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On Optimal Interpolation Triangle Incidences

E.F. D'Azevedo
R.B. Simpson

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On Optimal Interpolation Triangle Incidences

E. F. D'Azevedo†

R. B. Simpson†

ABSTRACT

We study the problem of determining the optimal incidences for triangulating a given set of vertices, for the model problem of interpolating a convex quadratic surface by piecewise linear functions. An exact expression for the maximum error is derived, and the optimality criterion is minimization of the maximum error. The optimal incidences are shown to be derivable from an associated Delaunay triangulation and hence are computable in $O(N \log N)$ time, for N vertices.

Key words. optimal mesh, triangulation incidence, surface approximation

AMS(MOS) subject classifications. 65D05, 65L50

1. Introduction

In this paper, we study the question of an optimal choice of edge incidences for triangulating a given set of points. The study uses a model approximation problem, piecewise linear interpolation of a convex quadratic surface and our optimality criterion is that the optimal choice of incidences minimizes the maximum error in any triangle. We establish that the optimal incidence problem can be transformed to an equivalent Delaunay triangulation problem, showing in particular that at least this model optimal incidence problem can be solved in time $O(N \log N)$ for N interpolation points.

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† Department of Computer Science, University of Waterloo, Waterloo, Ontario, Canada. This work has been supported by the Natural Science and Engineering Council of Canada and the government of Ontario through the Information Technology Research Centre.

General triangles have two independent length scales associated with them, e.g. the longest edge, and the length of the perpendicular from this edge to the opposite vertex. It is common to regard the local error over a triangle T as depending on one length scale, (the 'size' of T, typically denoted 'h') and to impose a geometric condition on the triangulation, i.e. small angles should be avoided. Strang and Fix [15] developed an error bound which depends on the reciprocal of the sine of the minimum interior angle. However, Aziz and Babuska[3] showed that actually small angles do not play a crucial role in approximation properties, but that limiting the largest angle was necessary and sufficient for convergence. Indeed, for a convex surface in which the curvature in the principal direction is markedly different from the curvature in the perpendicular direction, incidences producing triangles with small angles are appropriate, and are present in an optimal triangulation incidence, as shown in example 2 below.

A commonly used incidence relation for a set of vertices is the geometrically defined Delaunay triangulation[9, 12, 13]. In Section 1, we give an example of quadratic functions, and a series of sets of vertices, for which the Delaunay triangulation can be arbitrarily far from optimal. Representations of the error in linear interpolation for a general triangle and general quadratic function are surprisingly complicated, e.g.[4-7]. In Section 2, we use a geometric argument to develop both analytic and geometric descriptions of this error, and of its maximum, over a general triangle. In Section 3, the main result of the paper is presented, i.e. that an optimal triangle incidence for this interpolation problem can be obtained from a Delaunay triangulation of a transform of the given vertex set. For this result, we establish a new geometric optimality property of the Delaunay triangulation concerning minimizing the maximum circumcircle of the triangulation.

For the quadratic model problem, the determination of optimal incidences is based on a criterion that is locally applied to a transform of the given vertices (See Section 3). For more general smooth functions, this procedure can be used locally to determine appropriate triangle incidences if an adequate estimate for the Hessian matrix of the data function is available[1, 2, 14].

1.1. Example 1

To see the influence of the choice of triangle incidence on the accuracy of approximation, let us look at an example of piecewise linear approximation of the quadratic function

$$f(x,y) = \lambda_1 x^2 + \lambda_2 y^2 \quad , \quad \lambda_1 \gg \lambda_2 > 0$$

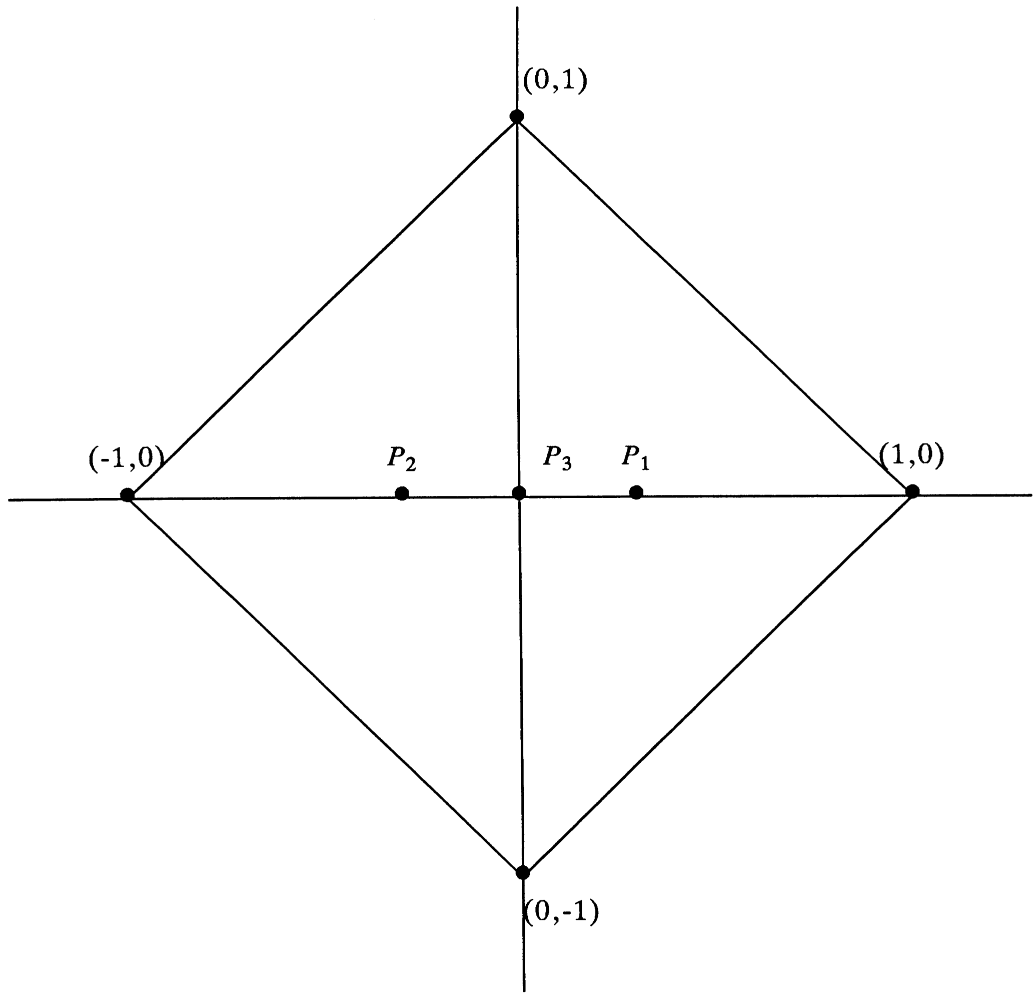

FIG. 1.1

Consider the interpolation error over the square (see figure 1.1), if the y-axis is chosen as a diagonal, maximum interpolation error is

$$E_y = (\lambda_1 + \lambda_2)^2/(4\lambda_1)$$

and occurs at P_1 (P_2). (Expressions for the errors are developed in section 2 below.) If the x-axis is chosen, maximum interpolation error is

$$E_x = \lambda_1$$

at the origin (P_3) (see (2.5) below). The ratio of the error from these two incidences is

$$\frac{E_x}{E_y} = \frac{\lambda_1}{(\lambda_1 + \lambda_2)^2/(4\lambda_1)} = \frac{4}{(1+\lambda_2/\lambda_1)^2} \leq 4$$

