The Grid Spatial Stability and an Adaptive Scheme for the Diffusion-Convection Equations

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ABSTRACTS

The concept of grid spatial stability is introduced and used to judge the convergence of the finite difference schemes for convection - diffusion problems. The higher (2nd or 3rd) order upwind schemes are found not convergent at high Reynolds numbers. A relaxation of the scheme leads to a time dependent problem.

An adaptive third order scheme is then suggested and demonstrated to be comparable to the fourth order upwind compact scheme. In the numerical experiments with several model Burgers Equations this scheme produces satisfactory accuracy with a modest number of grid points, particularly when the Reynolds number is high.

1. Introduction

In the finite difference approach for the solution of diffusion - convection equations in fluid mechanics it is attractive to have a "universal" scheme which is accurate enough (say, free of numerical diffusion) to give a correct flow picture and yet able to handle a large range of Reynolds numbers (say, for Reynolds numbers up to 10^6 or higher) without introducing too many grid points. In addition it is desirable if the scheme can be easily programmed.

In searching for such a "universal" scheme the first problem one is encountered with is the limitation of the grid Reynolds number Rg =Reuh, where Re is the Reynolds number, u and h the velocity and grid size respectively. Such a limitation is due entirely to the numerical schemes and the solution procedure. If it is violated non-physical oscillations (wiggles) will occur in the numerical solutions (see [1] and [2]). In an unsteady problem the non-physical oscillations receive an explanation of temporal instability [2], which can be described by the standard stability analysis. In a steady state problem, there are various explanations. For example, Roache [1] analyzes the generating mechanism of the non-physical wiggles; Ciment, Leventhal and Weinberg [4] conduct a standard linear spatial stability analysis to their operator compact implicit scheme; by studying the error growth in various finite difference schemes, Leonard [5-6] points out that the non-physical wiggles in a numerical solution are directly analogous to the oscillatory nature of the marginal stability in the dynamic system. He also introduces the concept of "feedback sensitivity", which sheds some light on the nature of the problem. However,the "feedback sensitivity" is still a qualitative concept. For example, it does not describe exactly why the non-physical oscillations occur when $Rg>2$ and standard central difference scheme is used for the convective term.

In Section 2 of the present paper we attempt to give a systematic analysis, which is termed Grid Spatial Stability (GSS), to the cause of wiggles in the numerical solutions for a steady state problem; and then we consider several examples and show that in order to overcome the grid spatial instability a natural and appropriate way is to convert the steady state problem to a pseudo-unsteady one.

The second problem one confronts in searching for a "universal" scheme is the problem of accuracy. As is well-known, for example, the first -order upwind scheme for the convective term is once announced a scheme that circumvents the oscillatory phenomena in the numerical solution. However, due to its low order of accuracy, there is a strong inherent artificial diffusion which distorts the flow picture [14] and makes the scheme unacceptable when the grid Reynolds number is high. Later several schemes (the upwind corrected schemes) were presented, such as Dennis and Chang[15], Kholsa and Rubin[10], to improve the problem with the artificial diffusion.

We postulate some major requirements for the proposed "universal" scheme:

(i) There is no or weak limitation to the Reynolds number, the scheme should produce stable wiggle-free solution in any case.

(ii) In order to be consistent with the central differencing diffusion term the convective term in the scheme should be of at least 2nd or 3rd order accuracy, since the error for the diffusion term is:

$$O(h^2)/Re = uh/RgO(h^2) = O(h^3)$$

(iii) When a boundary layer is confronted, there is no overshoot or undershoot and an accurate numerical outer solution may be achieved.

(iv) The scheme is compact enough and there is a tight coupling between the grid points used for the convection and diffusion terms.

As a matter of fact research in this topic has been carried out via different methods. For example, in [4] an operator compact implicit (O.C.I.) is presented and later modified to an upwind compact scheme (see Christie[8]). Leonard [5-6] presents second and third order schemes called QUICK and QUICKEST, which work well for high Reynolds numbers, though there are still overshoots or undershoots when a sudden change is experienced; Kawamura et al[13] use a 5-point third order scheme to compute a few typical problems in fluid dynamics up to the range of turbulence flow.

Following the above general guide-lines, in Section 3 of this paper we present an adaptive scheme which is basically of third order accuracy, and "boundary layer fitting " — special cares are taken when there is a sudden change in the velocity.

In Section 4 numerical experiments with the adaptive scheme for one-dimentional Burgers equations are conducted and compared to the exact solutions or the numerical results with a fourth order upwind compact scheme [8].

2. The Grid Spatial Stability

We first introduce the concept of grid spatial stability (GSS) for a (linearized) finite difference scheme. This is cast in a way similar to the method used by Lin[7] in the theory of hydrodynamic stability and to the Von Neumann method of stability[12].

Assume that

$$L(u_i)=f_i \tag{1}$$

$$i=1,2...,n$$

is the finite difference equation approximating a steady-state diffusion- convection equation, and U_i the exact solution of (1).

Let

$$u_i=U_i + e_i$$

be the perturbed values of U_i with error e_i. If e_i do not grow with the solution process of (1) the finite difference scheme is said to be spatially stable. One sees that such a spatial stability is directly equivalent to the convergence of the solution procedure. Different solution procedures may lead to different spatial stabilities. However, a commonly occurring solution procedure (for example, the Jacobi iterative procedure) involves expressing u_i in terms of the u values at the adjacent grid nodes :

$$u_i = \Lambda(...u_{i-2},u_{i-1},u_{i+1},u_{i+2},...) \tag{2}$$

For convenience the corresponding spatial stability is termed "Grid Spatial Stability"(GSS).

Following Lin[7] and the Von Neumann method of temporal stability analysis (for example, see Richtmyer and Morton[12]), a Fourier modal analysis may be conducted for the errors e_i. Therefore when considering anyone of the Fourier modes one may write :

$$\cdots\cdots\cdots\cdots$$

$$e_{i-2} = exp\,[\sqrt{-1}(i-2)\alpha]$$

$$e_{i-1} = exp\,[\sqrt{-1}(i-1)\alpha]$$

$$e_{i+1} = exp\,[\sqrt{-1}(i+1)\alpha]$$

$$e_{i+2} = exp\,[\sqrt{-1}(i+2)\alpha]$$

$$\cdots\cdots\cdots\cdots$$

(where $\alpha=\pi k/n$, $k=0,1,2,\ldots$ are the phase angles of the Fourier modes of the errors)

and in particular, e_i being associated with the complex valued amplification factor G:

$$e_i = exp\,[\sqrt{-1}i\,\alpha]G$$

By substituting $\ldots e_{i-2}$, e_{i-1} , e_i , e_{i+1} e_{i+2} ,...in

$$L(e_i)=0 \tag{3}$$

which is derived from (1), G can be solved. $|G|\leq 1$ indicates that the scheme is spatially stable or the solution procedure is stable and convergent; while $|G| > 1$ corresponds to the spatial instability.

The GSS criterion is based on the local grid pointwise structure of the finite difference scheme and the very first concept of stability. This is in contrast to the conventional stability analysis (for example, see [4]) which generally leads to a problem of finding the eigenvalues of a global matrix and determining whether the spectral radius is greater than unity or not. Except for a few typical cases the latter is often a more difficult problem than the GSS criterion. In addition the GSS provides a convenient way to consider the spatial stability/convergence of the scheme at the boundaries. Even in a time dependent problem with an implicit scheme the GSS may also be used to judge the convergence of the solution procedure at the given time level.

