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On Consistent Equiarea Triangulations

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Research Report CS-87-60
October 1987

On Consistent Equiarea Triangulations

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1. Introduction

A consistent triangulation of a polygon in the plane is a collection of triangles (closed sets) that:

- cover the polygon
- intersect at most in a common vertex, or along a common edge

Consistent triangulations form a basic geometric discretization mechanism, as used in surface approximation, or the finite element method (see e.g. Strang and Fix, 1973). The corresponding discretization basis for an interval $a \leq x \leq b$ is a partition into subintervals, $x_i \leq x \leq x_{i+1}$ with $x_1 = a$ and $x_{N+1} = b$. A useful technique for generating partitions of an interval is the equidistribution of a positive weight function, $\phi(x)$, typically derived from the approximation errors of the numerical method being served , [Babuska and Rheinboldt (1979), deBoor (1973), Pereyra and Sewell (1975)]. An equidistributing partition of the interval $a \leq x \leq b$ into N subintervals is one for which the 'weight' of each subinterval is the same, i.e.

$$\int_{x_i}^{x_{i+1}} \phi \, dt = \frac{1}{N} \int_a^b \phi \, dt \quad \text{for } i = 1, \ldots, N$$

There are simple algorithms for constructing such a partition for any N (e.g.Hyman and Naughton, (1985)).

A straightforward generalization of these ideas to triangulations would define, for a positive weight function, $\phi(x, y)$ defined over a region R, an equidistributing triangulation of R into N triangles, Δ_i, $i = 1$ to N , as one for which the 'weight' of each triangle was the same, i.e.

$$\int\int_{\Delta_i} \phi \, dsdt = \frac{1}{N} \int\int \phi \, dsdt \quad \text{for } i = 1, \ldots, N$$

In this paper, we show that such triangulations do not necessarily exist, for arbitrary N, by showing that the unit square cannot be partitioned into five triangles of equal area, consistently. This is an observation of geometric interest aside from the motivation from mesh generation. Clearly, there are such triangulations of the square for all even N, so it raises the question of what is the least odd N, if any, for which an equiarea, consistent triangulation of the unit square exists.

2. Basic relations

There are some basic constraints imposed on the number of vertices, edges, and triangles of any consistent triangulation, T. These have appeared in the literature, e.g. J.A. George [1972], but their derivations are brief so we give them here for completeness.

For a consistent triangulation, let

N_T = the number of triangles

N_B = the number of boundary edges

N_E = the number of interior edges

N_V = the number of vertices

Lemma 2.1

2.1) $3N_T = 2\,N_E + N_B$

2.2) $N_V = 1 + N_B + N_E - N_T$

Proof

The first relation, 2.1), follows from redundantly counting the edges in T. Each triangle has three edges, hence the left hand side; however, every internal edge is incident on two triangles and hence has been counted twice, while every boundary edge is incident on one only, hence the right hand side.

The second relation can be developed from Euler's formula for a 3 dimensional simply connected polytope. Consider T as lying in the (x,y) plane, centred on (0,0) and introduce P = (0,0,1). If we join each boundary edge of T to P by a triangular face, we obtain a polytope. Euler's formula for a simply connected polytope in general is

2.3) $N_F + N_P - N_E = 2$

for N_F = the number of faces, N_P = the number of vertices and N_E = the number of edges.

In the case of the polytope we have embedded T in,

2.4) $N_F = N_T + N_B$

$N_P = N_V + 1$

$N_E = 2\,N_B + N_E$

So Euler's formula gives 2.27.

3. Equiarea Consistent Triangulation of the Unit Square into Five Triangles

Lemma 3.1

There is no equiarea consistent triangulation of a rectangle into three triangles.

One of the triangle vertices must lie in the interior of a side of the rectangle. The triangle with base on the opposite side of the rectangle has area = 1/2 the rectangle area.

Lemma 3.2

In a consistent triangulation of the square into five triangles, at least one edge of the square must be an edge of the triangulation and there can be at most one interior vertex of the triangulation.

>From 2.1) we see that the number of boundary edges in T must be odd. If every edge of the square were broken into at least two boundary edges then we have have $N_b \geq 9$ hence from 2.1) $N_E \leq 3$ which is not possible.

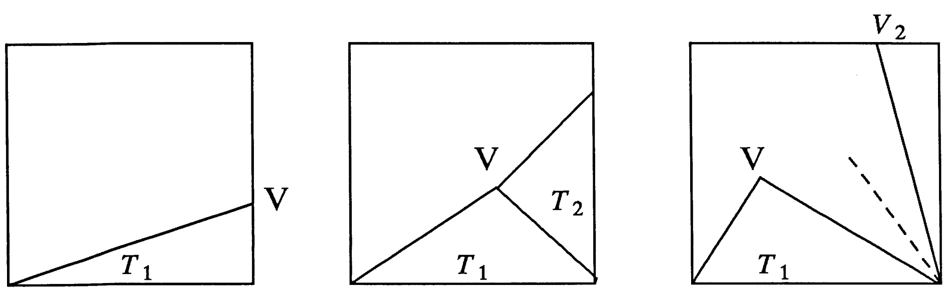
>From 2.1), $N_B \geq 5$, so that $N_E \leq 5$. But $N_T = 5$, so $N_E - N_T \leq 0$, and from 2.2) $N_v \leq 1 + N_B$. Since there must be N_B vertices on the boundary to define N_B boundary edges, there can be at most one more vertex of the triangulation in the interior.