Consider the same interpolation problem with extra nodes, $(ih,1)$, $(-ih,1)$,$(ih,-1)$,$(-ih,1)$, $i \in \{1,2, \cdots ,N\}$ and $h = 1/N$. A standard choice of triangle incidences for a given set of vertices is the Delaunay triangulation introduced in section 3, along with some of its geometric optimality properties. If a Delaunay triangulation incidence is chosen, including the x-axis as shown in figure 1.2, the maximum interpolation error is dominated by E_x (see figure 1.2). Consider the triangle incidences in figure 1.3. The maximum interpolation error for these triangle incidences is

$$E_h = (h^2\lambda_1 + 4\lambda_2)/4$$

Then the ratio of errors,

$$\frac{E_h}{E_x} = \frac{(h^2\lambda_1 + 4\lambda_2)}{(4\lambda_1)}$$

$$= h^2/4 + \lambda_2/\lambda_1$$

For h small (N large),

$$\frac{E_h}{E_x} \approx \lambda_2/\lambda_1$$

Hence the Delaunay triangle incidence can be arbitrarily far from optimal with respect to minimizing the maximum error.

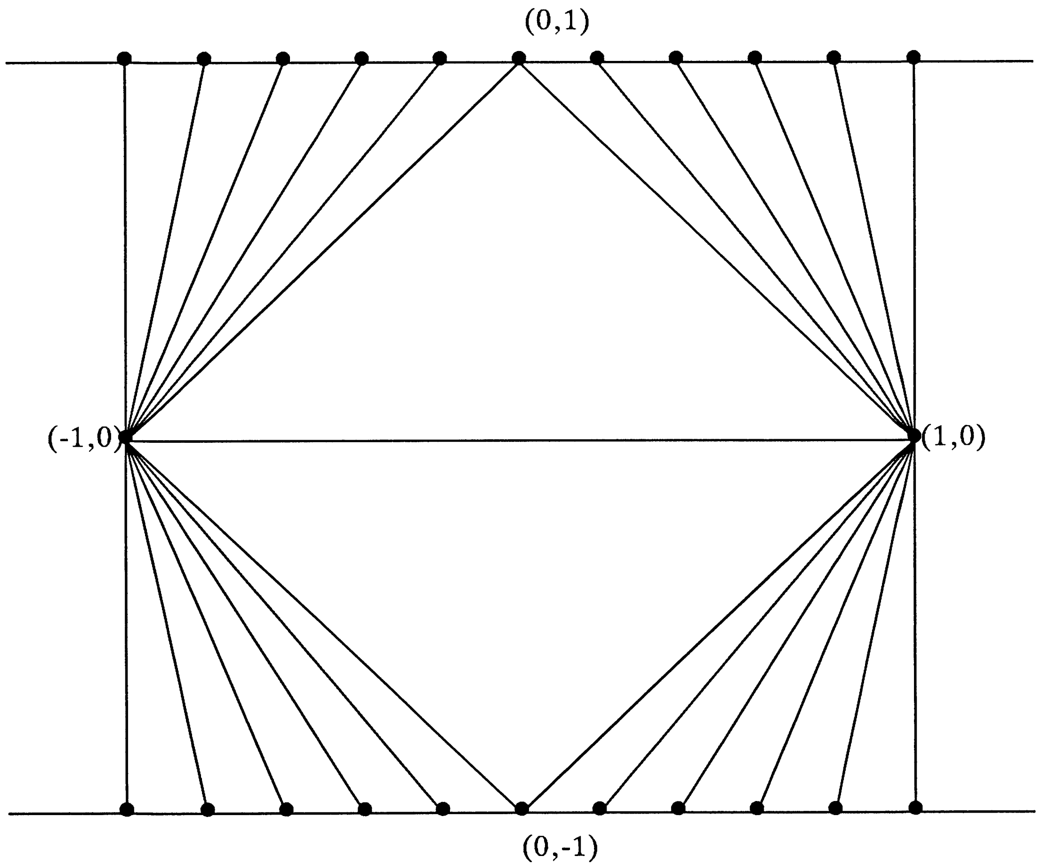

FIG. 1.2

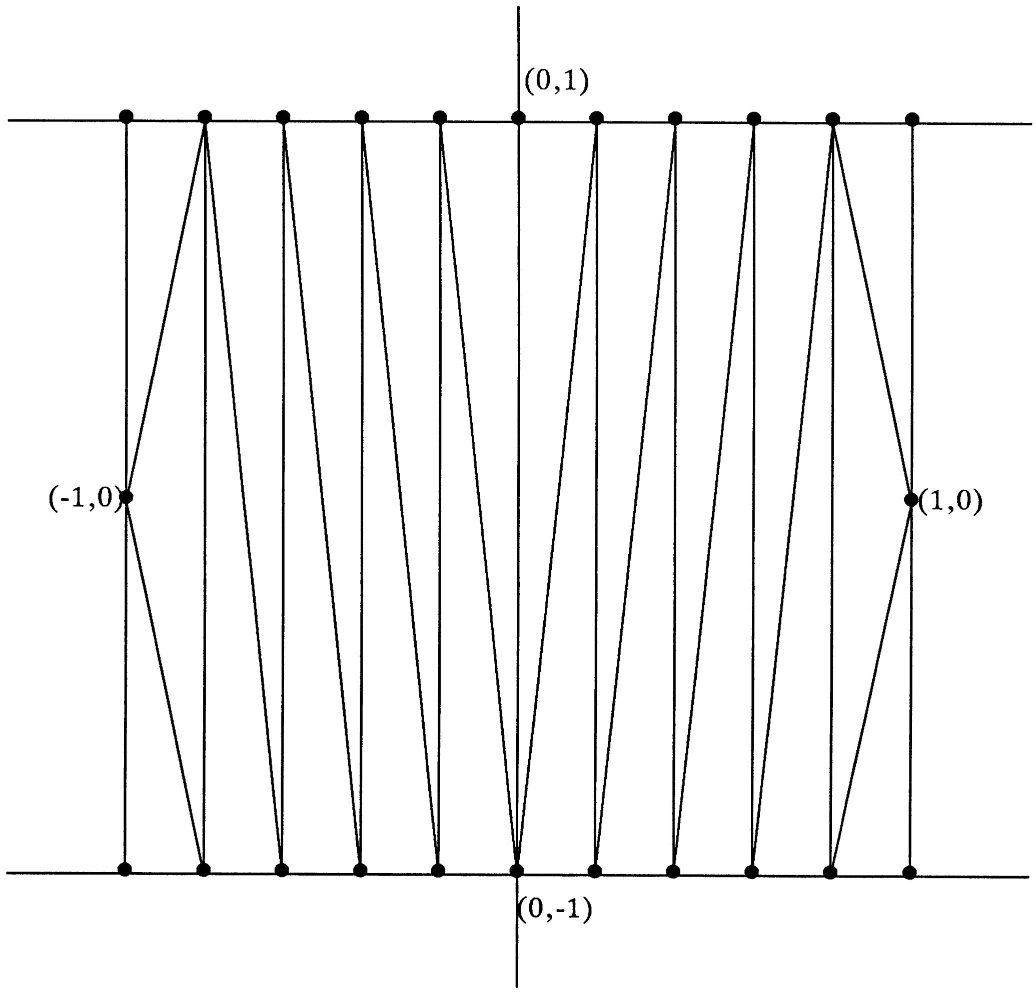

FIG. 1.3

1.2. Example 2

Here is a specific example of interpolation of a convex function with markedly different curvature. The data function to be interpolated is $f(x,y) = 100x^2 + y^2$. Figure 1.4 shows the Delaunay triangulation for an arbitrary set of points over the unit square and figure 1.5 shows the optimal triangulation incidence determined in this work. The errors of interpolation for both triangulations are displayed in figure 1.6, where the maximum interpolation error over each triangle is plotted in ascending order. Note that the maximum error for the Delaunay triangulation is nearly six times larger than the one for the optimal triangulation.

