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Adaptive Implicit Criteria for Two Phase Flow with Gravity and Capillary Pressure

Research Report

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ABSTRACT

Based on monotonicity conditions, the switching criteria for an adaptive implicit discretization for two phase flow in a porous medium with gravity and capillary pressure are derived. These criteria can be applied to multi-dimensional flows provided an easily checked condition is satisfied. Use of the monotonicity conditions is demonstrated for some example problems in one and two dimensions.

Introduction

Multiphase flow problems in porous media arise in oil reservoir simulation [1], contamination of groundwater with organic pollutants [2] and geothermal energy extraction. Recently, adaptive implicit methods have been used to solve reservoir simulation problems [3-9]. The basic idea of these techniques is to use a fully implicit discretization only in those cells undergoing large flow rates, while using an IMPES (implicit pressure, explicit saturation) method elsewhere. This permits large timesteps with less computational work than a fully implicit discretization. Full details of the adaptive implicit method can be obtained in references [5,10]. Various heuristic methods have been used as criteria for switching from IMPES to fully implicit and vice versa. In reference [5], a conservative strategy is proposed, based on changes observed over a timestep, and allowing only switching from IMPES to fully implicit. Other authors have proposed an alternative criteria which permits backward switching [8], but this criteria does not necessarily give a monotone discretization.

All the above methods for adaptive implicit switching are based on the concept of preventing stepwize instability. In other words, given a fixed grid size, these methods attempt to ensure that the solution remains bounded as the number of timesteps becomes large. However, as will be demonstrated, it is possible to obtain bounded solutions which produce non-physical local maxima and minima. This problem can be avoided by using monotonicity requirements as adaptive implicit criteria.

Note that a true adaptive implicit discretization is used in [5,8,10]. This is not simply zeroing the elements of the Jacobian as suggested by several authors [4,6].

The objective of this article is to examine some two phase flow problems, and to determine the monotonicity conditions. Stability will generally follow from monotonicity. A monotone discretization will prevent the appearance of non-physical local maxima and minima.

2. Formulation

Recently, Sammon [11] has used monotonicity arguments to demonstrate that phase upstream weighting used for one dimensional, two phase flow with gravity but zero capillary pressure, converges to the physically correct solution satisfying the entropy condition. Similar arguments will be extended in the following to determine the monotonicity conditions for two phase flow with gravity and capillary pressure. Of course, the inclusion of a finite capillary pressure means that the system of equations, even in one dimension, is not purely hyperbolic. However, we are only interested in adaptive implicit criteria, not convergence issues. Since these conditions can be applied on a cell by cell basis, it is possible to determine criteria for two dimensional problems if certain tests are satisfied.

For definiteness in the following, we take the two phases to be oil *(o)* and water *(w)*. The equations for two phase, incompressible flow are given by [1]:

$$\frac{\partial}{\partial t}(S_o \phi) = \nabla \cdot [\lambda_o \nabla P_o - \rho_o g \nabla D] \tag{1}$$

$$\frac{\partial}{\partial t}(S_w \phi) = \nabla \cdot [\lambda_w \nabla P_w - \rho_w g \nabla D]$$

where:

S_ℓ	=	saturation of phase $\ell =$ oil, water
ϕ	=	porosity
λ_ℓ	=	$\dfrac{K K_{r\ell}}{\mu_\ell}$
$K_{r\ell}$	=	relative permeability of phase ℓ
μ_ℓ	=	viscosity of phase ℓ
K	=	absolute permeability
D	=	depth
g	=	gravitational acceleration
P_ℓ	=	pressure of phase ℓ
ρ_ℓ	=	density of phase ℓ

The oil and water pressures are related through the capillary pressure, which is a function of water saturation S_w:

$$P_w = P_o + P_c(S_w) \tag{2}$$

while the relative permeability $K_{r\ell}$ is an experimentally determined function of phase saturation S_ℓ [1]. Equations (1) are discretized using a cell centered finite difference method on a Cartesian grid [12]. It is assumed that cell edges are perpendicular to the line joining the cell centers. With these assumptions, a compact expression can be obtained for the discretized equations, which is valid for any number of dimensions. If cell i has a set of nearest neighbours N_i, then the discretized equations can be written as:

$$S_{oi}^{N+1} - S_{oi}^{N} = \sum_{j \in N_i} \frac{\theta_j \lambda_{o,j+1/2}^{M}}{\Delta x_{j+1/2}} [(P_{oj}^{N+1} - P_{oi}^{N+1}) - \rho_o g (D_j - D_i)] \tag{3}$$

$$S_{wi}^{N+1} - S_{wi}^{N} = \sum_{j \in N_i} \theta_j \frac{\lambda_{w,j+1/2}}{\Delta x_{j+1/2}} [(P_{oj}^{N+1} - P_{oi}^{N+1}) - \rho_w g (D_j - D_i) - (P_{cj}^{M} - P_{ci}^{M})] \tag{4}$$

where

$$\theta_j = \frac{\Delta t \; A_{j+1/2}}{V_i \phi_i}$$

V_i = volume of the i'th cell

$A_{j+1/2}$ = interface area between cell i and cell j

$\Delta x_{j+1/2}$ = $(\Delta x_i + \Delta x_j)/2$

Δx_i = cell width in the direction from cell i to cell j.

For convenience, the phase potentials $\psi_{\ell j}$ are defined as:

$$\psi_{oj} = \{P_{oj}^{N+1} - P_{oi}^{N+1} - \rho_o g (D_j - D_i)\}/\Delta x_{j+1/2} \tag{5}$$

$$\psi_{wj} = \{P_{oj}^{N+1} - P_{oi}^{N+1} - \rho_w g (D_j - D_i) - (P_{cj}^{M} - P_{cj}^{M})\}/\Delta x_{j+1/2}$$

The notation:

$$\lambda_{\ell, j+1/2}$$

refers to the value of λ_ℓ evaluated at the upstream point, depending on the sign of the phase potential ψ_ℓ. Note that in equations

(3-4), if $\forall\, j$, $M=N$, then an IMPES method is used, while if $\forall\, j$, $M=N+1$, then a fully implicit discretization is specified.