In order to illustrate how the GSS can be used to justify convergence of a scheme or a "wiggle-free" solution,let us consider a few popular schemes:

(i) The Burgers equation with central difference for the convective term:

In this case (3) has the form:

$$u\frac{e_{i+1}-e_{i-1}}{2h}=\frac{e_{i+1}-2e_i+e_{i-1}}{\mathrm{Re}h^2}$$

where Re=Reynolds number, h=grid size.

or

$$Rg(e_{i+1}-e_{i-1})/2=e_{i+1}-2e_i+e_{i-1}$$

where Rg=Reuh

$$G=\frac{\sqrt{-1}\,Rg\,\sin\alpha}{2}+\cos\alpha$$

In order that $|G|\leq 1$, we must have $Rg\leq 2$. Moreover, if the grid stability is considered around the boundary, (3) should be modified to account for the boundary condition. For example, if u_0 is specified at $x=x_0$, then $e_0=0$; we have

$$u_0\frac{e_2}{2h}=\frac{e_2-2e_1}{\mathrm{Re}h^2}$$

$$1/2\ Rg\,exp[2\sqrt{-1}\alpha]=exp[2\sqrt{-1}\alpha]-2exp[\sqrt{-1}\alpha]\ G$$

$$G=\frac{Rg/2-1}{2}$$

In order that $|G|\leq 1$ it is necessary to have $Rg\leq 6$.

On the other hand if $\partial u/\partial x$ is specified at $x=x_0$, we have,

$$\frac{e_1-2e_0+e_{-1}}{\mathrm{Re}\ h^2}=0$$

or

$$|G|=|\cos\alpha|\leq 1 \qquad \text{for } \textit{any Re}.$$

This explains the phenomenon, as pointed out by Roache [1], that a derivative boundary condition can cure the "wiggle" effect created by a Dirichlet boundary condition when central differencing is used for the convective term.

(ii)The Burgers equation with the first order upwind differencing for the convective term:

Equation (3) has the form

$$u>0, \qquad u\,\frac{e_i - e_{i-1}}{h} = \frac{e_{i-1} - 2e_i + e_{i+1}}{\mathrm{Re}\; h^2}$$

$$u<0 \qquad u\,\frac{e_{i+1} - e_i}{h} = \frac{e_{i-1} - 2e_i + e_{i+1}}{\mathrm{Re}\; h^2}$$

or

$$Rg(\,G - exp(+\sqrt{-1}\;\alpha) = 2cos\,\alpha - 2G \qquad Rg = uhRe$$

Thus

$$|G| = \left|\frac{2cos\,\alpha + Rg exp(\;\sqrt{-1}\alpha)}{2 + Rg}\right| \leq 1 \quad \text{for } \textit{all Rg}$$

(iii) The elliptic equation (say, in two dimensional space):

For simplicity, assume $\Delta x = \Delta y = h$, the five point scheme leads to:

$$e_{i-1,j} + e_{i+1,j} + e_{i,j-1} + e_{i,j+1} - 4e_{i,j} = 0$$

or

$$exp[-\sqrt{-1}\alpha] + exp[\sqrt{-1}\alpha] + exp[-\sqrt{-1}\beta] + \exp[\sqrt{-1}\beta] - 4G = 0$$

$$|G| = \left|\frac{\cos\alpha + \cos\beta}{2}\right| \leq 1 \qquad \textit{spatially stable};$$

In other words the five point scheme is always convergent (in the general sense).

(iv) A linear equation system with the coefficient matrix diagonally dominated:

$$Au = b \qquad with\ A = (a_{ij})$$

where $u=(u_1, u_2, \ldots\ldots, u_n)$ is the unknown vector.

$$|G| = \frac{\left|\sum_{j=1, j\neq i}^{n} a_{ij}\ exp[j\sqrt{-1}\alpha]\right|}{a_{ii}} \leq 1$$

The system of linear equations is spatially stable.

(v)The Burgers equation with fourth order central differencing for the convective term:

$$u\ \frac{-e_{i+2}\ +8e_{i+1}\ -8e_{i-1}\ +e_{i-2}}{12h} = \frac{e_{i+1}\ -2e_i\ +e_{i-1}}{\text{Re}\ h^2}$$

$$G = \cos\alpha - \sqrt{-1}\ Rg\frac{(8sin\,\alpha - sin\,2\alpha)}{12}$$

$$|G| \leq 1, \qquad \text{if } Rg \leq 6/5.$$

From the above examples (i),(ii) and (v) it seems that an upwind scheme for the convective term could help to relax the limitation to Rg. Unfortunately this is not always true. Consider the following example:

(vi) The Burgers equation with second order one-sided upwind scheme for the convective term. Say,u>0, (3) has the following form:

$$u\ \frac{3e_i\ -4e_{i-1}\ +e_{i-2}}{2h} = \frac{e_{i+1}\ -2e_i\ +e_{i-1}}{\text{Re}h^2}$$

Then

$$G = \frac{2cos\,\alpha - Rg(-4cos\,\alpha + cos\,2\alpha)/2 - Rg\sqrt{-1}(4sin\,\alpha - sin\,2\alpha)/2}{3/2Rg+2}$$

When Rg is large, for a range of α (say, $\alpha \approx \pi$), |G|=5/3>1 , the scheme is unstable. However, if a relaxation method is applied to the solution procedure, the new amplification factor becomes

$$\xi = (1-\omega)+\omega G = 1-\omega(1-G)$$

where ω is the relaxation factor.

Then

$$\xi = 1 + \omega \frac{2(\cos\alpha-1)-Rg(3-4\cos\alpha+\cos 2\alpha)/2-Rg/2\sqrt{-1}(4\sin\alpha-\sin 2\alpha)}{3/2Rg+2} \quad \text{(A)}$$

and ξ can hopefully be less than 1 for large Rg if ω is small enough, details are given in the next section.

Moreover, if we let the time step size

$$\Delta t = \frac{\omega}{(3u)/(2h)+2/(Reh^2)}$$

in the time dependent problem

$$u_t = -uu_x + u_{xx} \ /Re$$

with forward differencing in time and the same upwind scheme for the convective term, and apply the conventional Von Neumann temporal stability analysis to it, we can see that the amplification factor has exactly the same form (A). This manifests the equivalence of such a time dependent problem to the above relaxed steady-state problem, and the GSS analysis may be replaced by the conventional Von Neumann temporal stability analysis.

3. The Adaptive Scheme for the Diffusion-Convection Equations

As is well known, in spite of its attractive convergence (spatial stability), the first order upwind differencing is not acceptable for the convective term, particularly when the Reynolds number is high. It produces false results due to the artificial diffusion(e.g., see [14]). On the other hand, the conventional central differencing is free of artificial diffusion but limited by the spatial stability condition $Rg \leq 2$. In order to remove the limitation we have shown in the previous section by the GSS criterion that a straightforward second order upwind scheme does not work but a relaxed one which is equivalent to a time dependent problem may hopefully work. For this reason we only need to consider the time dependent problems. A steady state problem is first relaxed to a pseudo- time dependent problem and the steady state solution (if exists) can be achieved when time t is large enough and the numerical solution practically does not change.