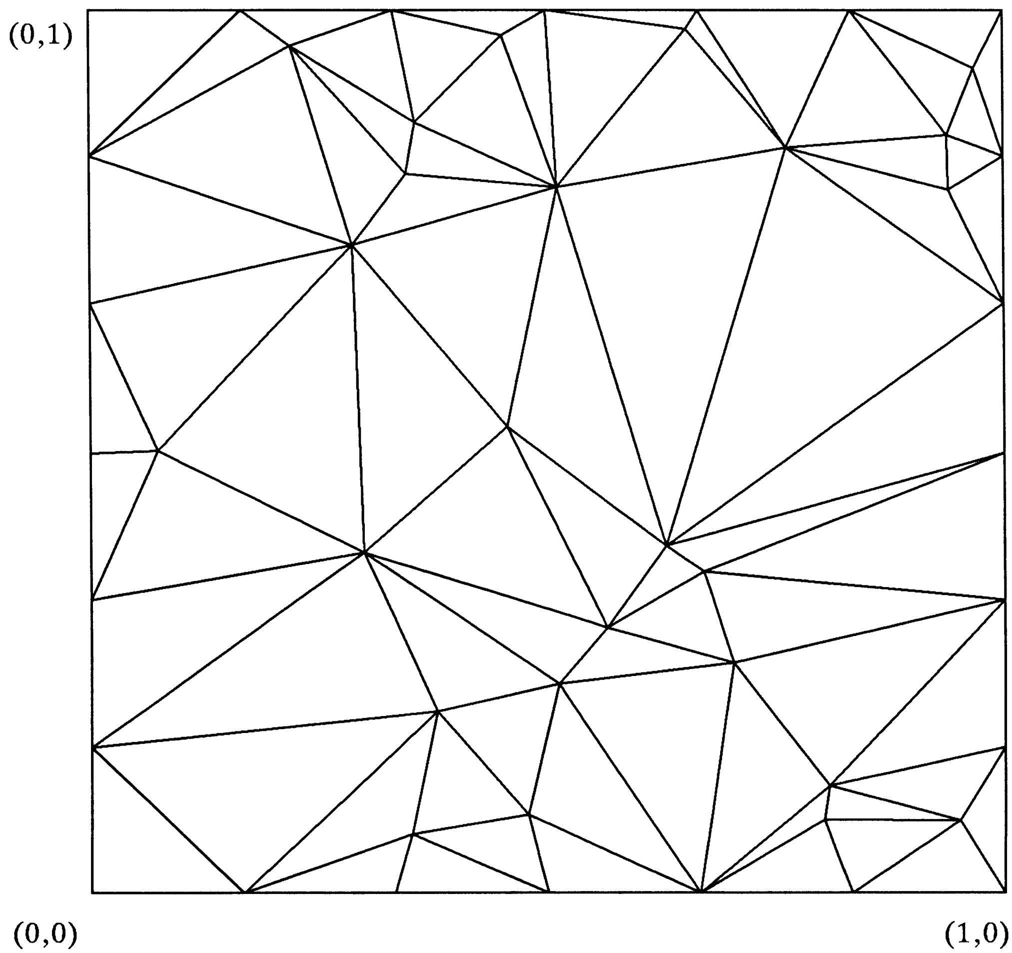

FIG. 1.4

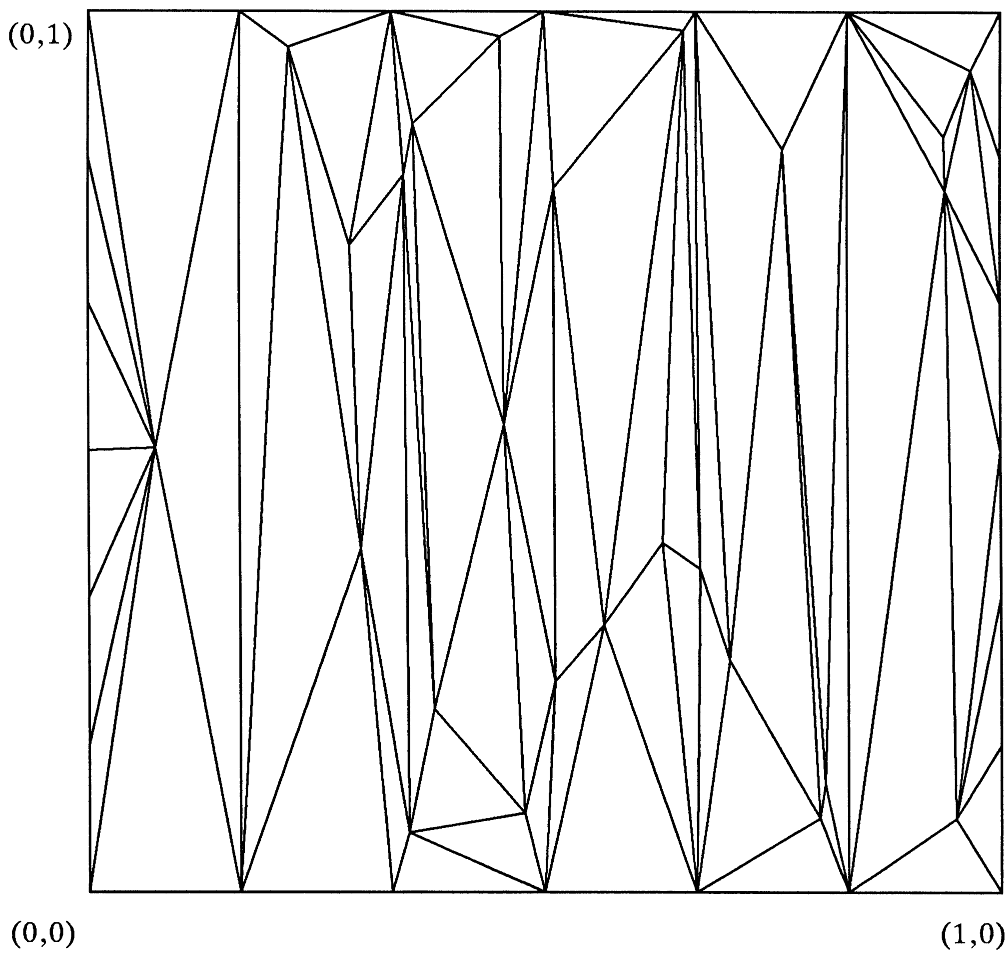

FIG. 1.5

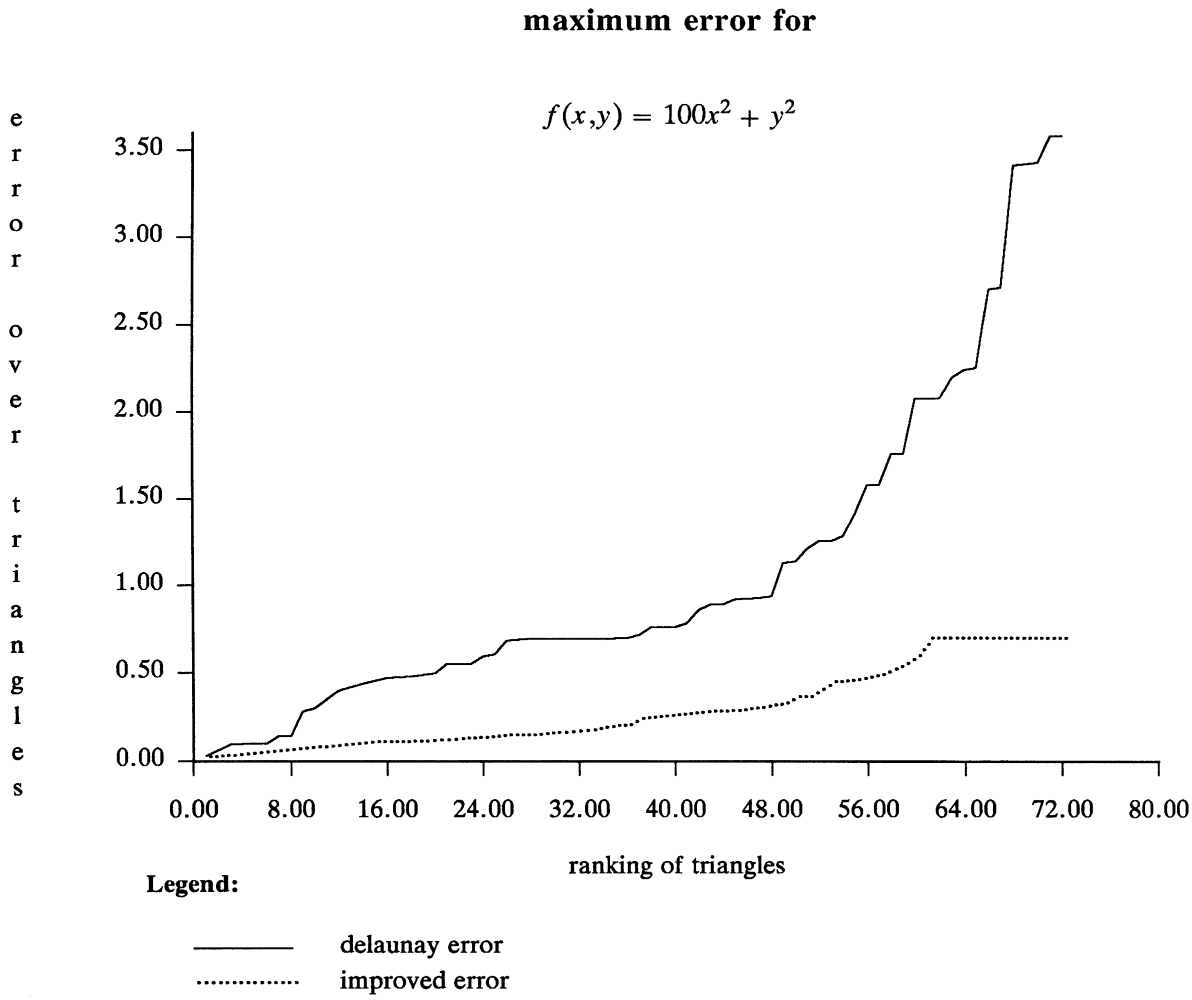
maximum error for
$f(x,y) = 100x^2 + y^2$
error over triangles
3.50
3.00
2.50
2.00
1.50
1.00
0.50
0.00
0.00
8.00
16.00
24.00
32.00
40.00
48.00
56.00
64.00
72.00
80.00
ranking of triangles
Legend:
delaunay error
improved error

FIG. 1.6

2. Model Problem

2.1. Error Expressions

Error expressions to describe the interaction between the parameters of a general triangle, and the relevant parameters of a general function for even piecewise linear interpolation are surprisingly complicated (e.g.[4-7]). In our discussion, we model a general function by a convex quadratic polynomial; i.e. let

$$f(\mathbf{x}) = \mathrm{a} + \mathbf{d}^t\mathbf{x} + \mathbf{x}^t H\mathbf{x} \tag{2.1}$$

where H is symmetric with eigenvalues $\lambda_1 \geq \lambda_2 > 0$, and without any further loss of generality, we may assume that the coordinate axes are aligned with the principal axes of H so that

$$f(x,y) = \mathrm{a} + d_1x + d_2y + \lambda_1x^2 + \lambda_2y^2 \tag{2.2}$$