To avoid a profusion of subscripts, the following convention will be observed. The notation λ_ℓ will be used to signify:

$$\lambda_\ell = \lambda^M_{\ell, j+½} \tag{6}$$

with the sub/superscripts understood. Similarily, the notation P_{oi} will be used to mean:

$$P_{oi} = P^{N+1}_{oi} \tag{7}$$

and

$$P_{ci} = P^M_{ci}$$

$$\triangle x = \triangle x_{j+½}$$

Consequently, if equations (3) and (4) are added together, and noting that:

$$S_o + S_w = 1$$

then:

$$0 = \sum_j \frac{\theta_j}{\triangle x} \left[(\lambda_o + \lambda_w)(P_{oj} - P_{oi}) \right. \tag{8}$$

$$- (\rho_o\, g\, \lambda_o + \rho_w\, g\, \lambda_w)(D_j - D_i)$$

$$\left. - \lambda_w (P_{cj} - P_{ci}) \right]$$

where we have used the notation of equations (6-7). Equations (3) and (8) are the starting point for further analysis. If the system is one dimensional, or if the function

$$\lambda_o/(\lambda_o + \lambda_w)$$

is constant over N_i (this will be discussed later), then the pressure can be eliminated from equation (3) using equation (8), and the system could, in principle, be solved for S_{oi}^{N+1}:

$$S_{oi}^{N+1} = g_i(S_{oi}^N, S_{oj}^N, S_{oj}^{N+1}) \; ; \; j \in N_i \tag{9}$$

Consequently, if $g_i(S_{oi}^N, S_{oj}^N, S_{oj}^{N+1})$ is a monotone function of its arguments in the interval [a,b], that is if:

$$\frac{\partial g_i}{\partial S_{oi}^N} \geq 0 \; ; \; \frac{\partial g_i}{\partial S_{oj}^N} \geq 0 \; ; \; \frac{\partial g_i}{\partial S_{oj}^{N+1}} \geq 0 \tag{10}$$

$$\{S_{oj}^N, \; S_{oj}^{N+1}, \; S_{oi}^N\} \in [a, b]$$

then bounds for the maximum and minimum values of S_{oi}^{N+1} are given by:

$$\max \; \{S_{oi}^{N+1}\} \leq g_i(b, b, b) \tag{11}$$

$$\min \; \{S_{oi}^{N+1}\} \geq g_i(a, a, a)$$

(The inequality signs are necessary if the arguments of g_i are not independent). If, for example, an IMPES type discretization is used, and:

$$g_i(a, a, a) = a$$

then:

$$S_{oi}^{N+1} \in [a, b] \;\text{ if }\; S_{oi}^N, S_{oj}^N \in [a, b]$$

and hence the discretization is stable.

The desirability of a monotone scheme is easily demonstrated. Suppose that an IMPES scheme is being used with:

$$S_{oi}^N = S_{oj}^N = a$$

and suppose that:

$$(S_{oi}^{N+1})^* = g_i(a, a, a)$$

Now consider an initial state with:

$$S_{oi}^N = a + \alpha \ ; \ S_{oj}^N = a$$

$$\alpha << 1, \ \alpha > 0$$

then:

$$S_{oi}^{N+1} = (S_{oi}^{N+1})^* + \left.\frac{\partial g_i}{\partial S_{oi}^N}\right|_a \alpha + 0(\alpha^2)$$

Suppose that g_i is not monotone, with:

$$\frac{\partial g_i}{\partial S_{oi}^N} < 0$$

then for α sufficently small:

$$S_{oi}^{N+1} < (S_{oi}^{N+1})^*$$

In other words, a small positive perturbation of S_{oi}^N has produced a value of S_{oi}^{N+1} which is less than the unperturbed value. This is physically absurd.

In the following, the conditions for monotonicity will be determined. As will be demonstrated, a monotone discretization will generally lead to a stable scheme with desireable properties. It will also be shown that, under certain assumptions, the fully implicit method with phase upstream weighting is monotone, as is the

IMPES method for a sufficiently small timestep. Consequently, we will attempt to derive switching criteria that will ensure that the adaptive implicit method also has this same monotonicity property.

3. Monotonicity Conditions

To avoid having to work out the conditions for various special cases, the most general expressions will be obtained. This requires somewhat involved but straightforward algebra.

As discussed above, we will need the derivatives of equations (3) and (8) with respect to S_{oK}^M. Differentiating equation (3) and using the notation of equations (5-7) gives:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oK}^{L}} = \delta_{i,K}\ \delta_{L,N} \tag{12}$$

$$+ \sum_j \theta_j \left[\lambda_o{}'\ \psi_{oj} + \lambda_o \left(\frac{\partial\, P_{oj}}{\partial\, S_{oK}^L} - \frac{\partial\, P_{oi}}{\partial\, S_{oK}^L} \right) / \triangle x \right]$$

where $\delta_{i,j}$ is the Kronecker delta, and:

$$\lambda_\ell{}' = \left. \frac{d\,\lambda_\ell}{d\, S_o} \right|_{S_o\, =\, S_{o,j+1/2}^M} \delta_{\, j+1/2,K}\ \ \delta_{\, M,L} \tag{13}$$

Differentiating equation (8) with respect to S_{oK}^L gives:

$$0 = \sum_j \frac{\theta_j}{\Delta x} \left\{ (\lambda_o{}' + \lambda_w{}')\, (P_{oj} - P_{oi}) \right. \tag{14}$$

$$+ (\lambda_o + \lambda_w) \left(\frac{\partial\, P_{oj}}{\partial\, S_{oK}^L} - \frac{\partial\, P_{oi}}{\partial\, S_{oK}^L} \right)$$

$$- (\lambda_o{}' \rho_o\, g + \lambda_w{}' \rho_w\, g)\, (D_j - D_i)$$

$$- \lambda_w' \, (P_{cj} - P_{ci})$$

$$- \lambda_w \, (P_{cj}' - P_{ci}')\}$$

where:

$$P_{ci}' = \frac{d\,P_c}{d\,S_o} \Bigg|_{S_o \,=\, S_{oK}^L} \delta_{i,K} \;\; \delta_{L,M} \tag{15}$$

It is convenient to define the discrete total mobility:

$$\begin{aligned} \lambda_t &= \lambda_o + \lambda_w \\ &= \lambda_{o,j+½}^M + \lambda_{w,j+½}^M \end{aligned} \tag{16}$$

Note that the discrete λ_t is not necessarily equal to the continuous λ_t evaluated at a point, since the upstream point for the water phase may not be the upstream point for the oil phase.

