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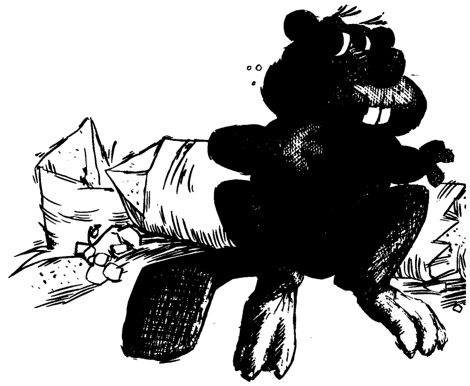

A New Measure of Presortedness

Vladimir Estivill-Castro
Derick Wood

Data Structuring Group
Research Report CS-87-58

October 15, 1987

A New Measure of Presortedness *

Vladimir Estivill-Castro † Derick Wood†

October 19, 1987

Abstract

A new measure of presortedness is presented which we call $Par(X)$. It is proved that this measure is distinct from other common measures of presortedness. We design a comparison-based sorting algorithm that sorts arbitrary lists in $O(n \log n)$ comparisons and, moreover, is optimal with respect to this new measure.

1 Introduction

We present a new measure of presortedness that is derived from the study of parallel-sorting algorithms [6]. This measure which is based on the idea of presortedness presented in [3] we call $Par(X)$.

Intuitively, a list is p–sorted if, for each element x in the list, deleting the p elements immediately before and after x, the resulting list has x in its correct sorted position. We first introduce the formal definitions and the fundamental properties of $Par(X)$ in Section 2. Then, in Section 3, we compare $Par(X)$ with the measures: $Inv(X)$, the number of inversions in X; $Runs(X)$, the number of runs in X; $Exc(X) = |X|-$the number of cycles in the permutation corresponding to X, and $Rem(X)$ the minimum number of elements that need to be removed to obtain a sorted list. It is proved that $Par(X)$ is not equivalent to any of these measures. We say these functions measure presortedness since nearly sorted lists have small measure. Mehlhorn [5], who coined the term presorted, called a list presorted if it has a small $Inv(X)$ value.

If m is a measure of presortedness, an m–optimal comparison-based algorithm is an algorithm that sorts all lists, but performs particularly well for lists having small m. A definition for optimality with respect to $Inv(X)$

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was suggested by Mehlhorn in [5], the idea was studied empirically by Cook and Kim [2], but the formal concept is due to Mannila [4]. An m–optimal algorithm performs optimally with respect to a measure of presortedness if it performs as well as any sorting algorithm that uses as input not only the list X but also the value $m(X)$.

We present a Par–optimal algorithm in Section 5, while Section 4 provides a lower bound for any algorithm that sorts p–sorted lists of length n. Our algorithm is an $O(n \log n)$ comparison-based sorting algorithm that performs optimally with respect to the $Par(X)$ measure.

For a list X, $|X|$ denotes its length and for a set S, $\|S\|$ denotes its cardinality. Let $X = \langle x_1, \ldots, x_n \rangle$ and $Y = \langle y_1, \ldots, y_m \rangle$ be two lists; then their catenation is denoted by XY and is defined as $\langle x_1, \ldots, x_n, y_1, \ldots y_m \rangle$. We denote the empty list as $\langle\rangle$.

2 Definitions

Let $X = \langle x_1, x_2, \ldots, x_n \rangle$ be a list of length n of elements x_i from some linear order, that is, for all $i, j \in \{1, 2, \ldots, n\}$, $x_i \leq x_j$ or $x_j \leq x_i$. We also call X an *n–list.*

Definition 2.1 X *is p–*sorted *if and only if, for all $i, j \in \{1, 2, \ldots, n\}$, $i - j > p$ implies $x_j \leq x_i$.*

The following are immediate properties of p–sortedness:

1. If a list is p–sorted, then it is $(p + i)$–sorted, for all $i \geq 0$.
2. p–sorted does not imply $(p - 1)$–sorted.
3. A list is 0–sorted if and only if it is sorted (completely sorted).
4. The definition of being p–sorted is equivalent to:

   X is p–sorted if and only if the following two conditions are satisfied:

   (a) There is no $j < i - p$ such that $x_i < x_j$.