The error function for the linear interpolation is a quadratic form in x,y with the same quadratic terms. We represent it as

$$E(x,y) = \lambda_1x^2 + \lambda_2y^2 + b_1x + b_2y + c \tag{2.3}$$

where the values of b_1, b_2, and c depend on the coordinates of the triangle vertices. Now $E(x,y)$ vanishes at the triangle vertices. Since the level curves of $E(x,y)$ are ellipses, then the zero error level curve of $E(x,y)$ is a circumscribing ellipse of the triangle. We will denote the ellipse as $e(T)$ for triangle T. The equation of the ellipse for the level curve of value $-K$, has the form

$$\begin{aligned}
&E(x,y) = -K \\
&\Rightarrow \lambda_1x^2 + \lambda_2y^2 + b_1x + b_2y + c = -K \qquad (2.4)\\
&\Rightarrow \lambda_1(x + \frac{b_1}{2\lambda_1})^2 + \lambda_2(y + \frac{b_2}{2\lambda_2})^2 = E - K \quad \text{where } E = \frac{b_1^2}{4\lambda_1} + \frac{b_2^2}{4\lambda_2} - c \\
&\Rightarrow (x + \frac{b_1}{2\lambda_1})^2/(\frac{(E-K)}{\lambda_1}) + (y + \frac{b_2}{2\lambda_2})^2/(\frac{(E-K)}{\lambda_2}) = 1
\end{aligned}$$

The parameters b_1, b_2, and c can be explicitly computed by requiring the ellipses for $K = 0$ to be the circum-ellipse of T.

