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Research Report CS-88-10
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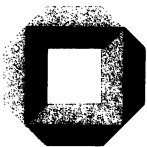
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*Exploring the Properties of
Uniform B-splines Using the
Fourier Integral*

Eng-Wee Chionh

CS-88-10

March, 1988

Exploring the Properties of Uniform B-splines Using the Fourier Integral

Eng-Wee Chionh

University of Waterloo
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Abstract

Many properties of uniform B-splines can be readily derived and studied by using the Fourier integral and convolution. The significance of some of these properties to computer graphics, computer-aided geometric design, signal processing, and image processing is discussed.

1 Introduction

The terms *spline function* and *B-spline* were introduced by Schoenberg in 1946 and 1967 respectively, though they are known earlier without being so called. Due to the many nice structural properties and excellent approximation powers of spline functions, there has been a rapid development of theory and applications of spline functions beginning in the sixties. They have been applied with tremendous success in numerical approximation problems. A recursive definition for B-splines, one of the classes of spline basis functions, was found independently by Cox (1972) and de Boor (1972), who mentioned that Lois Mansfield had also discovered this recursion. This formulation of B-splines greatly facilitates their numerical computation. B-splines were applied to computer-aided geometric design by Riesenfeld (1973), and signal processing by Hou and Andrews (1978). Both of these uses of B-splines are solutions to approximation problems in specific application domains. In computer-aided geometric design, the approximation problem is to construct a curve or surface with certain desirable properties using a given set of control vertices. In signal processing, the approximation problem is to reconstruct the original signal from the signal samples. When the signals are images, this becomes the image reconstruction application. The approximation process is known as interpolation filtering ¹ in signal processing. It is done by convolving the signal samples with an interpolation filter, in this case a B-spline ².

Given the wide applicability of B-splines, it is useful to study their properties. Using the Fourier integral and convolution, we will show that many important properties of B-splines can be readily derived and studied. The significance of some of these properties to computer graphics, computer-aided geometric design, signal processing, and image processing will be discussed. Due to the significance of the function $\frac{\sin \pi x}{\pi x}$ in signal processing, the

¹Interpolation filtering and low-pass filtering are different. In interpolation filtering, signal samples are convolved with an interpolation filter; in low-pass filtering, the original signal is convolved with a low-pass filter.

²Note that when B-splines of order higher than 2 are used, the reconstructed signal will not interpolate the signal samples.

use of B-splines in the integration of various forms of $\frac{\sin \pi x}{\pi x}$ is given. Ease of integration is an immediate consequence of the approach we take to study B-splines, namely through use of the Fourier integral and convolution.

We assume the readers are familiar with B-splines, Fourier integrals, and convolutions. For readers who need information on these subjects, the following references are recommended: Bartels, Betty, & Barsky (1987) for splines and their applications in computer graphics and computer-aided geometric design; Schumaker (1981) for the mathematics of splines and a brief but comprehensive account of the historical development and other representations of B-splines (pages 182–183); Bracewell (1986) and Papoulis (1977) for Fourier integrals, convolutions, and signal processing; Pratt (1978) and Gonzalez & Wintz (1987) for digital image processing.

All conventions, definitions, notations, basic theorems of the Fourier integral and of convolution, and basic properties of important functions are given in the Appendix. It is worth emphasizing our convention that at a point of discontinuity x , $f(x)$ is defined to be $\frac{f(x_-) + f(x_+)}{2}$. Bartels etc., de Boor, Cox, and Schumaker do not adopt an explicit convention for function values at points of discontinuity, but one can be “inferred” from their definition of x_+^0 , which is $f(x) = f(x_+)$. A convention for function values at points of discontinuity is necessary in order to interpret formulas consistently that involve derivatives of B-splines. For example, it will be shown that $B'_{0,2}(\bar{u}) = B_{0,1}(\bar{u}) - B_{1,1}(\bar{u})$. A convention for the value of $B'_{0,2}(1)$ is required as $B_{0,2}(\bar{u})$ is not differentiable at $\bar{u} = 1$ but $B_{0,1}(\bar{u})$ and $B_{1,1}(\bar{u})$ are defined for all values of \bar{u} (See Figure 1).

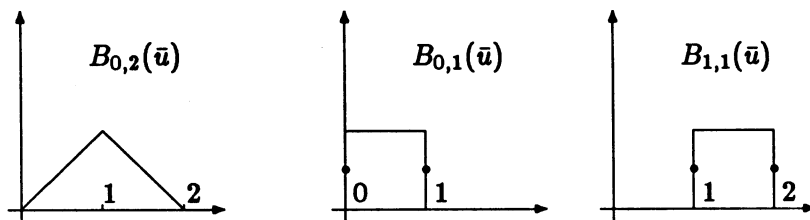


Figure 1: Graph of $B_{0,2}(\bar{u})$, $B_{0,1}(\bar{u})$, and $B_{1,1}(\bar{u})$.

2 Some Observations

Those familiar with Fourier integral and convolution will notice the resemblance of the shapes of the B-splines and the repeated convolutions of a rectangle function. Indeed, we will show that the uniform B-splines of integer knots are the repeated convolutions of the unit rectangle function $\mu(x)$ on the unit interval $(0, 1)$. This would not be too surprising if one knows that Schoenberg (1946) defined the following function in his studies of spline curves:

$$M_k(x) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{2 \sin(u/2)}{u} \right)^k e^{jux} du \quad k \geq 1$$

Since $\Pi(x) \leftrightarrow \frac{\sin(\omega/2)}{\omega/2}$, it can be seen that $M_k(x)$ is simply $\Pi(x)^{*k}$, i.e. $\Pi(x)$ convolved with itself $k - 1$ times, where $\Pi(x)$ is the unit rectangle function defined on the unit interval $(-\frac{1}{2}, \frac{1}{2})$. With this definition of $M_k(x)$, Schoenberg established that

$$M_k(x) = \frac{1}{(k-1)!} \Delta^k x_+^{k-1}$$

where x_+^{k-1} is the one-sided power function and Δ^k is the k th order unit step central difference.

The approach taken in this paper is exactly the opposite of that of Schoenberg. We will prove that:

$$(-1)^k k [0, \dots, k : t] (\bar{u} - t)_+^{k-1} = \mu(\bar{u})^{*k} \quad k \geq 1$$

But by the definition of B-splines given by Bartels etc. (1987), the left hand side of the above equation is a uniform B-spline with knots $0, \dots, k$. Note that due to the difference of convention for function values at points of discontinuity, when $k = 1$, our B-spline differs from that of Bartels etc., de Boor, Cox, and Schumaker at $\bar{u} = 0, 1$.

Without loss of generality, we only study uniform B-splines with integer knots. It is advantageous to be able to express the uniform B-splines either as divided differences or repeated convolutions. It seems that divided differences facilitate numerical computations but repeated convolutions facilitate analytical studies as will be shown in later sections.

3 Divided Difference and Convolution

Divided difference and convolution are closely related by the Dirac delta function $\delta(x)$. Several relationships between divided difference and convolution are established in this section. They will be needed later to establish other results.

