MATHEMATICAL MODELS of REASONING

Competence Models of Reasoning about Propositions in English & Their Relationship to the Concept of Probability

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ABSTRACT

Reasoning may be separated into two kinds: *analytic* and *synthetic.* Analytic reasoning deduces incontestable consequences from prior hypotheses. Synthetic reasoning invents hypotheses. I advance the thesis that the probabilistic logic outlined here accurately describes the *standards of inference* for analytic propositional logic as it is found in ordinary English. The value of probabilistic logic as a normative prescription for reasoning is not considered in this paper.

There are, in fact, an infinite number of probabilistic logics, each one differing in its choice of a set of probabilities. We offer an axiomatization of the inference and consistency rules for such a family of probabilistic logics. Many of these logics use non-numerical probabilities, but each one retains the capabilities of the classical binary propositional logic.

How probabilities get assigned to complex propositions cannot be "too simple"—probabilistic logic cannot be compositional in Frege's sense. On the other hand, probabilities cannot be "too complex" as values—the "set of all possible worlds in which A is true" cannot be used as a probability value for the proposition A.

The relationships between probabilistic logic and Johnson-Laird's theory of "mental models", Barwise and Perry's theory of "situation semantics", and Montague's approach to the logic of propositions in English are explored.

Cox's Theorem has been used to support the claim that classical real-valued probability theory is somehow inevitable. The examples of non-numerical probabilistic logics given in this paper show that this is not so. Nevertheless, I show that Cox's Theorem can be strengthened—this new version shows why it is very difficult indeed to avoid re-inventing ordinary probability theory.

Even if we set our empirical motivations aside, it is difficult to conceive of any theory of analytic reasoning that uses "probabilities" and that does not satisfy the axioms presented in this paper. Many proposed theories for reasoning with "probabilities", besides confusing analysis with synthesis, have fallen victim to Cox's Theorem. They are either not suited for propositional logic (Zadeh's "fuzzy logic"), or unwittingly re-invent ordinary probability theory (the Dempster-Shafer theory), or else have mathematical shortcomings (MYCIN).

I INTRODUCTION

This paper's primary aim is to *describe* (by presenting a set of axioms for a probabilistic logic) the *standards of deductive inference* for the logic of propositions as it is found in eveyday English. I feel that we have given too little credit to people's ability to *recognise* sound arguments. I am, therefore, mainly empirically motivated. As a corollary to this aim, we will exhibit non-numerical forms of probabilistic logic that are also mathematically cogent. We will also examine the impact of the claim that there is no psychological validity to the distinction between deductive and non-deductive inference.

A *normative theory* of reasoning is one that prescribes standards to which any being must adhere for its own good, according to philosophers. A *descriptive theory* of reasoning, on the other hand, is a scientific (psychological) model of how people reason in daily life. Borrowing CHOMSKY's terms, a descriptive theory can be divided into two parts:

(1) *a competence theory* — an idealised set of *standards* to which human reasoning adheres, and

(2) *a performance theory* — a description of how the competence theory is *implemented.*

The exact placement of the boundary between competence and performance is a matter of scientific intuition—roughly, it is between *what is being implemented* and *how it's being implemented.* What is crucial, however, is that the combined descriptive theory must account for actual observations. The performance theory introduces auxiliary assumptions, about resource limitations for example, that modify the predictions of the competence theory so that they come into line with empirical fact.

Consider an electronic pocket calculator that can only add. PEANO's theory of arithmetic is an example of a competence theory—it is the ultimate standard calculator designers adhere to. But pocket calculators, we all know, only implement finite-precision arithmetic. Auxiliary assumptions are therefore needed to explain our observations when large numbers are added. Even further implementation details are needed to account for the calculator's behaviour when the power supply runs low. These auxiliary assumptions may be included as part of the competence theory, or as part of the performance theory. This choice depends on the purposes for which the theory is being formulated and is usually a matter of scientific judgement.