   (b) There is no $j > i + p$ such that $x_i > x_j$.

Definition 2.2 *Let $X = \langle x_1, \ldots, x_n \rangle$ be a list. Y is said to be an* m–subsequence *of X if $Y = \langle x_{i(1)}, x_{i(2)}, \ldots, x_{i(m)} \rangle$ and $i : \{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots n\}$ is injective and monotonically increasing. We say that Y is an* m–sublist *of a* n–list *X if $Y = \langle x_i, x_{i+1}, \ldots, x_{i+m-1} \rangle$, for some i, $1 \leq i \leq n - m + 1$.*

Igarashi and Wood [3] provided a useful equivalent local condition for p–sortedness.

Theorem 2.1 *A list X is p–sorted, if and only if, every $(2p+2)$–sublist of X is p–sorted. Moreover, for every $p \geq 0$, there is a list X satisfying the following two conditions:*

1. *X is not p–sorted.*
2. *Every $(2p+1)$–sublist of X is p–sorted.*

This theorem is the fundamental tool to prove that a given list is p–sorted.

In the examples and the following definitions we assume without loss of generality that the x_i are nonnegative integers.

We now define the new measure of presortedness.

Definition 2.3 *Let $N^{<N}$ denote the set of all finite sequences of nonnegative integers. Define Par:$N^{<N} \rightarrow N$ by:*
$Par(X) = p$ if and only if $p = min\{q | X$ is q–sorted $\}$.

The following are some basic properties of $Par(X)$:

1. For all $X = \langle x_1, x_2, \ldots, x_n \rangle$, $0 \leq Par(X) \leq n-1$.
2. $Par(X) = 0$ if and only if X is sorted.
3. For all $X = \langle x_1, x_2, \ldots, x_n \rangle$, $Par(X) = n-1$ if and only if $x_n < x_1$.

We now give the five axioms, proposed by Mannila [4], that a measure of presortedness must satisfy and we verify that Par satisfies them.

Definition 2.4 *Letting $m : N^{<N} \rightarrow N$ be some function, we say that m is* a measure of presortedness, *if and only if:*

1. *If X is in ascending order, $m(X) = 0$.*
2. *If $X = \langle x_1, x_2, \ldots, x_n \rangle$, $Y = \langle y_1, y_2, \ldots, y_n \rangle$ and $x_i \leq x_j$ if and only if $y_i \leq y_j$ for all $i, j \in \{1, 2, \ldots, n\}$, then $m(X) = m(Y)$.*
3. *If Y is a subsequence of X then $m(Y) \leq m(X)$.*
4. *If $X \leq Y$ (that is, every element of X is no greater than every element of Y), then $m(XY) \leq m(X) + m(Y)$.*
5. *For all x in N, $m(\langle x \rangle X) \leq |X| + m(X)$.*

Lemma 2.2 *$Par(X)$ is a measure of presortedness.*

Proof: We verify the five axioms for $Par(X)$.

1. This was already established as an immediate property.

2. Suppose $Par(X) = p$ and $Par(Y) = q$ and, without loss of generality, assume $p < q$. Since Y is not $(q-1)$–sorted, (q must be positive, since $0 \leq p < q$) there exist y_i and y_j such that $i - j = q$, and $y_i > y_j$. This implies $x_i > x_j$ and $i - j = q > p$. Since X is p–sorted we must have $x_i \leq x_j$, and we have obtained a contradiction.