There has been a constant effort in searching for such schemes: Leith[16], Leonard[5,6] applied polynomial or exponential function interpolation to form their schemes with an accuracy up to second or third order; Kawamura et al[13] obtained a five-point scheme of third order accuracy with attractive numerical results, which turns out to be a fourth order central differencing scheme plus a fourth order difference multiplied by a constant. These schemes produce good results particularly when Re is high (though some still have certain artificial diffusion), however, when the unknown function is subject to a sudden change (e.g., a boundary layer), they tend to produce undesirable wiggles (see [6]).

Following the guide-lines postulated in Section 1, the purpose of the present Section is to present an adaptive scheme which is basically of third order accuracy but "boundary layer fitting"——around a boundary layer it becomes a one-sided second order upwind scheme and avoids the wiggles. This procedure can be understood as a " computational noise filtering".

We first consider the one-sided second order upwind scheme and its temporal stability, then modify it to a third order upwind scheme and consider the stability as well. Finally the criterion for switching from the third order scheme to the one-sided second order scheme is presented.

The one-dimensional Burgers Equation with a source term S(x) is given as:

$$u_t = -Uu_x + u_{xx}/Re + S(x) \tag{4}$$

where u is the unknown velocity, U=U(u,x) and S=S(x) are given functions; Re is the Reynolds number.

(i) the one-sided second order upwind scheme:

for $U \geq 0$,

$$u_x = \frac{3u_m - 4u_{m-1} + u_{m-2}}{2h}$$

for $U<0$,

$$u_x = \frac{-3u_m + 4u_{m+1} - u_{m+2}}{2h}$$

where m=1,2,...,n replaces i, i is reserved for $\sqrt{-1}$, and h the grid size.

Let $c = U\,\Delta t/h$ be the Courant number and

$$\nu = \frac{\Delta t}{\mathrm{Re}h^2} = \frac{c}{Rg}$$

where Rg is the grid Reynolds number.

The finite difference approximation to (4) is

$$u^{l+1}{}_m = u^l{}_m - c/2(3u^l{}_m - 4u^l{}_{m-1} + u^l{}_{m-2}) + \nu(u^l{}_{m+1} - 2u^l{}_m + u^l{}_{m-1}) + s(x_m) \tag{5a}$$

for $U \geq 0$;

$$u^{l+1}{}_m = u^l{}_m - c/2(-3u^l{}_m +$$

$$+4u^l{}_{m-1} - u^l{}_{m-2}) + \nu(u^l{}_{m+1} - 2u^l{}_m + u^l{}_{m-1}) + s(x_m) \qquad (5b)$$

for U<0; where the integer l denotes t=lΔt.

For both U>0 and U<0 the Von Neumann stability analysis yields

$$G = 1 - 2\nu(1-\cos\alpha) - c/2(1-\cos\alpha)^2 \mp ic/2[4\sin\alpha - \sin 2\alpha] \qquad (6)$$

G is a non-trivial function of the phase α, Figure 1 shows the polar stability diagrams for the most severe case Re=∞, ($\nu=0$) with the Courant number c=.2, .1 and .05. One can see that the diagrams lie well within the unit circle and the scheme is stable. However, one could have trouble when α is close to 0. When $\alpha=0$, G =1. From (6), when $\alpha \simeq 0$, by Taylor expansion of G ,

$$|G| = \sqrt{(1-(\nu\alpha)^2 - c/8\alpha^4)^2 + (c\alpha)^2}$$

or

$$|G| \simeq 1 + 1/2(c^2 - 2\nu)\alpha^2$$

When $\nu=0$ (the most severe case),

$$|G| = 1 + 1/2(c\alpha)^2 > 1.$$

Fortunately, provided c is small enough (say, c≤.1) |G| is greater than unity only by a second order infinitesimal which involves a small range of α; and there is no disasterous error growth.

The one-sided scheme is appropriate for such a grid point at which u experiences a dramatic change (i.e. a boundary layer exists). It does not need any information from the other side of the grid point and yields formally second order accuracy. On the other hand, for a regular grid point such a scheme does not provide the maximum accuracy. For example, if U>0, the scheme (5a) utilizes 4 grid points x_{m-2}, x_{m-1}, x_m and x_{m+1}; while the one-sided scheme only takes the u values at the first 3 points. For better accuracy and tighter coupling between the convection and diffusion terms in the equation it is necessary to construct a third order upwind scheme for the convection term.

(ii) the third order upwind scheme

The third order scheme for the convection term is constructed by combining the above one-sided upwind scheme and the central difference scheme in a way that their second order error terms just cancel out. For example, for U>0

$$\frac{3u_m-4u_{m-1}+u_{m-2}}{2h}=(u_x)_m-\frac{h^2(u_{xxx})_m}{3}+O(h^3)$$

and

$$\frac{u_{m+1}-u_{m-1}}{2h}=(u_x)_m+\frac{h^2(u_{xxx})_m}{6}+O(h^4)$$

By eliminating the h^2 terms one obtains the third order scheme for U>0,

$$\frac{2u_{m+1}+3u_m-6u_{m-1}+u_{m-2}}{6h}=(u_x)_m+O(h^3) \tag{7a}$$

for U<0, similarly,

$$\frac{-2u_{m-1}-3u_m+6u_{m+1}-u_{m+2}}{6h}=(u_x)_m+O(h^3) \tag{7b}$$

In the third order scheme (7a) and (7b) are used to replace the second upwind convective terms in (5a) and (5b) respectively.

A stability analysis similar to that for the second order upwind scheme may be carried out:

$$G=1-2\nu(1-\cos\alpha)-c/3(1-\cos\alpha)^2-i\,sgn(U)c\frac{[8\sin\alpha-\sin2\alpha]}{6}$$

where sgn(U) is the Kronecker function of U. Figure 2 shows the half polar stability diagram for Re=∞ and c=.2, .1 and .05. One sees that the diagrams lie well within the unit circle except around α=0. A Taylor expansion at α=0 gives

$$|G|=1+(c^2/2-\nu)\alpha^2 \qquad (\textit{to 2nd infinitesmal})$$

A strict condition for $|G|\leq 1$ is $c^2/2\leq\nu$ or Rg$\leq$2/c. Nevertheless by the same argument mentioned for the second order scheme, such a $|G|$ value will not cause a disasterous error growth, in general, c=.1 is quite appropriate for Re up to ∞ Table I lists the $|G|$

values vs. α (in radian) for c=.1 and Re=∞ one sees $|G|$ is greater than 1 only by an amount of the order 10^{-4} for $\alpha \neq 0$.

With the third order scheme the convective term is more consistent with the diffusion term as pointed out in 1. However, despite its higher accuracy the scheme still produces spurious wiggles when u is subject to a sudden change (see Fig. 3) since it is a two-sided scheme. In order to circumvent this one needs to consider the one-sided scheme in (i). Although the formal accuracy is lower the one-sided scheme yields wiggle-free solutions. An appropriate criterion is needed to switch from the third order scheme to the second order one-sided scheme.

