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Exact Solutions for a Generalized Nonlinear Schrödinger Equation

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Research Report CS-88-42
October, 1988

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Exact Solutions for a Generalized Nonlinear Schrödinger Equation

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ABSTRACT

Exact solutions of a generalized nonlinear Schrödinger equation

$$iu_t + u_{xx} + q_c |u|^2 u + q_q |u|^4 u + iq_m |u|^2_x u + iq_u |u|^2 u_x = 0$$

are obtained by transforming the equation to one whose coefficients satisfy a simple algebraic relationship. The transformed equation also yields conservation laws and sufficient conditions for boundedness and blow-up of solutions for the original equation.

1. Introduction

A generalized nonlinear Schrödinger equation (GNLS)

$$iu_t + u_{xx} + q_c |u|^2 u + q_q |u|^4 u + iq_m |u|^2_x u + iq_u |u|^2 u_x = 0 \tag{1.1}$$

governs the modulation of a quasi-monochromatic wave train in a weakly nonlinear, dispersive medium. Here $u(x,t)$ is the complex wave amplitude, $i^2=-1$, and the parameters q_c, q_q, q_m, and q_u are real constants. We are considering the initial value problem for (1.1) with $u(x,0)=u_0(x)$ specified. Equation (1.1) was derived independently by Johnson [1] and Kakutani & Michihiro [2] to describe the behaviour of the Stokes wave near the state of modulational instability. (Although Johnson's equation includes a term in $u\varsigma_t$ where $\varsigma_x = |u|^2$, this may be eliminated as shown in [2] .) The GNLS also has wide applicability as a model equation for a large class of evolutionary systems where the relevant time and space scales are greater than those captured by the usual nonlinear Schrödinger equation with a cubic nonlinearity [3]. It contains, as special cases, the nonlinear Schrödinger equations

$$iu_t + u_{xx} + q_c |u|^2 u = 0 \tag{1.2}$$

$$iu_t + u_{xx} + iq_m(|u|^2u)_x = 0 \tag{1.3}$$

$$iu_t + u_{xx} + iq_u|u|^2u_x = 0 \tag{1.4}$$

$$iu_t + u_{xx} + q_c|u|^2u + q_q|u|^4u = 0. \tag{1.5}$$

Eq. (1.2) is the well-known cubic Schrödinger equation, which has important applications in nonlinear optics [4], plasma physics [5], and fluid dynamics [6]. The derivative nonlinear Schrödinger equation, Eq. (1.3), governs the propagation of nonlinear Alfvén waves [7], while Eq. (1.4) describes the self-modulation of the complex amplitude of solutions to the Benjamin-Ono equation [8]. Eq. (1.5), the cubic-quintic Schrödinger equation, governs the propagation of light beams in an inhomogeneous medium when the nonlinear polarization contains susceptibilities of third and fifth orders [9], and also describes the boson gas with 2- and 3-body interactions [10].

Eqs. (1.2)-(1.4) are particularly interesting, being completely integrable and consequently possess an infinite number of invariants of motion. For certain initial conditions, they can be solved by the Inverse Scattering Transform [11] and, for those initial conditions which decay rapidly as $x \rightarrow \pm\infty$, they admit soliton solutions, solitary waves which preserve their shapes and velocities upon interaction. In general, however, the GNLS (1.1) does not belong to the class of completely integrable equations. Enns *et al* [12] demonstrated numerically that the solitary waves of the cubic-quintic Schrödinger equation do not exhibit the stability of solitons (unless q_q=0), although they found that quasi-soliton behaviour persisted over a large region of the parameter space. Outside this region, they observed that the solitary wave interaction resulted in dispersive or explosive behaviour, depending both on the initial condition and on the choice of parameters q_c and q_q. As the GNLS represents, in some sense, a correction to the cubic Schrödinger equation, a study of the behaviour of its solutions may provide a more realistic assessment of the physical stability of solitary waves.

In this paper, we use a gauge transformation to obtain a new generalized equation from which exact solutions to (1.1) are found. The solutions include solitary wave forms (although these are not solitons in general) as well as singular and oscillatory functions. The transformed system is also used to find three invariants of (1.1) from which sufficient conditions for boundedness and blow-up of solutions are derived. A Galilean invariance of the GNLS suggests a rate for the blow-up which agrees with numerical experiment.

2. Gauge Transformations Between GNLS Equations

Consider a gauge transformation

$$u(x,t) = \psi(x,t)\exp i\theta(x,t), \tag{2.1}$$

where $\psi(x,t)$ is a complex valued function, and where $\theta(x,t)$ is real and defined by

$$\theta_x = 2\delta\,|\psi|^2$$

$$\theta_t = \delta\,[4\mathrm{Im}(\psi\overline{\psi}_x) - (2q_m + q_u + 8\delta)\,|\psi|^4].$$

(Here $\overline{\psi}$ denotes the complex conjugate of ψ.) Under (2.1), the GNLS transforms to a new GNLS for $\psi(x,t)$

$$i\psi_t + \psi_{xx} + Q_c\,|\psi|^2\psi + Q_q\,|\psi|^4\psi + iQ_m\,|\psi|^2_x\psi + iQ_u\,|\psi|^2\psi_x = 0 \tag{2.2}$$

with

$$Q_c = q_c$$

$$Q_q = q_q + 2\delta\,q_m - \delta\,q_u + 4\delta^2$$

$$Q_m = q_m + 4\delta$$

$$Q_u = q_u.$$

The transformation parameter, δ, is an arbitrary constant and introduces one degree of freedom in the choice of Q_q and Q_m. We note that the compatibility condition $\theta_{xt}=\theta_{tx}$ is satisfied by the field equation for ψ, and θ is therefore well defined by the theory of conservative vector fields. The transformation preserves the modulus of the solution, $|\psi| = |u|$, and works both ways.

The above transformation is a generalization of that used by Calogero & Eckhaus [13] and Kundu [14] (their analyses correspond to the choice $\delta=(q_u-q_m)/4$) to transform the GNLS (1.1), subject to the constraint $4q_q = {q_m}^2 - q_m q_u$, to the mixed nonlinear Schrödinger equation, which is completely integrable [15] and has N-soliton solutions [16]. Solutions for those GNLS equations which satisfy the integrability condition are then constructed from known solutions for the mixed NLS equation. For example, if $q_u = 0$, so that $4q_q = {q_m}^2$, and if $q_c > 0$, a solitary wave solution corresponding to the one-hump soliton of the cubic NLS is given explicitly by

$$u(x,t) = \left(\frac{2\alpha}{q_c}\right)^{1/2}\mathrm{sech}(\sqrt{\alpha}(x-ct)+\mu)\exp\, i\phi(x,t) \tag{2.3}$$

$$\phi(x,t) = \frac{c}{2}(x-bt) - \frac{\sqrt{\alpha}q_m}{q_c}\tanh(\sqrt{\alpha}(x-ct)+\mu) + \nu$$

$$\alpha = \frac{c}{2}\left(\frac{c}{2} - b\right) > 0$$

where b, c, μ, and ν are arbitrary constants. We note that the GNLS solution differs from the cubic Schrödinger soliton only by the peculiarities of its phase and that the interaction of the solitary waves (2.3) should exhibit the same clean interactions as the cubic Schrödinger solitons.

The choice $\delta = -\frac{1}{8}(2q_m + q_u)$ forces $2Q_m + Q_u = 0$ in (2.2) so that the resulting equation for ψ is

$$i\psi_t + \psi_{xx} + Q_c |\psi|^2\psi + Q_q |\psi|^4\psi + iQ_m |\psi|^2_x\psi + iQ_u |\psi|^2\psi_x = 0 \tag{2.4}$$

where

$$\begin{aligned} Q_c &= q_c \\ Q_q &= q_q - \frac{1}{16}(2q_m + q_u)(2q_m - 3q_u) \\ Q_m &= -\frac{1}{2}Q_u = -\frac{1}{2}q_u \\ Q_u &= q_u. \end{aligned} \tag{2.5}$$

Noting that $2Q_m + Q_u = 0$, we rewrite Eq. (2.3) in the equivalent, but more convenient, form

$$i\psi_t + \psi_{xx} + F(\psi)\psi = 0, \tag{2.6}$$

where the real function $F(\psi)$ is given by

$$F(\psi) = Q_c |\psi|^2 + Q_q |\psi|^4 + Q_u \mathrm{Im}(\psi\overline{\psi}_x).$$

Hereafter, a reference to the transformed system refers to Eq. (2.6), corresponding to the above choice of δ.

3. Exact Solutions

The GNLS (1.1) has a plane wave solution with constant amplitude a,

$$u(x,t) = a \exp i(\kappa x - \omega t).$$

The real dispersion relation, $\omega = \kappa^2 + q_u |a|^2\kappa - q_c |a|^2 - q_q |a|^4$, depends on the amplitude, and on the three parameters q_c, q_q, and q_u.

We now obtain new exact solutions of (1.1) using the transformed equation (2.6). Writing $\psi(x,t)$ as

$$\psi(x,t) = f(x-ct)\exp ig(x-bt) \tag{3.1}$$

where f and g are real functions, with

$$g(y) = \frac{c}{2}y + \nu,$$

and where b, c, and ν are arbitrary constants, allows us to easily calculate $\theta(x,t)$ as

$$\theta(x,t) = 2\delta\int f^2(x-ct)d(x-ct) = -\frac{1}{4}(2q_m + q_u)\int f^2(x-ct)d(x-ct).$$

Solutions to (1.1) are then readily obtained, using the relation (2.1), once $\psi(x,t)$ is determined. Substituting (3.1) into the transformed GNLS (2.6), we get a cnoidal wave type equation for $f(x-ct)$, which can be integrated up once to obtain

$$(f')^2 + \frac{Q_q}{3}f^6 + \frac{1}{2}\left(Q_c - \frac{c}{2}Q_u\right)f^4 + \frac{c}{2}\left(b - \frac{c}{2}\right)f^2 + C = 0$$

where C is an arbitrary constant. Defining $z = f^2$, z then satisfies

$$(z')^2 + \frac{4Q_q}{3}z^4 + (2Q_c - cQ_u)z^3 + (2bc - c^2)z^2 + 4Cz = 0 \tag{3.2}$$

and may be solved for in terms of the elliptic functions. If $C=0$, these solutions can be expressed in terms of the elementary functions and include oscillatory, singular, phase jump, and solitary wave solutions. These are four-parameter families of solutions, with arbitrary constants b, c, ν, and μ. The parameters b and c, representing, respectively, the speeds of the carrier and envelope waves of ψ, partly determine the form of the solution. We list below some interesting cases.