At the center, $(-b_1/(2\lambda_1), -b_2/(2\lambda_2))$ of $e(T)$, $|E(x,y)|$ attains a maximum value which can be expressed as :

$$E = \frac{\left(D_{12}\, D_{23}\, D_{31}\right)}{16\ \lambda_1 \lambda_2\, A^2} \quad \text{where} \quad D_{ij} = \left(\lambda_1 (x_i - x_j)^2 + \lambda_2 (y_i - y_j)^2\right) \tag{2.5}$$

and (x_i, y_i) $i = 1,2,3$ are vertices of the triangle, and A is area of the triangle. The details of the derivation can be found in appendix A.

The maximum interpolation error for a triangle T will be denoted as

$$E_{\max}(T) = \max_{(x,y)\, \in\, T} |p_f(x,y) - f(x,y)| \tag{2.6}$$

where $p_f(x,y)$ is the linear interpolant of f at the vertices of T. In the case that the center of $e(T)$ lies in T (including the boundary of T), $E_{\max}(T) = E$ and we note that the area of $e(T)$ is $A_1 = \pi E / \sqrt{\lambda_1 \lambda_2}$, so that

$$E_{\max}(T) = E = \sqrt{\lambda_1 \lambda_2} A_1 / \pi \tag{2.7}$$

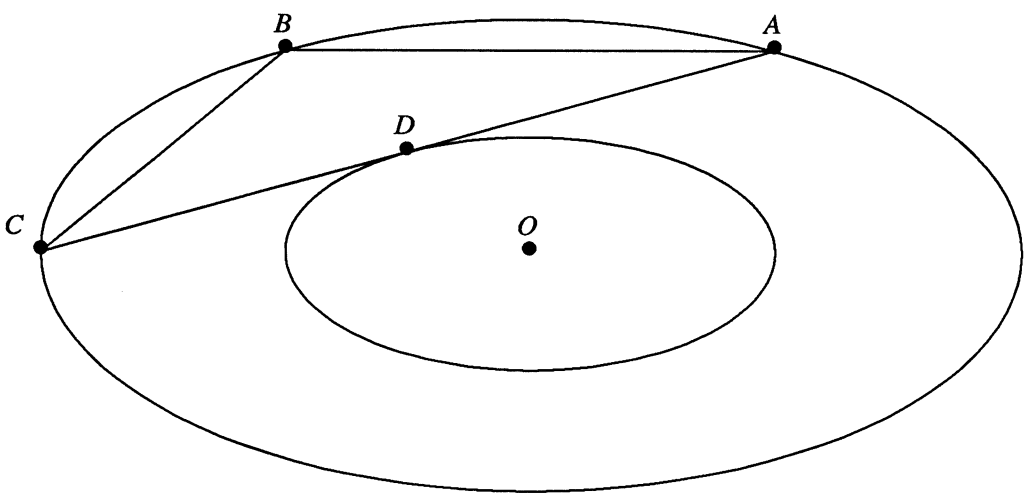

FIG. 2.1

If the center of $e(T)$ is not in T, (see figure 2.1), $E_{max}(T)$ is attained on an edge of T. Geometrically, $E_{max}(T)$ is the level of the osculating level curve (ellipse) tangent to this side, $|E(x,y)| = E_{max}(T)$. The area of the tangent ellipse is from (2.4):

$$\pi(E - E_{max}(T))/\sqrt{\lambda_1\lambda_2} = A_2$$

Thus $E_{max}(T)$ can be expressed in terms of the ratio, $\rho = A_2/A_1$ of these areas, and E, as

$$E_{max}(T) = (1-\rho)E \tag{2.8}$$

Consider a rescaling along x-axis by $(\lambda_1/\lambda_2)^{1/2}$ to transform the elliptical level curves to concentric circles (see figure 2.2). Since the area of an ellipse is directly proportional to the area of the corresponding circle in the transformed plane,

$$\begin{aligned} E_{max}(T) &= (1-\rho)E \\ &= \left(1 - \frac{\pi(OD)^2}{\pi(OA)^2}\right)E \\ &= \left(\frac{OA^2 - OD^2}{OA^2}\right)E \\ &= \left(\frac{AD^2}{OA^2}\right)E \end{aligned} \tag{2.9}$$

where D is midpoint of AC tangent to the inner circle.

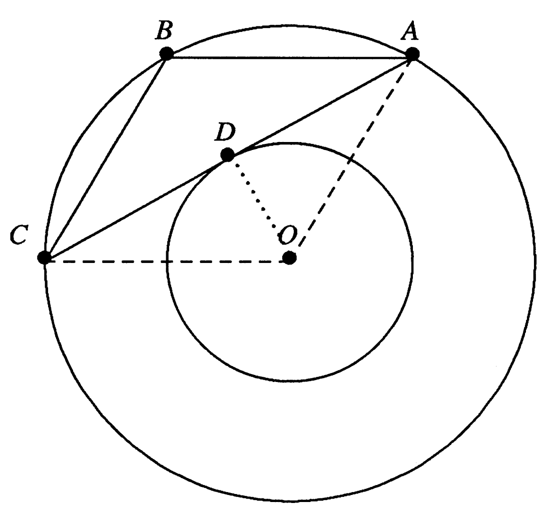

FIG. 2.2

From this result in the transformed plane, we gain a geometric interpretation of the maximum error. Let $\bigcirc(T)$ denote the circumcircle of the transformed triangle T and $|\bigcirc(T)|$ denote the area of this circumcircle. If the transformed image of a triangle has no obtuse angle, then its maximum error is proportional to the area of its circumscribing circle. If there is an obtuse angle in the transformed triangle, the maximum error occurs along the longest edge and the error is proportional to the area of circle with the longest edge as diameter.

3. Delaunay Triangulation

The Delaunay triangulation of a fixed set of vertices, and its related figure, the Voronoi diagram, are much studied geometric constructions. (See [9, 12, 13]) Here, we briefly review some properties which are relevant for optimal interpolation incidence. The Delaunay triangulation selects triangle incidences that maximize the minimum angle in the triangulation. Lawson[9] proposed an algorithm for converting an arbitrary triangulation to a Delaunay one by repeated application of a local edge swapping procedure. In it, for each interior edge of the current triangulation, the two neighboring triangles are examined. If they form a convex quadrilateral, and if the replacement of the examined edge by the other diagonal of this quadrilateral would increase the minimum angle, then a swap of diagonals is made. Lawson showed that this criterion for picking the diagonal in a convex quadrilateral is characterized by the property that the circumcircles of either of the two triangles thus formed do not contain the fourth vertex of the quadrilateral[1]. This criterion is referred to as the empty circle criterion. Lawson showed that the repeated application of his edge examination/edge swapping procedure terminates in a Delaunay triangulation.

3.1. Optimal Incidence for the Model Problem

We shall define a triangle incidence to be globally optimal if it minimizes the maximum interpolation error. Each interior edge in a triangulation is associated with a quadrilateral with that edge as diagonal. We shall also define a triangular mesh incidence to be locally optimal if for each convex quadrilateral associated by an interior edge in the triangulation, the incidence minimizes the maximum interpolation error over the quadrilateral.

Here we show that the problem of constructing a locally optimal mesh incidence for N points, can be transformed to the problem of generating a Delaunay triangulation which provides an $O(N \log N)$ algorithm for solving this problem. We also show that a locally optimal

[1] If the fourth vertex lies on the boundary of a circumcircle, then no change in the minimum angle occurs from edge swapping and either choice of edge results in a Delaunay triangulation.

incidence is globally optimal.