We first establish Lemmas 1, 2, 3, and 4. These four lemmas give us the divided difference of an arbitrary function $f(\bar{u})$ in terms of convolution.

Lemma 1

$$[i, \dots, i+k:t]f(\bar{u}-t) = [0, \dots, k:t]f(\bar{u}-i-t)$$

where $k \geq 0$.

Proof

By the definition of divided difference.

■

Lemma 2

$$[i, \dots, i+k:t]f(\bar{u}-t) = ([0, \dots, k:t]f(\bar{u}-t)) * \delta(\bar{u}-i)$$

where $k \geq 0$.

Proof

Use Lemma 1 and note that $[0, \dots, k:t]f(\bar{u}-i-t)$ is a linear combination of terms of the form $f(\bar{u}-i-t)$, $0 \leq t \leq k$, each of which can be written as $f(\bar{u}-t) * \delta(\bar{u}-i)$.

■

Lemma 3

$$[0, \dots, k+1 : t]f(\bar{u} - t) = [0, \dots, k : t]f(\bar{u} - t) * \frac{\delta(\bar{u} - 1) - \delta(\bar{u})}{k+1}$$

where $k \geq 0$.

Proof

This is simply the definition of divided difference rewritten using Lemma 2.

Note that $h(\bar{u}) * \delta(\bar{u}) = h(\bar{u})$ for any $h(\bar{u})$.

■

Lemma 4

$$[0, \dots, k : t]f(\bar{u} - t) = \frac{(-1)^k}{k!} f(\bar{u}) * (\delta(\bar{u}) - \delta(\bar{u} - 1))^{*k}$$

where $k \geq 0$. Note the convention that $h(\bar{u})^{*0} = \delta(\bar{u})$ for any $h(\bar{u})$.

Proof

By mathematical induction and Lemma 3.

■

Lemma 5 shows that $(\delta(\bar{u}) - \delta(\bar{u} - 1))^{*k}$ is a linear combination of some translations of the Dirac delta function $\delta(\bar{u})$. An immediate consequence of Lemma 5 is Corollary 1, which shows that the divided difference of a function at knots $0, \dots, k$ is a linear combination of some translations of that function.

Lemma 5

$$(\delta(\bar{u}) - \delta(\bar{u} - 1))^{*k} = \sum_{i=0}^k \binom{k}{i} (-1)^i \delta(\bar{u} - i)$$

where $k \geq 0$.

Proof

Use mathematical induction and note that $\delta(\bar{u} - i) * \delta(\bar{u} - j) = \delta(\bar{u} - i - j)$.

■

Corollary 1

$$[0, \dots, k : t] f(\bar{u} - t) = \frac{(-1)^k}{k!} \sum_{i=0}^k \binom{k}{i} (-1)^i f(\bar{u} - i)$$

where $k \geq 0$.

Proof

By Lemmas 4 and 5.

■

We will now establish Theorem 1, which can be considered as a special case of Lemma 6. Theorem 1 is essential for the subsequent sections, it relates the divided difference of a function to the derivative of that function (if it exists) and the repeated convolution of the rectangle function $\mu(\bar{u})$.

Lemma 6

$$[0, \dots, k : t]f(\bar{u} - t) = \frac{(-1)^k}{k!} f^{(m)}(\bar{u}) * \mu(\bar{u})^{*m} * (\delta(\bar{u}) - \delta(\bar{u} - 1))^{*(k-m)}$$

where $0 \leq m \leq k$.

Proof

By Lemma 4 and using $\mu'(\bar{u}) = \delta(\bar{u}) - \delta(\bar{u} - 1)$, $f(\bar{u}) * g'(\bar{u}) = f'(\bar{u}) * g(\bar{u})$.

■

Theorem 1

$$[0, \dots, k : t]f(\bar{u} - t) = \frac{(-1)^k}{k!} f^{(k)}(\bar{u}) * \mu(\bar{u})^{*k}$$

where $k \geq 0$.

Proof

Let $m = k$ in Lemma 6.

■

4 Uniform B-Splines by Repeated Convolutions

There are many equivalent definitions for B-splines. We shall take the definition of divided difference as the standard definition for B-splines, then show that this is equivalent to one by repeated convolution when the knot sequences are uniform. For notational brevity we assume integer knots. The following definition for B-splines by divided difference is taken from Bartels etc. (1987, page 146).

Definition 1

Given knots $\bar{u}_0, \dots, \bar{u}_m, \dots, \bar{u}_{m+k}$ with $i \leq m$ and $\bar{u}_i < \bar{u}_{i+k}$, $B_{i,k}(\bar{u})$, the B-spline of order k associated with the knots $\bar{u}_i, \dots, \bar{u}_{i+k}$ is defined to be

$$B_{i,k}(\bar{u}) \equiv (-1)^k (\bar{u}_{i+k} - \bar{u}_i) [\bar{u}_i, \dots, \bar{u}_{i+k} : t] (\bar{u} - t)_+^{k-1}$$

Applying the above definition to the knots $i, \dots, i+k$, we have the following definition for uniform B-spline with integer knots.

Definition 2

The uniform B-spline of order k with knots $i, \dots, i+k$ is

$$B_{i,k}(\bar{u}) = (-1)^k k [i, \dots, i+k : t] (\bar{u} - t)_+^{k-1}$$

where $k \geq 1$.

From this definition, it can be seen that $B_{i,k}(\bar{u})$ is a piecewise polynomial by Corollary 1 and the fact that \bar{u}_+^n is a piecewise polynomial. The constituent polynomials of $B_{0,k}(\bar{u})$, $1 \leq k \leq 5$ are given in the Appendix.

From Definition 2, we see that for any integer i , there is a uniform B-spline $B_{i,k}(\bar{u})$ of order k . The following theorem relates these uniform B-splines: each $B_{i,k}(\bar{u})$ is a translation of $B_{0,k}(\bar{u})$. Thus without loss of generality, we need only study the properties of $B_{0,k}(\bar{u})$.

Theorem 2

$$\begin{aligned} B_{i,k}(\bar{u}) &= B_{0,k}(\bar{u}) * \delta(\bar{u} - i) \\ &= B_{0,k}(\bar{u} - i) \end{aligned}$$

where $k \geq 1$.

Proof

By Lemma 2 and the definition of $B_{i,k}(\bar{u})$.

■

The following theorem is a main result. It shows that the uniform B-spline $B_{0,k}(\bar{u})$ is simply the repeated convolution of the rectangle function $\mu(\bar{u})$.

Theorem 3

$$B_{0,k}(\bar{u}) = \mu(\bar{u})^{*k}$$

where $k \geq 1$.

Proof

Let $f(\bar{u}) = \bar{u}_+^{k-1}$ in Theorem 1. Note that $(\bar{u}_+^{k-1})^{(k-1)} = (k-1)! U(\bar{u})$ and $U'(\bar{u}) = \delta(\bar{u})$, where $U(\bar{u})$ is the Heaviside unit step function.

■

Since

$$\mu(\bar{u}) \leftrightarrow \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-\frac{j\omega}{2}}$$

we have

$$B_{0,k}(\bar{u}) = \mu(\bar{u})^{*k} \leftrightarrow \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-\frac{j\omega}{2}} \right)^k$$

Using this result, many properties of $B_{0,k}(\bar{u})$ can be derived by examining the corresponding properties in the frequency domain.