$\mathbf{Q_q < 0}$**:** Defining

$$\alpha = -\frac{4Q_q}{3}$$

$$\beta = -\frac{3}{4Q_q}(cQ_u - 2Q_c)$$

$$\gamma = -\frac{3}{4Q_q}(c^2 - 2bc)$$

equation (3.2) becomes

$$(z')^2 = \alpha z^2(z^2 + \beta z + \gamma) = \alpha z^2(z - r_1)(z - r_2).$$

The values of r_1 and r_2 determine the form of the solution z; solitary waves arise if r_1 and r_2 are real and $r_1 > r_2 > 0$. The solution to (1.1) is then

$$u(x,t) = \left(\frac{r_1 r_2}{r_1 + (r_1 - r_2)\sinh^2(\chi)}\right)^{1/2} \exp i\phi(x,t) \tag{3.3}$$

$$\phi(x,t) = -\frac{(2q_m + q_u)}{4}\left(-\frac{3}{Q_q}\right)^{1/2}\tanh^{-1}\left(\left(\frac{r_2}{r_1}\right)^{1/2}\tanh(\chi)\right) + \frac{c}{2}(x - bt) + \nu$$

$$\chi = \left(\frac{-r_1 r_2 Q_q}{3}\right)^{1/2} (x-ct) + \mu.$$

$\mathbf{Q_q > 0}$: Now with

$$\alpha = \frac{4Q_q}{3}$$

$$\beta = \frac{3}{4Q_q}(cQ_u - 2Q_c)$$

$$\gamma = \frac{3}{4Q_q}(c^2 - 2bc)$$

equation (3.2) becomes

$$(z')^2 = \alpha z^2(-z^2 + \beta z + \gamma) = \alpha z^2(r_1 - z)(z - r_2).$$

If r_1 and r_2 are real, with $r_1 > 0 > r_2$, a solitary wave solution exists and the corresponding solution for (1.1) is

$$u(x,t) = \left(\frac{r_1 r_2}{r_2 + (r_2 - r_1)\sinh^2(\chi)}\right)^{1/2} \exp i\phi(x,t)$$

$$\phi(x,t) = -\frac{(2q_m + q_u)}{4}\left(\frac{3}{Q_q}\right)^{1/2} \tan^{-1}\left(\left(-\frac{r_1}{r_2}\right)^{1/2} \tanh(\chi)\right) + \frac{c}{2}(x - bt) + \nu$$

$$\chi = \left(\frac{-r_1 r_2 Q_q}{3}\right)^{1/2} (x-ct) + \mu.$$

If r_1 and r_2 are real, with $r_1 > r_2 > 0$, then z is oscillatory and

$$u(x,t) = \left(\frac{r_1 r_2}{r_1 + (r_2 - r_1)\cos^2(\chi)}\right)^{1/2} \exp i\phi(x,t)$$

$$\phi(x,t) = -\frac{(2q_m + q_u)}{4}\left(\frac{3}{Q_q}\right)^{1/2} \tan^{-1}\left(\left(\frac{r_2}{r_1}\right)^{1/2} \tan(\chi)\right) + \frac{c}{2}(x - bt) + \nu$$

$$\chi = \left(\frac{r_1 r_2 Q_q}{3}\right)^{1/2} (x-ct) + \mu.$$

We note that, in contrast to the cubic NLS, the GNLS can have solitary wave solutions for both positive and negative values of Q_q.

Assume $q_u \neq 0$. Then the GNLS is invariant under the Galilean transformation

$$x^* = A^2(x + 2A^2Bt) \tag{3.4}$$
$$t^* = A^4t$$
$$u^* = \frac{1}{A}u(x,t)\exp\, iA^2B(x+A^2Bt)$$

where

$$B = \frac{q_c}{q_u}\left(1-\frac{1}{A^2}\right)$$

and A is an arbitrary nonzero real constant. Then, since

$$iu^*_{t^*} + u^*_{x^*x^*} + q_c\,|u^*|^2u^* + q_q\,|u^*|^4u^* + iq_m\,|u^*|^2_{x^*}u^* + iq_u\,|u^*|^2u^*_{x^*} = 0,$$

we can construct new solutions for the GNLS using the relations (3.4).

We can also use this invariance to reduce the GNLS to an ordinary differential equation. If we think of the solutions $u(x,t)$ as defining a manifold (u,x,t) on which the GNLS is satisfied, then (u^*,x^*,t^*) must also lie on this manifold, and $u^*=u(x^*,t^*)$

$$\frac{1}{A}u\,\exp\, iA^2B(x+A^2Bt) = u(A^2(x+2A^2Bt),A^4t). \tag{3.5}$$

Differentiating Eq. (3.4) with respect to A and then setting $A=1$, and hence $B=0$, we get a first order linear partial differential equation for u

$$\left(2\,i\,\beta\,x - 1\right)u = 2\,(x+2\,\beta\,t)\,u_x + 4\,t\,u_t \tag{3.6}$$

where we have, for convenience, defined $\beta=q_c/q_u$. The characteristic equations for (3.5) are

$$\frac{du}{(2\,i\,\beta\,x - 1)\,u} = \frac{dx}{2\,(x+\beta\,t)} = \frac{dt}{4\,t}$$

which may be solved to yield the two invariants of the transformation

$$\eta = t^{-1/2}(x - 2\,\beta\,t) \tag{3.7}$$

$$v = t^{1/4}u\,\exp\, -i\beta\,(x-\beta\,t). \tag{3.8}$$

Therefore the general solution of (3.5) has the form

$$\Gamma(v,\eta) = 0$$

where $\Gamma(\cdot,\cdot)$ is an arbitrary function. If we define the functional dependence as $v=v(\eta)$, we see that the necessary and sufficient form for $u(x,t)$ to be invariant under the Galilean transformation is given by

$$u(x,t) = t^{-1/4} v\left(t^{-1/2}(x-2\beta t)\right) \exp\, i\,\beta(x-\beta\, t) \tag{3.9}$$

where $v(\eta)$ must satisfy the GNLS when reduced by (3.9) to the ordinary differential equation

$$-iv - 2i\eta v' + 4v'' + 4q_q\,|v|^4 v + 4iq_m(\,|v|^2)'v + 4iq_u\,|v|^2 v' = 0. \tag{3.10}$$

The general solution of (3.10) will give, according to (3.9), the general solution of the GNLS invariant under the Galilean transformation (3.4).

We now introduce a change of variables and let

$$\tilde{x} = x\ ;\ \tilde{t} = t_0 - t\ ;\ \tilde{u}(\tilde{x},\tilde{t}) = u(x,t) \tag{3.11}$$

where t_0 is a real constant. Under (3.11), the resulting equation

$$-i\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}\tilde{x}} + q_c\,|\tilde{u}|^2\tilde{u} + q_q\,|\tilde{u}|^4\tilde{u} + iq_m\,|\tilde{u}|^2_{\tilde{x}}\tilde{u} + iq_u\,|\tilde{u}|^2\tilde{u}_{\tilde{x}} = 0 \tag{3.12}$$

is invariant under the Galilean transformation

$$\begin{aligned} x^* &= A^2(\tilde{x} - 2A^2B\tilde{t}) \\ t^* &= A^4\tilde{t} \\ u^* &= \frac{1}{A}\tilde{u}(\tilde{x},\tilde{t}) \exp\, iA^2B(\tilde{x} - A^2B\tilde{t}) \end{aligned} \tag{3.13}$$

where A and B are defined as before. The invariants for the transformation (3.13) are now found to be

$$\tilde{\eta} = \tilde{t}^{-1/2}(\tilde{x} + 2\beta\tilde{t}) \tag{3.14}$$

$$\tilde{v} = \tilde{t}^{-1/4}\,\tilde{u}\, \exp -i\beta(\tilde{x}+\beta\tilde{t}). \tag{3.15}$$

and Eq. (3.12) reduces, if we use the functional dependence $\tilde{v} = \tilde{v}(\tilde{\eta})$, to the ODE

$$i\tilde{v} + 2i\tilde{\eta}\tilde{v}' + 4\tilde{v}'' + 4q_q\,|\tilde{v}|^4\tilde{v} + 4iq_m(\,|\tilde{v}|^2)'\tilde{v} + 4iq_u\,|\tilde{v}|^2\tilde{v}' = 0. \tag{3.16}$$

In terms of the original coordinates, we find that the necessary and sufficient form of $u(x,t)$ corresponding to this Galilean invariance is

$$u(x,t) = (t_0-t)^{-1/4}\tilde{v}\left((t^0-t)^{-1/2}(x+2\beta(t_0-t))\right) \exp\, i\beta(x+\beta(t_0-t)) \tag{3.17}$$

which is valid for $t<t_0$ and for $\tilde{v}$ a solution of (3.16).

4. Conservation Laws

Consider the GNLS transformed to (2.6) and assume that $\psi(x,t)$ is a rapidly decreasing function of x, $p(x)D_x\psi{=}0$ as $x\to\pm\infty$ for any polynomial $p(x)$. We have the three conservation laws

$$\int_{-\infty}^{\infty} |\psi|^2\,dx = E_0 \tag{4.1}$$

$$\int_{-\infty}^{\infty} [4\,\mathrm{Im}(\psi\overline{\psi}_x) + Q_u\,|\psi|^4]\,dx = E_1 \tag{4.2}$$

$$\int_{-\infty}^{\infty} [\,|\psi_x|^2 - \frac{1}{2}Q_c\,|\psi|^4 - \frac{1}{3}Q_q\,|\psi|^6]\,dx = E_2. \tag{4.3}$$

We note that these laws are also valid if ψ is periodic in x; in this case, integration is performed over the spatial period. For the purpose at hand, however, we consider only the case of rapidly decreasing functions on $\infty< x <\infty$; henceforth, integration with respect to x is performed over the real line, unless otherwise indicated.