(iii) the criterion for switching to the second order upwind scheme:

As is well known, the third order scheme is based on the local cubic polynomial interpolation. A sudden change in u will cause wiggles in the solution and spoil it, since the local cubic polynomial approximation quickly deteriorates. Such wiggles are characterized by the alternating ups and downs(the local extrema) at the successive grid points. Nevertheless, an exceptional case is, if u values are still monotonic around the boundary layer, say,

$u(x)<u(x+h)<u(x+2h)$; with $u(x+h)-u(x)>>h$ (there exists a boundary layer)

no wiggles will appear in the solution, and the local cubic polynomial or the third order scheme is still accurate. On the other hand, if u(x+h) is a local extremum the third order scheme is likely to yield a wiggle-like solution. In this case a second order one-sided scheme which involves only grid points outside the boundary layer (i.e., x, x+h and x+2h) is preferable.

The monotonicity property is a useful tool for filtering the computational noise and has been used by many authors for hyperbolic equations, particularly in handling shock wave computations; for example, see Godunov[17] , Van Leer[11], etc..

In practice in the neighborhood of a boundary layer, the monotonicity is checked by the sign of the product

$s=[(u(x)-u(x+h)][u(x+h)-u(x+2h)]$

If s is positive the monotonicity holds and the third order scheme is employed. Otherwise we switch to the second order upwind scheme. This adaptive scheme makes maximum use of the third order scheme and switches to the second order one-sided scheme just before the spurious wiggles appear in the solution. The third order scheme is resumed if the monotonicity holds again. In Fig.3 the solution of a Burgers equation with the adaptive scheme is shown as a comparison to the third order scheme result. As a matter of fact this solution agrees almost exactly with the theoretical solution. Both solutions are obtained through the pseudo- time dependent procedure.

In the next section several numerical experiments will be conducted and the results are compared to the theoretical soluitons (if available) or the numerical results by the fourth order upwind compact scheme (Christie [8]).

4. Numerical Results

In this section several versions of the Burgers equation, both steady and unsteady, linear and non-linear, with different boundary conditions are solved numerically with the present adaptive scheme. The results are compared to the theoretical exact solution or other existing numerical results.

(i)The first problem considered is the linear Burgers equation:

$$u_{xx}\,/Re-u_x = 0 \qquad 0\leq x\leq 1,\ u(0)=1,\ u(1)=0 \tag{8}$$

Different Re values from 5 to 10^6 are considered. The analytical solution

$$u=\frac{e^{Rex} - e^{\mathrm{Re}}}{1-e^{\mathrm{Re}}}$$

will be used for comparison. There is a sharp boundary layer near x=1 when Re is large. This model has been used by many authors for testing their schemes. In order to obtain consistent accuracy with the scheme the fictitious points outside the boundary should be evaluated by cubic polynomial extrapolation when necessary. In all the cases a modest number (10) of grid points are used. As mentioned above the solutions are obtained by a pseudo- time dependent procedure:

$$u_t = u_{xx}/Re - u_x$$

Table IIa gives the numerical solution for Re=5. The exact solution is also listed for comparison. With such a low Reynolds number the adaptive scheme is completely of third order accuracy. The steady state solution is obtained at t=4 (400 time steps with Δt=.01) with the max. error .002.

Table IIb shows a comparison of the results between the present method and the fourth order upwind compact scheme by Christie[8] (referred to as U.C. in the table) for Re=20 and 100. One sees that the two schemes yield quite comparable results. Numerical solutions with the present adaptive scheme for Re=1000, 10^4 and 10^6 can be found in Table IIc. Figures 4a- 4d show the good agreements of the numerical results with the analytical solutions (solid lines).

(ii) The second problem is a time dependent non-linear Burgers equation:

$$u_t = u_{xx}\ /Re - uu_x \tag{9}$$

$0 \leq x \leq 1$, $t > 0$ and $u(0,t) = u(1,t) = 0$; $u(x,0) = \sin(\pi x)$.

Cole[9] predicted the appearance of a sharp boundary layer as Re increases and presented the analytical solution in a form of infinite series. Benton and Platzman[17] describe the analytical solutions for a class of Burgers equations of this type. Christie[8] gives some numerical results using his fourth order upwind compact scheme. They will be used for comparison with the present method.

For Re=10, 1000 and 10000, t=.5 and n (number of grid subintervals) =10 and 40 (Δt is changed accordingly to keep $\Delta t/\ \Delta x$=.1) (9) is solved numerically by the adaptive scheme. The results are listed in Table IIIa. Figure 5 shows the curves for different Re numbers. The solid lines represent those curves with n=40. One sees the results with n=10 and n=40 agree quite well. It is interesting to compare the present numerical results with those obtained by the upwind compact scheme of Christie[8]. In Table IIIb we list the numerical solutions by both schemes at x=.9 and t=.5. The results with the upwind compact scheme (U.C.) are taken directly from [8]. It can be seen that the adaptive scheme presents more consistent results with different n when Re is high (Re=10000). To further compare the two schemes in Table IIIc we present the results in the case Re=10000, t=.5 and n=160 around the boundary layer. Although the numerical evaluation of the analytical solution is not directly available, the adaptive scheme does yield a sharper boundary layer at x=1.0 than the upwind compact scheme; there is a 23% difference at the grid point next to x=1.0. It seems that the result with the present adaptive scheme could be better.

(iii) The third problem we have investigated is another non-linear Burgers equation:

$$u_{xx}/Re-uu_x=u_t \qquad -.5\leq x\leq .5 \tag{10}$$

subject to u(-.5, t)=1, u(.5 t)=-1. Initially u(x,0) is a straight line passing through the points (-.5,1) and (.5, -1).

The analytical (steady-state) solution is

$$u=-\alpha tanh(\alpha/2 \text{ Re } x)$$

where α satisfies αtanh[Reα/4]=1 and must be evaluated numerically by Newton's method.

Table IV lists the exact and numerical steady-state solutions for Re=10, 100 and 10^6 respectively. Here Δx=.05 and Δt=.005. The maximum error is around the magnitude of 10^{-3} even when Re is high. In Figure 6 we present the curves for Re=10, 100 and 10^6. The solid lines represent the exact solutions. It can be seen that both analytical and numerical solutions agree quite well with each other. It should be noted that the symmetry property of u (about x=0) is used in the numerical solutions.

5. Conclusion

In consideration of both accuracy and convergence for a steady state convection dominated problem the Grid Spatial Stability shows the non-convergence of a higher order upwind scheme. However a relaxation of the scheme proves to be equivalent to the solution procedure of a pseudo- time dependent one. An adaptive upwind scheme has been suggested to handle both time dependent and steady-state problems (via the pseudo- time dependent procedure). The explicit scheme is generally of third order accuracy in space but could become second order accurate at a thin boundary layer. It has been demonstrated that the present scheme yields numerical results comparable to those by the fourth order upwind compact scheme; and there is practically no limitation to the Reynolds number, and no parameters to choose as well. (In the U.C. method one has to choose a parameter)

In programming, similar to the first order upwind differencing, there is a little more work than the regular central differencing. In addition, the scheme necessitates extrapolation for the fictitious values outside the boundary points. However, comparing to the advantages of the adaptive scheme the little extra work seems worthwhile.

In principle there is no difficulty to extend the present adaptive scheme to two or three dimensional flow problems. The work is underway and will probably form the subject of another paper.