Theorem3.1

There is no equiarea consistent triangulation of the unit square into five triangles.

Proof

The proof proceeds by contradiction of a case by case examination of possible equiarea triangulations. Let us locate the unit square in the standard position in the first quadrant, with the edge that is also a triangle side on the x axis. Let us denote this triangle by T_1. Since all triangles must have area = 1/5, T_1 must have a vertex V at $(x, 2/5)$ for $0 \leq x \leq 1$.

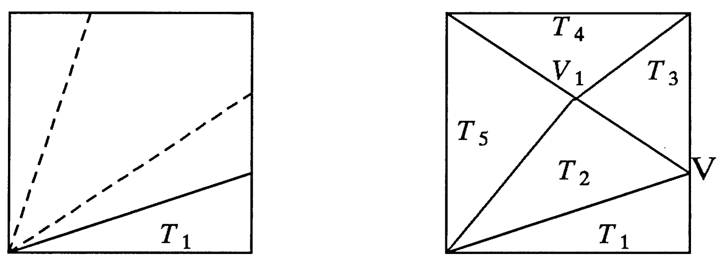


case 1 case 2 case 3

Figure 3.1

We can assume that either $x = 1$, or $0 < x < 1$, since the case $x = 0$ could be reflected in the line $x = 0.5$ to produce an equivalent case where $x = 1.0$. The case $x = 1$ (or equivalently $x = 0$) we will designate as case 1; the case in which two triangles are incident on (1,0), we designate as case 2 and the case of three (or more) triangles incident on (1,0), we designate as case 3, (see Figure 3.1). Note for case 3 that since there can be at most one interior vertex, the (right most) interior edge incident on (1,0) must end on the top edge of the square,at a vertex designated V_2.

In case 1, there may be no other edges incident on (0,0), in which case there must be a second triangle, T_2, with vertices (0,0), (1,2/5), (0,2/5), and , by Lemma 3.1, the remaining rectangle cannot be triangulated into three equiarea triangles. Alternatively, in case 1, there may be at least one additional interior edge incident on (0,0). We identify the case where there is at least one additional edge which is not incident on an interior vertex as case 1.1, and the case where there is only one interior edge and it is incident on an interior vertex as case 1.2 (see Figure 3.2).



case 1.1 case 1.2

Figure 3.2

In case 1.1., either there is no interior vertex, or it lies on one side of the additional edge incident on (0,0). If it lies above, then the additional edge terminates at (1,4/5), if it lies below, then it terminates at (2/5,1) and if there is no internal vertex it must have one of these two possible configurations (see dashed lines, Figure 3.2, case 1.1).

In any event, the result is a quadrilateral that must be decomposed into three equal area triangles introducing one vertex on the boundary of the square, which can easily be seen to be impossible.

In case 1.2, recourse to Lemma 2.1 will show that the topology of the possible triangulations must be as in Figure 3.2. In this topology, if area $(T_5) =$ area $(T_4) = 1/5$, then $V_1 = (2/5,3/5)$ and area $(T_3) = 9/50$.

Returning to case 2, then, we designate the upper vertex of T_2 as $V_1 = (1,y)$ and note that x and y are constrained by

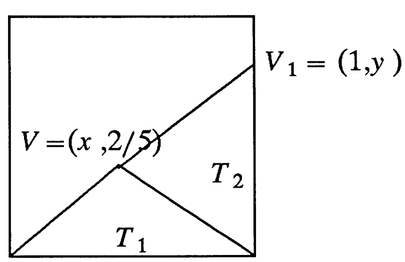


Figure 3.3

3.1) $(1-x)y/2 = area\ (T_2) = 1/5$

Now $x \geq 2/5$, or there is no triangle with its base lying on the side (0.,0.) to (0.,1.) of area $= 1/5$. From 3.1) then, we conclude that

3.2) $y = 2/5(1-x) \geq 2/3$

so, in particular, the interior vertex $V = (x,2/5)$ cannot be connected to (1.0,1.0) since the resulting triangle V, V_1,(1.0,1.0) would have area < 1/5. Consequently, in case 2.1, there would have to be a triangle containing the corner of the square at (1.0,1.0). Since the height of this triangle is at most 1, this implies $y \leq 3/5$, which contradicts 3.2). Hence case 2.1 does not lead to a possible equiarea triangulation.

Returning to case 2.2, it is straightforward to see that the remaining quadrilateral formed by (0,0),(3/5,2/5),(1,1), and (0,1) cannot be triangulated into three equal area triangles, by introducing a vertex on a side of the square.

For case 3, if there is no vertex of the triangulation on the side of the square (1.0,0) to (1.0,1.0) , then this case can be reduced to case 1 by a rotation through 90 degrees.

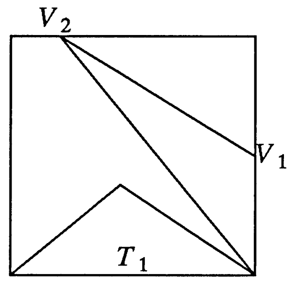


Figure 3.4

However, if there is a vertex , V_1, on this side, as indicated in Figure 3.4 , then it must be at (1.0, 0.5) , in which case V_2 must be at (1/5,1). But, clearly , the remaining five sided portion of the square cannot be triangulated by two triangles, so case 3 does not lead to a possible equiarea consistent triangulation, and the theorem is established.

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