Proof of (4.1)

Multiplying (2.6) by $\overline{\psi}$ and subtracting the complex conjugate of the resulting equation,

$$|\psi|^2_{\,t} = 2[\mathrm{Im}(\psi\overline{\psi}_x)]_x \tag{4.4}$$

which, when integrated over the real line, observing the vanishing boundary conditions, yields (4.1).

Proof of (4.2)

Multiplying (2.6) by $\overline{\psi}$, adding the complex conjugate of the resulting equation, and differentiating the sum with respect to x,

$$i[\overline{\psi}_x\psi_t+\overline{\psi}\psi_{xt}-\psi_x\overline{\psi}_t-\psi\overline{\psi}_{xt}] + [\overline{\psi}\psi_{xxx}+\psi\overline{\psi}_{xxx}] + |\psi_x|^2_{\,x} + 2[F(\psi)]_x\,|\psi|^2 + 2F(\psi)\,|\psi|^2_{\,x} = 0. \tag{4.5}$$

Now, multiplying (2.6) by $2\overline{\psi}_x$, and adding the complex conjugate of the resulting equation, one obtains

$$2i[\overline{\psi}_x\psi_t-\psi_x\overline{\psi}_t] + 2\,|\psi_x|^2_{\,x} + 2F(\psi)\,|\psi|^2_{\,x} = 0$$

which when subtracted from (4.5) gives

$$[\mathrm{Im}(\psi\overline{\psi}_x)]_{\,t} = -\frac{1}{2}[\overline{\psi}\psi_{xxx}+\psi\overline{\psi}_{xxx}] + \frac{1}{2}|\psi_x|^2_{\,x} - [F(\psi)]_x\,|\psi|^2. \tag{4.6}$$

Integration over the real line, observing the vanishing boundary conditions, yields

$$\frac{d}{dt}\int \mathrm{Im}(\psi\overline{\psi}_x)dx = -Q_u\int |\psi|^2[\mathrm{Im}(\psi\overline{\psi}_x)]_x dx. \tag{4.7}$$

Multiplying (4.4) by $|\psi|^2$, we find

$$|\psi|^4{}_t = 4|\psi|^2[\mathrm{Im}(\psi\overline{\psi}_x)]_x$$

which, together with (4.7), gives the conservation law (4.2).

Proof of (4.3)

Multiplying the GNLS (2.3) by $\overline{\psi}_t$, adding its complex conjugate and integrating with respect to x,

$$\int(\overline{\psi}_t\psi_{xx}+\psi_t\overline{\psi}_{xx})dx = \int[Q_c|\psi|^2|\psi|^2{}_t+Q_q|\psi|^4|\psi|^2{}_t+Q_u\mathrm{Im}(\psi\overline{\psi}_x)|\psi|^2{}_t]\,dx = 0$$

which may be simplified to

$$\begin{aligned}\frac{d}{dt}\left[\int\left(|\psi_x|^2-\frac{1}{3}Q_q|\psi|^6-\frac{1}{2}Q_c|\psi|^4\right)dx\right] &= Q_u\int|\psi|^2{}_t\mathrm{Im}(\psi\overline{\psi}_x)dx\\ &= 2Q_u\int\mathrm{Im}(\psi\overline{\psi}_x)[\mathrm{Im}(\psi\overline{\psi}_x)]_x dx\\ &= 0\end{aligned}$$

using (4.4) and in view of the vanishing boundary conditions. □

In terms of our original equation (1.1) the conservation laws are

$$\int_{-\infty}^{\infty}|u|^2dx = E_0$$

$$\int_{-\infty}^{\infty}[2\,\mathrm{Im}(u\overline{u}_x) - q_m|u|^4]\,dx = E_1$$

$$\int_{-\infty}^{\infty}\left(|u_x|^2 - \frac{1}{2}(2q_m+q_u)|u|^2\mathrm{Im}(u\overline{u}_x) - \frac{1}{2}q_c|u|^4 + \frac{1}{6}[q_m(2q_m+q_u)-2q_q]|u|^6\right)dx = E_2.$$

(Here, E_1 is half that of (4.2)).

5. Boundedness and Blow-up of Solutions

It is well known [4], [17] that solutions of the nonlinear Schrödinger equation with a power nonlinearity

$$iu_t + u_{xx} + q\,|u|^p u = 0,$$

evolving from a smooth initial condition, are bounded for all time if $p=2$, but may blow up in finite time if $p \geq 4$. We can determine sufficient conditions for boundedness and blow-up of solutions of the transformed GNLS (2.6), which may be tested *a priori* given the values of Q_c, Q_q, Q_m, and Q_u and the initial condition $\psi_0(x)=\psi(x,0)$. From these we can then obtain conditions for the boundedness and blow-up of solutions to the GNLS (1.1).

In what follows, we assume that $\psi(x,t)$ is a rapidly decreasing function of x, and that ψ, while it exists, is sufficiently smooth so that all relevant integrals are well defined. We let $I(t)$ denote $\int |\psi_x|^2 dx$ for convenience.

Theorem 1

Let $M=\min(4E_0^2, 8|Q_u|^{-1}E_0)$. Then the solution to the transformed GNLS (2.6) will remain bounded if $MQ_q<3$.

Proof of Theorem 1 :

Since $\psi^2 = 2\int_{-\infty}^{x} \psi\psi_x dx$

$$|\psi|^2 \leq 2\int_{-\infty}^{\infty} |\psi||\psi|_x dx \leq 2\left(\int |\psi|^2 dx\right)^{1/2}\left(\int |\psi_x|^2 dx\right)^{1/2} = 2E_0^{1/2}I^{1/2} \qquad (5.1)$$

using Cauchy's inequality. Then

$$\int |\psi|^4 dx \leq 2E_0^{1/2}I^{1/2}\int |\psi|^2 dx \leq 2E_0^{3/2}I^{1/2} \qquad (5.2)$$

$$\int |\psi|^6 dx \leq 2E_0^{1/2}I^{1/2}\int |\psi|^4 dx \leq 4E_0^2 I. \qquad (5.3)$$

From the second conservation law (4.2) we have (assuming $Q_u \neq 0$)

$$\int |\psi|^4 dx \leq 4|Q_u|^{-1}E_0^{1/2}I^{1/2} + |Q_u^{-1}E_1| \qquad (5.4)$$

and hence

$$\int |\psi|^6 dx \leq 2E_0^{1/2}I^{1/2}\int |\psi|^4 dx \leq 8|Q_u|^{-1}E_0 I + 2|Q_u^{-1}E_1|E_0^{1/2}I^{1/2}. \qquad (5.5)$$

We can use the above inequalities, together with the conservation laws, to find sufficient conditions for the boundedness of I. This boundedness also implies the boundedness of $||\psi||_\infty$ by (5.1).

From the conservation law (4.3)

$$I - \frac{1}{3}Q_q\int |\psi|^6 dx = E_2 + \frac{1}{2}Q_c\int |\psi|^4 dx$$
$$\leq E_2 + |Q_c| E_0^{3/2} I^{1/2}$$

by (5.2). If $Q_q < 0$ then $I \leq E_2 + |Q_c| E_0^{3/2} I^{1/2}$, which is impossible if I is unbounded. If $Q_q > 0$ then from (5.3)

$$\left(1 - \frac{4}{3}Q_q E_0^2\right) I \leq E_2 + \frac{1}{2}|Q_c| E_0^{3/2} I^{1/2}$$

and I must be bounded if $1 - \frac{4}{3}E_0^2 Q_q > 0$. Similarly, from (5.5),

$$\left(1 - \frac{8}{3|Q_u|}Q_q E_0\right) I \leq E_2 + \left(|Q_c| E_0 + \frac{2|E_1| Q_q}{3|Q_u|}\right) E_0^{1/2} I^{1/2}$$

which again implies that I is bounded if $1 - \frac{8}{3|Q_u|}Q_q E_0 > 0$. So I, and hence $||\psi||_\infty$ is bounded if either $4Q_q E_0^2 < 3$ or $8Q_q E_0 < 3|Q_u|$. □

Corollary 1

Assume $u(x,t)$ is a smooth and rapidly decreasing function of x. Let $M = \min(4E_0^2, 8|q_u|^{-1}E_0)$. Then the solution to the GNLS (1.1) will remain bounded if

$$M\left(16q_q - (2q_m + q_u)(2q_m - 3q_u)\right) \leq 48. \qquad \square$$

Proof of Corollary 1 :

Replace the coefficients for the transformed system (2.3) with the corresponding ones of the original system (1.1), and the result follows immediately since $|u| = |\psi|$. □

The condition for boundedness is automatically satisfied if $Q_q \leq 0$ or, in terms of the original parameters, if $16q_q \leq (2q_m + q_u)(2q_m - 3q_u)$. Otherwise it imposes a smallness condition on the L^2 energy of the system. The presence of the nonlinear derivative term increases the boundedness space; if this is absent we can find sufficient conditions for blow-up of the solution.