From Theorem 3, Theorems 4 and 5 can be readily derived. They show that higher order uniform B-splines can be computed from lower order uniform B-splines either by convolution or by integration.

Theorem 4

$$B_{0,k}(\bar{u}) = B_{0,k-m}(\bar{u}) * B_{0,m}(\bar{u})$$

where $0 < m < k$.

Proof

By Theorem 3 and the associativity of convolution.

■

Theorem 4 says that higher order B-splines can be obtained recursively from lower order B-splines by convolution. For example, we have

$$\begin{aligned} B_{0,5}(\bar{u}) &= B_{0,1}(\bar{u}) * B_{0,4}(\bar{u}) \\ &= B_{0,2}(\bar{u}) * B_{0,3}(\bar{u}) \\ &= B_{0,2}(\bar{u}) * B_{0,2}(\bar{u}) * B_{0,1}(\bar{u}) \\ &= \dots \end{aligned}$$

Theorem 5

$$B_{0,k+1}(\bar{u}) = \int_{\bar{u}-1}^{\bar{u}} B_{0,k}(v) dv$$

where $k \geq 1$.

Proof

By Theorem 4,

$$\begin{aligned} B_{0,k+1}(\bar{u}) &= B_{0,k}(\bar{u}) * \mu(\bar{u}) \\ &= \int_{-\infty}^{+\infty} B_{0,k}(v) \mu(\bar{u} - v) dv \end{aligned}$$

But $\mu(\bar{u} - v) = 1$ when $0 < \bar{u} - v < 1$ and equals zero elsewhere except at $\bar{u} = 0, 1$, hence the theorem is proved.

■

Thus higher order B-splines can be obtained recursively from lower order B-splines by integration. This theorem is useful in computing the algebraic expressions for the constituent polynomials of a uniform B-spline. In algebraic manipulation systems such as the *Maple* (Char, Geddes, Gonnet, & Watt 1985), programming an integration is straightforward. Since the value of an integration is not affected by ignoring function values at a finite number of points, this way of obtaining algebraic expressions has the advantage of not having to treat knot points specially.

5 Properties of $B_{0,k}(\bar{u})$

Many properties of uniform B-splines are derived in this section. Each property is given in the form of a theorem. The significance of each theorem is discussed following its proof. The proof of these theorems clearly shows the usefulness of the convolution definition.

Theorem 6

$$B_{0,k}(\bar{u}) \text{ is } C^{k-2}$$

where $k \geq 1$.

Proof

Clearly $B_{0,1}(\bar{u})$ is C^{-1} . For $k \geq 2$ write Theorem 5 as

$$B_{0,k+1}(\bar{u}) = \int_{-\infty}^{\bar{u}} B_{0,k}(v) dv - \int_{-\infty}^{\bar{u}-1} B_{0,k}(v) dv$$

hence

$$B'_{0,k+1}(\bar{u}) = B_{0,k}(\bar{u}) - B_{0,k}(\bar{u} - 1)$$

The proof follows by mathematical induction and the fact that $B_{0,2}(\bar{u})$ is C^0 .

■

This theorem reveals the continuity of curves and surfaces when control vertices are approximated by uniform B-splines. Knowledge of continuity is essential in controlling the smoothness of the curves and surfaces in computer aided geometric design. Smoothness is important in design for either functional or aesthetic reasons.

Theorem 7

$B_{0,k}(\bar{u})$ is symmetric about $k/2$

where $k \geq 1$.

Proof

$B_{0,k}(\bar{u})$ is symmetric about $k/2$ iff $B_{0,k}(\bar{u} + \frac{k}{2}) = B_{0,k}(\bar{u}) * \delta(\bar{u} + \frac{k}{2})$ is symmetric about the origin, i.e. an even function. But a function is even iff its Fourier integral is real, indeed

$$\begin{aligned} B_{0,k}(\bar{u}) * \delta(\bar{u} + k/2) &\leftrightarrow \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-j\frac{\omega}{2}} \right)^k e^{jk\omega/2} \\ &\leftrightarrow \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^k \end{aligned}$$

is real.

■

Theorem 7 is useful when $B_{0,k}(\bar{u} + \frac{k}{2})$ is viewed as an interpolation filter. It allows the interpolated value at any point x to be interpreted as either the sum at x of filters centered at each sample point weighted by the respective sample values or the sum of filter values at each sample point weighted by the respective sample values of the filter centered at x (Smith, 1983, page 248). An example would clarify the above comment. Consider a triangular filter $f(x)$ covering 5 sample points. From Figure 2, the sum at x_0 of contributing weighted filters centered at the sample points a, b, c, d is

$$Af(x_0 - a) + Bf(x_0 - b) + Cf(x_0 - c) + Df(x_0 - d) \quad (1)$$

From Figure 3, the sum of weighted filter values at contributing sample points a, b, c, d of the filter centered at x_0 is

$$Af(a - x_0) + Bf(b - x_0) + Cf(c - x_0) + Df(d - x_0) \quad (2)$$

Clearly Equations 1 and 2 are equal when $f(x)$ is even, i.e. symmetric about the origin.

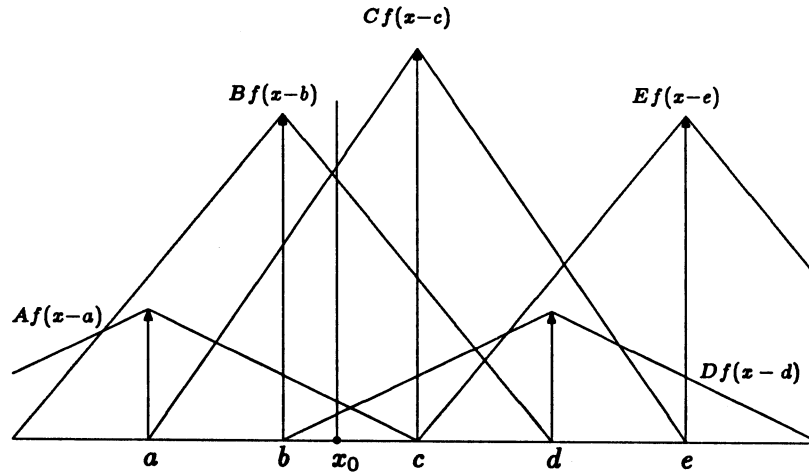


Figure 2: Filters centered at sample points weighted by the sample values.