It is easy to argue, therefore, that the phrase "I have a competence theory for X" is just an embellished way of saying "I have an incomplete theory for X". But I feel this criticism is ungenerous. There is merit to making competence theories of complex phenomena simple by ignoring the "inessential" elements, even if this means a loss of empirical accuracy. GALILEO's theory of free falling objects, $d = at^2/2$, was suggestive and useful because it was simple. But this equation completely ignored air friction, an empirical fact that DA VINCI had already used to design parachutes.

G. FREGE believed mathematical logic to be a normative theory — that it was a universal prescription for doing mathematics [MACNAMARA 1986, pp.14-17]. He attacked Psychologism (an idea advanced by J.S. MILL, among others) which asserted that logic was *merely* a description of how people reasoned, when they were careful. FREGE's view has predominated.

[J. MACNAMARA 1986], however, resurrects the view that logic may be a competence theory for human reasoning. But MACNAMARA is careful to reconcile his view with FREGE's by pointing out that logic may still enjoy normative status—"logic is a normative prescription" and "logic is a scientific description" are not mutually exclusive opinions. I am indebted to [MACNAMARA 1986] for raising and explaining this point.

[MACNAMARA 1986] is primarily concerned with the logic of terms and how this helps us understand children's acquisition of language. We, on the other hand, present a competence theory for the deductive logic of propositions as it is found in English, and go on to show how this is related to theories of non-deductive reasoning (induction in the widest sense).

A competence theory for reasoning can be split into two parts: an *analytic* theory and a *synthetic* theory. The terms *analytic* and *synthetic* are taken from [POLYA 1957]. (Later on we will subdivide synthetic reasoning into two varieties, one of which is very closely related to analytic reasoning.) Analysis derives *necessary consequences* from an initial set of hypotheses. Synthetic reasoning, on the contrary, is satisified with inferences which are *consistent* with some collection of hypotheses and facts. For example, synthetic reasoning as practised by scientists is concerned with finding a useful language, and a set of hypotheses (a scientific model) in this language that is consistent with the known facts and with currently favored general scientific principles. Synthesis, therefore, is a goal- and value-driven process.

Two commonly held opinions are that mathematical logic is "the only" philosophically defensible theory of analytic reasoning, and that probability and statistics are, on the other hand, tools for synthetic reasoning. [FINE 1973], an extensive survey of theories of probability, does little to dispel these opinions. R. CARNAP is well known for his attempt to develop a normative theory of *inductive logic* or a *theory of confirmation* based on some calculus of probabilities.

I find these opinions unhelpful for understanding how speakers of English reason. I will treat probabilistic logic strictly as a way of *describing* what conclusions are taken as being beyond dispute by speakers of

English. A general descriptive theory of analytic reasoning as it is found in English must, I believe, subsume ordinary logic, and possibly classical probability theory. This, of course, forces a reconsideration of some of the conventions associated with the hypothetico-deductive method and with Bayesian statistics. (Is the language in which hypotheses are stated compositional in FREGE's sense? Are probabilities necessarily numerical?)

Some further points must be made at the outset—

(1) I am not concerned with defending probabilistic logic as a normative theory of reasoning in this paper. But, following MACNAMARA, this remains an open possibility.

(2) Though they must be related, a description of the logic of English is not necessarily the same thing as a description of the internal process of the reasoning—the first is just a surface manifestation of the second.

(3) I am willing to speculate only a very little about how the internal process of reasoning may actually be implemented—I rely on P. N. JOHNSON-LAIRD's theory of mental models for insights into this subject. In people, all kinds of inference, both synthetic and analytic, may be the product of a single process, even though it is convenient to make certain distinctions. Such distinctions are justifiable as long as we are only describing the *standards* that reasoning adheres to.

(4) Since I am concerned with descriptive theories, I accept that revisons are inevitable. My primary goal in this paper is to give a clear description of one such theory—it is easier to modify a theory when you understand what it is saying.