Lemma 1

Define $y(t) = \int x\,[\mathrm{Im}(\psi\bar{\psi}_x) + \frac{1}{4}Q_u|\psi|^4]\,dx$. Then $\dot{y}(t) = -2E_2 - \frac{1}{2}Q_c\int |\psi|^4 dx$. □

Proof of Lemma 1 :

From (4.6) we have

$$[\mathrm{Im}(\psi\overline{\psi}_x)]_t = -\frac{1}{2}[\overline{\psi}\psi_{xxx} + \psi\overline{\psi}_{xxx}] + \frac{1}{2}|\psi_x|^2_x - [F(\psi)]_x |\psi|^2.$$

Multiplying by x and integrating over the real line,

$$\frac{d}{dt}\int x\,\mathrm{Im}(\psi\overline{\psi}_x)dx = -\frac{1}{2}\int x[\overline{\psi}\psi_{xxx}+\psi\overline{\psi}_{xxx}]dx + \frac{1}{2}\int x\,|\psi_x|^2_x dx - \int x[F(\psi)]_x |\psi|^2 dx.$$

Integrating each term on the right hand side by parts, we have

$$\int x(\overline{\psi}\psi_{xxx}+\psi\overline{\psi}_{xxx})dx = 3\int |\psi_x|^2 dx$$

$$\int x\,|\psi_x|^2_x dx = -\int |\psi_x|^2 dx$$

$$\int x\,|\psi|^2[F(\psi)]_x dx = -\frac{1}{2}Q_c\int |\psi|^4 dx - \frac{2}{3}Q_q\int |\psi|^6 dx + \frac{1}{4}Q_u\int x\,|\psi|^4_t dx$$

so

$$\frac{d}{dt}\int x[\mathrm{Im}(\psi\overline{\psi}_x) + \frac{1}{4}Q_u|\psi|^4]dx = -2\int |\psi_x|^2 dx + \frac{2}{3}Q_q\int |\psi|^6 dx + \frac{1}{2}Q_c\int |\psi|^4 dx$$

$$= -2E_2 - \frac{1}{2}Q_c\int |\psi|^4 dx. \qquad \square$$

<u>*Theorem 2*</u>

Assume $\psi(x,t)$ is a rapidly decreasing function of x, and assume ψ satisfies

$$i\psi_t + \psi_{xx} + Q_c|\psi|^2\psi + Q_q|\psi|^4\psi = 0$$

with smooth initial condition $\psi_0(x)$. Then ψ will blow up in time T where $0 \le T \le T^*$, for some finite time T, if $E_2 < 0$ and $Q_c \le 0$. Furthermore,

$$T^* \le \frac{1}{2E_2}\left[\int x\,\mathrm{Im}(\psi_0\overline{\psi}_{0,x})dx - \left([\int x\,\mathrm{Im}(\psi_0\overline{\psi}_{0,x})dx]^2 - E_2\int x^2|\psi_0|^2 dx\right)^{1/2}\right]. \qquad \square$$

Proof of Theorem 2

We first note that the conditions $E_2 < 0$ and $Q_c \le 0$ imply that $\frac{4}{3}Q_q E_0{}^2 > 1$. From the conservation law (4.3)

$$E_2 = I - \frac{1}{3}Q_q\int |\psi|^6 dx - \frac{1}{2}\int |\psi|^4 dx$$

$$I - \frac{1}{3}Q_q\int |\psi|^6 dx = E_2 + \frac{1}{2}Q_c\int |\psi|^4 dx < 0$$

Since $\int |\psi|^6 dx \leq 4E_0{}^2 I$,

$$\left(1 - \frac{4}{3}Q_q E_0{}^2\right) I < 0$$

so $\frac{4}{3}Q_q E_0{}^2 > 1$.

Now

$$\begin{aligned}\frac{d}{dt}\int x^2|\psi|^2 dx &= \int x^2 |\psi|^2{}_t\, dx \\ &= 2\int x^2[\mathrm{Im}(\psi\bar{\psi}_x)]_x\, dx \\ &= -4\int x\,\mathrm{Im}(\psi\bar{\psi}_x) dx \\ &= -4y(t).\end{aligned}$$

Since $\dot{y}(t) = -2E_2 - \frac{1}{2}Q_c\int |\psi|^4 dx > -2E_2$, $y(t)$ grows at least linearly and there exists some finite time T such that

$$\lim_{t\to T}\int x^2|\psi|^2 dx = 0$$

where $0 \leq T \leq T^*$ and

$$T^* \leq \frac{1}{2E_2}\left[\int x\,\mathrm{Im}(\psi_0\bar{\psi}_{0,x})dx - \left([\int x\,\mathrm{Im}(\psi_0\bar{\psi}_{0,x})dx]^2 - E_2\int x^2|\psi_0|^2 dx\right)^{1/2}\right].$$

By Weyl's inequality,

$$\left(\int |\psi|^2 dx\right)^2 \leq 4\left(\int x^2|\psi|^2 dx\right)\left(\int |\psi_x|^2 dx\right)$$

so

$$I \geq \frac{E_0{}^2}{4\int x^2|\psi|^2 dx}.$$

Hence $\lim_{t\to T} I = \infty$.

This implies that $||\psi||_\infty$ blows up in finite time. From the third conservation law, since $E_2 < 0$ and $Q_c \leq 0$,

$$\frac{1}{3}Q_q\int |\psi|^6 dx \geq I$$

so $\lim\limits_{t\to T}\int |\psi|^6 dx = \infty$. Since

$$\int |\psi|^6 dx \leq ||\psi||_\infty^4 \int |u|^2 dx = ||\psi||_\infty^4 E_0$$

it follows that $\lim\limits_{t\to T} ||\psi||_\infty = \infty$.

□

Corollary 2

Let $u(x,t)$ be a solution of the GNLS with $u(x,0)=u_0(x)$. Then, if $q_u=0$, u will blow up in finite time T, where $0<T<T^*$, if $E_2<0$ and $q_c \leq 0$. Furthermore

$$T^* \leq \frac{1}{2E_2}\left(\int x\left(\mathrm{Im}(u_0\overline{u}_{0,x}) - \frac{1}{4}(2q_m+q_u)|u_0|^4\right)dx\right.$$
$$\left. - \left(\left(\int x\left(\mathrm{Im}(u_0\overline{u}_{0,x}) - \frac{1}{4}(2q_m+q_u)|u_0|^4\right)dx\right)^2 - E_2\int x^2|u_0|^2 dx\right)^{1/2}\right).$$

□

Proof of Corollary 2 :

The result follows immediately if we replace ψ by u in the blow-up theorem. □

The conditions for boundedness and blow-up agree with the intuitive notion that if the quintic nonlinearity dominates the dispersion, the solutions will experience explosive growth. Physically of course, the wave amplitude cannot grow indefinitely; the assumptions under which the mathematical equation is derived become invalid as the solution blows up, and new physical processes must be included in the model. Nevertheless, the blow-up according to (1.1) is of physical interest as it explains the formation of "spikes" which have been experimentally observed (in the context of optics, for example, the early stages of blow-up correspond to the self-focussing of laser beams; see [18] for a comprehensive review of blow-up phenomena for nonlinear Schrödinger equations). The form of the similarity solution (3.17) suggests a growth rate

$$||u||_\infty \propto (t_0-t)^{-1/4},$$

the same as has been suggested for the nonlinear Schrödinger equation with a simple quintic nonlinearity [19].

6. Numerical Experiments

We present below some numerical experiments to illustrate the results presented above. The problems were solved using a pseudo-spectral split-step discretization and all experiments were run on a SUN 3/160 computer. The accuracy of the computed solutions was checked by monitoring the change in the theoretically conserved quantities E_0, E_1, and E_2.

(i) The integrable GNLS equation

$$iu_t + u_{xx} + |u|^2 u + |u|^4 u - 2|u|^2_x u = 0.$$

is solved for the initial condition

$$u(x,0) = \frac{1}{\sqrt{2}}\operatorname{sech}\left(\frac{1}{2}(x-15)\right)\exp i\left(\frac{1}{2}(x-15)+\tanh\left(\frac{1}{4}(x-15)\right)\right)$$

$$+\ \frac{1}{2}\operatorname{sech}\left(\frac{1}{2\sqrt{2}}(x-35)\right)\exp i\left(-\frac{1}{4}(x-35)+\tanh\left(\frac{1}{2\sqrt{2}}(x-35)\right)\right)$$

which corresponds to two initially well-spaced solitons of the form (2.3), the one initially on the right moving left with unit speed, the one on the left moving right with speed $1/2$. Theoretically, the two solitary waves should emerge from their interaction with their shapes and velocities unchanged, although they may be displaced from the position they would have occupied had the collision not occurred. The elastic collision of the waves (Figure 1) nicely exhibits the stability of the solitons.

(ii) The computed solitary wave solution for the choice of coefficients

$$iu_t + u_{xx} - \frac{1}{2}|u|^2 u - \frac{7}{4}|u|^4 u - |u|^2_x u - 2|u|^2 u_x = 0$$

has the form

$$u(x,t) = \left(\frac{4}{4+3\sinh^2(x-2t-15)}\right)^{1/2}\exp i\phi(x,t)$$

$$\phi(x,t) = 2\tanh^{-1}\left(\frac{1}{2}\tanh(x-2t-15)\right) + x - 15.$$

For the initial condition $u_0(x)$ corresponding to the above, the numerical output (Figure 2) agrees with the predicted solution of a bell-shaped solitary wave propagating to the right with speed 2.

(iii) The derived conditions for boundedness and blow-up of solutions are supported by numerical experiments, the results of which are presented in Figures 3 and 4. The initial condition for

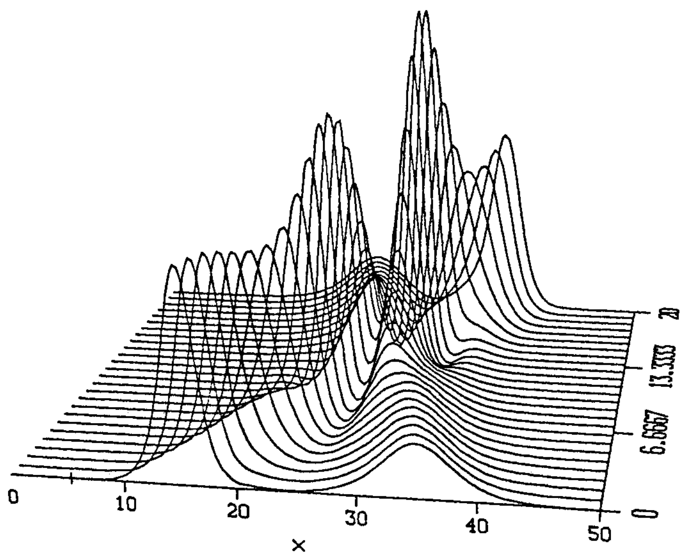

Figure 1 : Soliton interaction for $iu_t + u_{xx} + |u|^2u + |u|^4u - 2|u|^2_x u = 0.$