3. Let $X = \langle x_1, \ldots, x_n\rangle$ and $Y = \langle x_{i(1)}, \ldots, x_{i(s)}\rangle$, for some s, $1 \leq s \leq n$, where $i{:}\{1, \ldots, s\} \to \{1, \ldots, n\}$ is such that $1 \leq j < k \leq s$ implies $i(j) < i(k)$. We wish to prove that $k - j > p$ implies $x_{i(j)} \leq x_{i(k)}$. Since i is and injection and monotonically increasing, $k - j \leq i(j) - i(k)$. Hence, $p < k - j$ implies $p < i(k) - i(j)$. In other words, $x_{i(j)} \leq x_{i(k)}$ as desired.

4. If $X < Y$ we claim that $Par(XY) = max\{Par(X), Par(Y)\}$. Let $Par(X) = p$, $Par(Y) = q$ and $r = max\{p, q\}$. We first prove that $Par(XY) \leq r$. We write $XY = \langle z_1, z_2, \ldots, z_{n+m}\rangle$, where $z_i = x_i$ if $1 \leq i \leq n$ and $z_i = y_{i-n}$ if $n+1 \leq i \leq n+m$. We must show that if $i - j > r$, then $z_j \leq z_i$. Suppose $i - j > r$.

Case 1: If $i \leq n$ then $j \leq i$ and $z_j = x_j \leq x_i = z_i$, because $Par(X) = p \leq r$.

Case 2: If $n+1 \leq j$, then $z_i = y_{i-n}$, $z_j = y_{j-n}$, and $(i-n)-(j-n) = i-j$. Thus $z_j \leq z_i$, because $Par(Y) = q \leq r$.

Case 3: If $j \leq n$ and $n+1 \leq i$, then $z_j \in X$ and $z_i \in Y$. Since $X \leq Y$, we conclude that $z_j \leq z_i$.

Notice that $Par(XY) \leq max\{Par(X), Par(Y)\}$ is enough to verify the axiom since $Par(X) \geq 0$ and $Par(Y) \geq 0$. The reader can now verify that $Par(XY) \geq \max\{Par(X), Par(Y)\}$.

5. By the first basic property, $Par(\langle x\rangle X) \leq |\langle x\rangle X| - 1 = |X| \leq Par(X) + |X|$, since $Par(X) \geq 0$. □

The concept of an optimal algorithm with respect to a measure of presortedness was given in a general form by Mannila [4].

Definition 2.5 *Let m be a measure of presortedness, and S a sorting algorithm which uses $T_S(X)$ comparisons on input X. We say that S is* optimal *with respect to m (or m–*optimal*) if, for some $c > 0$, we have, for all $X = \langle x_1, x_2, \ldots, x_n\rangle$:*

$$T_S(X) \leq c \cdot \max\{|X|, \log(||below(X, m)||)\}$$

where $\text{below}(X, m) = \{\pi | \pi$ *is a permutation of* $\{1, \ldots, n\}$ *and* $m(\pi(X)) \leq m(X)\}$.

3 Comparing Measures of Presortedness

We now compare our measure of presortedness with other measures.

Definition 3.1 *The number of inversions of* $X = \langle x_1, \ldots, x_n \rangle$ *is denoted by* $Inv(X)$ *and defined by:*
$Inv(X) = \|\{(i,j) | 1 \leq i < j \leq n \text{ and } x_i > x_j\}\|$.

Lemma 3.1 *For all* $X \in N^{<N}$, $Par(X) \leq Inv(X)$.

Proof: Let $Par(X) = p$. If $p = 0$, X is sorted, so $Inv(X) = 0$ and we are done.

If $p \neq 0$, since X is not $(p-1)$–sorted, there exists x_i such that $x_{i+p} < x_i$. Now consider the $p-1$ elements $x_{i+1}, x_{i+2}, \ldots, x_{i+p-1}$. If x_{i+s} with $s \in \{1, 2, \ldots, p-1\}$ is such that:

Case 1: $x_{i+s} > x_i$. Then $x_{i+s} > x_i > x_{i+p}$ and $i + p > i + s$, so we have an inversion.