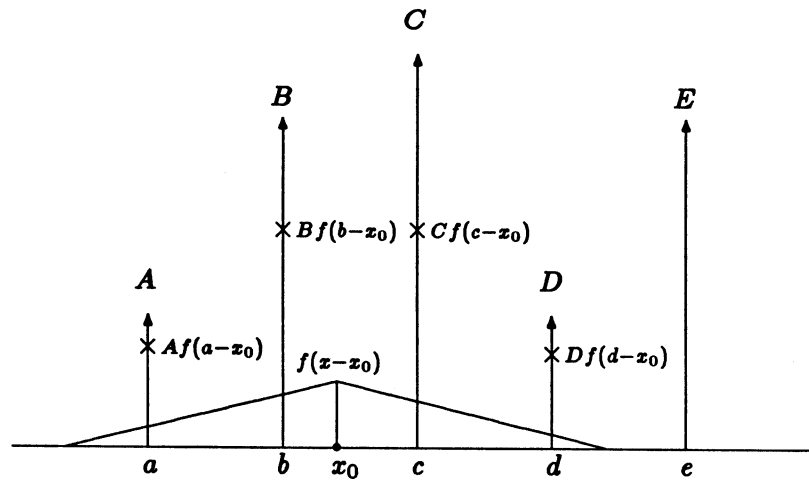


Figure 3: Weighted filter values at sample points of the filter centered at x_0 .

Theorem 8

$$\begin{aligned}
B_{0,k}(\bar{u}) &> 0 & 0 < \bar{u} < k \\
B_{0,k}(\bar{u}) &= 0 & \bar{u} \leq 0 \text{ or } \bar{u} \geq k
\end{aligned}$$

where $k \geq 2$.

Proof

By mathematical induction and Theorem 5.

■

Theorem 8 gives the non-negativity and locality properties. Locality is important in computer graphics and computer-aided geometric design: manipulation of a control vertex only affects a portion of the curve or surface “near” that vertex rather than the entire curve or surface. Non-negativity is desirable in image processing. It ensures that the intensities of images recovered from non-negative samples by a uniform B-spline interpolation filter would not be negative.

Theorem 9

$$\sum_{i=-\infty}^{\infty} B_{i,k}(\bar{u}) = 1$$

where $k \geq 1$.

Proof

By Theorem 2

$$\begin{aligned} \sum_{i=-\infty}^{\infty} B_{i,k}(\bar{u}) &= \sum_{i=-\infty}^{\infty} B_{0,k}(\bar{u}) * \delta(\bar{u} - i) \\ &= B_{0,k}(\bar{u}) * \sum_{i=-\infty}^{\infty} \delta(\bar{u} - i) \\ &\leftrightarrow \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}} \right)^k 2\pi \sum_{i=-\infty}^{\infty} \delta(\omega - 2i\pi) \end{aligned}$$

The last expression means function $\left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}} \right)^k$ is sampled at $\{2i\pi\}_{i=-\infty}^{\infty}$, which gives $2\pi\delta(\omega)$. This proves the theorem since $1 \leftrightarrow 2\pi\delta(\omega)$.

■

Any collection of functions whose sum is uniformly equal to one is said to have the *partition of unity* property. This and the non-negativity property of Theorem 8 ensure the convex hull property, which is important in computer graphics and computer-aided geometric design. Convex hull property means that any B-spline curve or surface will be entirely contained within the convex hull of its control vertices.

The partition of unity property also provides translation invariance. Thus in general, B-spline geometric modeling is affine invariant. This means that the affine transformation (Gans, 1969) of the approximated curve or surface is the approximated curve or surface of the transformed control vertices. Since affine transformations are one-to-one and onto mappings between Euclidean spaces of the same dimensions, being affine invariant means that the B-spline geometric modeling is coordinate independent.

Theorem 10

$$\int_{-\infty}^{+\infty} B_{0,k}(\bar{u}) d\bar{u} = 1$$

where $k \geq 1$.

Proof

Since

$$B_{0,k}(\bar{u}) \leftrightarrow \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-j\frac{\omega}{2}} \right)^k$$

By Fourier integral theorem we have

$$\left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-j\frac{\omega}{2}} \right)^k = \int_{-\infty}^{+\infty} B_{0,k}(\bar{u}) e^{-j\omega \bar{u}} d\bar{u}$$

The theorem follows by letting $\omega = 0$.

■

Thus we can use $B_{0,k}(\bar{u} + \frac{k}{2})$ as a convolution filter ³ without having to perform normalization. However, when the knot spacings are not unity as assumed, the integral value is ϵ , which is the knot spacing for knots $0, \epsilon, \dots, k\epsilon$. In this case a simple normalization is required.

³Note that $B_{0,k}(\bar{u} + \frac{k}{2})$ is symmetric about the origin. See Theorem 7.

Theorem 11

$$B_{0,k+1}(\bar{u}) = \frac{\bar{u}}{k} B_{0,k}(\bar{u}) + \frac{k+1-\bar{u}}{k} B_{1,k}(\bar{u})$$

where $k \geq 1$.

Proof

Consider

$$\begin{aligned} -j\bar{u}(B_{0,k}(\bar{u}) * (\delta(\bar{u}) - \delta(\bar{u} - 1))) &\leftrightarrow \frac{d}{d\omega} \left[\left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-\frac{j\omega}{2}} \right)^k (1 - e^{-j\omega}) \right] \\ &\leftrightarrow 2j \frac{d}{d\omega} \left[\frac{\sin^{k+1}(\frac{\omega}{2})}{(\frac{\omega}{2})^k} e^{-\frac{j(k+1)\omega}{2}} \right] \\ &\leftrightarrow j \left[(k+1) \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-\frac{j\omega}{2}} \right)^k e^{-j\omega} - \right. \\ &\quad \left. k \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-\frac{j\omega}{2}} \right)^{k+1} \right] \end{aligned}$$

But the inverse Fourier integral of the above is

$$j((k+1)B_{0,k}(\bar{u}) * \delta(\bar{u} - 1) - kB_{0,k+1}(\bar{u}))$$

hence the theorem is proved.

■

Theorem 11 is a special case of the Cox/de Boor B-spline recurrence when the knots are $0, \dots, k$. This recurrence greatly facilitates the numerical computations of B-splines.

Theorem 12

$$B_{0,k}^{(m+n)}(\bar{u}) = \sum_{i=0}^m \binom{m}{i} (-1)^i B_{0,k-m}^{(n)}(\bar{u} - i)$$

where $0 \leq m < k$.

Proof

By Theorem 3 we have

$$B_{0,k}(\bar{u}) = B_{0,k-m}(\bar{u}) * \mu(\bar{u})^{*m}$$

But $(f(\bar{u}) * g(\bar{u}))' = f'(\bar{u}) * g(\bar{u}) = f(\bar{u}) * g'(\bar{u})$ and $\mu'(\bar{u}) = \delta(\bar{u}) - \delta(\bar{u} - 1)$, hence

$$B_{0,k}^{(m+n)}(\bar{u}) = B_{0,k-m}^{(n)}(\bar{u}) * (\delta(\bar{u}) - \delta(\bar{u} - 1))^{*m}$$

By Lemma 5, the theorem is proved.

■

In curve and surface design, it is necessary to perform derivative calculations. This theorem provides a general recurrence formulation for the higher derivatives of higher order B-splines in terms of the lower derivatives of lower order B-splines.

Theorem 13

$B_{0,k}(\bar{u})$ approaches the Gaussian distribution function for large k .

Proof

Use Theorem 3 and the central-limit theorem. See Bracewell (1978, page 168) for more detail.

■

This theorem describes the asymptotic behavior of B-splines in terms of their orders.