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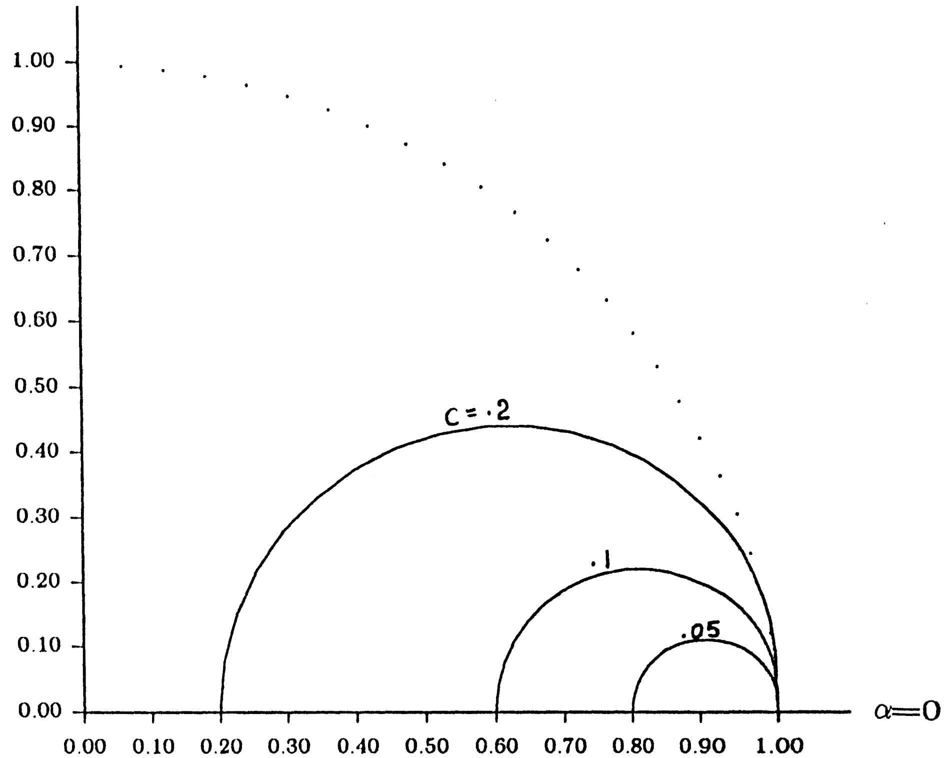