Case 2: $x_{i+s} < x_i$. Then $i + s > i$ and we have an inversion.

For each $s \in \{1, 2, \ldots, p-1\}$ we have an inversion and since $x_{i+p} < x_i$, we have at least p inversions. Therefore $Inv(X) \geq p = Par(X)$. □

Lemma 3.2 *There is no* $c > 0$ *such that, for all* $X \in N^{<N}$, $Inv(X) \leq c \cdot Par(X)$.

Proof: Let $X = \langle n, n-1, \ldots, 1 \rangle$. Then $Inv(X) = n(n-1)/2$ which is $\Theta(n^2)$ while $Par(X) = n - 1$. □

Definition 3.2 *The number of maximal ascending subsequences of* X *is* $\|\{i | 1 \leq i < n \text{ and } x_{i+1} < x_i\}\| + 1$. *Since this is trivially not a measure of presortedness, we define* Runs(X) *to be this value less one.*

Lemma 3.3 *1. There is no* $c > 0$ *such that, for all* $X \in N^{<N}$, $Par(X) \leq c \cdot Runs(X)$.

2. There is no $c > 0$ *such that, for all* $X \in N^{<N}$, $Runs(X) \leq c \cdot Par(X)$.

Proof: 1. Let $X = \langle n, 2, 3, \ldots, n-1, 1 \rangle$ then $Par(X) = n-1$ but $Runs(X) = 2$.

2. Consider $X = \langle 2, 1, 4, 3, 6, 5, \ldots, n, n-1 \rangle$ then $Par(X) = 1$ while $Runs(X) = \lfloor n/2 \rfloor$. □

Definition 3.3 *The length of the largest ascending subsequence is denoted by $Las(X)$ and is defined by: $Las(X) = max\{t | \exists i(1), i(2), \ldots, i(t) \mid 1 \leq i(1) < i(2) < \cdots < i(t) \leq n$ and $x_{i(1)} < \cdots < x_{i(t)}\}$. Since $Las(X) \neq 0$ when X is sorted, we define $Rem(X) = |X| - Las(X)$. $Rem(X)$ is a measure of presortedness.*

Lemma 3.4 *For all lists X of length n, $Las(X) \geq \lceil n/(Par(X)+1) \rceil$.*

Proof: Consider $x_1, x_{1+p+1}, x_{1+2(p+1)}, \ldots, x_{1+k(p+1)}$ with $1+k(p+1) \leq n < 1+(k+1)(p+1)$. If X is p–sorted, the above sequence is in ascending order and has length $k+1$, but

$$n < 1 + (k+1)(p+1)$$

$$\Rightarrow n \leq (k+1)(p+1)$$

$$\Rightarrow n/(p+1) \leq k+1 \leq Las(X)$$

and $Las(X)$ is an integer so $Las(X) \geq \lceil n/(p+1) \rceil$.

Now consider the list $X = \langle p+1, p, p-1, \ldots, 2, 1, 2(p+1), 2(p+1)-1, \ldots \rangle$. This list is easily seen to be p–sorted and not $(p-1)$–sorted (using Theorem 2.1), so $Par(X) = p$ and $Las(X) = \lceil n/(p+1) \rceil$, proving that the bound of the above Lemma is tight. On the other hand $X = \langle p+1, 1, 2, 3, \ldots, p, 2(p+1), 2(p+1)-p, 2(p+1)-p+1, 2(p+1)-p+2, \ldots, 2(p+1)-1, 3(p+1), 3(p+1)-p, 3(p+1)-p+1, \ldots 3(p+1)-1, \ldots \rangle$ is such that $Par(X) = p$ and $Las(X) \geq p\lfloor n/(p+1) \rfloor$. This shows that, in general, we can have examples with strict inequality and with $Las(X)$ being far from $\lceil n/(Par(X)+1) \rceil$. □

Lemma 3.5 *1. $Rem(X) \leq |X|(1 - 1/(Par(X)+1))$.*

2. There is no $c > 0$ such that, for all $X \in N^{<N}$ $Par(X) \leq c \cdot Rem(X)$.

3. There is no $c > 0$ such that, for all $X \in N^{<N}$ $Rem(X) \leq c \cdot Par(X)$.

Proof: 1. $Rem(X) = |X| - Las(X) \leq |X| - |X|/(Par(X)+1)$.