6 Evaluation of $\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k dx$

The interpolating or filtering function $\frac{\sin \pi x}{\pi x}$ is significant. It can be thought of either as the perfect reconstruction filter for samples of a bandlimited signal satisfying the Nyquist sampling rate condition, or as the ideal low-pass convolution filter. In this section we relate $B_{0,k}(\bar{u})$ to the evaluation of integrals of variants of $\frac{\sin \pi x}{\pi x}$ in more general forms than those given in the mathematical handbooks by Beyer (1981) and Gradshteyn & Ryzhik (1980). It is interesting to note that $\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^{2k} dx$ provides an indication of the attenuation of signal energy after convolving a signal with a B-spline of order k . For if $f(x), g(x)$ are the signals before and after the convolution with a B-spline $B_{0,k}(x + \frac{k}{2})$, then by Rayleigh's theorem, the signal energy after convolution is

$$\begin{aligned} \int_{-\infty}^{+\infty} |g(x)|^2 dx &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |G(\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |F(\omega)|^2 \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2k} d\omega \\ &\leq |F(\omega_{max})|^2 \int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^{2k} dx \end{aligned}$$

where $f(x) \leftrightarrow F(\omega), g(x) \leftrightarrow G(\omega)$ and note that $B_{0,k}(x + \frac{k}{2}) \leftrightarrow \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^k$.

In image processing, loss of energy due to the attenuation of high frequencies makes images look fuzzy.

We begin by giving the Fourier transform pair of the uniform B-spline $B_{0,k}(\bar{u})$.

Lemma 7

$$B_{0,k}(\bar{u}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}} \right)^k e^{j\omega\bar{u}} d\omega \quad (3)$$

$$\left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}} \right)^k = \int_{-\infty}^{+\infty} B_{0,k}(\bar{u}) e^{-j\omega\bar{u}} d\bar{u} \quad (4)$$

where $k \geq 1$.

Proof

This is simply an application of the Fourier integral theorem to $B_{0,k}(\bar{u})$.

■

With Lemma 7, the following interesting theorem can be obtained.

Theorem 14

$$\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k dx = B_{0,k}\left(\frac{k}{2}\right)$$

where $k \geq 1$.

Proof

Let $\bar{u} = \frac{k}{2}$ in Equation 3 and change the variable ω to $2\pi x$ in the integration.

■

Since $B_{0,k}(\frac{k}{2})$ can be easily computed by the Cox/de Boor B-spline recurrence, we have a general formula for the evaluation of $\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k dx$.

An interesting consequence of Theorem 14 is Corollary 2, which says that $\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k dx$ is rational. Rational values of this integral for $1 \leq k \leq 28$ are given in the Appendix.

Corollary 2

$$\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k dx \text{ is rational.}$$

Proof

By the divided difference definition, $B_{0,k}(\frac{k}{2})$ is the value of a piecewise polynomial with rational coefficients at $\bar{u} = \frac{k}{2}$.

■

The following corollary is given because these integrals are often encountered in signal processing.

Corollary 3

$$\int_{-\infty}^{+\infty} \frac{\sin \pi x}{\pi x} dx = 1$$

$$\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^2 dx = 1$$

Proof

Since $B_{0,1}(\frac{1}{2}) = B_{0,2}(1) = 1$.

■

Lemma 8 and Theorem 15 are given mainly for the motivation of providing more information regarding the definite integrals of variants of $\frac{\sin \pi x}{\pi x}$. Other definite integrals of variants of $\frac{\sin \pi x}{\pi x}$ can be found in mathematical handbooks such as Beyer (1981) and Gradshteyn & Ryzhik (1980).

Lemma 8

$$B_{0,k}(\bar{u}) = \frac{\bar{u}^{k-1}}{(k-1)!} \quad 0 \leq \bar{u} \leq 1$$

$$B_{0,k}(\bar{u}) = \frac{(k-\bar{u})^{k-1}}{(k-1)!} \quad k-1 \leq \bar{u} \leq k$$

where $k \geq 2$, if $k = 1$ then the equations are not true at end points.

Proof

By mathematical induction and Theorems 5 and 8.

■

Theorem 15

$$\int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k \cos(\pi x(2\bar{u} - k)) dx = \frac{\bar{u}^{k-1}}{(k-1)!} \quad 0 \leq \bar{u} \leq 1$$

$$= \frac{(k-\bar{u})^{k-1}}{(k-1)!} \quad k-1 \leq \bar{u} \leq k$$

where $k \geq 2$, if $k = 1$ then the equations are not true at end points.

Proof

Use Lemmas 7 and 8, change the variable ω to $2\pi x$ and note that the imaginary part of the integration is odd, thus equals 0.

■

The next theorem relates the maximum value of $B_{0,2k}(\bar{u})$ and the area under the curve of $B_{0,k}(\bar{u})$.

Theorem 16

$$B_{0,2k}(k) = \int_{-\infty}^{+\infty} B_{0,k}(\bar{u})^2 d\bar{u}$$

where $k \geq 1$.

Proof

By Rayleigh's theorem, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} B_{0,k}(\bar{u})^2 d\bar{u} &= \int_{-\infty}^{+\infty} |B_{0,k}(\bar{u})|^2 d\bar{u} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-\frac{i\omega}{2}} \right)^k \right|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(\frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} \right)^{2k} d\omega \\ &= B_{0,2k}(k) \end{aligned}$$

The last equality is obtained by changing variable $\omega = 2\pi x$ and using Theorem 14.

■

Theorem 16 provides an alternative means of evaluating the value of $B_{0,2k}(k)$ when $B_{0,k}(\bar{u})$ is known. But this is mainly of theoretical interest because $B_{0,2k}(k)$ can be efficiently computed from the Cox/de Boor recurrence.

7 Conclusion

The results presented in this paper are not new. But the main purpose of this paper is to illustrate the intimacy between uniform B-splines with integer knots and repeated convolutions of a rectangle function. It might be that some are more familiar with the divided difference approach but some are more accustomed to the convolution approach. If this is indeed the case, this paper serves to unite these two approaches coherently through the focus of studying the properties of uniform B-splines. In doing so, many important properties of uniform B-splines are explicitly proved and their significance commented. The paper achieves this purpose with one unifying theme and a single tool—the Fourier integral.