In one dimension, each term of the sum in equation (14) is identically zero, since equation (8) is a difference form of:

$$\nabla \cdot V_t = 0$$

$$V_t = \lambda_t \, \nabla \, P_o - (\rho_o g \lambda_o + \rho_w g \lambda_w) \, \nabla \, D$$

$$- \lambda_w \, \nabla \, P_c$$

and hence implies that $V_t =$ constant, and thus:

$$\frac{\partial \, V_t}{\partial \, S_{oK}^M} = 0$$

Consequently, in one dimension, we can multiply each term of equation (14) under the summation sign by:

$$
\begin{aligned}
f_{oj} &= \lambda_o / (\lambda_o + \lambda_w) \\
&= \lambda^M_{o,j+1/2} / (\lambda^M_{o,j+1/2} + \lambda^M_{w,j+1/2})
\end{aligned} \tag{17}
$$

to obtain:

$$
\begin{aligned}
\sum_j \frac{\theta_j}{\Delta x} \lambda_o \left(\frac{\partial P_{oj}}{\partial S^L_{oK}} - \frac{\partial P_{oi}}{\partial S^L_{oK}} \right) \quad (18) \\
= \sum_j \frac{\theta_j}{\Delta x} \frac{\lambda_o}{\lambda_t} \{ (\lambda_o' \rho_o g + \lambda_w' \rho_w g)(D_j - D_i) \\
- (\lambda_o' + \lambda_w')(P_{oj} - P_{oi}) \\
+ \lambda_w' (P_{cj} - P_{ci}) \\
+ \lambda_w (P_{cj}' - P_{ci}') \}
\end{aligned}
$$

In two or more dimensions, equation (14) cannot in general be multiplied under the summation sign by f_{oj} (equation (17)), since each term of the sum in equation (18) is not necessarily zero, nor is f_{oj} a constant independent of j. However, in two or more dimensions, the monotonicity condition (for backward switching) will not be applied unless:

$$
\left| \max \{ f_{o,j+1/2} \} - \min \{ f_{o,j+1/2} \} \right| < \beta \ ; \tag{19}
$$

$$
\beta << 1 \ ; \quad j \in N_i
$$

which implies that:

$$
f_{o,j+1/2} \sim \text{constant} \ ; \quad j \in N_i
$$

In practice, this is not unduly restrictive, since f_{oj} represents the discrete fractional flow. If f_{oj} is rapidly changing over the set of nearest

neighbour cells, then this indicates the presence of a shock front, and hence an implicit treatment will almost certainly be required. On the other hand, ahead or behind the shock, $f_{o,j+½}$ will be sensibly constant over the set of nearest neighbour cells.

Note that in general:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oK}^{M}} \neq 0\ ;\ \ K \notin N_i$$

since the pressures P_{oj}^{N+1}, P_{oi}^{N+1} will depend on saturations not in N_i. However, if the system is one dimensional, or if equation (19) is satisfied, then S_{oi}^{N+1} does not explicity depend on S_{oK}^{M}, $K \notin N_i$.

Consequently, assuming that multiplication of equation (14) by $f_{o,j+½}$ within the summation sign is valid, then equation (18) can be used to eliminate the pressure derivatives from equation (12). This gives a compact expression for the derivatives:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oK}^{L}} = \delta_{i,K}\, \delta_{L,N} \tag{20}$$

$$+ \sum_j \theta_j \left\{ \frac{\lambda_o{}'\lambda_w\, \psi_{oj}}{\lambda_t} - \frac{\lambda_w{}'\lambda_o\, \psi_{wj}}{\lambda_t} \right.$$

$$\left. + (P_{cj}{}' - P_{ci}{}')/\triangle x \right\}$$

Noting that:

$$\lambda_\ell \psi_\ell = \lambda^M_{\ell,j+1/2} \psi_{\ell j}$$
$$= \lambda^M_{\ell j} \max(0, \psi_{\ell j})$$
$$+ \lambda^M_{\ell i} \min(0, \psi_{\ell j})$$

which follows from the definition of upstream weighting then:

$$\psi_{\ell j} \lambda_\ell' = \frac{\partial \lambda_\ell}{\partial S^L_{oK}} \psi_{\ell j}$$
$$= (\lambda^M_{\ell j})' \max(0, \psi_{\ell j}) \delta_{j,K} \delta_{L,M}$$
$$+ (\lambda^M_{\ell i})' \min(0, \psi_{\ell j}) \delta_{i,K} \delta_{L,M}$$

Consequently, it is easy to evaluate equation (20) for the various derivatives of interest. For example, if cell i is an IMPES cell, then

$$\left. \frac{\partial S^{N+1}_{oi}}{\partial S^M_{oj}} \right|_{j \in N_i, j \neq i} \tag{21}$$
$$= \sum_i \theta_j \{ \lambda_{oj}' \lambda_w \max(\psi_{oj}, 0)/\lambda_t$$
$$- \lambda_{wj}' \lambda_o \max(\psi_{wj}, 0)/\lambda_t$$
$$+ P_{cj}' / \Delta x \};$$

Physically realizable relative permeabilities and capillary pressure curves obey the following conditions [1].

$$\frac{d\,\lambda_0}{d\,S_o} \geq 0 \qquad (22)$$

$$\frac{d\,\lambda_w}{d\,S_o} \leq 0$$

$$\frac{d\,P_c}{d\,S_o} \geq 0$$

Consequently, from equation (21):

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oj}^{M}} \geq 0 \qquad (23)$$

$$j \in N_i \quad , \quad j \neq i$$

$$\forall\ \theta_j\ ;$$

$$\lambda_{\ell j} = \lambda_{\ell j}^{N}\ ,\ \lambda_{\ell j}^{N+1} \quad ; \quad \lambda_{\ell i} = \lambda_{\ell i}^{N}$$

If cell i is an implicit cell then:

$$\lambda_{\ell i} = \lambda_{\ell i}^{N+1}$$

Consequently:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oi}^{N}} = 1 + \frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oi}^{N}} \sum_j \theta_j \,\{\lambda_o{}'\lambda_w \min\,(0,\, \psi_{oj})/\lambda_t \qquad (24)$$

$$- \lambda_w{}'\lambda_o \min\,(0,\, \psi_{wj})/\lambda_t$$

$$- P_{ci}{}'/\Delta x\}$$

Recalling equation (22), all the terms multiplying θ_j in equation (24) are always negative or zero. Consequently:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oi}^{N}} \geq 0\ ; \tag{25}$$

$$\forall\ \theta_j\ \ ;\ \ \lambda_{\ell i} = \lambda_{\ell i}^{N+1}$$

If cell i is implicit, then equation (21) becomes:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oj}^{M}} \Bigg|_{j \in N_i, j \neq i}$$

$$= \sum_{i} \theta_j \{\lambda_{oj}{}' \lambda_w \max(\psi_{oj}, 0)/\lambda_t$$

$$-\lambda_{wj}{}' \lambda_o \max(\psi_{wj}, 0)/\lambda_t$$

$$+P_{cj}{}' / \triangle x\}\ ;$$

$$+\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oj}^{M}} \sum_{j} \theta_j \{\lambda_o{}' \lambda_w \min(0, \psi_{oj})/\lambda_t$$

$$-\lambda_w{}' \lambda_o \min(0, \psi_{wj})/\lambda_t$$

$$-P_{ci}{}'/\triangle x\}$$

using equation (22), the above equation also implies that:

$$\frac{\partial\, S_{oi}^{N+1}}{\partial\, S_{oj}^{M}} > 0\ j \in N_i\ \ ,\ \ j \neq i$$

$$\forall\ \theta_j\ \ ;\ \ \lambda_{\ell i} = \lambda_{\ell i}^{N+1}$$

If cell i is an IMPES cell with

$$\lambda_{\ell i} = \lambda_{\ell i}^{N}$$

then:

$$\frac{\partial S_{oi}^{N+1}}{\partial S_{oi}^{N}} = 1 + \sum_j \theta_j \{\lambda_o'\lambda_w \min(0, \psi_{oj})/\lambda_t - \lambda_w'\lambda_o \min(0, \psi_{wj})/\lambda_t - P_{ci}'/\triangle x\} \tag{26}$$