2. Let $X = \langle n, 2, 3, \ldots, n-1, 1 \rangle$ then $Rem(X) = 2$ while $Par(X) = n-1$.

3. Let $X = \langle 2, 1, 4, 3, \ldots \rangle$ then $Rem(X) = \lfloor n/2 \rfloor$ but $Par(X) = 1$. □

Definition 3.4 *We now consider $Exc(X) = n-$ the number of cycles in the permutation of $\{1, 2, \ldots, n\}$ corresponding to X. Exc is also a measure of presortedness.*

Lemma 3.6 *There is no $c > 0$ such that, for all $X \in N^{<N}$, $Par(X) \leq c \cdot Exc(X)$ and there is no $d > 0$ such that, for all $X \in N^{<N}$, $Exc(X) \leq d \cdot Par(X)$.*

Proof: Let $X = \langle n, 2, 3, \ldots, n-1, 1\rangle$ then the cycles of the permutation are $(1\ n)(2)(3)\ldots(n-1)$ and then $Exc(X) = 1$ while $Par(X) = n-1$, on the other hand if $X = \langle 2, 1, 4, 3, 6, 5, \ldots, n, n-1\rangle$ we have $Par(X) = 1$ but $Exc(X) \geq \lfloor n/2 \rfloor$. □

We have compared the measure $Par(X)$ with the most common measures of presortedness and have shown:

Theorem 3.7 *$Par(X)$ is not equivalent to any of the measures of presortedness $Inv(X), Rem(X), Runs(X)$ or $Exc(X)$, but, for all $X \in N^{<N}$:*

1. *$Par(X) \leq Inv(X)$.*
2. *$Rem(X) \leq |X|(1 - 1/(Par(X)+1))$.*

We conclude that $Par(X)$ measures global presortedness and does not recognize local presortedness. In this sense $Par(X)$ is similar to $Inv(X)$.

4 A Lower Bound

We claim that any comparison-based algorithm that sorts any p–sorted list of length n requires $\Omega(\max\{n, n\log(p+1)\})$ comparisons.

To show this, consider the set

$$A_i = \{1+i(p+1), 2+i(p+1), 3+i(p+1), \ldots, p+i(p+1), p+1+i(p+1)\}$$

Let B_i be any permutation of the elements of A_i. We build a list X by catenation of the B_i:

$$X = B_0 B_1 B_2 \cdots B_{\lfloor n/(p+1)\rfloor - 1}$$

It can be directly verified that X is p–sorted. Further, this shows that there are at least $(p+1)!^{\lfloor n/(p+1)\rfloor}$ p–sorted lists. Hence, we conclude:

Theorem 4.1 *There are at least $(p+1)!^{\lfloor n/(p+1)\rfloor}$ n–lists X with $Par(X) \leq p$.*

Therefore, any comparison-based algorithm that sorts p–sorted lists requires at least $\lfloor n/(p+1)\rfloor \log((p+1)!)$ comparisons, that is, $\Omega(n\log(p+1))$ comparisons.

5 An Optimal Algorithm

We propose a comparison-based algorithm called the *Try–to–Merge Sort* that given a list as input produces the corresponding sorted list as output.

Try–to–Merge Sort is an $O(n \log n)$ worst-case sorting algorithm, but if the input list X is p–sorted, then we can certify that the algorithm requires $O(\lceil \log(p+1)+1 \rceil n)$ comparisons.