8 Acknowledgement

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A Appendix

This appendix contains information for the following:

1. Conventions, definitions, and notations
2. Basic theorems of Fourier integral and convolution
3. Cox/de Boor B-spline recurrence
4. Dirac delta function $\delta(x)$
5. Replicating/sampling function $\Psi(x)$
6. Heaviside unit step function $U(x)$
7. Rectangle function $\Pi(x)$ on interval $(-\frac{1}{2}, \frac{1}{2})$
8. Rectangle function $\mu(x)$ on interval $(0, 1)$
9. Values of $B_{0,k}(\frac{k}{2})$ and $\int_{-\infty}^{+\infty} B_{0,k}(\bar{u})^2 d\bar{u}$
10. Algebraic expressions for $B_{0,k}(\bar{u})$

• \equiv	Definition
• $j \equiv \sqrt{-1}$	The imaginary unit
• $\overline{a + bj} = a - bj$	Complex conjugate
• $-\infty < \bar{u} < \infty$	Parameter variable for spline basis functions
• $-\infty < \omega < \infty$	Independent variable for angular frequency
• $B_{i,k}(\bar{u})$	Uniform B-splines with integer knots $i, \dots, i+k$
• $\binom{n}{k}$	n choose k
• $f'(x), f^{(n)}(x)$	First and n th derivatives of function $f(x)$
• C^k	Set of functions with k continuous derivatives
• C^{-1}	Set of functions with a finite number of jump discontinuities
• $f(x) \equiv \frac{f(x_-) + f(x_+)}{2}$	Function value at point of discontinuity x
• $f(x) * h(x) \equiv \int_{-\infty}^{+\infty} f(u)h(x-u)du$	Convolution of $f(x)$ and $h(x)$
• $f(x) \leftrightarrow F(\omega)$	Fourier transform pair
• $f(x)^{*k} \equiv f(x)^{*(k-1)} * f(x)$ $f(x)^{*0} \equiv \delta(x)$	Self convolution
• $\Delta^k f(x) \equiv \Delta^{k-1}(\Delta(x))$ $\Delta f(x) \equiv f(x + \frac{1}{2}) - f(x - \frac{1}{2})$	k th unit step central difference
• $[\bar{u}_i, \dots, \bar{u}_{i+k} : t]f(t)$ $\equiv \begin{cases} \frac{[\bar{u}_{i+1}, \dots, \bar{u}_{i+k} : t]f(t) - [\bar{u}_i, \dots, \bar{u}_{i+k-1} : t]f(t)}{\bar{u}_{i+k} - \bar{u}_i} & \text{if } \bar{u}_i \neq \bar{u}_{i+k} \\ \frac{1}{k!} \frac{d}{dt} f(t) _{t=\bar{u}_i} & \text{if } \bar{u}_i = \dots = \bar{u}_{i+k} \end{cases}$	k th divided difference
• $[\bar{u}_i : t]f(t) \equiv f(\bar{u}_i)$	
• $\text{rect}(x)$	Rectangle function
$f(x) \equiv \begin{cases} c \neq 0 & a < x < b \\ \frac{c}{2} & x = a, b \\ 0 & \text{otherwise} \end{cases}$	
• $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}, \quad -\infty < x < \infty$	Gaussian distribution function
• x_+^k	Truncated power function
$x_+^0 \equiv \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases} \quad \text{for } k > 0, \quad x_+^k \equiv \begin{cases} 0 & x < 0 \\ x^k & x \geq 0 \end{cases}$	

Table 1: Conventions, Definitions, and Notations

Theorems of Fourier integral and convolution	
Transform and inverse transform	$F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-j\omega x} dx$ $f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega x} d\omega$
Symmetry	$F(x) \leftrightarrow 2\pi f(-\omega)$
Conjugate functions	$\overline{f(x)} = \overline{F(-\omega)}$
Scaling (for any real $a \neq 0$)	$f(ax) = \frac{1}{ a } F\left(\frac{\omega}{a}\right)$
Shifting (for any real a)	$f(x - a) = e^{-ja\omega} F(\omega)$ $e^{ja\omega} f(x) = F(\omega - a)$
Modulation	$f(x) \cos(\omega_0 x) = \frac{1}{2} (F(\omega + \omega_0) + F(\omega - \omega_0))$
Derivatives	$(-jx)^n f(x) \leftrightarrow F^{(n)}(\omega)$ $f^{(n)}(x) \leftrightarrow (j\omega)^n F(\omega)$
Convolution	$f_1(x) * f_2(x) \leftrightarrow F_1(\omega) F_2(\omega)$ $f_1(x) f_2(x) \leftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$
Parseval's formula	$\int_{-\infty}^{+\infty} f_1(x) \overline{f_2(x)} dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F_1(\omega) \overline{F_2(\omega)} d\omega$
Rayleigh's theorem	$\int_{-\infty}^{+\infty} f(x) ^2 dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) ^2 d\omega$
Moment theorem	$F^{(n)}(0) = (-j)^n \int_{-\infty}^{+\infty} x^n f(x) dx$
Derivative of convolution	$(f(x) * g(x))' = f'(x) * g(x) = f(x) * g'(x)$
Convolution algebraic rules	$f(x) * g(x) = g(x) * f(x)$ $f(x) * (g(x) * h(x)) = (f(x) * g(x)) * h(x)$ $f(x) * (g(x) + h(x)) = f(x) * g(x) + f(x) * h(x)$

The following Cox/de Boor B-spline recurrence is taken from Bartels, etc. (1987) pages 182–183. The definition for $B_{i,1}(\bar{u})$ is modified at $\bar{u} = \bar{u}_i, \bar{u}_{i+1}$ to conform to our convention for function values at points of discontinuity.

Given knots $\bar{u}_0, \dots, \bar{u}_m, \dots, \bar{u}_{m+k}$ where $k \geq 1$:

for any $i = 0, \dots, m$

$$B_{i,1}(\bar{u}) = \begin{cases} 0 & \bar{u} < \bar{u}_i \\ \frac{1}{2} & \bar{u} = \bar{u}_i \\ 1 & \bar{u}_i < \bar{u} < \bar{u}_{i+1} \\ \frac{1}{2} & \bar{u} = \bar{u}_{i+1} \\ 0 & \bar{u} > \bar{u}_{i+1} \end{cases}$$

and for any $r = 2, \dots, k$

$$B_{i,r}(\bar{u}) = \frac{\bar{u} - \bar{u}_i}{\bar{u}_{i+r-1} - \bar{u}_i} B_{i,r-1}(\bar{u}) + \frac{\bar{u}_{i+r} - \bar{u}}{\bar{u}_{i+r} - \bar{u}_{i+1}} B_{i+1,r-1}(\bar{u})$$

with the convention that

$$\frac{\bar{u} - \bar{u}_i}{\bar{u}_{i+r-1} - \bar{u}_i} B_{i,r-1}(\bar{u}) = 0 \quad \text{if} \quad \bar{u}_{i+r-1} - \bar{u}_i = 0$$

and

$$\frac{\bar{u}_{i+r} - \bar{u}}{\bar{u}_{i+r} - \bar{u}_{i+1}} B_{i+1,r-1}(\bar{u}) = 0 \quad \text{if} \quad \bar{u}_{i+r} - \bar{u}_{i+1} = 0$$

Figure 4: Cox/de Boor B-spline recurrence.

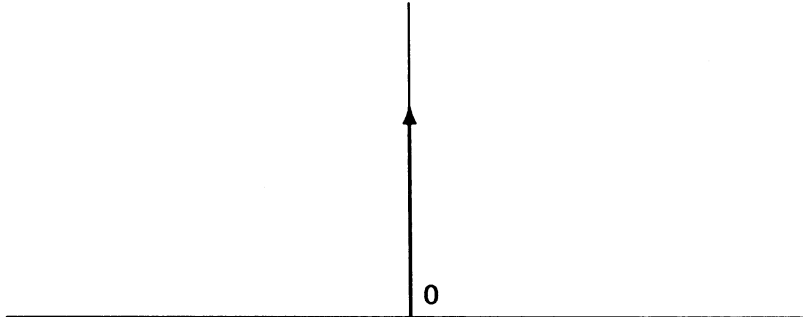


Figure 5: Graph of $\delta(x)$.