Since all terms multiplying θ_j in equation (26) are negative, the montonicity condition is:

$$\left| \sum_j \theta_j \{\lambda_o'\lambda_w \min(0, \psi_{oj})/\lambda_t - \lambda_w'\lambda_o \min(0, \psi_{wj})/\lambda_t - P_{ci}'/\triangle x\} \right| < 1 \tag{27}$$

4. Application of Monotonicity Conditions

The results from Section 3 can be summarized as follows:

$$\frac{\partial S_{oi}^{N+1}}{\partial S_{oj}^{M}} \geq 0 \tag{28}$$

$$j \neq i; \quad M = N, N+1; \quad \forall\ \theta_j$$

If cell i is a fully implicit cell, then:

$$\frac{\partial S_{oi}^{N+1}}{\partial S_{oi}^{N}} \geq 0 \quad \forall\ \theta_j \tag{29}$$

and if cell i is an IMPES cell, then:

$$\frac{\partial S_{oi}^{N+1}}{\partial S_{oi}^{N}} \geq 0 \tag{30}$$

if equation (27) is satisfied. In more than one dimension, these conditions are valid if:

$$f_{o,j+½} \sim \text{constant} \ , \quad j \in N_i \tag{31}$$

In the following, we will assume that the flow is either one dimensional or equation (31) holds. Consequently, a fully implicit cell ys gives rise to a monotone discretization, while an IMPES cell so monotone for a sufficiently small timestep (from equation). We will require that the selection of fully implicit-IMPES in the adaptive implicit method also ensures that the discretization is monotone. We will demonstrate an important property of a monotone discretization in the following.

Recalling that S_{oi}^{N+1} can, in principle, be written as:

$$S_{oi}^{N+1} = g_i \ (S_{oi}^{N}, S_{oj}^{N}, S_{oj}^{N+1})$$

then if:

$$S_{\min} = \min \ \{S_{oj}^{M}, S_{oi}^{N}\} \qquad j \in N_i \ ; \quad M = N, N+1$$

$$S_{\max} = \max \ \{S_{oj}^{M}, S_{oi}^{N}\} \qquad j \in N_i \ ; \quad M = N, N+1$$

and if g_i is a monotone function of its arguments in $[S_{\min}, S_{\max}]$, then:

$$g_i\,(S_{\min},\ldots) \leq S_{oi}^{N+1} \leq g_i\,(S_{\max},\ldots)$$

In the case of an IMPES cell ($\lambda_{oi} = \lambda_{oi}^{N}$), the bounds can be determined by setting:

$$S_{oi}^{N} = S_o^* = \text{constant}$$

and setting all functions of S_{oi}^{N} and S_{oj}^{M} to constants in equations (3) and (4). This gives:

$$S_{oi}^{N+1} = S_o^* + \frac{\lambda_o \lambda_w}{\lambda_t} (\rho_w - \rho_0) g \sum_j \frac{\theta_j}{\Delta x} (D_j - D_i) \tag{32}$$

On a regular cartesian grid, the sum:

$$\sum_j \frac{\theta_j}{\Delta x} (D_j - D_i) \tag{33}$$

vanishes for all interior cells. Consequently, for these cells:

$$S_{\min} \leq S_{oi}^{N+1} \leq S_{\max} \tag{34}$$

In the case of a fully implicit cell ($\lambda_{oi} = \lambda_{oi}^{N+1}$), the bounds can be determined by setting:

$$S_{oi}^{N} = S_o^* = \text{constant} \tag{35}$$

and setting all functions of S_{oi}^{N} and S_{oj}^{M} to constants in equations (3) and (4), as for the IMPES case. However, in this situation, we have an implicit equation for S_{oi}^{N+1} since:

$$\lambda_{oi}^{N+1},\ \lambda_{wi}^{N+1},\ P_{ci}^{N+1}$$

are functions of S_{oi}^{N+1}. Assuming that equation (33) is identically zero, then one solution for S_{oi}^{N+1} is clearly given by:

$$S_{oi}^{N+1} = S_o^*$$

as for the IMPES case (this is verified by direct substitution). If the system is one dimensional, or f_{oi} = constant, $\forall j,\ j \in N_i$, then equation (3) can be written as:

$$S_{oi}^{N+1} = S_{oi}^{N} + \gamma(S_{oi}^{N+1},\ S_{oj}^{M}) \tag{36}$$

where we have imagined using equation (4) to eliminate the pressure. From equations (12) and (24) it follows that:

$$\frac{\partial \gamma}{\partial S_{oi}^{N+1}} \leq 0$$

so that the right hand side of (36) is a decreasing function of S_{oi}^{N+1}, and the left hand side of equation (36) is a straight line function of S_{oi}^{N+1}, so that there is at most one solution for S_{oi}^{N+1}. Consequently, equation (34) is valid for interior implicit cells, as well as IMPES cells.

Note that $S_{\min}$ and $S_{\max}$ can contain values of S_{oj}^{N+1}, as can condition (27). These are, of course, unknown values at the beginning of a timestep. However, as will be discussed later, estimates of these quantities are easily obtained in an adaptive implicit method.

Equation (34) implies that the values of S_{oi}^{N+1} is bounded by the maximum and minimum value of $\{S_{oi}^{N},\ S_{oj}^{M}\}$. This property means that in the absence of source terms, no new local maxima or minima can appear during the time evolution of the system.

If the values of S_{oi}^{N} are bounded by:

$$0 \leq S_{oi}^{N},\ S_{oj}^{N} \leq 1$$

and $M = N$ in equation (34), then equation (34) obviously implies stability. If S_{oj}^{N+1} appears in equation (34), then cell j is an implicit cell. Assuming that:

$$\lambda_o \equiv 0,\ \ S_o \leq 0$$

$$\lambda_w \equiv 0,\ \ S_w \leq 0$$

then if equations (3) and (4) are written for cell j, and if:

$$S_{oK}^N, S_{wK}^N \in [0, 1], \forall K$$

then it is impossible to solve equations (3) and (4) unless:

$$S_{oj}^{N+1} \in [0, 1]$$

consequently, the extreme values of $S_{\min}$ and $S_{\max}$ appearing in equation (34) are bounded by $[0, 1]$, and hence monotonicity implies stability.