Fig. 1 Polar Stability Diagram for the 2nd Order Scheme

Re=∞, c=.2, .1 and .05

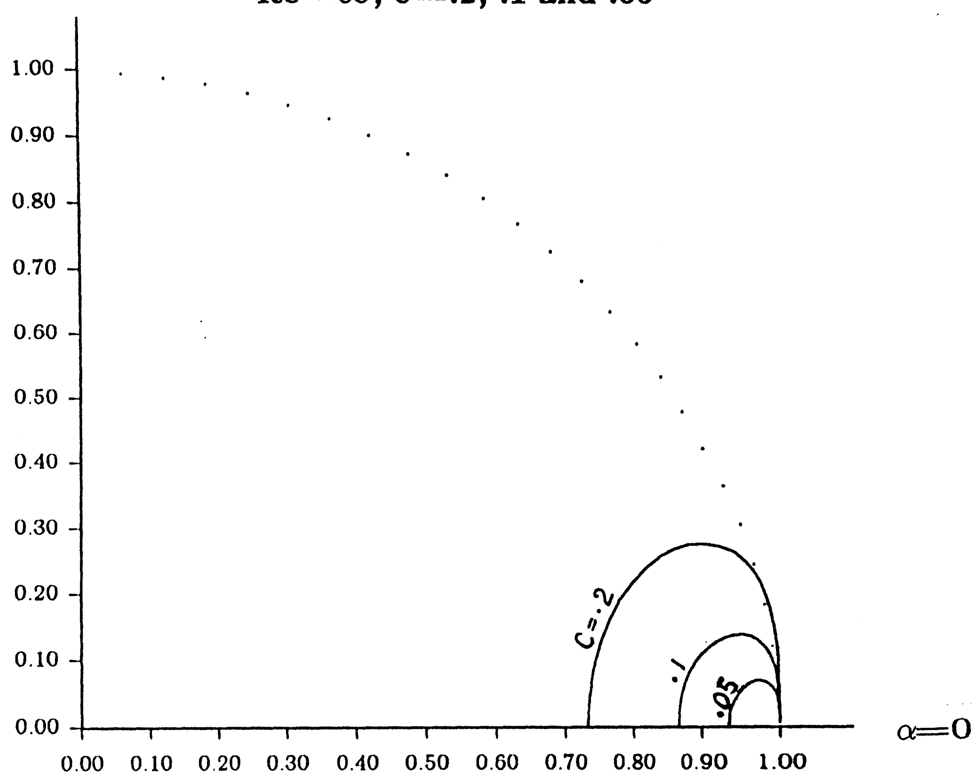


Fig. 2 Polar Stability Diagram for the 3rd Order Scheme

Re=∞, c=.2, .1 and .05

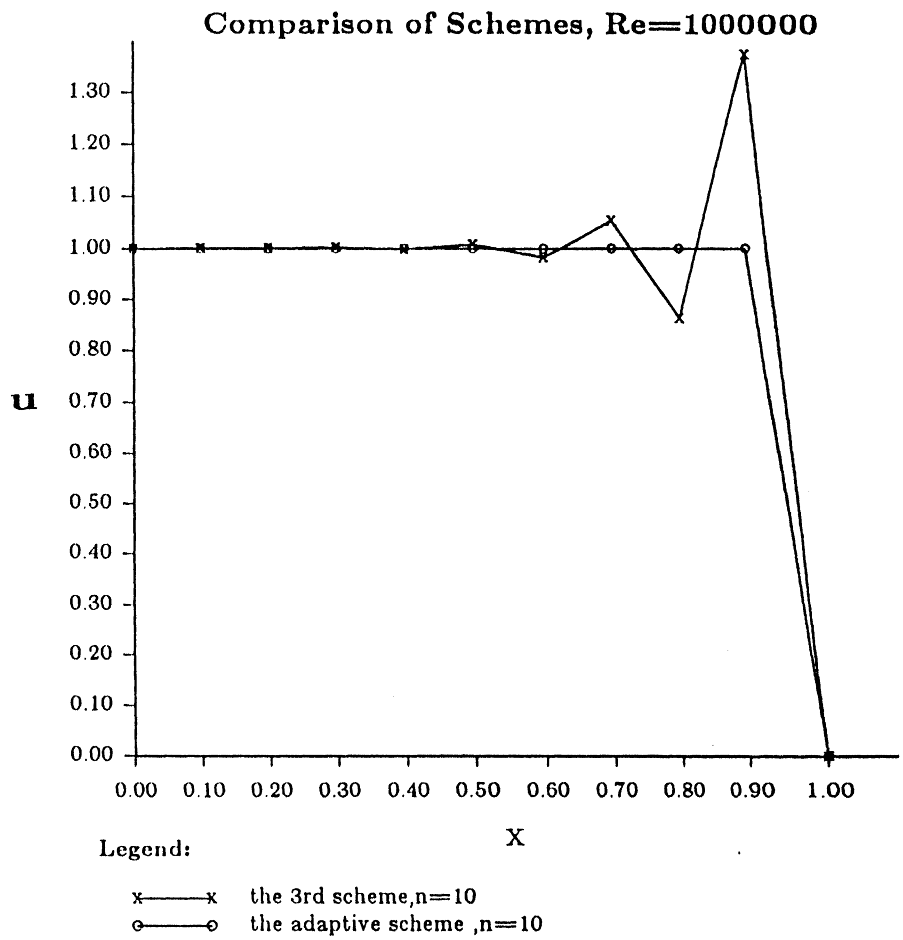


Fig. 3 Comparison of Schemes, $Re=10^6$

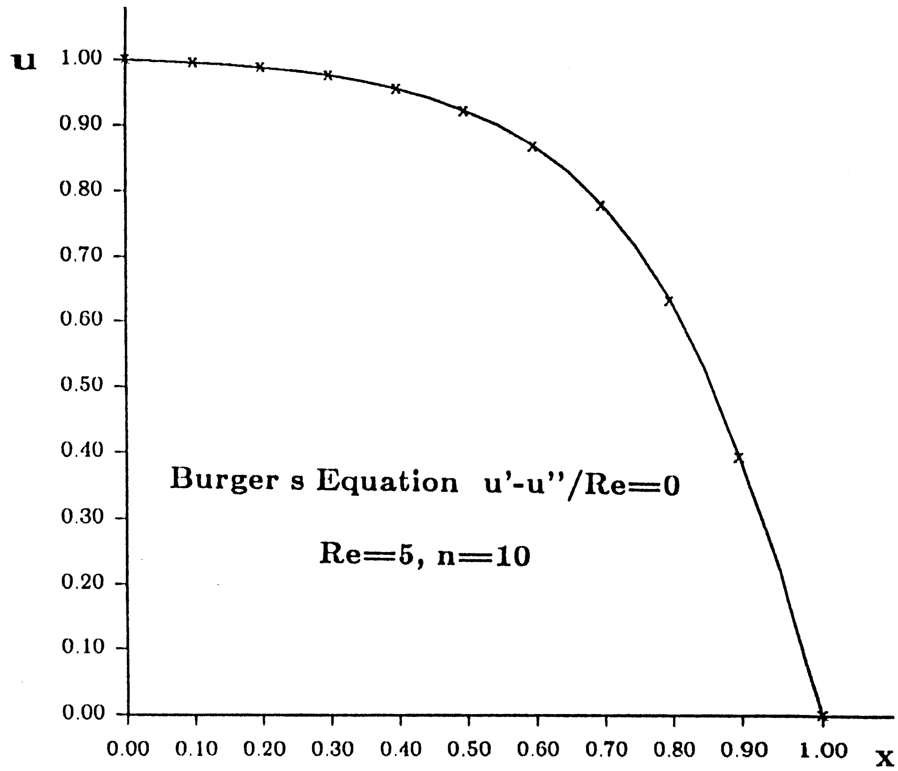

u
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
x
Burger s Equation u'-u''/Re=0
Re=5, n=10

Fig. 4a

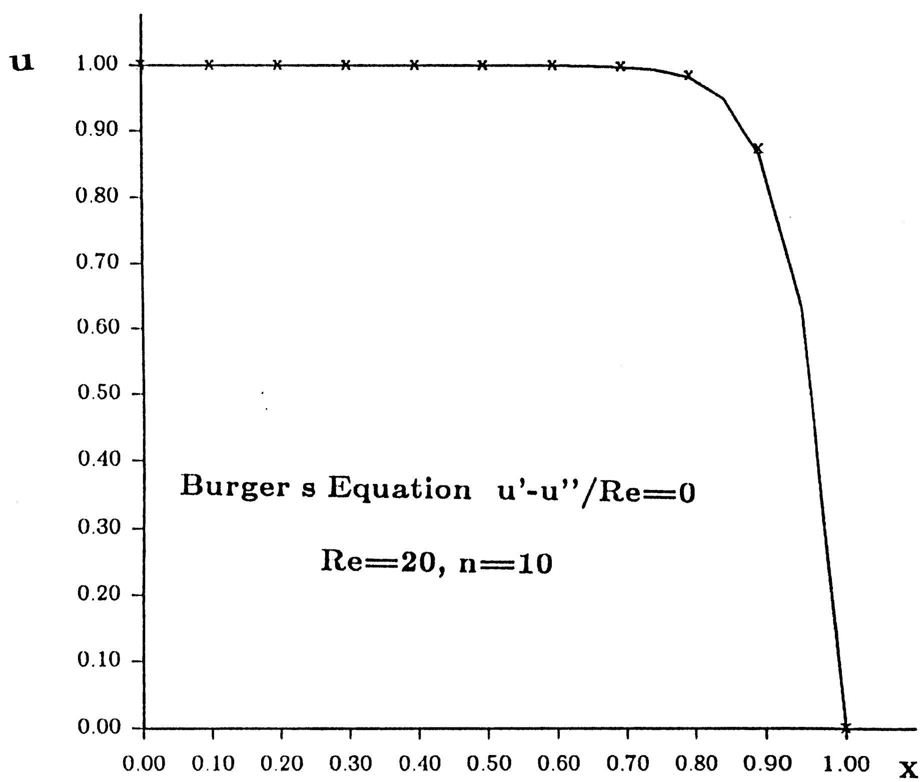

u
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
x
Burger s Equation u'-u''/Re=0
Re=20, n=10

Fig. 4b

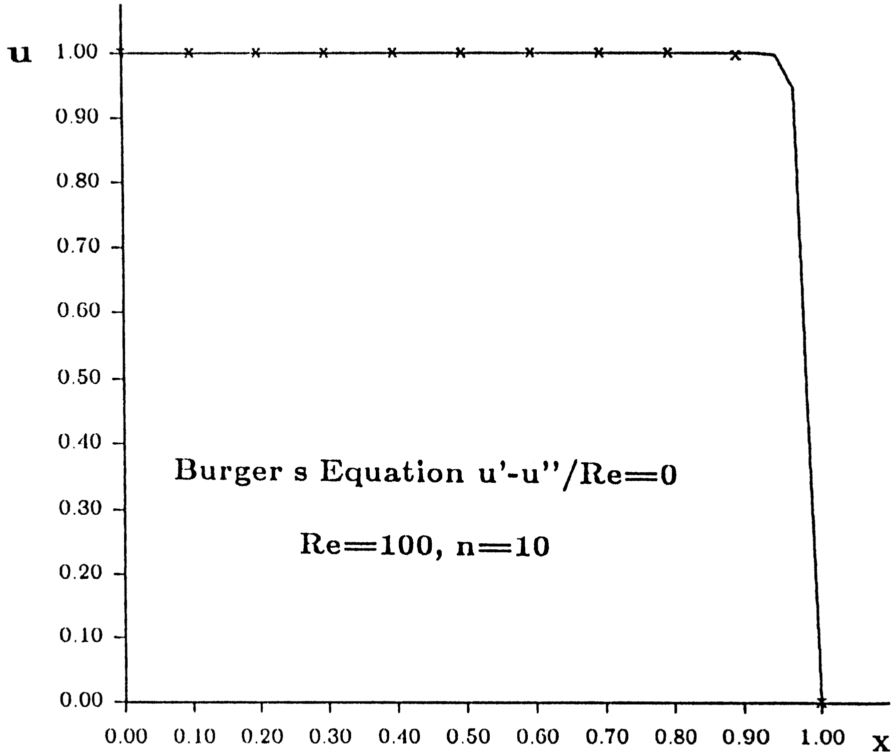

u
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
Burger s Equation u'-u''/Re=0
Re=100, n=10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
x