Dirac delta function $\delta(x)$	
Fourier transform pair	$\delta(x) \leftrightarrow 1 \leftrightarrow 2\pi \delta(\omega)$
Unit area	$\int_{-\infty}^{+\infty} \delta(x) dx = 1$
Sampling	$f(x)\delta(x) = f(0)\delta(x)$
Sifting	$\int_{-\infty}^{+\infty} f(x)\delta(x) dx = f(0)$
Shifting	$f(x) * \delta(x - a) = f(x - a)$ $f(bx) * \delta(x - a) = \delta(b(x - a))$
Convolution identity	$f(x) * \delta(x) = f(x)$
Derivative of $U(x)$	$U'(x) = \delta(x)$
Derivative of $\mu(x)$	$\mu'(x) = \delta(x) - \delta(x - 1)$

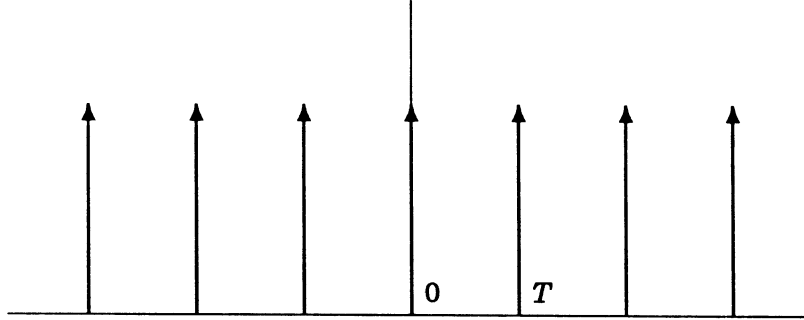


Figure 6: Graph of $\Psi(x) = \sum_{i=-\infty}^{\infty} \delta(x - iT)$.

Replicating/sampling function $\Psi(x)$	
Fourier transform	$\sum_{i=-\infty}^{\infty} \delta(x - iT) \leftrightarrow \omega_0 \sum_{i=-\infty}^{\infty} \delta(\omega - i\omega_0)$ <p style="text-align: center;">where $\omega_0 = \frac{2\pi}{T}$</p>
Almost zero everywhere	$\Psi(x) = 0 \quad x \neq iT$
Local unit area	$\int_{(i-\frac{1}{2})T}^{(i+\frac{1}{2})T} \Psi(x) dx = 1$
Sampling	$\Psi(x)f(x) = \sum_{i=-\infty}^{\infty} f(iT)\delta(x - iT)$
Replicating	$\Psi(x) * f(x) = \sum_{i=-\infty}^{\infty} f(x - iT)$

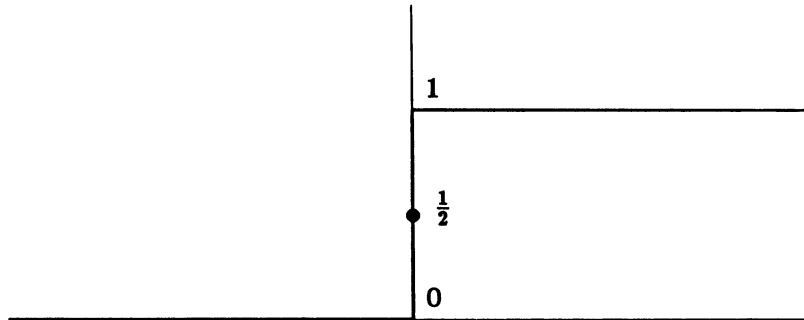


Figure 7: Graph of $U(x)$.

Heaviside unit step function $U(x)$	
Fourier transform	$U(x) \leftrightarrow \pi \delta(\omega) + \frac{1}{j\omega}$
Differentiation	$U'(x) = \delta(x)$
Divided difference	$U(x) - U(x - 1) = \mu(x)$
Truncated power function	$x_+^0 = U(x)$

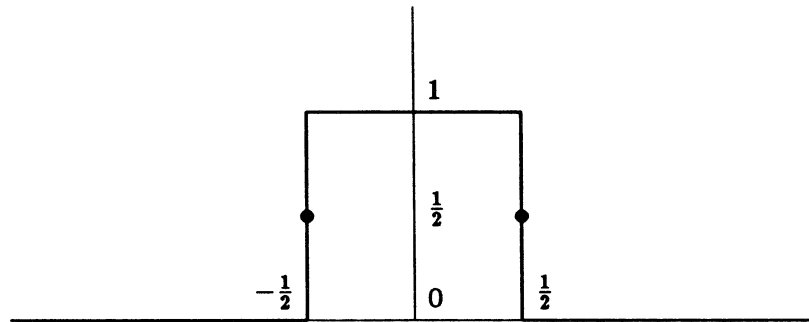


Figure 8: Graph of $\Pi(\bar{u})$.

Unit rectangle function $\Pi(x)$	
Fourier transform	$\Pi(x) \leftrightarrow \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}}$
Differentiation	$\Pi'(x) = \delta(x + \frac{1}{2}) - \delta(x - \frac{1}{2})$
Translation	$\Pi(x) = \mu(x) * \delta(x + \frac{1}{2})$
B-spline	$\Pi(x) = B_{0,1}(x + \frac{1}{2})$

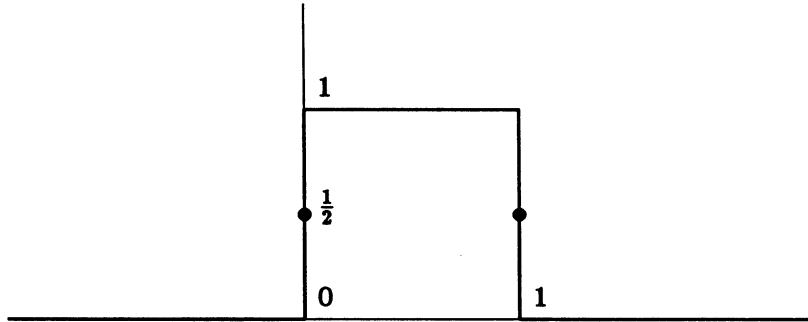


Figure 9: Graph of $\mu(x)$.