Letting $X = \langle x_1, \ldots, x_n \rangle$ define : $X_{even} = \langle x_2, x_4, \ldots, x_{2\lfloor n/2 \rfloor} \rangle$ and $X_{odd} = \langle x_1, x_3, \ldots, x_{2\lceil n/2 \rceil} \rangle$. We claim:

Theorem 5.1 *1. If X is p-sorted then X_{even} and X_{odd} are $\lfloor p/2 \rfloor$-sorted. Moreover, for every $p > 1$, there is a p-sorted list X such that X_{even} and X_{odd} are not $(\lfloor p/2 \rfloor - 1)$-sorted.*

2. For any $n \in N$, there is a list X such that $Par(X) > n$ and X_{even}, X_{odd} are both sorted.

Proof: 1. Let b_i and b_j be elements of X_{even} such that $j - i > \lfloor p/2 \rfloor$. We must prove that $b_j \geq b_i$. Now $b_j = x_{2j}$, $b_i = x_{2i}$, and $2j - 2i = 2(j-i)$. Since $j - i$ is an integer, $j - i > p/2$, and we obtain $2j - 2i > 2(p/2) = p$. Thus, since X is p–sorted, $x_{2j} \geq x_{2i}$, that is, $b_j \geq b_i$. The claim for X_{odd} is proved similarly.

Now, let $p > 1$. If p is odd, let $X = \langle x_1, x_2, \ldots, x_{p+3} \rangle$, where $x_i = 2 + i$, for $i = 1, 2, \ldots, p-1$, $x_p = 1$, $x_{p+1} = 2$, $x_{p+2} = 3+p+2$, and $x_{p+3} = 3+p+3$, and if p is even, let $X = \langle x_1, \ldots, x_{p+4} \rangle$, where $x_i = 2 + i$, for $i = 1 \ldots p$, $x_{p+1} = 1$, $x_{p+2} = 2$, $x_{p+3} = 3 + p + 3$, and $x_{p+4} = 3 + p + 4$. We claim that X is p-sorted and X_{even} and X_{odd} are not $(\lfloor p/2 \rfloor - 1)$–sorted.

Suppose p is even, $(p = 2k)$, x_1 and x_{p+1} belong to X_{odd}, more precisely, x_1 is the first element of X_{odd} and x_{p+1} is the $\lceil (p+1)/2 \rceil$ element in X_{odd}. But $\lceil (p+1)/2 \rceil - 1 = k > \lfloor p/2 \rfloor - 1$, and $x_1 = 2 + 1 = 3 > 1 = x_{p+1}$, hence X_{odd} is not $(\lfloor p/2 \rfloor - 1)$–sorted. All other cases are verified similarly.

2. Finally, let $n \in N$ and define $X = \langle x_1, \ldots x_{2n+3} \rangle$ by: $x_{2i} = i$, for $i = 1, 2, \ldots, n+1$, and $x_{2i-1} = n + 2i$, for $i = 1, 2, \ldots, n+2$. It can be verified that $Par(X) = 2n + 1$ and X_{even} and X_{odd} are sorted. □

We assume that we have a boolean function $merge(X_1, X_2, X)$ that attempts to merge the two lists X_1 and X_2 as if they are sorted. If X_1 and X_2 are sorted, it returns *true* and their merge is X, otherwise it returns *false* and X is undefined. If the input lists X_1 and X_2 have lengths n_1 and n_2, then the merging algorithm has complexity $O(n_1 + n_2)$.

5.1 Sorting Algorithm

Try–to–Merge Sort(X:list, n:integer) { $|X| = n$ }
Input: $X = \langle x_1, x_2, \ldots, x_n \rangle$.
Output: The elements in X in ascending order.

begin

if $merge(X_{even}, X_{odd}, X)$ **then** $\{successful\}$
else
begin
Try–to–Merge Sort$(X_{even}, \lfloor n/2 \rfloor)$;
Try–to–Merge Sort$(X_{odd}, \lceil n/2 \rceil)$;
$merge(X_{even}, X_{odd}, X)$
end
end

5.2 Algorithm Correctness

We assume that the procedure that performs the *merge* operation is correct.