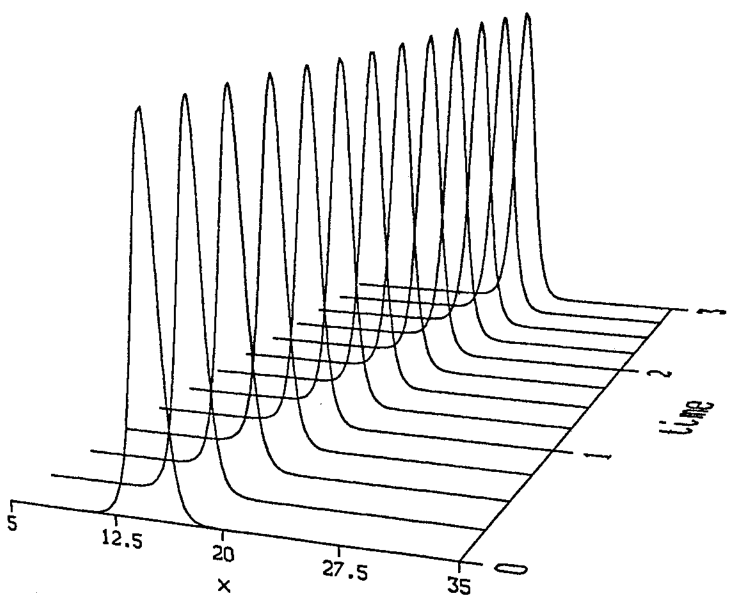

Figure 2 : Solitary wave solution for $iu_t + u_{xx} - \frac{1}{2}|u|^2u - \frac{7}{4}|u|^4u - |u|^2_x u - 2|u|^2u_x = 0.$

both cases was the Gaussian function $u(x,0) = e^{-x^2}$.

In the first test case, with the choice of parameters $q_c = -2$, $q_q = -1$, and $q_m = q_u = 0$, the solution remains bounded (Figure 3) in accordance with Theorem 1. In the second case, the parameters were chosen to be $q_c = -2$, $q_q = 20$, and $q_m = q_u = 0$, so that $E_2 = -2.6844673$ and the predicted time of blow-up, according to Theorem 2, was $T^* \approx 1.7$. This bound appears to be quite loose, as the impending blow-up is well under way (Figure 4) by time $t = 0.07$.

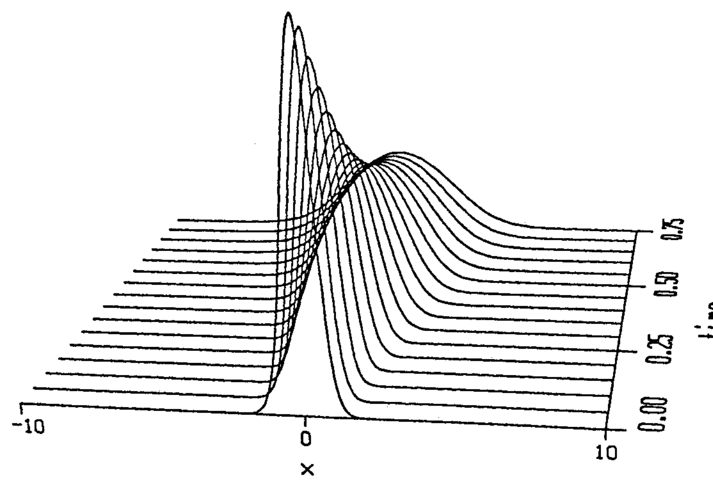

Figure 3 : Decay for $iu_t + u_{xx} - 2|u|^2u - |u|^4u = 0$.

The $(t_0 - t)^{-1/4}$ growth rate suggested by the the similarity solution seems to hold even for this problem (Figure 5) during the early stages of the blow-up, $0.035 \leq t \leq 0.055$.

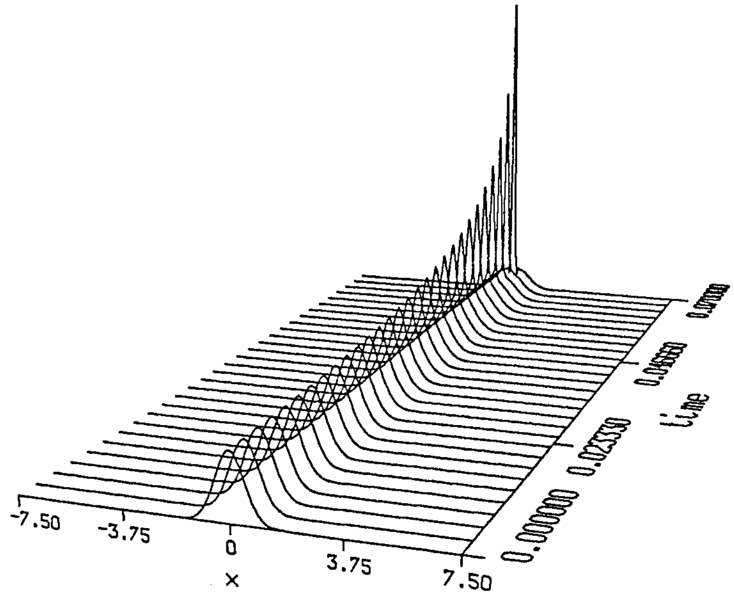

Figure 4 : Blow-up for $iu_t + u_{xx} - 2|u|^2u + 20|u|^4u = 0.$

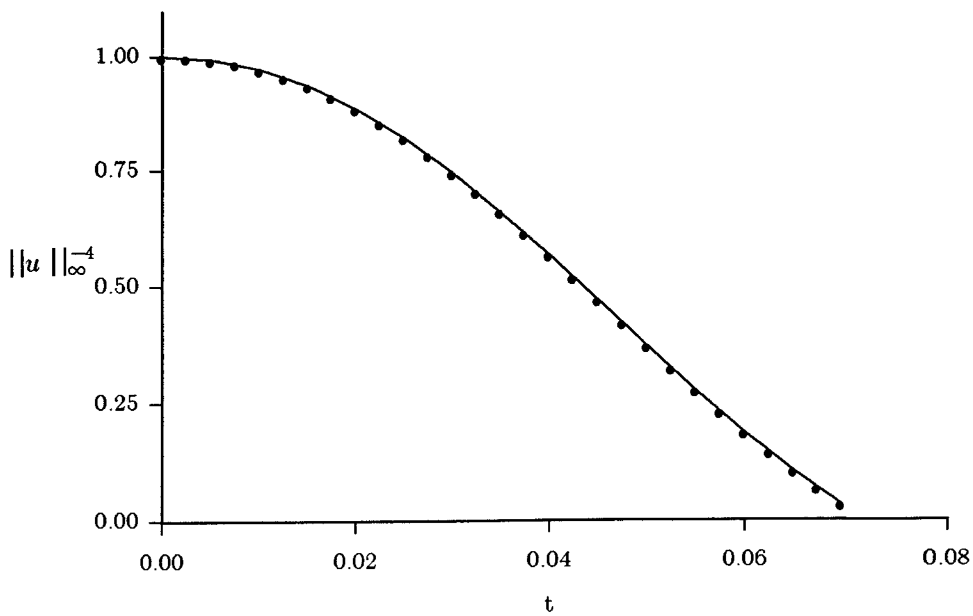

Figure 5 : Spike evolution for $iu_t + u_{xx} - 2|u|^2u + 20|u|^4u = 0.$

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7. References

1. Johnson, R.S., "On the modulation of water waves in the neighbourhood of $kh \approx 1.363$," *Proc. R. Soc. London A*, **357** pp. 131-141 (1977).

2. Kakutani, T. and Michihiro, K., "Marginal State of Modulational Instability - Note on Benjamin- Feir Instability," *J. Phys. Soc. Japan*, **52**(12) pp. 4129-4137 (December 1983).

3. Calogero, F. and Eckhaus, W., "Nonlinear evolution equations, rescalings, model PDEs and their integrability: I," *Inverse Problems*, **3** pp. 229-262 (1987).

4. Strauss, W.A., "The Nonlinear Schrödinger Equation," pp. 452-465 in *Contemporary Developments in Continuum Mechanics*, ed. L. A. Medeiros, North-Holland Publishing Company (1978).

5. Zakharov, V.E. and Shabat, A.B., "Exact Theory of Two-Dimensional Self-Focusing and One- Dimensional Self-Modulation of Waves in Nonlinear Media," *Soviet Physics JETP*, **34**(1) pp. 62-29 (January 1972).

6. Hasimoto, H. and Ono, H., "Nonlinear Modulation of Gravity Waves," *J. Phys. Soc. Japan*, **33**(3) pp. 805-811 (September 1972).

7. Kaup, D. and Newell, A.C., "An exact solution for a derivative nonlinear Schrödinger equation[a)]," *J. Math. Phys.*, **19**(4) pp. 798-801 (April 1978).

8. Tanaka, M., "Nonlinear Self-Modulation Problem of the Benjamin-Ono Equation," *J. Phys. Soc. Japan*, **51**(8) pp. 2686-2692 (August 1982).

9. Pushkarov, K. I., Pushkarov, D. I., and Tomov, I. V., "Self-action of light beams in nonlinear media: soliton solutions," *Opt. Quantum Electron.*, **11** pp. 471-478 (1979).

10. Barashenkov, I.V. and Makhankov, V.G., "Soliton-like "Bubbles" in a System of Interacting Bosons," *Physics Letters A*, **128**(1,2) pp. 52-55 (March 1988).

11. Calogero, F. and Degasperis, A., *Spectral Transform and Solitons I,* North-Holland, New York (1982).

12. Cowan, S., Enns, R.H., Rangnekar, S.S., and Sanghera, S.S., "Quasi-soliton and other behaviour of the nonlinear cubic-quintic Schrödinger equation," *Can. J. Phys.*, **64** pp. 311-315 (1986).

13. Calogero, F. and Eckhaus, W., "Necessary Conditions for integrability of nonlinear PDEs," *Inverse Problems*, **3** pp. L27-L32 (1987).

14. Kundu, A., "Exact solutions to higher order nonlinear equations through gauge transformation," *Physica D*, **25** pp. 399-406 (1987).