Equation (33) does not vanish for edge cells on a regular cartesian grid. For these boundary cells, it is not possible to obtain a "maximum" principle as given by equation (34). Of course, this is because physically, new local maxima and minima can appear in boundary cells. (Imagine a trivial example with two vertical cells, each with $S_{oi}^N = .5$, and oil less dense than water). However, we would still like to ensure that the adaptive implicit method is always monotone, since the fully implicit method is always monotone, and the IMPES method is also monotone for a sufficiently small timestep.

As far as stability for edge cells is concerned, if equation (27) is satisfied for $S_{oi}^N, S_{oj}^M \in [0, 1]$, then since

$$\lambda_o \lambda_w \sum_j \frac{\theta_j}{\Delta x} (D_j - D_i) = 0$$

at $S_o = 0$ or $S_o = 1$, then $S_{oi}^{N+1} \in [0, 1]$. However, satisfying equation (27) (for IMPES cells) over the entire range $[0, 1]$ will be quite restrictive in general for practical problems. Of course, applying such a test over $[0, 1]$ would also be computationally expensive.

An alternative for edge cells would be to apply equation (27), and then check that the bound on S_{oi}^{N+1} given by equation (32) is in $[0, 1]$. However, this bound is quite crude, and will probably be too pessimistic. Unfortunately, it is not possible to obtain sharper bounds for edge cells without additional assumptions.

The results derived above will be used to aid in the development of switching criteria. Some heuristic reasoning is required to develop practical tests. This will be described in the next section.

5. A Practical Stability Test

The whole idea of the adaptive implicit method is to produce a stable, physically reasonable solution while saving computational work. Consequently, a simple, inexpensive method of applying the tests derived in this article is required. As discussed elsewhere [5], an extremely cautious approach is necessary for practical problems. In other words, it is desireable to err on the side of too much implicitness, rather than too little.

The following method is used to switch cells from fully implicit to IMPES. After each Newton iteration, only cells which satisfy:

$$\left| \max \left(f_{o,j+1/2}\right) - \min \left(f_{o,j+1/2}\right) \right| < \mathrm{tol}_1 \; ; \quad j \in N_i \tag{37}$$

$$\max \left| S_{oi}^{N} - S_{oj}^{M} \right| < \mathrm{tol}_2 \; ; \quad j \in N_i \tag{38}$$

$$\left| S_{oi}^{N+1} - S_{oi}^{N} \right| < \mathrm{tol}_3 \tag{39}$$

are considered as candidates for switching from fully implicit to IMPES. Equation (37) ensures that multiplication of equation (14) by $f_{o,j+1/2}$ under the summation sign is valid (this is required only for multi-dimensional problems). Equation (38) ensures that:

$$\left| S_{\min} - S_{\max} \right| < \mathrm{tol}_2$$

so that equation (27) does not have to be evaluated over a range of saturation values. After each Newton, iteration, condition (27) is evaluated for fully implicit cells satisfying the above conditions. Note that the most recent estimates of $\lambda_{\ell,j+1/2}^{N+1}$ are available if required in equation (27). If equation (27) is satisfied, then the cell is switched to the IMPES state. For edge cells where the gravity sum (equation (33)) is non-zero, the switched cell is monitored for the next few iterations to ensure that equation (39) is satisfied. The philosophy behind this latter check can be stated as follows: in implicit edge cells, if equation (39) is satisfied, this indicates that gravity equilibration has occurred. Consequently, if large changes are observed after switching to IMPES, this is symptomatic of non-physical overshoot and possible instability. This approach is preferred since equation (32) tends to be unduly pessimistic.

The method for switching from IMPES to fully implicit is a combination of the method used previously [5] and the montonicity condition described in this work. If either:

$$|S_{oi}^{N+1} - S_{oi}^{N}| > \text{tol}_3 \tag{40}$$

or condition (27) is violated, then the cell is switched to a fully implicit state. The tolerance in equation (40) is set to a fraction of the timestep selector norm [5], and will catch rapidly growing instabilities. Condition (27) indicates a physically unreasonable non-monotone discretization, which can be avoided by using a fully implicit method. Note that condition (27) is rigorously correct only in one dimension, or if equation (37) is satisfied. However, in accordance with our cautious approach, condition (27) is always checked.

Consequently, the strategy used previously [5] is augmented with an additional check to trigger IMPES to fully implicit switches. As well, a new capability is added to switch from implicit to IMPES. The switch from implicit to IMPES occurs only when the discretization is monotone.

6. One Dimensional Example

Consider the one dimensional slab shown in Figure 1, with the physical properties given in Table 1. The slab has an initial water saturation of $S_w = .2$, which is the critical saturation. Water is injected halfway up the slab, so that initially some of the water flows downward, while oil flows upward, establishing a counter current flow regime. The domain of Figure 1 was discretized with 20 cells in the vertical direction.

Two runs were carried out. One run used the switching criteria described in a previous article [5]. In this case, a timestep selector norm of 20% (saturation change) was specified, while the implicit trigger (tol_3) was set at 5%. A norm of 20% would be typical for a problem of this type.

The saturation profile after ten days of injection is shown in Figure 2a. The local maximum at a depth of $10m$ is due to the injection of fluid at this point. However, the local maxima at depths of 4 and $16m$ are clearly non-physical. These cells are just on the boundary of the fully implicit region. If the run is continued, these local maxima (which are at the 4% level) persist. The existence of these local maxima is easily explained. The large derivatives of the capillary pressure curve near $S_w = .2$ (see Table 1a) results in a non-monotone IMPES discretization (from equation (27)). After a small local maximum evolves, the strongly non-linear capillary pressure curve has a much smaller derivative, and hence the discretization becomes monotone, and the solution remains bounded. A similar effect can also be observed in problems where the fractional flow curve has a large derivative over a small saturation range. This occurs in any problem where the ratio of the viscosities of the two fluids is large. Consequently, there is no disastorous instability in the stepwise sense; the variables always remain bounded due to the non-linear nature of the problem. However, the non-physical saturation maxima remain. Eventually, if a large enough timestep is taken, an implicit switch is triggered for the cells in question, and a better behaved solution is obtained.

Figure 2b shows the simulation results for a run using the criteria suggested in this paper. Note the disappearance of the non-physical local maxima. In this case, condition (27) was violated by an order of magnitude for the non-monotone cells, and hence an implicit method was used. The results were in good agreement with a fully implicit (in all cells) run. The values used for the tolerances in equations (38-39) were $tol_1 = tol_2 = .20$.

The relative permeabilities and capillary pressure curves used were typical of those found in non-aqueous phase groundwater contamination [13].