Lemma 5.2 *For all lists X of length n,* Try–to–Merge Sort(X, n) *returns X in sorted order.*

Proof: We prove correctness by induction on the length of X.

Basis: If $|X| = 0$ then $X = \langle\rangle$, so X_{even} and X_{odd} are also empty, and as $merge(X_{even}, X_{odd}, X)$ is true, it yields $X = \langle\rangle$. Clearly the empty list is the correct result.

If $|X| = 1$ then $X_{odd} = X$ and $X_{even} = \langle\rangle$, $merge(X_{even}, X_{odd}, X)$ is true. It returns X_{odd} which is the correct result since a list of length one is always sorted.

Induction Step: Assume that the algorithm works correctly for input lists of length less than n, and we are given X of length $n \geq 2$. By the assumptions about procedure $merge(X_1, X_2, X)$ we have two cases according to whether this procedure returns *true* or *false*.

Case 1: If the merge is successful, that is, procedure *merge* returns *true*, then it returns X is sorted order.

Case 2: If the merge is unsuccessful, the algorithm performs the 'else' part. Since $n \geq 2$, $n > \lceil n/2 \rceil \geq \lfloor n/2 \rfloor$ and, by the induction hypothesis, the recursive calls to *Try–to–Merge Sort* return X_{even} and X_{odd} in sorted order. Hence, in the inner call to procedure *merge* they will be merged successfully producing as output the lists of elements in X in sorted order.

□

5.3 Algorithm Complexity

We now show that if we execute *Try–to–Merge Sort*(X,n), where n is the length of the list X, then the algorithm performs $O(n \log n)$ comparisons in the worst case.

By a worst case analysis the number of comparisons that the algorithm performs satisfies the recurrence relation:
$T(1) = 1$
$T(n) = 2T(n/2) + cn$, for some constant $c > 0$.
This has growth rate $\Theta(n \log n)$; see [1].

5.4 Par(X)–optimality

We claim that

Lemma 5.3 *If X is $(2^i - 1)$–sorted, then the maximum depth of recursion of* Try–to–Merge Sort$(X, |X|)$ *is i.*

Proof: We prove this claim by induction on i.

Basis: If $i = 0$ then X is $2^0 - 1 = 0$–sorted. In this case X is sorted, so X_{even} and X_{odd} are sorted, therefore the merging is successful. X is returned in ascending order and no recursive calls are made.

Induction Step: Assume that, for some $i \geq 1$, if $k < i$ and X is $(2^k - 1)$–sorted, then the depth of recursion of the call *Try–to–Merge Sort*(X,$|X|$) is no greater than k.

Let X be $(2^i - 1)$–sorted, and suppose we call *Try–to–Merge Sort*(X,$|X|$). If the merging on the odd and even subsequences of X is successful, then we produce the desired result with no recursive calls. But, in the worst case, the merging is unsuccessful and we call recursively:

1. *Try–to–Merge Sort*(X_{even},$\lfloor |X|/2 \rfloor$)
2. *Try–to–Merge Sort*(X_{odd},$\lceil |X|/2 \rceil$)

By Theorem 5.1 X being $(2^i - 1)$–sorted implies that X_{even} is $\lfloor \frac{2^i - 1}{2} \rfloor$–sorted, that is, X_{even} is $\lfloor 2^{i-1} - 1/2 \rfloor = (2^{i-1} - 1)$–sorted, and similarly X_{odd} is $(2^{i-1} - 1)$–sorted. By the induction hypothesis the above two recursive calls have a depth of recursion of at most $i - 1$. Therefore the maximum depth of recursion of the original call is bounded by i.

