set of orientations $\mathcal{O}$ "grows" to include all possible orientations, the set $\mathcal{O}$-*hull*$(\mathbf{P})$ "grows" to the (normal) convex hull of $\mathbf{P}$.

Lemma 7.1 $\forall \mathcal{O}_1, \mathcal{O}_2, \mathbf{P}$; $\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P})) \subseteq (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})$.

Proof: See [16]. □

This result also holds if we replace $\mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P}))$ by $\mathcal{O}_2\text{-}hull(\mathcal{O}_1\text{-}hull(\mathbf{P}))$.

Simple counter-examples show that these results are best possible, in that, there exist sets for which the respective converses are false. However, we can strengthen Lemma 7.1 considerably by restricting $\mathbf{P}$ to be connected.

Theorem 7.2 (The Decomposition Theorem) *If* $\mathbf{P}$ *is connected, then* $\forall \mathcal{O}_1, \mathcal{O}_2$,

$$\begin{aligned}(\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P}) &= \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P})) \\ &= \mathcal{O}_2\text{-}hull(\mathcal{O}_1\text{-}hull(\mathbf{P})) \\ &= \mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P})\end{aligned}$$

Proof: See [16]. □

Corollary 7.3 *If* $\mathbf{P}$ *is connected and* $\mathcal{O} = \bigcup \mathcal{O}_i$, *then* $(\bigcup \mathcal{O}_i)\text{-}hull(\mathbf{P}) = \bigcup(\mathcal{O}_i\text{-}hull(\mathbf{P}))$.

This corollary verifies Toussaint and Sack's observation [22] that the (normal) convex hull is the union of the "visibility hulls" over all directions of visibility. The Decomposition Theorem bears a strong resemblance to the double integration rule where if $f(x,y)$ is continuous, then $\int\int f(x,y)dxdy = \int\int f(x,y)dydx$.

Sack [18] showed, in the orthogonal case, that the horizontal hull of the vertical hull of an orthogonal polygon (or alternately the vertical hull of the horizontal hull) is equivalent to the union of both hulls. It was taken as self-evident that the union is the smallest horizontally and vertically convex polygon enclosing the orthogonal polygon. Corollary 7.3 validates that assumption.

This decomposition result immediately yields an algorithm to find the hull of any connected set given that we can find the hull in one direction. However, connected $\mathcal{O}$-convex sets have considerably more structure than this which we can exploit to construct optimal algorithms to find the hull of any connected set (see [14] for the special case of finite $\mathcal{O}$, see [13] for the general case).

8 Conclusions

We have shown that $\mathcal{O}$-convex sets contain both convex sets and orthogonally convex sets as sub-classes and that the properties of both can be explained as special cases of the properties of $\mathcal{O}$-convex sets. The main characteristic of convex sets that we have lost in the generalization to $\mathcal{O}$-convex sets is *connectivity*: a convex set is always connected.

Connected $\mathcal{O}$-convex sets enjoy the properties of convex sets, if we replace line by stairline and recognize that a stairsegment joining two points is not necessarily unique. In the following we assume that $\mathbf{P}$ is a *connected* $\mathcal{O}$-convex set.

Simple Connectedness. If $\mathcal{O}$ is non-empty, then $\mathbf{P}$ is simply connected (Lemma 3.3). Indeed, the connected components of any $\mathcal{O}$-convex set are simply connected once $\mathcal{O}$ is non-empty (Theorem 3.4 together with Lemma 3.3).

Line Intersection. The intersection of $\mathbf{P}$ and any $\mathcal{O}$-line is either empty or a connected set, by definition. This holds even if $\mathbf{P}$ is not connected. One aspect of this property for convex sets is that lines are themselves convex. We can obtain the needed analogy by saying that the intersection of any two $\mathcal{O}$-convex sets is again $\mathcal{O}$-convex (Lemma 3.1) (although observe that the intersection of two connected $\mathcal{O}$-convex sets may not be connected).

Intersection. $\mathbf{P}$ is the intersection of all $\mathcal{O}$-convex sets which contain it (Lemma 3.2). This holds even if $\mathbf{P}$ is not connected.

Separation. If $p \notin \mathbf{P}$, then there exists a stairline separating p and $\mathbf{P}$ (Theorem 4.2 and Corollary 5.3).

Halfplane Intersection. $\mathbf{P}$ is the intersection of all stair-halfplanes which contain it (Corollary 5.5).

Visibility. If $p, q \in \mathbf{P}$, then there exists a stairsegment in $\mathbf{P}$ connecting p and q (Theorem 6.1).

Restricted-orientation convexity is a generalization of orthogonal convexity which has itself been separately defined in computational geometry, digital picture processing, VLSI design and combinatorics [1,17,25]. Restricted-orientation convexity serves as a useful vantage point to survey and unify many scattered results and observations in the literature of computational geometry. We have shown that restricted-orientation convexity is a reasonable generalization of convexity since properties analogous to those of normal convex sets hold for these more general "convex" sets.

It may be argued that since computational geometry concerns itself with figures in $\Re^n$ that it is not necessary to develop the theory of $\mathcal{O}$-convex sets in as general a setting as is possible. There are two telling rejoinders to this point of view, the first being a purely practical one. To take but one pertinent example, the history of algorithms for finding the convex hull of a simple polygon illustrates that unaided *geometric intuition* is not sufficiently powerful to avoid egregious errors. There have been several algorithms proposed over time (and accepted as correct) which were later shown to be incorrect. Any theoretical machinery that may aid insight is desirable. Secondly, there is a well-demonstrated synergism between theoretical investigations and practical problems, in that practice suggests new areas for theory and in turn a developing theory suggests a broadening and sharpening of practique. Finally, if any further justification were needed, we submit that the study of restricted-orientation convexity is of sufficient interest and importance in its own right.

Besides the above justifications we believe that this material will be beneficial in at least two practical areas (restricted-orientation VLSI design and restricted-orientation robotic path problems) and that it is of continuing theoretical interest as evidenced by further work in "starshapedness", "visibility", the computation of nearness of "convex" polygons, etc. [13].

It is our opinion that, while the practical concerns from which computational geometry grew will continue to change and expand, the broad outlines of computational geometry that serve to delineate it from classical geometry and combinatorial geometry are now sufficiently well defined that it can now, in its turn, give impetus to the development of new directions of geometry.

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