The rescaling of the x-axis introduced at the end of section 2, which results in error ellipses being mapped into circles, will be used here to define the transform plane, for which we do not explicitly introduce coordinates.

Theorem 1

A locally optimal interpolation triangulation incidence of N vertices is defined by a Delaunay triangulation in the transformed plane (and hence is computable in $O(N\log N)$ time which is optimal).

To simplify our discussion, we shall use the notation that all references to angles are based on the labelling in figure 3.1. Moreover, we shall always assume vertex A to be exterior to $\bigcirc(\triangle BCD)$. The diagonal BD is the incidence selected by the empty circle criterion. By elementary geometry,

$$\angle CAD = \theta_1 < \theta_2 = \angle CBD$$

$$\angle BAC = \phi_1 < \phi_2 = \angle BDC$$

$$\angle DCA = \gamma_1 < \gamma_2 = \angle DBA$$

$$\angle ACB = \eta_1 < \eta_2 = \angle ADB$$

Lemma 1

Given a convex quadrilateral ABCD with vertex A exterior of $\bigcirc(\triangle BCD)$, then $\max(|\bigcirc(\triangle ABC)|\ ,\ |\bigcirc(\triangle ADC)|) \geq \max(|\bigcirc(\triangle BCD)|\ ,\ |\bigcirc(\triangle ABD)|)$.

Proof of Lemma 1

case 1

Assume that $\theta_2 \leq \pi/2$. Let E be the midpoint of CD, O_1, O_2 be centers for $\bigcirc(\triangle ADC)$, and $\bigcirc(\triangle BCD)$ respectively. The O_1 and O_2 lie on the perpendicular to DC through E, so

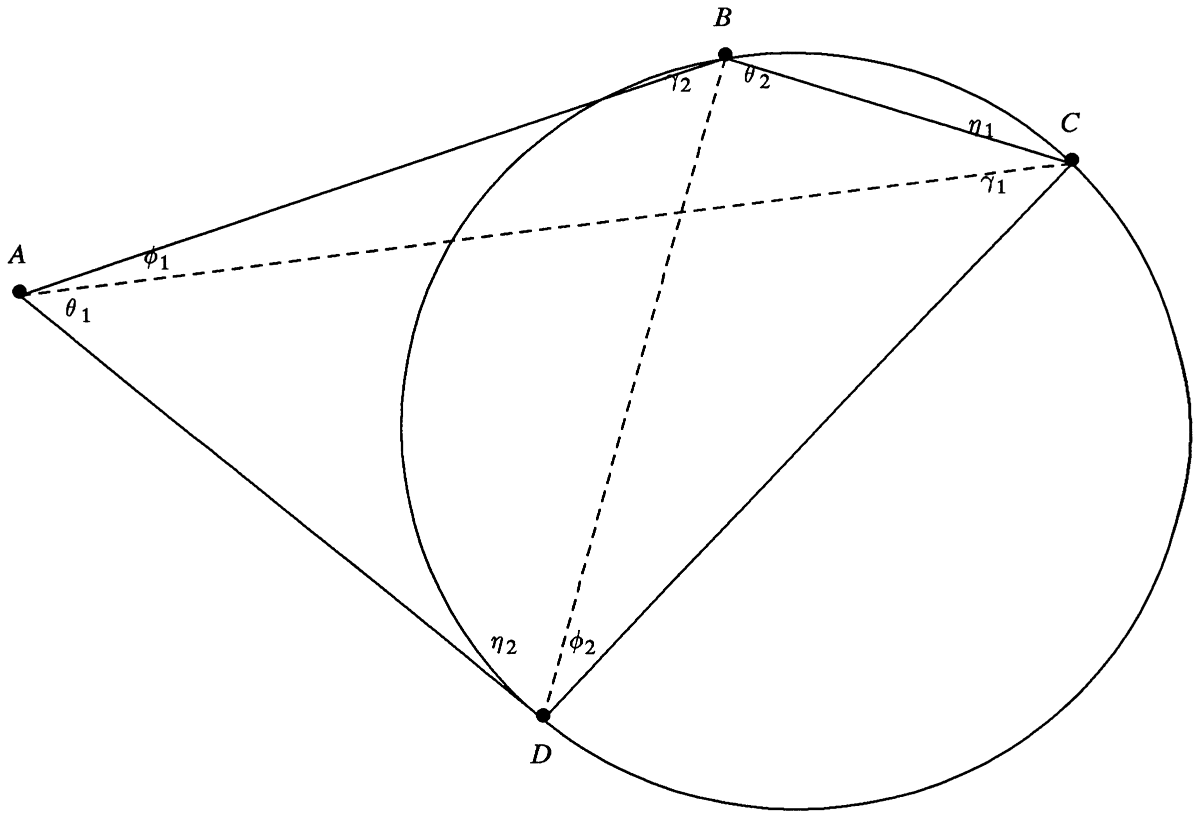

FIG. 3.1

O_1, O_2 and E are collinear. Let A', B' be points where this line meets $\bigcirc(\Delta ADC)$ and $\bigcirc(\Delta BCD)$.

$$\angle EA'C = \theta_1/2$$

$$\angle EB'C = \theta_2/2$$

$$\tan(\theta_2/2) = \frac{EC}{EB'} = \frac{EC}{EO_2 + O_2B'} = \frac{EC}{EO_2 + O_2C} \tag{3.1}$$