Fig. 4c

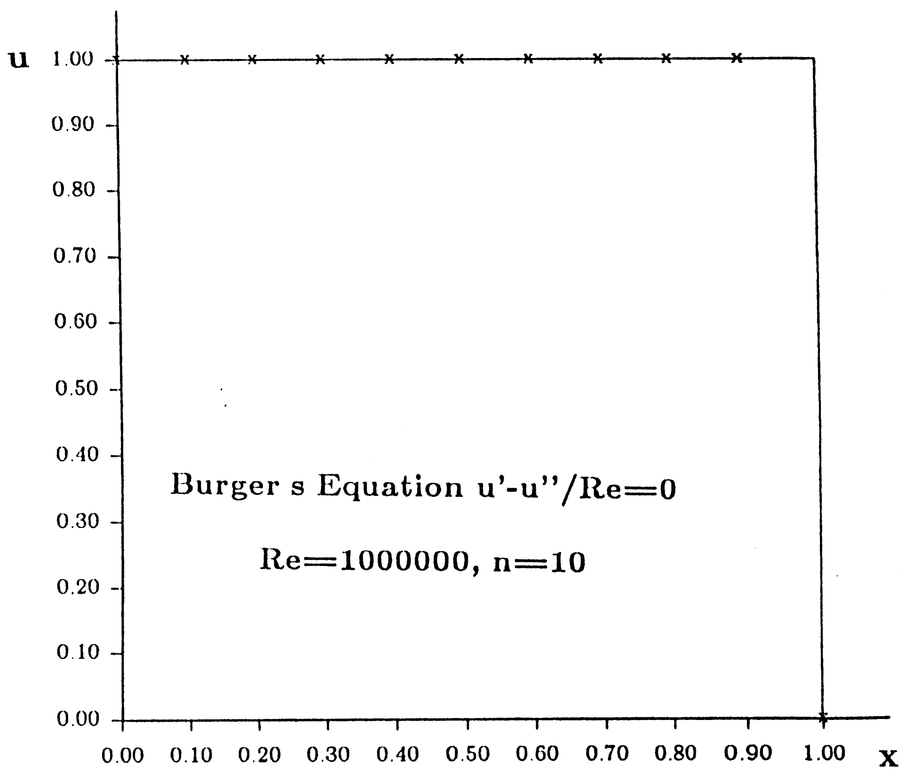

u
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
Burger s Equation u'-u''/Re=0
Re=1000000, n=10
0.00
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
0.90
1.00
x

Fig. 4d

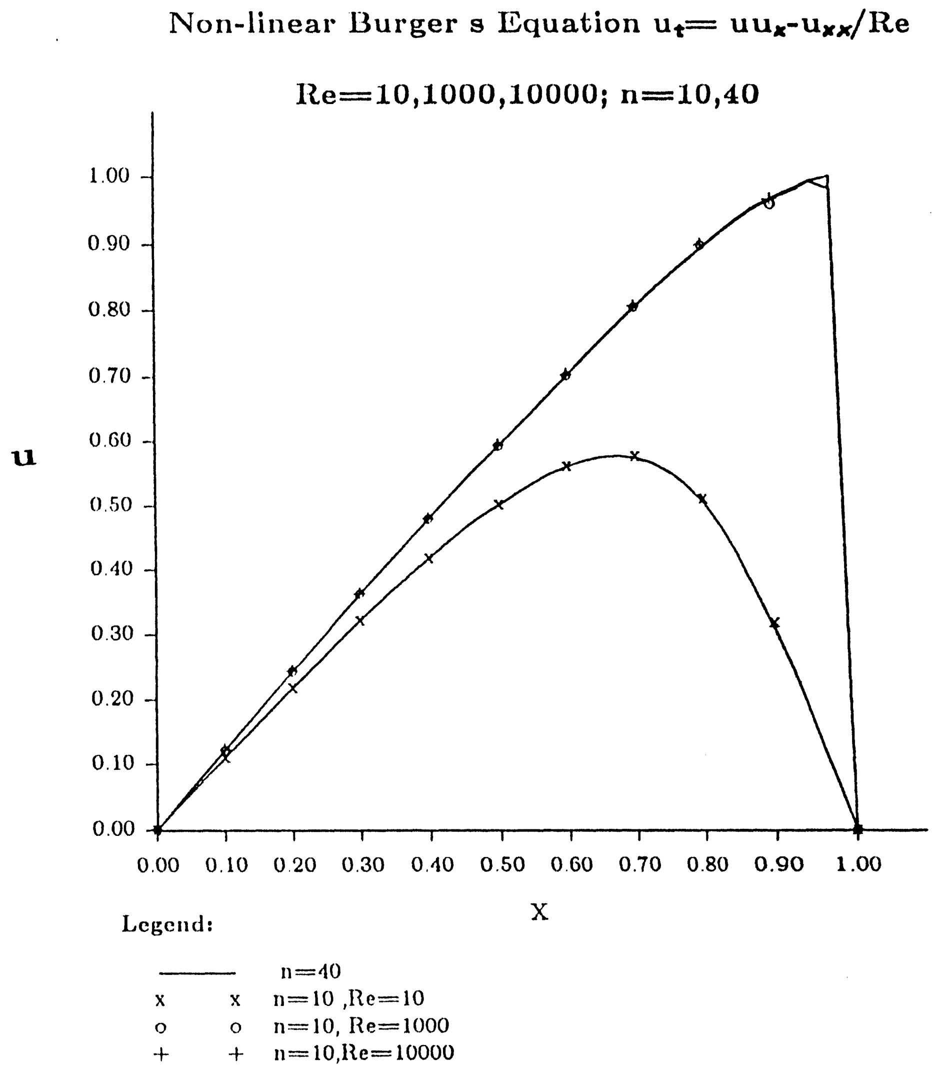
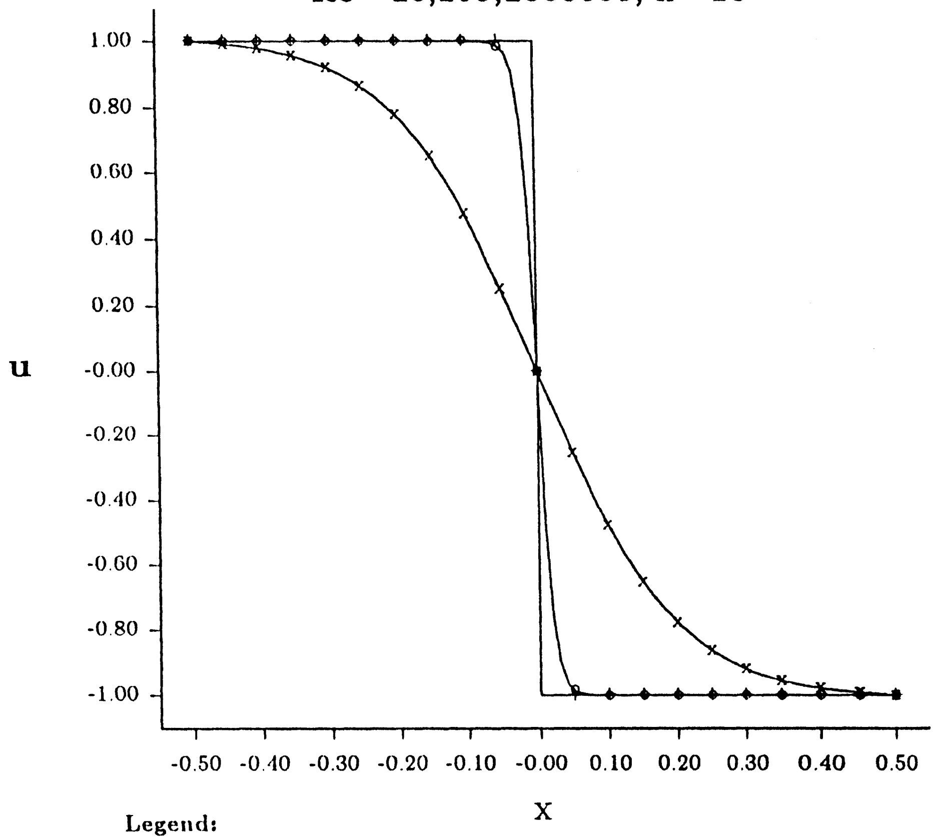