7. Waterflood Example

A one quarter five spot [1], two phase, incompressible water injection problem is shown in Figure 3. The relative permeabilities and capillary pressure data are taken from reference [8]. Water is injected at a constant rate in the top left corner, and a constant pressure is specified in the lower right corner. The initial water saturation is the critical saturation. A 10×10 grid is used for this run. Other pertinant data are given in Table 2.

Figure 4 shows the location of the implicit cells at various times, using the switching criteria developed in this work. The implicit region tracks the water front quite closely as it moves through the reservoir. Table 3 gives the run statistics for a fully implicit, adaptive implicit (forward switching only), and an adaptive implicit simulation with forward and back switching. Both adaptive implicit runs used the criteria of this work. Note that the average degree of implicitness was 26% for the run with forward and back switching, which is significantly smaller than the 51% obtained using forward switching only. A reasonable reduction in CPU time is also obtained. This reduction should be larger for finer grids since a smaller proportion of the total number of cells will be required to track the front.

The backward switching was particularly useful when water broke through to the producer. Because of the large non-linearity, small timesteps were required. Consequently, almost all cells, except those near the producer, reverted to the IMPES state. This makes the small timesteps during water breakthrough comparatively cheap. After breakthough, the timesteps quickly increased, and cells became fully implicit again.

All three runs gave very similar results for saturations and oil production versus time.

8. Conclusions

Previous switching criteria for adaptive implicit methods were based on attempting to detect stepwize instability, and ensuring that the solution remained bounded. While this approach can detect extreme instabilities, it is unable to detect a non-monotone discretization which can give rise to a bounded but nonphysical solution with new local maxima.

The monotonicity conditions for two phase, incompressible flow with gravity and capillary pressure were developed in this work. This criteria is rigorously valid for one dimensional flow, and for multi-dimensional problems provided the discrete fractional flow is constant over the set of neighbour cells. For cells which are not on the domain boundaries, the monotonicity condition ensures that non-physical local maxima and minima cannot occur.

Some approximations were introduced in order to yield a practical monotonicity test. The test was demonstrated for both one and two dimensional example problems. The one dimensional example demonstrated the advantages of detecting non-monotone, non-physical behaviour, while the two dimensional example showed the benefits of being able to switch cells from IMPES to fully implicit, and vice versa.

In many situations, an implicit method is required only near sharp fronts. This makes the adaptive implicit method, with the criteria developed here, extremely attractive for two phase flow problems. Note that phase upstream weighting is used in this study. This gives a monotone discretization in counter current (oil phase flowing in opposite direction to water phase) flow situations. Many groundwater contamination problems can be reasonably approximated by two phase flow [14], so the methods developed here may be used unchanged for this application.

For full three phase flow problems encountered in oil reservoir simulation, in many cases the problem domain can be divided into regions where only two phases are active. However, the full three phase monotonicity conditions await further analysis.

References

[1] K. Aziz and A. Settari, "Petroleum Reservoir Simulation", Applied Science, London.

[2] L.M. Abriola and G.F. Pinder, "A multi-phase approach to modelling of groundwater contamination, 1. Equation development", Water Resour. Res. **21** (1985) 11-18.

[3] G.W. Thomas and D.H. Thurnau, "Reservoir simulation using an adaptive implicit method" Soc. Pet. Eng. J. **23** (1983) 750-768.

[4] W.I. Bertiger and F.J. Kelsey, "Inexact Newton methods" Proc. 8th SPE Symposium on Reservoir Simulation, Dallas, 1985.

[5] P.A. Forsyth and P.H. Sammon, "Practical considerations for adaptive implicit methods in reservoir simulation", J. Comp. Phys. **62** (1986) 265-281.

[6] T.B. Tan, "Implementation of an improved adaptive implicit method in a thermal compositional simulator", Proc. 9th SPE Symposium on Reservoir Simulation, San Antonio, 1987.

[7] J.E. Killough and C.A. Kossack, "Fifth comparative solution project: evaluation of miscible flood simulators", Proc. 9th SPE Symposium on Reservoir Simulation, San Antonio, 1987.

[8] L.S.K. Fung, D. Collins, L. Nghiem, "An adaptive implicit switching criterion based on numerical stability analysis", Proc. 9th SPE Symposium on Reservoir Simulation, San Antonio, 1987.

[9] P. Quandalle and J.C. Sabathier, "Typical features of a new multipurpose reservoir simulator", Proc. 9th SPE Symposium on Reservoir Simulation, San Antonio, 1987.

[10] P.A. Forsyth, and P.H. Sammon, "Local mesh refinement and modelling of faults and pinchouts", SPE J. Form. Eval. **1** (1986) 275-286.

[11] P.H. Sammon, "An analysis of upstream differencing" submitted to Soc. Pet. Eng. J. Res. Eng. (1986).

[12] D.W. Peaceman, "Fundamentals of Reservoir Simulation", Elsevier, Amsterdam, 1977.

[13] C.R. Faust, "Transport of immiscible fluids within and below the unsaturated zone: a numerical model", Water Resour. Res. **21** (1985) 587-596.

[14] M. Osborne and J. Sykes, "Numerical modelling of immiscible organic transport at the Hyde Park landfill", Water Resour. Res. **22** (1986) 25-33.

Figure Captions

(1) Region for one dimensional example.

(2a) Saturation profile at ten days, old switching criterion [5].

(2b) Saturation profile at ten days, new switching criterion.

(3) One quarter five spot water injection problem.

(4a) Implicit map for water injection problem, .10 pore volumes injected.

(4b) Implicit map for water injection problem, .30 pore volumes injected.

(4c) Implicit map for water injection problem, 1.0 pore volumes injected.

Table 1

Data for one dimensional example.

Absolute permeability, K	$10^{-12}\ m^2$
Porosity, ϕ	.3
Initial saturations	
	$S_w = .2$
	$S_o = .8$
Discretization	
	$nx = 1 \quad \triangle x = 1.0m$
	$ny = 20 \quad \triangle y = 1.0m$
Densities	
	$\rho_o = 864\ Kg/m^3$
	$\rho_w = 1000\ Kg/m^3$
Viscosities	
	$\mu_o = 1\text{cp}$
	$\mu_w = 1\text{cp}$
Initial Pressure	$100\ kpa$

Table 1a

Relative permeability and capillary pressure data for one dimensional example.

S_w	K_{rw}	K_{ro}	P_{cow} (kpa)
.2	0.0	.68	103.0
.3	.04	.55	27.6
.4	.10	.43	10.3
.5	.18	.31	7.58
.6	.30	.20	7.44
.7	.44	.12	7.30
.8	.60	.05	7.17
.9	.80	0.0	7.03
1.0	1.0	0.0	6.89

Table 2

Data for one quarter five spot problem.