Now from ΔO_2EC we have

$$(O_2C)^2 = (O_2E)^2 + (EC)^2$$

Thus (3.1) can be rewritten as

$$\tan(\theta_2/2) = \frac{EC}{O_2C + ((O_2C)^2 - (EC)^2)^{1/2}}$$

Similarly

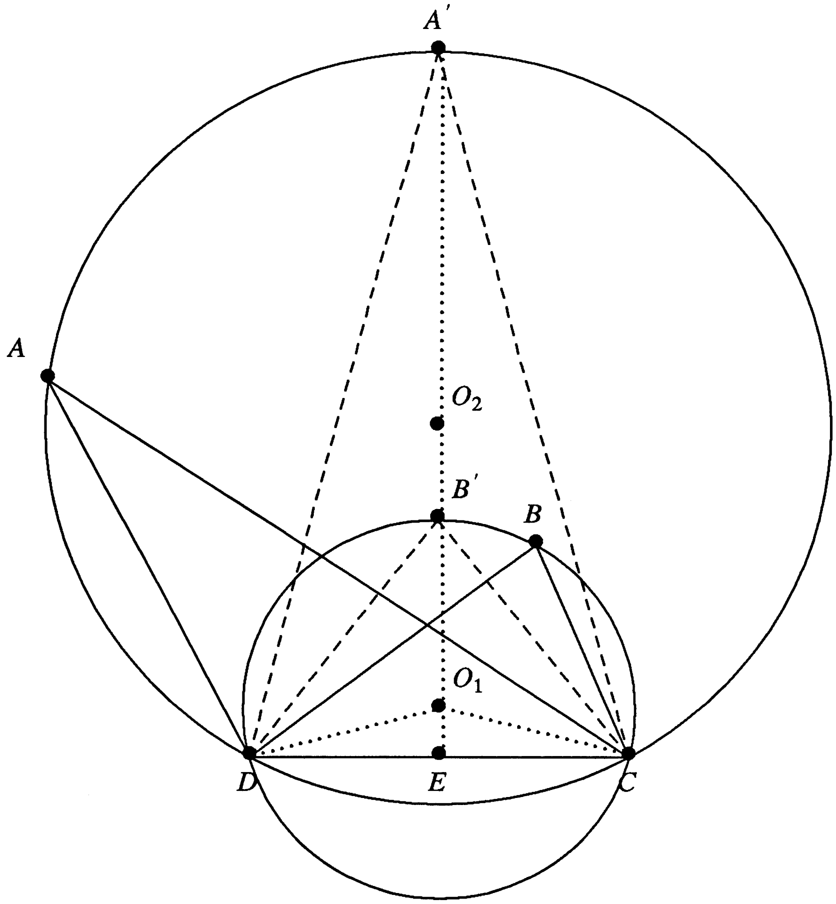

FIG. 3.2

$$\tan(\theta_1/2) = \frac{EC}{O_1C + ((O_1C)^2 - (EC)^2)^{1/2}} \tag{3.2}$$

$$\theta_1 \leq \theta_2 \leq \pi/2 \implies \tan(\theta_1/2) \leq \tan(\theta_2/2)$$

Therefore

$$\frac{EC}{O_1C + ((O_1C)^2 - (EC)^2)^{1/2}} \leq \frac{EC}{O_2C + ((O_2C)^2 - (EC)^2)^{1/2}}$$

$$\implies O_2C + ((O_2C)^2 - (EC)^2)^{1/2} \leq O_1C + ((O_1C)^2 - (EC)^2)^{1/2} \tag{3.3}$$

Since $x + (x^2 - a)^{1/2}$ is a monotone increasing function of x, thus (3.3) implies $O_2C \leq O_1C$. Therefore $\bigcirc(\Delta ADC)$ has larger radius than $\bigcirc(\Delta BCD)$.

Similarly if $\phi_2 \leq \pi/2$ then $\bigcirc(\Delta ABC)$ has larger radius than $\bigcirc(\Delta BCD)$.

case 2

Assume $\theta_2 > \pi/2$. From ΔBCD, we have $\phi_2 + \theta_2 < \pi$, thus $\phi_2 \leq \pi/2$ if $\theta_2 > \pi/2$. From the result in **case 1**, $|\bigcirc(\Delta ABC)| \geq |\bigcirc(\Delta BCD)|$. Similarly, by symmetry, C would be exterior to $\bigcirc(\Delta ABD)$.

Thus either $\bigcirc(\Delta ACD)$ or $\bigcirc(\Delta ABC)$ would be larger than $\bigcirc(\Delta ABD)$. Since cases 1 and 2 are exhaustive, the lemma is proved.

Corollary

The empty circle criteria applied to a convex quadrilateral selects the diagonal which minimizes the maximum circumcircle of the corresponding triangles.

Lemma 2

Given a convex quadrilateral ABCD with vertex A exterior to ΔBCD the empty circle criterion selects the triangulation incidence which minimizes the area of the circle corresponding to the maximum interpolation error error over the quadrilateral.

Proof of Lemma 2

Recall from Section 2.3 that the maximum interpolation error for a triangle is proportional to the area of the circumcircle of the transformed image of the triangle, if this image contains no obtuse angle. Otherwise, the maximum interpolation error is proportional to the area of the circle with diameter equal to the edge opposite the obtuse angle, i.e. the longest edge of the image triangle.

The first case was dealt with in Lemma 1 and its corollary in which we established that the diagonal BD is selected by the empty circle criterion. We now carry out a case by case study of configurations of image triangles with obtuse angles to show that the empty circle criterion also minimizes the maximum error circles. Hence we shall consider only cases where ΔABC or ΔACD contain an obtuse angle.

case 1

Assume $\phi_1 \geq \pi/2$. (See figure 3.3) We show that BC is the longest edge and it determines the error circle, regardless of the choice of diagonals. The area of the error circle for ΔABD is $\pi(BD/2)^2$, and for ΔBCD is $\pi(BC/2)^2$. Note that $\phi_2 \geq \phi_1 \geq \pi/2$. $|\bigcirc(\Delta ABC)|$ is $\pi(BC/2)^2$, and $|\bigcirc(\Delta ADC)|$ is $\pi(AC/2)^2$. Now $\phi_1 < \phi_2$, thus $|BC| > |BD|$. Since BD

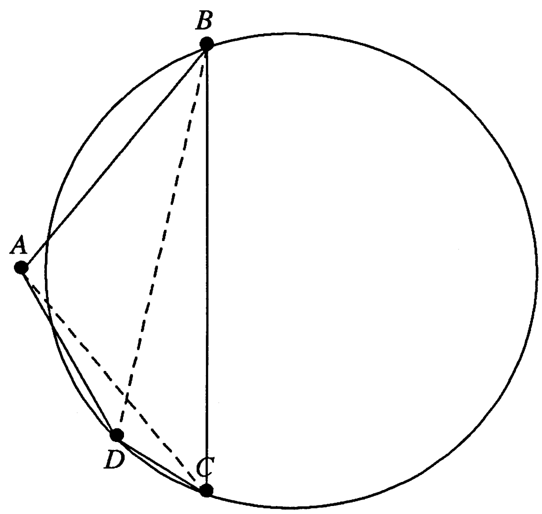

FIG. 3.3

cannot be the longest chord, the lemma holds. By symmetry, same applies if θ_1, η_1, γ_1 is obtuse.

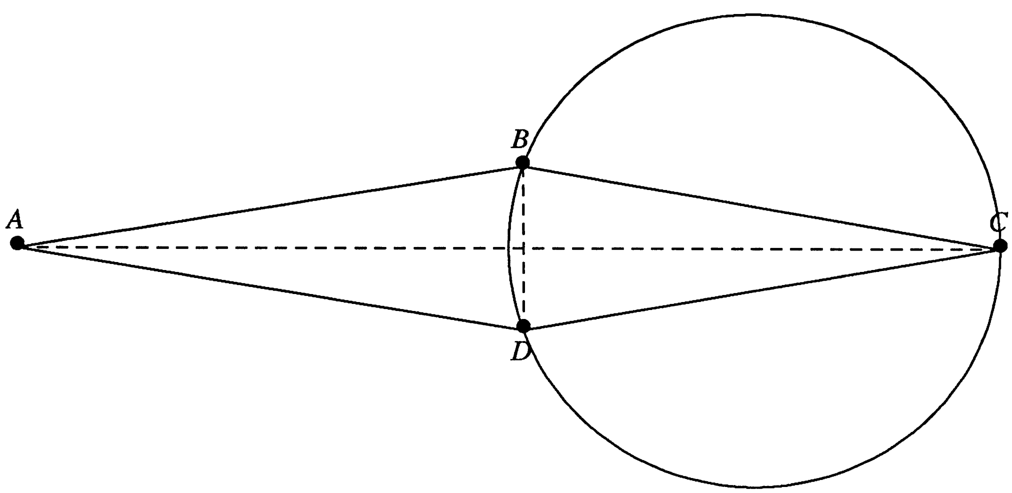

FIG. 3.4

case 2

Assume $\angle ABC$ and $\angle ADC$ are both obtuse. (See figure 3.4) The error circle for each of ΔABC and ΔACD would be the circle with AC as diameter. Let this circle be denoted by Γ. Both vertices B, C are contained in Γ. If triangle BCD has an obtuse angle, its error is indicated by its longest edge which is entirely interior to Γ. If ΔBCD has no obtuse angle, its circumcircle is smaller than Γ. Hence error for ΔBCD is smaller than error for triangle ABC. Same reasoning applies for ΔABD. Therefore diagonal BD should be chosen, the circle criterion holds.