15. Wadati, M., Konno, K., and Ichikawa, Y., "A Generalization of Inverse Scattering Method," *J. Phys. Soc. Japan*, **46**(6) pp. 1965-1966 (June 1979).

16. Eichhorn, H., "Bäcklund Transformation and N-Soliton Solution for the Generalized Nonlinear Schrödinger Equation," *Annalen der Physik,* **44**(4) pp. 261-264 (1987).

17. Glassey, R.T., "On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations," *J. Math. Phys.,* **18**(9) pp. 1794-1797 (September 1977).

18. Rasmussen, J. J. and Rypdal, K., "Blowup in Nonlinear Schroedinger Equations - I. A General Review," *Physica Scripta,* **33** pp. 481-497 (1986).

19. Rypdal, K. and Rasmussen, J. J., "Blow-up in Nonlinear Schroedinger Equations-II. Similarity structure of the blow-up singularity," *Physica Scripta,* **33** pp. 489-504 (1986).

Exact Solutions for a Generalized Nonlinear Schrödinger Equation

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ABSTRACT

Exact solutions of a generalized nonlinear Schrödinger equation

$$iu_t + u_{xx} + q_c |u|^2 u + q_q |u|^4 u + iq_m |u|^2_x u + iq_u |u|^2 u_x = 0$$

are obtained by transforming the equation to one whose coefficients satisfy a simple algebraic relationship. The transformed equation also yields conservation laws and sufficient conditions for boundedness and blow-up of solutions for the original equation.

1. Introduction

A generalized nonlinear Schrödinger equation (GNLS)

$$iu_t + u_{xx} + q_c |u|^2 u + q_q |u|^4 u + iq_m |u|^2_x u + iq_u |u|^2 u_x = 0 \tag{1.1}$$

governs the modulation of a quasi-monochromatic wave train in a weakly nonlinear, dispersive medium. Here $u(x,t)$ is the complex wave amplitude, $i^2=-1$, and the parameters q_c, q_q, q_m, and q_u are real constants. We are considering the initial value problem for (1.1) with $u(x,0)=u_0(x)$ specified. Equation (1.1) was derived independently by Johnson [1] and Kakutani & Michihiro [2] to describe the behaviour of the Stokes wave near the state of modulational instability. (Although Johnson's equation includes a term in $u\varsigma_t$ where $\varsigma_x = |u|^2$, this may be eliminated as shown in [2] .) The GNLS also has wide applicability as a model equation for a large class of evolutionary systems where the relevant time and space scales are greater than those captured by the usual nonlinear Schrödinger equation with a cubic nonlinearity [3]. It contains, as special cases, the nonlinear Schrödinger equations

$$iu_t + u_{xx} + q_c |u|^2 u = 0 \tag{1.2}$$

$$\theta_x = 2\delta\,|\psi|^2$$

$$\theta_t = \delta\,[4\mathrm{Im}(\psi\bar{\psi}_x) - (2q_m + q_u + 8\delta)\,|\psi|^4].$$

(Here $\bar{\psi}$ denotes the complex conjugate of ψ.) Under (2.1), the GNLS transforms to a new GNLS for $\psi(x,t)$

$$i\psi_t + \psi_{xx} + Q_c\,|\psi|^2\psi + Q_q\,|\psi|^4\psi + iQ_m\,|\psi|^2{}_x\psi + iQ_u\,|\psi|^2\psi_x = 0 \tag{2.2}$$

with

$$Q_c = q_c$$

$$Q_q = q_q + 2\delta\,q_m - \delta\,q_u + 4\delta^2$$

$$Q_m = q_m + 4\delta$$

$$Q_u = q_u\,.$$

The transformation parameter, δ, is an arbitrary constant and introduces one degree of freedom in the choice of Q_q and Q_m. We note that the compatibility condition $\theta_{xt}=\theta_{tx}$ is satisfied by the field equation for ψ, and θ is therefore well defined by the theory of conservative vector fields. The transformation preserves the modulus of the solution, $|\psi| = |u|$, and works both ways.

The above transformation is a generalization of that used by Calogero & Eckhaus [13] and Kundu [14] (their analyses correspond to the choice $\delta=(q_u-q_m)/4$) to transform the GNLS (1.1), subject to the constraint $4q_q = q_m{}^2 - q_m q_u$, to the mixed nonlinear Schrödinger equation, which is completely integrable [15] and has N-soliton solutions [16]. Solutions for those GNLS equations which satisfy the integrability condition are then constructed from known solutions for the mixed NLS equation. For example, if $q_u = 0$, so that $4q_q = q_m{}^2$, and if $q_c > 0$, a solitary wave solution corresponding to the one-hump soliton of the cubic NLS is given explicitly by

$$u(x,t) = \left(\frac{2\alpha}{q_c}\right)^{1/2} \mathrm{sech}(\sqrt{\alpha}(x-ct)+\mu)\exp\, i\phi(x,t) \tag{2.3}$$

$$\phi(x,t) = \frac{c}{2}(x-bt) - \frac{\sqrt{\alpha}q_m}{q_c}\tanh(\sqrt{\alpha}(x-ct)+\mu) + \nu$$

$$\alpha = \frac{c}{2}\left(\frac{c}{2} - b\right) > 0$$

where b, c, μ, and ν are arbitrary constants. We note that the GNLS solution differs from the cubic Schrödinger soliton only by the peculiarities of its phase and that the interaction of the solitary waves (2.3) should exhibit the same clean interactions as the cubic Schrödinger solitons.

$$\theta(x,t) = 2\delta \int f^2(x-ct)d(x-ct) = -\frac{1}{4}(2q_m + q_u)\int f^2(x-ct)d(x-ct).$$

Solutions to (1.1) are then readily obtained, using the relation (2.1), once $\psi(x,t)$ is determined. Substituting (3.1) into the transformed GNLS (2.6), we get a cnoidal wave type equation for $f(x-ct)$, which can be integrated up once to obtain

$$(f')^2 + \frac{Q_q}{3}f^6 + \frac{1}{2}\left(Q_c - \frac{c}{2}Q_u\right)f^4 + \frac{c}{2}\left(b - \frac{c}{2}\right)f^2 + C = 0$$

where C is an arbitrary constant. Defining $z = f^2$, z then satisfies

$$(z')^2 + \frac{4Q_q}{3}z^4 + (2Q_c - cQ_u)z^3 + (2bc - c^2)z^2 + 4Cz = 0 \tag{3.2}$$

and may be solved for in terms of the elliptic functions. If $C=0$, these solutions can be expressed in terms of the elementary functions and include oscillatory, singular, phase jump, and solitary wave solutions. These are four-parameter families of solutions, with arbitrary constants b, c, ν, and μ. The parameters b and c, representing, respectively, the speeds of the carrier and envelope waves of ψ, partly determine the form of the solution. We list below some interesting cases.

$\mathbf{Q_q < 0}$: Defining

$$\alpha = -\frac{4Q_q}{3}$$

$$\beta = -\frac{3}{4Q_q}(cQ_u - 2Q_c)$$

$$\gamma = -\frac{3}{4Q_q}(c^2 - 2bc)$$

equation (3.2) becomes

$$(z')^2 = \alpha z^2(z^2 + \beta z + \gamma) = \alpha z^2(z - r_1)(z - r_2).$$

The values of r_1 and r_2 determine the form of the solution z; solitary waves arise if r_1 and r_2 are real and $r_1 > r_2 > 0$. The solution to (1.1) is then

$$u(x,t) = \left(\frac{r_1 r_2}{r_1 + (r_1 - r_2)\sinh^2(\chi)}\right)^{1/2} \exp\, i\phi(x,t) \tag{3.3}$$

$$\phi(x,t) = -\frac{(2q_m + q_u)}{4}\left(-\frac{3}{Q_q}\right)^{1/2} \tanh^{-1}\left(\left(\frac{r_2}{r_1}\right)^{1/2} \tanh(\chi)\right) + \frac{c}{2}(x - bt) + \nu$$

$$x^* = A^2(x + 2A^2Bt) \tag{3.4}$$
$$t^* = A^4t$$
$$u^* = \frac{1}{A}u(x,t)\exp\, iA^2B(x+A^2Bt)$$

where

$$B = \frac{q_c}{q_u}\left(1-\frac{1}{A^2}\right)$$

and A is an arbitrary nonzero real constant. Then, since

$$iu^*_{t^*} + u^*_{x^*x^*} + q_c\,|u^*|^2u^* + q_q\,|u^*|^4u^* + iq_m\,|u^*|^2_{x^*}u^* + iq_u\,|u^*|^2u^*_{x^*} = 0,$$

we can construct new solutions for the GNLS using the relations (3.4).

We can also use this invariance to reduce the GNLS to an ordinary differential equation. If we think of the solutions $u(x,t)$ as defining a manifold (u,x,t) on which the GNLS is satisfied, then (u^*,x^*,t^*) must also lie on this manifold, and $u^*=u(x^*,t^*)$

$$\frac{1}{A}u\,\exp\, iA^2B(x+A^2Bt) = u(A^2(x+2A^2Bt), A^4t). \tag{3.5}$$

Differentiating Eq. (3.4) with respect to A and then setting A=1, and hence B=0, we get a first order linear partial differential equation for u

$$\left(2\,i\,\beta\,x - 1\right)u = 2\,(x+2\,\beta\,t)\,u_x + 4\,t\,u_t \tag{3.6}$$

where we have, for convenience, defined $\beta = q_c/q_u$. The characteristic equations for (3.5) are

$$\frac{du}{(2\,i\,\beta\,x - 1)\,u} = \frac{dx}{2\,(x+\beta\,t)} = \frac{dt}{4\,t}$$

which may be solved to yield the two invariants of the transformation

$$\eta = t^{-1/2}(x - 2\,\beta\,t) \tag{3.7}$$

$$v = t^{1/4}u\,\exp\, -i\beta\,(x-\beta\,t). \tag{3.8}$$

Therefore the general solution of (3.5) has the form

$$\Gamma(v,\eta) = 0$$

where $\Gamma(\cdot,\cdot)$ is an arbitrary function. If we define the functional dependence as $v=v(\eta)$, we see that the necessary and sufficient form for $u(x,t)$ to be invariant under the Galilean transformation is given by

4. Conservation Laws

Consider the GNLS transformed to (2.6) and assume that $\psi(x,t)$ is a rapidly decreasing function of x, $p(x)D_x\psi=0$ as $x\to\pm\infty$ for any polynomial $p(x)$. We have the three conservation laws

$$\int_{-\infty}^{\infty} |\psi|^2 dx = E_0 \qquad (4.1)$$

$$\int_{-\infty}^{\infty} [4\,\mathrm{Im}(\psi\overline{\psi}_x) + Q_u |\psi|^4]\, dx = E_1 \qquad (4.2)$$

$$\int_{-\infty}^{\infty} [\, |\psi_x|^2 - \frac{1}{2}Q_c |\psi|^4 - \frac{1}{3}Q_q |\psi|^6]\, dx = E_2. \qquad (4.3)$$

We note that these laws are also valid if ψ is periodic in x; in this case, integration is performed over the spatial period. For the purpose at hand, however, we consider only the case of rapidly decreasing functions on $\infty < x < \infty$; henceforth, integration with respect to x is performed over the real line, unless otherwise indicated.