This completes our proof. $\square$

Using this lemma, and the observation that at each level of recursion we perform at most $|X|$ comparisons, we conclude:

Theorem 5.4 *If X is p–sorted and $|X| = n$, then* Try–to–Merge Sort(X, n) *requires $O(\lceil \log(p + 1) + 1 \rceil n)$ comparisons in the worst case.*

To prove that *Try–to–Merge Sort* is *Par*–optimal we need:

Lemma 5.5 *There is a $d > 0$ such that, for all $n > 1$ and for all m with $1 < m \leq n$,*

$$\log(m) + 1 \leq d\lfloor n/m \rfloor \frac{\log(m!)}{n}$$

Proof: We claim that $d = \max\{8, 4\frac{\log(3)+1}{\log(3)-1}\}$

Let $k = \lfloor n/m \rfloor$. Then $n/m < k+1$, so $1/n > 1/m(k+1)$, $n/m \geq 1$, and $k/(k+1) \geq 1/2$. Hence,

$$d\lfloor n/m \rfloor \frac{\log(m!)}{n} > d\frac{k}{k+1}\frac{\log(m!)}{m} \geq \frac{d}{2}\frac{\log(m!)}{m}$$

Case 1: $m = 2$.

Since $d \geq 8$, $\frac{d}{2}\frac{\log(m!)}{m} \geq 2 = \log(m) + 1$ as claimed.

Case 2: $m \geq 3$.

$$\frac{d}{2}\frac{\log(m!)}{m} = \frac{d}{2}\log((m!)^{1/m}) > \frac{d}{4}(\log(m) - \log 2) = (d/4)(\log(m) - 1)$$

By the definition of d and because $m \geq 3$

$$(d/4)(\log(m) - 1) \geq \frac{\log(3)+1}{\log(3)-1}(\log(m) - 1) \geq \log(m) + 1$$

□

Theorem 5.6 Try–to–Merge Sort *is Par–optimal.*

Proof: Let $T_{DMS}(X)$ be the number of comparisons that *Try–to–Merge Sort* performs on input X. Since, by Lemma 4.1, there are at least $(Par(X)+1)!^{\lfloor |X|/(Par(X)+1) \rfloor}$ lists in $below(X, Par)$, we have

$$\lfloor |X|/(Par(X)+1) \rfloor \log((Par(X)+1)!) \leq \log(||below(X, Par)||) \qquad (1)$$

and by Theorem 5.4 there is an $e > 0$ such that, for all $X \in N^{<N}$,

$$T_{DMS}(X) \leq e(\log(Par(X)+1)+1)|X| \qquad (2)$$

Let $X \in N^{<N}$.

Case 1: $Par(X) = 0$. Since X is sorted if and only if $Par(X) = 0$

$$\log(||below(X, Par)||) = \log(||\{X\}||) = \log(1) = 0$$

Moreover, in this case, the algorithm performs a successful merge. Hence, there is a $c_1 > 0$ independent of X such that

$$T_{DMS}(X) \leq c_1|X| = c_1 \max\{|X|, \log(||below(X, Par)||)\}$$

Case 2: $Par(X) > 0$. By Lemma 5.5, there is a $d > 0$ such that, for all X where $n = |X|$ and $m = Par(X) + 1$,

$$\log(Par(X) + 1) + 1 \leq d\lfloor\frac{|X|}{Par(X) + 1}\rfloor\frac{\log((Par(X) + 1)!)}{|X|}$$

Thus, there is a $d > 0$ such that

$$e(\log(Par(X)+1)+1)|X| \leq e \cdot d\lfloor\frac{|X|}{Par(X) + 1}\rfloor \log((Par(X)+1)!) \quad (3)$$

and by (1), (2), and (3), we conclude that there is a $c_2 = e \cdot d > 0$ such that

$$T_{DMS}(X) \leq c_2 \max\{|X|, \log(||below(X, Par)||)\}$$

Setting $c = \max\{c_1, c_2\}$, we conclude that there is a $c > 0$ such that, for all $X \in N^{<N}$,

$$T_{DMS}(X) \leq c \max\{|X|, \log(||below(X, Par)||)\}$$

and the theorem is proved. □

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