Absolute permeability, K	$10^{-13}\ m^2$
Porosity, ϕ	.3
Initial saturations	$S_w = .15$ $S_o = .85$
Discretization	$nx = 10 \quad \triangle x = 25\,m$ $ny = 10 \quad \triangle y = 25\,m$ $nz = 1 \quad \triangle z = 10\,m$
Depth	$D = 1000 - (i-1)2.5 + (j-1)\ 2.$ cell (1,1) is injector
Densities	$\rho_o = 864\ K_g/m^3$ $\rho_w = 1000\ K_g/m^3$
Viscosities	$\mu_o = 1\text{cp}$ $\mu_w = 1\text{cp}$

Table 2a

Injection/production data for one quarter five spot.

Injection	$500\, m^3/\text{day}$
Production	constant pressue $P_w = 1000\, kpa$

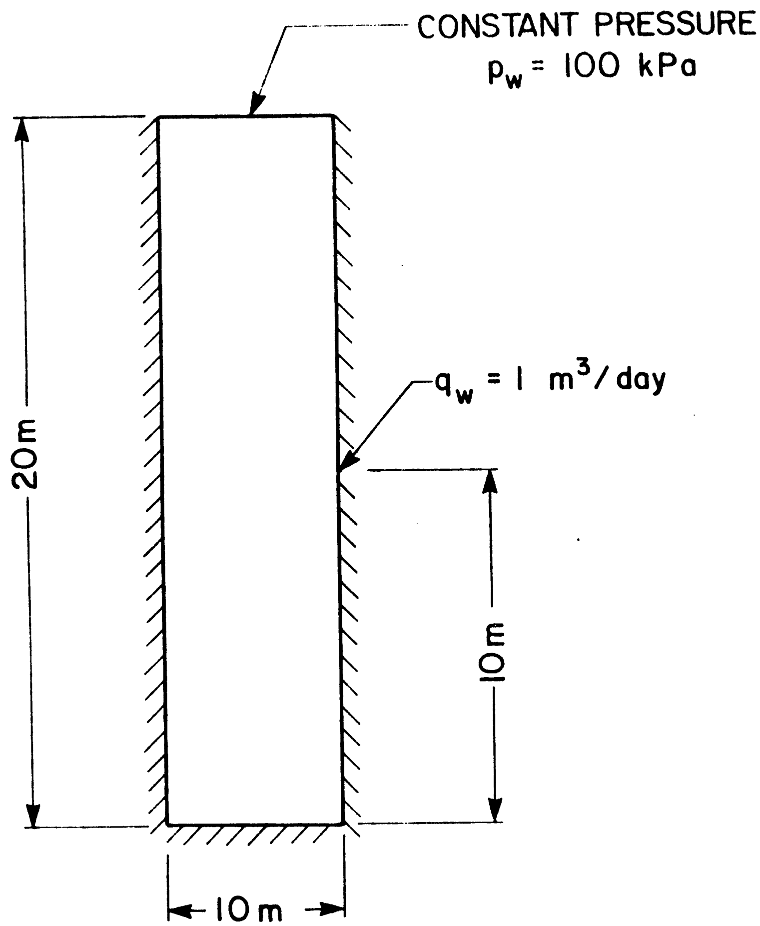
Relative permeability and capillary pressure from reference [8].

Table 3

Run statistics for the one quarter five spot water flood.

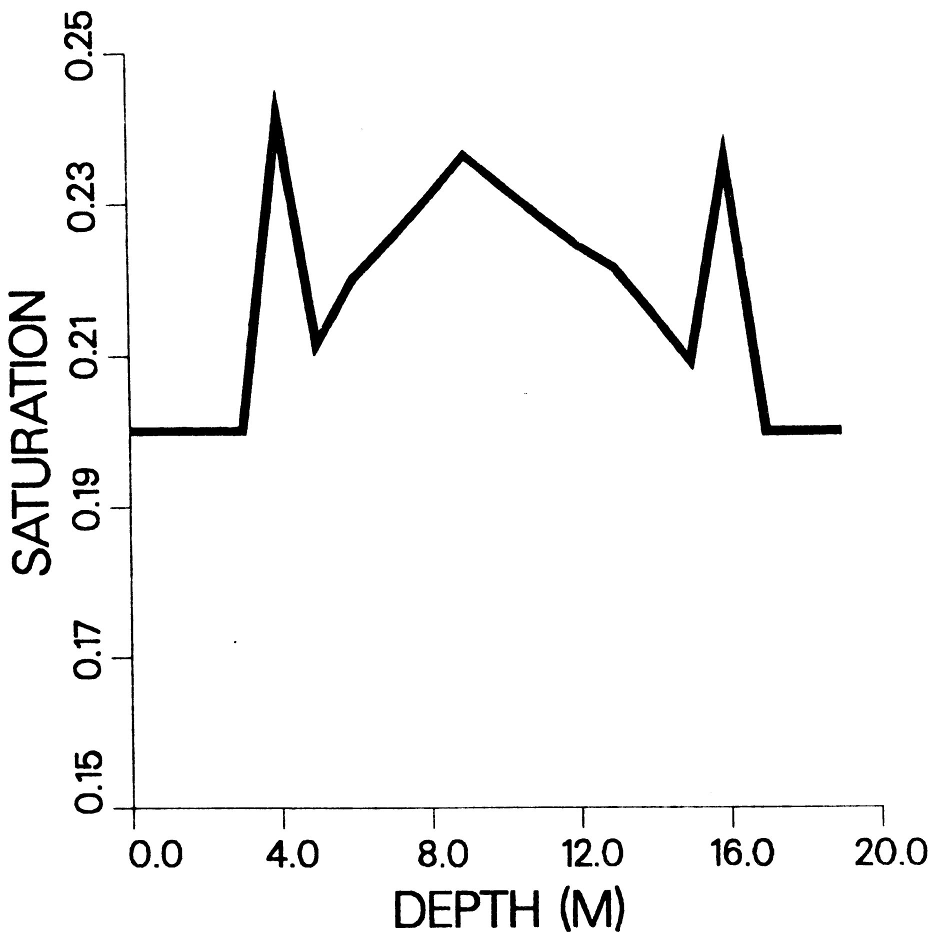
	Fully Implicit	Adaptive Implicit	
		Forward Switching Only	Forward and Back Switching
Average degree of implicitness	1.0	.51	.26
Total Newton Iterations	180	187	174
Normalized CPU time	1.0	.79	.63