Proof of (4.1)

Multiplying (2.6) by $\overline{\psi}$ and subtracting the complex conjugate of the resulting equation,

$$|\psi|^2_t = 2[\mathrm{Im}(\psi\overline{\psi}_x)]_x \qquad (4.4)$$

which, when integrated over the real line, observing the vanishing boundary conditions, yields (4.1).

Proof of (4.2)

Multiplying (2.6) by $\overline{\psi}$, adding the complex conjugate of the resulting equation, and differentiating the sum with respect to x,

$$i[\overline{\psi}_x\psi_t+\overline{\psi}\psi_{xt}-\psi_x\overline{\psi}_t-\psi\overline{\psi}_{xt}] + [\overline{\psi}\psi_{xxx}+\psi\overline{\psi}_{xxx}] + |\psi_x|^2_x + 2[F(\psi)]_x|\psi|^2 + 2F(\psi)|\psi|^2_x = 0. \quad (4.5)$$

Now, multiplying (2.6) by $2\overline{\psi}_x$, and adding the complex conjugate of the resulting equation, one obtains

$$2i[\overline{\psi}_x\psi_t-\psi_x\overline{\psi}_t] + 2|\psi_x|^2_x + 2F(\psi)|\psi|^2_x = 0$$

which when subtracted from (4.5) gives

$$[\mathrm{Im}(\psi\overline{\psi}_x)]_t = -\frac{1}{2}[\overline{\psi}\psi_{xxx}+\psi\overline{\psi}_{xxx}] + \frac{1}{2}|\psi_x|^2_x - [F(\psi)]_x|\psi|^2. \qquad (4.6)$$

Integration over the real line, observing the vanishing boundary conditions, yields

5. Boundedness and Blow-up of Solutions

It is well known [4], [17] that solutions of the nonlinear Schrödinger equation with a power nonlinearity

$$iu_t + u_{xx} + q\,|u|^p u = 0,$$

evolving from a smooth initial condition, are bounded for all time if $p=2$, but may blow up in finite time if $p \geq 4$. We can determine sufficient conditions for boundedness and blow-up of solutions of the transformed GNLS (2.6), which may be tested *a priori* given the values of Q_c, Q_q, Q_m, and Q_u and the initial condition $\psi_0(x)=\psi(x,0)$. From these we can then obtain conditions for the boundedness and blow-up of solutions to the GNLS (1.1).

In what follows, we assume that $\psi(x,t)$ is a rapidly decreasing function of x, and that ψ, while it exists, is sufficiently smooth so that all relevant integrals are well defined. We let $I(t)$ denote $\int |\psi_x|^2 dx$ for convenience.

Theorem 1

Let $M=\min(4E_0^2, 8\,|Q_u|^{-1}E_0)$. Then the solution to the transformed GNLS (2.6) will remain bounded if $MQ_q<3$.

Proof of Theorem 1 :

Since $\psi^2 = 2\int_{-\infty}^{x} \psi\psi_x dx$

$$|\psi|^2 \leq 2\int_{-\infty}^{\infty} |\psi|\,|\psi|_x dx \leq 2\left(\int |\psi|^2 dx\right)^{1/2}\left(\int |\psi_x|^2 dx\right)^{1/2} = 2E_0^{1/2}I^{1/2} \tag{5.1}$$

using Cauchy's inequality. Then

$$\int |\psi|^4 dx \leq 2E_0^{1/2}I^{1/2}\int |\psi|^2 dx \leq 2E_0^{3/2}I^{1/2} \tag{5.2}$$

$$\int |\psi|^6 dx \leq 2E_0^{1/2}I^{1/2}\int |\psi|^4 dx \leq 4E_0^2 I. \tag{5.3}$$

From the second conservation law (4.2) we have (assuming $Q_u \neq 0$)

$$\int |\psi|^4 dx \leq 4\,|Q_u|^{-1}E_0^{1/2}I^{1/2} + |Q_u^{-1}E_1| \tag{5.4}$$

and hence

$$\int |\psi|^6 dx \leq 2E_0^{1/2}I^{1/2}\int |\psi|^4 dx \leq 8\,|Q_u|^{-1}E_0 I + 2\,|Q_u^{-1}E_1|\,E_0^{1/2}I^{1/2}. \tag{5.5}$$

We can use the above inequalities, together with the conservation laws, to find sufficient conditions for the boundedness of I. This boundedness also implies the boundedness of $\|\psi\|_\infty$ by (5.1).

Proof of Lemma 1 :

From (4.6) we have

$$[\mathrm{Im}(\psi\bar{\psi}_x)]_t = -\frac{1}{2}[\bar{\psi}\psi_{xxx} + \psi\bar{\psi}_{xxx}] + \frac{1}{2}|\psi_x|^2_x - [F(\psi)]_x|\psi|^2.$$

Multiplying by x and integrating over the real line,

$$\frac{d}{dt}\int x\,\mathrm{Im}(\psi\bar{\psi}_x)dx = -\frac{1}{2}\int x[\bar{\psi}\psi_{xxx}+\psi\bar{\psi}_{xxx}]dx + \frac{1}{2}\int x|\psi_x|^2_x dx - \int x[F(\psi)]_x|\psi|^2 dx.$$

Integrating each term on the right hand side by parts, we have

$$\int x(\bar{\psi}\psi_{xxx}+\psi\bar{\psi}_{xxx})dx = 3\int|\psi_x|^2 dx$$

$$\int x|\psi_x|^2_x dx = -\int|\psi_x|^2 dx$$

$$\int x|\psi|^2[F(\psi)]_x dx = -\frac{1}{2}Q_c\int|\psi|^4 dx - \frac{2}{3}Q_q\int|\psi|^6 dx + \frac{1}{4}Q_u\int x|\psi|^4_t dx$$

so

$$\frac{d}{dt}\int x[\mathrm{Im}(\psi\bar{\psi}_x) + \frac{1}{4}Q_u|\psi|^4]dx = -2\int|\psi_x|^2 dx + \frac{2}{3}Q_q\int|\psi|^6 dx + \frac{1}{2}Q_c\int|\psi|^4 dx$$

$$= -2E_2 - \frac{1}{2}Q_c\int|\psi|^4 dx. \qquad \square$$

Theorem 2

Assume $\psi(x,t)$ is a rapidly decreasing function of x, and assume ψ satisfies

$$i\psi_t + \psi_{xx} + Q_c|\psi|^2\psi + Q_q|\psi|^4\psi = 0$$

with smooth initial condition $\psi_0(x)$. Then ψ will blow up in time T where $0 \le T \le T^*$, for some finite time T, if $E_2 < 0$ and $Q_c \le 0$. Furthermore,

$$T^* \le \frac{1}{2E_2}\left[\int x\,\mathrm{Im}(\psi_0\bar{\psi}_{0,x})dx - \left([\int x\,\mathrm{Im}(\psi_0\bar{\psi}_{0,x})dx]^2 - E_2\int x^2|\psi_0|^2 dx\right)^{1/2}\right]. \qquad \square$$

Proof of Theorem 2

We first note that the conditions $E_2 < 0$ and $Q_c \le 0$ imply that $\frac{4}{3}Q_q E_0{}^2 > 1$. From the conservation law (4.3)

$$E_2 = I - \frac{1}{3}Q_q\int|\psi|^6 dx - \frac{1}{2}\int|\psi|^4 dx$$

$$\frac{1}{3}Q_q\int |\psi|^6 dx \geq I$$

so $\lim\limits_{t\to T}\int |\psi|^6 dx = \infty$. Since

$$\int |\psi|^6 dx \leq ||\psi||_\infty^4 \int |u|^2 dx = ||\psi||_\infty^4 E_0$$

it follows that $\lim\limits_{t\to T} ||\psi||_\infty = \infty$.

□

Corollary 2

Let $u(x,t)$ be a solution of the GNLS with $u(x,0)=u_0(x)$. Then, if $q_u=0$, u will blow up in finite time T, where $0<T<T^*$, if $E_2 < 0$ and $q_c \leq 0$. Furthermore

$$T^* \leq \frac{1}{2E_2}\left(\int x\left(\operatorname{Im}(u_0\bar{u}_{0,x}) - \frac{1}{4}(2q_m+q_u)|u_0|^4\right)dx\right.$$

$$\left. - \left(\left(\int x\left(\operatorname{Im}(u_0\bar{u}_{0,x}) - \frac{1}{4}(2q_m+q_u)|u_0|^4\right)dx\right)^2 - E_2\int x^2|u_0|^2 dx\right)^{1/2}\right).$$

□

Proof of Corollary 2 :

The result follows immediately if we replace ψ by u in the blow-up theorem. □

The conditions for boundedness and blow-up agree with the intuitive notion that if the quintic nonlinearity dominates the dispersion, the solutions will experience explosive growth. Physically of course, the wave amplitude cannot grow indefinitely; the assumptions under which the mathematical equation is derived become invalid as the solution blows up, and new physical processes must be included in the model. Nevertheless, the blow-up according to (1.1) is of physical interest as it explains the formation of "spikes" which have been experimentally observed (in the context of optics, for example, the early stages of blow-up correspond to the self-focussing of laser beams; see [18] for a comprehensive review of blow-up phenomena for nonlinear Schrödinger equations). The form of the similarity solution (3.17) suggests a growth rate

$$||u||_\infty \propto (t_0-t)^{-1/4},$$

the same as has been suggested for the nonlinear Schrödinger equation with a simple quintic nonlinearity [19].