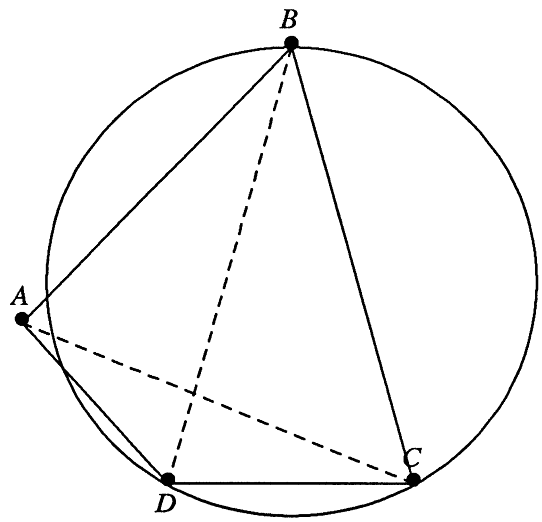

FIG. 3.5

case 3

Assume $\angle ABC \leq \pi/2$ is acute but $\angle ADC > \pi/2$ is obtuse, and ϕ_1, θ_1, η_1,γ_1 are all acute. (See figure 3.5) Error for ΔABC is proportional to the area of its circumcircle,

$$|\bigcirc(\Delta ABC)| = \frac{\pi(AC/2)^2}{\sin^2(\gamma_2+\theta_2)} = \frac{\pi(BC/2)^2}{\sin^2(\phi_1)} = \frac{\pi(AB/2)^2}{\sin^2(\eta_1)} \tag{3.4}$$

If η_2 is obtuse, $|\bigcirc(\Delta ABD)|$ is $\pi(AB/2)^2 < |\bigcirc(\Delta ABC)|$, otherwise it is bounded by the area of the circumcircle $\pi(AB/2)^2/\sin^2(\eta_1)$. If $\eta_2 \leq \pi/2$, then $\eta_1 \leq \eta_2$ and $\sin(\eta_1) \leq \sin(\eta_2)$. Therefore, the error for ΔABD is less than that of ΔABC.

Similarly, If ϕ_2 is obtuse, area for $\bigcirc(\Delta BCD)$ is $\pi(BC/2)^2 < |\bigcirc(\Delta ABC)|$, otherwise it is bounded by the area of its circumcircle $\pi(BC/2)^2/\sin^2(\phi_1)$. If $\phi_2 \leq \pi/2$, then $\phi_1 \leq \phi_2$ and $\sin(\phi_1) \leq \sin(\phi_2)$. Therefore, error for ΔBCD is less than that of ΔABC, and the empty circle criterion selects the incidence minimizing the maximum error.

Cases 1,2,3 exhausts all possibilities for an obtuse angle in ΔABC or ΔACD. The lemma is proved.

Proof of Theorem 1

For the image of the interpolation vertices in the transform plane, a triangulation which satisfies the empty circle criterion can be constructed in $O(N\log N)$ time (theorem 5.18, page 215 in [12]). The convex quadrilaterals of this Delaunay triangulation and of the triangulation induced on the interpolation vertices are in (1-1) correspondence. But by Lemmas 1 and 2, the incidences of the Delaunay triangulation minimize the interpolation error circles, with respect to diagonal interchanges, hence the induced triangulation is locally optimal.

Corollary *The locally optimal triangle incidence of Theorem 1 defines a globally optimal triangle incidence.*

Proof

Starting with any globally optimal triangle incidence, we can apply the local edge swapping procedure in the transformed plane to obtain a locally optimal incidence, with the same maximum error. However, the size of the maximum error circle is uniquely determined for Delaunay triangulation in the transformed plane, so the maximum error in the original globally optimal incidence cannot be smaller than that of a locally optimal incidence, hence a Delaunay triangulation also defineds a globally optimal incidence. Note however, a globally optimal triangulation incidence need not be a Delaunay triangulation.

The use of the transform plane appears in several related contexts in the literature. Nadler[10] uses it to establish the shape of triangulation for optimal L_2 linear interpolation for model quadratic data. Also in [11], Peraire et al. use it to support an adaptive remeshing scheme for the Finite Element Method for compressible flow computations.

4. Appendix A

Derivation of Interpolation Error

Let the interpolation error over the triangle be given by

$$E(x,y) = \lambda_1 x^2 + \lambda_2 y^2 + b_1 x + b_2 y + c \tag{A.1}$$

and by interpolation condition,

$$E(x_1,y_1) = E(x_2,y_2) = E(x_3,y_3) = 0$$

at the three vertices of the triangle. The unknows b_1, b_2, c can easily be obtained by solving the system of linear equations,

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ c \end{bmatrix} = \begin{bmatrix} -r_1 \\ -r_2 \\ -r_3 \end{bmatrix} \tag{A.2}$$

where $r_i = \lambda_1 x_i^2 + \lambda_2 y_i^2$. By Cramer's rule,

$$b_1 = \frac{\det \begin{bmatrix} -r_1 & y_1 & 1 \\ -r_2 & y_2 & 1 \\ -r_3 & y_3 & 1 \end{bmatrix}}{D}, \quad b_2 = \frac{\det \begin{bmatrix} x_1 & -r_1 & 1 \\ x_2 & -r_2 & 1 \\ x_3 & -r_3 & 1 \end{bmatrix}}{D}, \quad c = \frac{\det \begin{bmatrix} x_1 & y_1 & -r_1 \\ x_2 & y_2 & -r_2 \\ x_3 & y_3 & -r_3 \end{bmatrix}}{D} \tag{A.3}$$

where A is area of triangle and

$$D = \det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 2A.$$

Now by (2.4)

$$E_{\max} = \frac{b_1^2}{4\lambda_1} + \frac{b_2^2}{4\lambda_2} - c \tag{A.4}$$

The substitution of (A.3) into (A.4) and its simplification, was obtained through the algebraic computation system Maple[8],

$$E = \frac{\left(D_{12}\, D_{23}\, D_{31}\right)}{16\, \lambda_1 \lambda_2\, A^2} \quad \text{where} \quad D_{ij} = \left(\lambda_1 (x_i - x_j)^2 + \lambda_2 (y_i - y_j)^2\right). \tag{A.5}$$

E_{max} represents the global maximum interpolation error obtained at $(-b_1/2\lambda_1, -b_2/2\lambda_2)$. It can be shown by calculus that the local maximum error along the boundary is attained at the midpoint of each edge. The maximum error along edge (x_i,y_i) , (x_j,y_j) is $|D_{ij}/4|$.

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