Figure 1



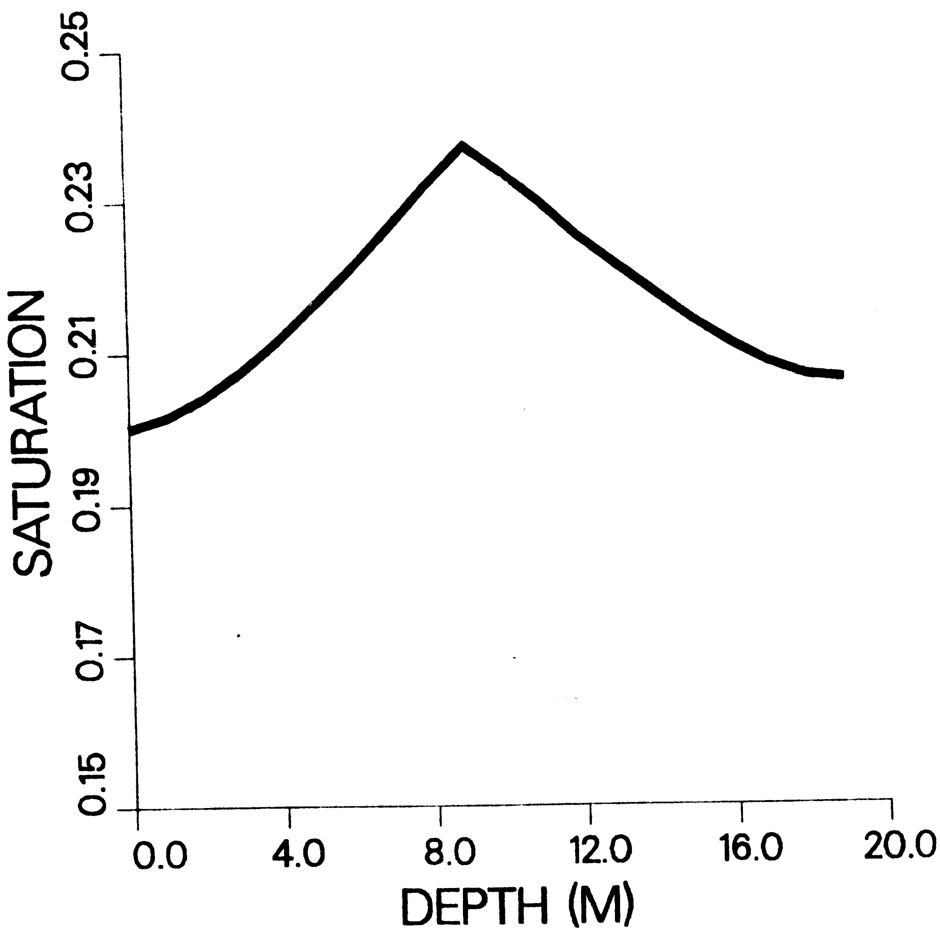
(1) Region for one dimensional example.

Figure 2a



(2a) Saturation profile at ten days, old switching criterion [5].

Figure 2b



(3)

(2b) Saturation profile at ten days, new switching criterion.

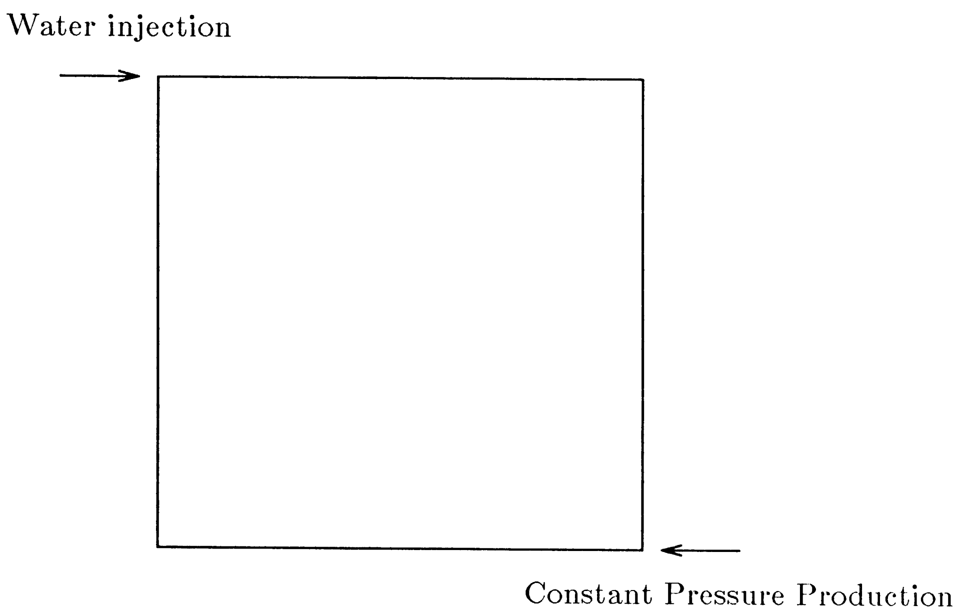


Figure 3

(3) One quarter five spot water injection problem.

I-implicit cell

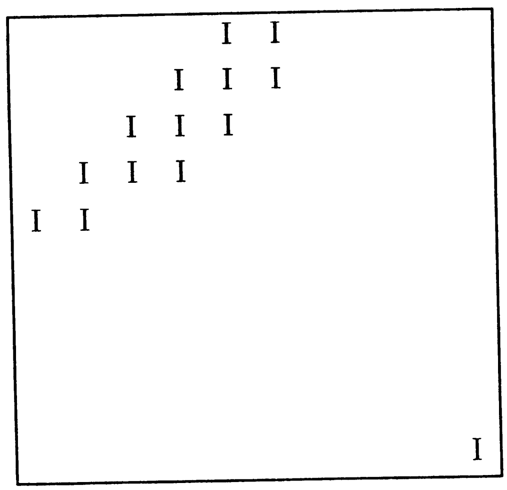


Figure 4a

(4a) Implicit map for water injection problem, .10 pore volumes injected.

I-implicit cell

								I	I
							I	I	I
						I	I	I	I
						I	I	I	
					I	I	I	I	
				I	I	I	I		
		I	I	I	I	I	I		
	I	I	I	I	I	I			
I	I	I	I	I					
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Figure 4b

(4b) Implicit map for water injection problem, .30 pore volumes injected.

I-implicit cell

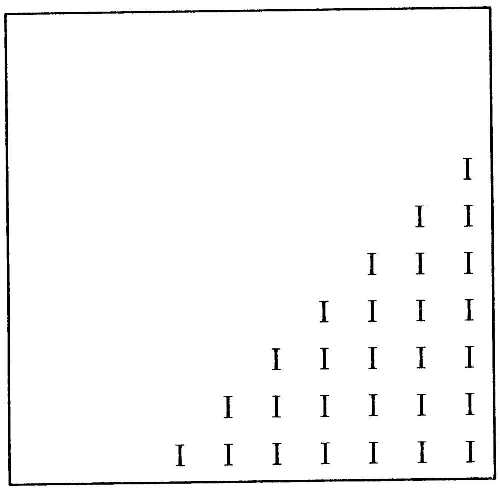

Figure 4c

(4c) Implicit map for water injection problem, 1.0 pore volumes injected.

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