Non-linear Burger s Equation $u_t = uu_x - u_{xx}/Re$
Re=10,1000,10000; n=10,40
u
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
0.00 0.10 0.20 0.30 0.40 0.50 0.60 0.70 0.80 0.90 1.00
X
Legend:
n=40
x x n=10 ,Re=10
o o n=10, Re=1000
+ + n=10,Re=10000

Fig. 5

Non-linear Burger s Equation $uu_x - u_{xx}/Re = 0$

Re=10,100,1000000; n=20



Legend:

———		Exact Solution
x	x	Re=10, n=20
o	o	Re=100, n=20
+	+	Re=1000000, n=20

Fig. 6

Table I

|G| vs. α (Re$=\infty$, c=.1)

α	|G|
.125664	1.000077
.251327	1.000283
.376991	1.000545
.502655	1.000748
.628319	1.000739
.753982	1.000340
.879646	.999357
1.005310	.997594
1.130973	.994867
1.256637	.991017
1.382301	.985924
1.507965	.979518
1.633628	.971794
1.759292	.962816
1.884956	.952727
2.010619	.941747
2.136283	.930176
2.261947	.918384
2.387610	.906799
2.513274	.895884
2.638938	.886116
2.764602	.877952
2.890265	.871796
3.015929	.867967
3.141593	.866667

Table IIa

Numerical and Exact Solution of (8)

x	numerical	exact
.0	1.0	1.0
.1	.996	.996
.2	.989	.988
.3	.977	.976
.4	.958	.957
.5	.926	.924
.6	.873	.871
.7	.784	.782
.8	.637	.636
.9	.396	.396
1.0	0	0
max. error	.0021	0

Table IIb

Numerical Solutions of (8)

Re	20	20	20	100	100	100
x	Exact	U.C.	Adaptive	Exact	U.C.	Adaptive
.0	1.0	1.0	1.0	1.0	1.0	1.0
.1	1.0	1.0	1.0	1.0	1.0	1.0
.2	1.0	1.0	1.0	1.0	1.0	1.0
.3	1.0	1.0	1.0	1.0	1.0	1.0
.4	1.0	1.0	1.0	1.0	1.0	1.0
.5	1.0	.999	1.0	1.0	1.0	1.0
.6	1.0	1.001	1.0	1.0	1.0	1.0
.7	.997	.993	.998	1.0	1.0	1.0
.8	.982	.990	.984	1.0	1.0	1.0
.9	.865	.839	.873	1.0	1.002	.996
1.0	0.	0.	0.	0.	0.	0.
max. error	0	.024	.008	0	.002	.004

Table IIc

Numerical Solutions of (8)

Re	1000	1000	10000	10000	1000000	1000000
x	exact	numerical	exact	numerical	exact	numerical
.0	1.0	1.0	1.0	1.0	1.0	1.0
.1	1.0	1.0	1.0	1.0	1.0	1.0
.2	1.0	1.0	1.0	1.0	1.0	1.0
.3	1.0	1.0	1.0	1.0	1.0	1.0
.4	1.0	1.0	1.0	1.0	1.0	1.0
.5	1.0	1.0	1.0	1.0	1.0	1.0
.6	1.0	1.0	1.0	1.0	1.0	1.0
.7	1.0	.999	1.0	1.0	1.0	1.0
.8	1.0	1.002	1.0	1.0	1.0	1.0
.9	1.0	.996	1.0	1.0	1.0	1.0
1.0	0.	0.	0.	0.	0.	0.
Max. error	0.	.004	0.	0.	0.	0.

Table IIIa

Numerical Solutions of (9) at t=.5

x	Re=10 n=10	,, n=40	Re=1000 n=10	,, n=40	Re=10000 n=10	,, n=40
0.	0.	0.	0.	0.	0.	0.
.1	.1095	.1096	.1216	.1217	.1216	.1217
.2	.2173	.2175	.2426	.2426	.2428	.2428
.3	.3211	.3214	.3621	.3621	.3624	.3623
.4	.4176	.4181	.4794	.4792	.4797	.4796
.5	.5013	.5019	.5931	.5929	.5936	.5934
.6	.5614	.5616	.7019	.7017	.7025	.7024
.7	.5771	.5756	.8044	.8032	.8056	.8042
.8	.5114	.5056	.8984	.8938	.8986	.8951
.9	.3180	.3101	.9588	.9662	.9654	.9685
1.0	0.	0.	0.	0.	0.	0.

Table IIIb

Comparison of Numerical solutions at t=.5, x=.9

	Re=10 n=10	,, n=20	,, n=40	,, n=160	Re=10000 n=10	,, n=20	,, n=40	,, n=160
Adaptive	.3180	/	.3101	/	.9654	/	.9685	.9665
U.C.	/	.3110	/	.3093	/	.891	/	.9590

Table IIIc

Comparison of Numerical Results

x	$Re=10^4$ n=160,t=.5 U.C.	$Re=10^4$ n=160,t=.5 Adaptive	$Re=10^6$ n=160,t=.5 Adaptive
.9	.959	.9665	.9667
	.962	.9701	.9703
	.965	.9735	.9737
	.968	.9768	.9770
	.971	.9799	.9802
	.973	.9829	.9831
	.976	.9856	.9859
	.978	.9882	.9886
.95	.980	.9907	.9910
	.981	.9929	.9932
	.983	.9948	.9952
	.984	.9966	.9970
	.985	.9981	.9985
	.985	.9993	.9997
	.988	1.0001	1.0007
	.761	.9901	1.0012
1.0	0.	0.	0.

Table IV

Comparison of Exact and Numerical Solutions of (10)

x	exact Re=10	numerical ,,	exact Re=100	numerical ,,	exact $Re=10^6$	numerical ,,
-.5	1.	1.	1.	1.	1.	1.
-.4	.9781	.9779	1.	1.	1.	1.
-.3	.9201	.9203	1.	1.	1.	1.
-.2	.7766	.7784	1.	1.	1.	1.
-.1	.4731	.4763	1.	1.0026	1.	1.
.0	0.	0.	0.	0.	0.	0.
.1	-.4731	-.4763	-1.	-1.0027	-1.	-1.
.2	-.7766	-.7784	-1.	-1.	-1.	-1.
.3	-.9201	-.9203	-1.	-1.	-1.	-1.
.4	-.9781	-.9779	-1.	-1.	-1.	-1.
.5	-1.	-1.	-1.	-1.	-1.	-1.
max. error		.0033		.0027		0

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