Unit rectangle function $\mu(x)$	
Fourier transform	$\mu(x) \leftrightarrow \frac{\sin \frac{\omega}{2}}{\frac{\omega}{2}} e^{-i\frac{\omega}{2}}$
Differentiation	$\mu'(x) = \delta(x) - \delta(x - 1)$
Divided difference	$\mu(x) = U(x) - U(x - 1)$
Translation	$\mu(x) = \Pi(x) * \delta(x - \frac{1}{2})$
B-spline	$\mu(x)^{*k} = B_{0,k}(x) \quad k \geq 1$

k	$B_{0,k}(\frac{k}{2}) = \int_{-\infty}^{+\infty} \left(\frac{\sin \pi x}{\pi x} \right)^k dx$	$B_{0,2k}(k) = \int_{-\infty}^{+\infty} B_{0,k}(\bar{u})^2 d\bar{u}$
1	1.00000 1	1.00000 1
2	1.00000 1	0.66667 $\frac{2}{3}$
3	0.75000 $\frac{3}{4}$	0.55000 $\frac{11}{20}$
4	0.66667 $\frac{2}{3}$	0.47937 $\frac{151}{315}$
5	0.59896 $\frac{115}{192}$	0.43042 $\frac{15619}{36288}$
6	0.55000 $\frac{11}{20}$	0.39393 $\frac{655177}{1663200}$
7	0.51102 $\frac{5887}{11520}$	0.36537 $\frac{27085381}{74131200}$
8	0.47937 $\frac{151}{315}$	0.34224 $\frac{2330931341}{6810804000}$
9	0.45292 $\frac{259723}{373440}$	0.32301 $\frac{12157712239}{37638881280}$
10	0.43042 $\frac{15619}{36288}$	0.30669 $\frac{37307713155613}{121845100408832}$
11	0.41096 $\frac{381773117}{928972800}$	0.29262 $\frac{339781108897078469}{1161157776629760000}$
12	0.39393 $\frac{655177}{1663200}$	0.28033 $\frac{75489558096433522049}{269291841030051840000}$
13	0.37884 $\frac{20646903199}{54499737600}$	0.26946 $\frac{11482547005345338463969}{42613214404755456000000}$
14	0.36537 $\frac{27085381}{74131200}$	0.25977 $\frac{3607856726470666022715979}{13888864094921367552000000}$
15	0.35324 $\frac{467168310097}{1322526965760}$	0.25105 $\frac{18497593486903125823791655511}{73681349947830849621196800000}$
16	0.34224 $\frac{2330931341}{6810804000}$	0.24315 $\frac{520679973964725199436393399689}{2141364232858834067116032000000}$
17	0.33221 $\frac{75920439315929441}{228532659683328000}$	0.23596 $\frac{53566213980441557762900314820839}{227014839707500300536423383040000}$
18	0.32301 $\frac{12157712239}{37638881280}$	0.22937 $\frac{731509401860533204925821188658871713}{3189243199501896583230447943680000000}$
19	0.31453 $\frac{5278968781483042969}{16783438527143608320}$	0.22330 $\frac{15405709916669781801989122003838993353}{68991243565044336071759296148275200000}$
20	0.30669 $\frac{37307713155613}{121645100408832}$	0.21769 $\frac{10572354363336924802260977429426060187229}{48566385007612960377714956523578327040000}$
21	0.29941 $\frac{9093099984535515162569}{30370031620545576960000}$	0.21248 —
22	0.29262 $\frac{339781108897078469}{1161157776629760000}$	0.20763 —
23	0.28628 $\frac{168702835448229388944396777}{589300093565066375331840000}$	0.20310 —
24	0.28033 $\frac{75489558096433522049}{269291841030051840000}$	0.19885 —
25	0.27473 $\frac{28597941405166726516864710559}{104093968527333324538616217600}$	0.19485 —
26	0.26946 $\frac{11482547005345338463969}{42613214404755456000000}$	— —
27	0.26448 $\frac{430374979754582929417781296799}{1627250590254128449978368000000}$	— —
28	0.25977 $\frac{3607856726470666022715979}{13888864094921367552000000}$	— —

Table 2: Values of $B_{0,k}(\frac{k}{2})$ and $\int_{-\infty}^{+\infty} B_{0,k}(\bar{u})^2 d\bar{u}$ by the *Maple* algebraic computation system (Char, Geddes, Gonnet, & Watt 1985).

$$\begin{aligned}
B_{0,1}(\bar{u}) &= \begin{cases} 0 & \bar{u} < 0 \\ \frac{1}{2} & \bar{u} = 0 \\ 1 & 0 < \bar{u} < 1 \\ \frac{1}{2} & \bar{u} = 1 \\ 0 & \bar{u} > 1 \end{cases} \\
B_{0,2}(\bar{u}) &= \begin{cases} 0 & \bar{u} < 0 \\ \bar{u} & 0 \leq \bar{u} < 1 \\ -\bar{u} + 2 & 1 \leq \bar{u} < 2 \\ 0 & \bar{u} > 2 \end{cases} \\
B_{0,3}(\bar{u}) &= \begin{cases} 0 & \bar{u} < 0 \\ \frac{1}{2}\bar{u}^2 & 0 \leq \bar{u} < 1 \\ -\bar{u}^2 + 3\bar{u} - \frac{3}{2} & 1 \leq \bar{u} < 2 \\ \frac{1}{2}\bar{u}^2 - 3\bar{u} + \frac{9}{2} & 2 \leq \bar{u} < 3 \\ 0 & \bar{u} > 3 \end{cases} \\
B_{0,4}(\bar{u}) &= \begin{cases} 0 & \bar{u} < 0 \\ \frac{1}{6}\bar{u}^3 & 0 \leq \bar{u} < 1 \\ -\frac{1}{2}\bar{u}^3 + 2\bar{u}^2 - 2\bar{u} + \frac{2}{3} & 1 \leq \bar{u} < 2 \\ \frac{1}{2}\bar{u}^3 - 4\bar{u}^2 + 10\bar{u} - \frac{22}{3} & 2 \leq \bar{u} < 3 \\ -\frac{1}{6}\bar{u}^3 + 2\bar{u}^2 - 8\bar{u} + \frac{32}{3} & 3 \leq \bar{u} < 4 \\ 0 & \bar{u} > 4 \end{cases} \\
B_{0,5}(\bar{u}) &= \begin{cases} 0 & \bar{u} < 0 \\ \frac{1}{24}\bar{u}^4 & 0 \leq \bar{u} < 1 \\ -\frac{1}{6}\bar{u}^4 + \frac{5}{6}\bar{u}^3 - \frac{5}{4}\bar{u}^2 + \frac{5}{6}\bar{u} - \frac{5}{24} & 1 \leq \bar{u} < 2 \\ \frac{1}{4}\bar{u}^4 - \frac{5}{2}\bar{u}^3 + \frac{35}{4}\bar{u}^2 - \frac{25}{2}\bar{u} + \frac{155}{24} & 2 \leq \bar{u} < 3 \\ -\frac{1}{6}\bar{u}^4 + \frac{5}{2}\bar{u}^3 - \frac{55}{4}\bar{u}^2 + \frac{65}{2}\bar{u} - \frac{655}{24} & 3 \leq \bar{u} < 4 \\ \frac{1}{24}\bar{u}^4 - \frac{5}{6}\bar{u}^3 + \frac{25}{4}\bar{u}^2 - \frac{125}{6}\bar{u} + \frac{625}{24} & 4 \leq \bar{u} < 5 \\ 0 & \bar{u} > 5 \end{cases}
\end{aligned}$$

Figure 10: Algebraic expressions for the constituent polynomials of $B_{0,k}(\bar{u})$ by the *Maple* algebraic computation system (Char, Geddes, Gonnet, & Watt 1985).

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