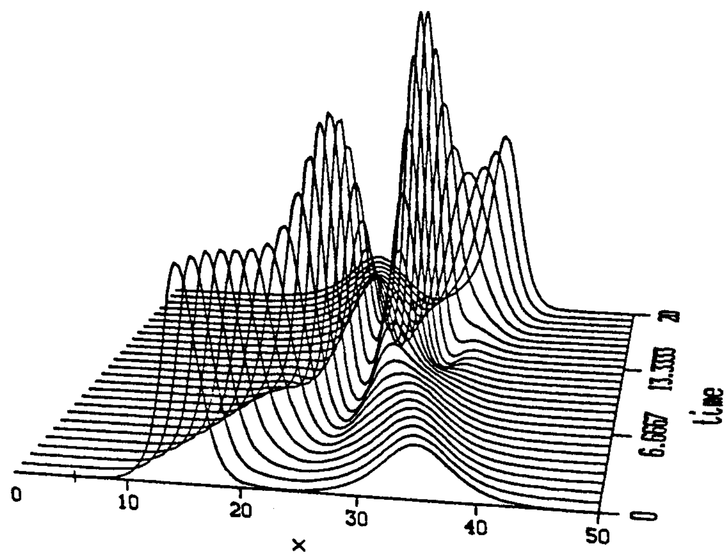

Figure 1 : Soliton interaction for $iu_t + u_{xx} + |u|^2u + |u|^4u - 2|u|^2_x u = 0.$

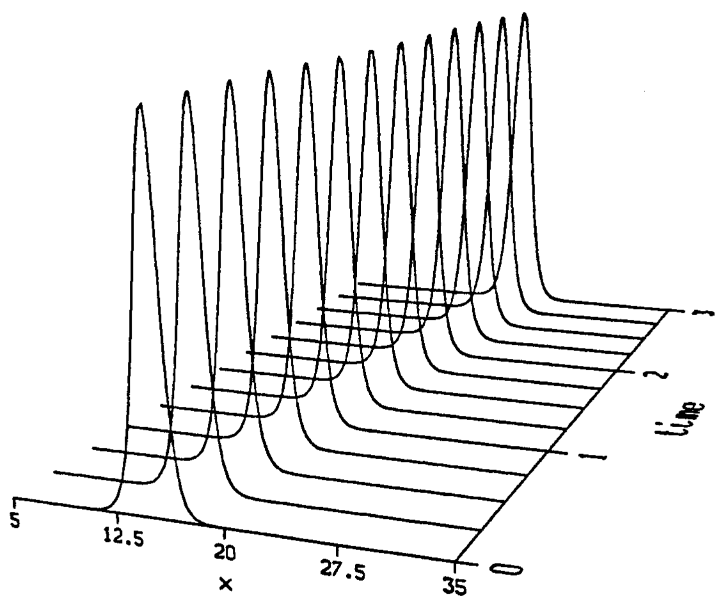

Figure 2 : Solitary wave solution for $iu_t + u_{xx} - \frac{1}{2}|u|^2u - \frac{7}{4}|u|^4u - |u|^2_x u - 2|u|^2u_x = 0.$

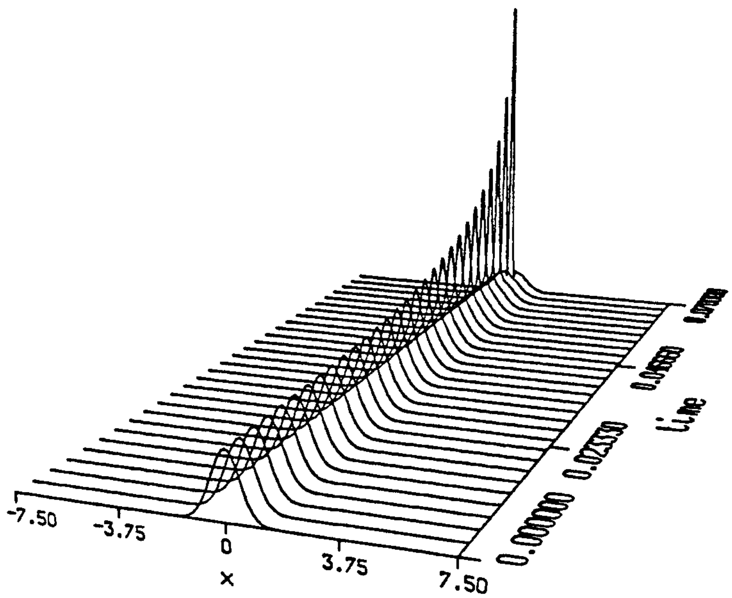

Figure 4 : Blow-up for $iu_t + u_{xx} - 2|u|^2u + 20|u|^4u = 0$.

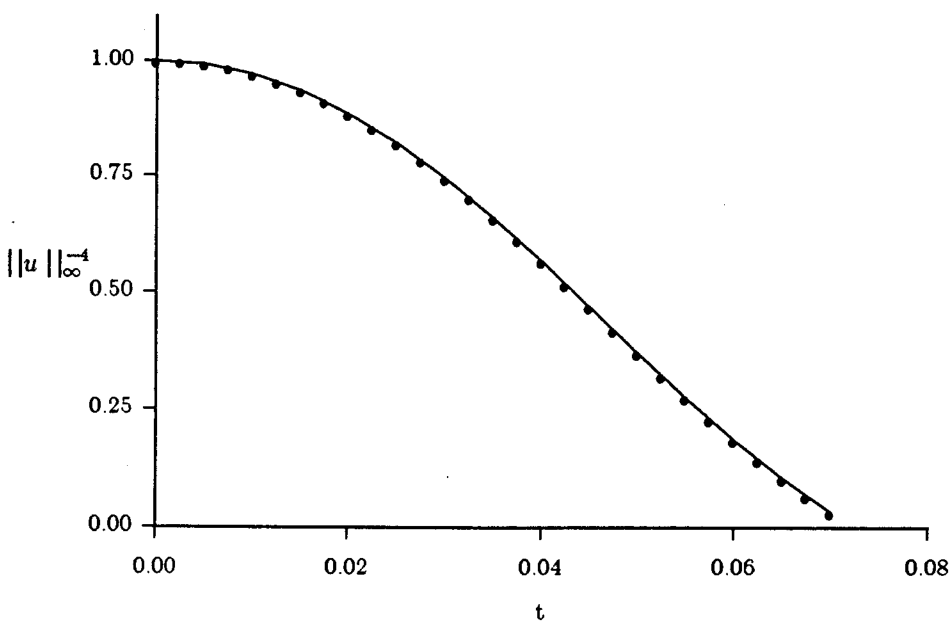

Figure 5 : Spike evolution for $iu_t + u_{xx} - 2|u|^2u + 20|u|^4u = 0$.

7. References

1. Johnson, R.S., "On the modulation of water waves in the neighbourhood of $kh \approx 1.363$," *Proc. R. Soc. London A*, **357** pp. 131-141 (1977).

2. Kakutani, T. and Michihiro, K., "Marginal State of Modulational Instability - Note on Benjamin- Feir Instability," *J. Phys. Soc. Japan*, **52**(12) pp. 4129-4137 (December 1983).

3. Calogero, F. and Eckhaus, W., "Nonlinear evolution equations, rescalings, model PDEs and their integrability: I," *Inverse Problems*, **3** pp. 229-262 (1987).

4. Strauss, W.A., "The Nonlinear Schrödinger Equation," pp. 452-465 in *Contemporary Developments in Continuum Mechanics*, ed. L. A. Medeiros, North-Holland Publishing Company (1978).

5. Zakharov, V.E. and Shabat, A.B., "Exact Theory of Two-Dimensional Self-Focusing and One- Dimensional Self-Modulation of Waves in Nonlinear Media," *Soviet Physics JETP*, **34**(1) pp. 62-29 (January 1972).

6. Hasimoto, H. and Ono, H., "Nonlinear Modulation of Gravity Waves," *J. Phys. Soc. Japan*, **33**(3) pp. 805-811 (September 1972).

7. Kaup, D. and Newell, A.C., "An exact solution for a derivative nonlinear Schrödinger equation[a)]," *J. Math. Phys.*, **19**(4) pp. 798-801 (April 1978).

8. Tanaka, M., "Nonlinear Self-Modulation Problem of the Benjamin-Ono Equation," *J. Phys. Soc. Japan*, **51**(8) pp. 2686-2692 (August 1982).

9. Pushkarov, K. I., Pushkarov, D. I., and Tomov, I. V., "Self-action of light beams in non-linear media: soliton solutions," *Opt. Quantum Electron.*, **11** pp. 471-478 (1979).

10. Barashenkov, I.V. and Makhankov, V.G., "Soliton-like "Bubbles" in a System of Interacting Bosons," *Physics Letters A*, **128**(1,2) pp. 52-55 (March 1988).

11. Calogero, F. and Degasperis, A., *Spectral Transform and Solitons I,* North-Holland, New York (1982).

12. Cowan, S., Enns, R.H., Rangnekar, S.S., and Sanghera, S.S., "Quasi-soliton and other behaviour of the nonlinear cubic-quintic Schrödinger equation," *Can. J. Phys.*, **64** pp. 311-315 (1986).

13. Calogero, F. and Eckhaus, W., "Necessary Conditions for integrability of nonlinear PDEs," *Inverse Problems*, **3** pp. L27-L32 (1987).

14. Kundu, A., "Exact solutions to higher order nonlinear equations through gauge transformation," *Physica D*, **25** pp. 399-406 (1987).

15. Wadati, M., Konno, K., and Ichikawa, Y., "A Generalization of Inverse Scattering Method," *J. Phys. Soc. Japan*, **46**(6) pp. 1965-1966 (June 1979).