II THE EVIDENCE FOR PROBABILISTIC LOGIC

This section and the next one develop a competence theory for analytic reasoning. Such a theory *describes* what people believe is the set of necessary consequences obtainable from a collection of initial hypotheses. The first step, undertaken in this section, is to describe and defend the choice of a suitable object language for this logic. I claim that this is the language of probabilistic logic.

A probabilistic logic is a calculus for manipulating the class of propositional attitudes exemplified by statements such as "If Q then P is likely." We are modelling a reasoner's attitudes towards propositions and how these attitudes depend on each other; we are not concerned with their objective truth. Ordinary binary logic uses only two propositional attitudes: "P is true" and "P is false". (These two attitudes are further collapsed, by classical logicians, into one by using the equation "P is false" = "not P is true.")

I adopt the following conventions:

(C1) [P|Q] denotes the conditional probability of P given that Q is true, Q is called the *antecedent* and P is called the *consequent,*

(C2) Q can never be an absurdity in the expression [P|Q],

(C3) [P] is shorthand for [P|**1**], and both denote the probability of P,

(C4) the symbol **1** denotes any tautology, and **0** denotes any absurdity,

(C5) *logical truth* is equated to probability 1, and *logical falsehood* is equated to probability 0,

(C6) &, v, and ~ denote the connectives "and", "or", and "not" respectively.

I begin first by showing that any kind of logic which accepts FREGE's simple notion of *compositionality* is inconsistent with the linguistic evidence. Compositionality in FREGE's sense says that the truth-value of a complex sentence such as A&B is some function of the truth-values of its two syntactic components: A, and B. I consider implication first, and then extend this analysis to the connective "&". At the same time I show that probabilistic logic agrees with the linguistic evidence.

Implication is the fundamental notion in any logic. English speakers believe the following conditionals all assert the same thing, namely that P is true whenever Q is—

(1) If Q then P.

(2) When Q, then P.

(3) P, given that Q.

(4) Assuming Q, P.

Probabilistic logic represents all of these by the formula "$[P|Q]=1$". Whenever I ask unsuspecting English speakers about assertions of the form "When ~P, then P", the overwhelming opinion is that this statement is absurd, independent of the choice of proposition P.

Classical logic, however, since it models conditionals by *material implications,* is inconsistent with this linguistic evidence. The material implication "$P \supset Q$" is false precisely when P is false and Q is true. It is a compositional connective. Hence, the formula "$(\sim P)\supset P$" is not absurd, but is true or false depending on whether P is true or false. Trained logicians seem to be able to convince themselves that "If today isn't Friday, then today is Friday" is true precisely on Fridays, and that "If $1\neq 1$, then $1=1$" is true on any day of the week.

This illustrates a profound difference between English on one hand, and classical logic on the other. Classical logic obeys FREGE's principle of compositionality, whereas neither English nor probabilistic logic obey this principle. In classical logic, unlike English and probabilistic logic, one may substitute any true statement by any other true statement with impunity. Both English and probabilistic logic are concerned with more general relationships between propositions than just relationships between their truth-values in the current world.

In addition to conditionals that denote certain entailment, it is easy to represent other kinds of English conditionals in probabilistic logic. We are familiar with such statements as "If it is cloudy then it will likely rain." This states a relationship between one proposition ("it is cloudy") and another ("it will rain") in terms of the probability "likely". Suppose the antecedent is true, then our belief in this conditional compels us to adopt the attitude that the consequent proposition is at least "likely". This argument is also easy to represent using probabilistic logic. The hypotheses are—

(H1) [it will rain | it is cloudy] = likely,

(H2) [it is cloudy] = 1.

If we accept these two hypotheses, we must accept that "[it will rain & it is cloudy]=likely". (This is the product rule for probabilities.) But since "[it will rain] ≥ [it will rain & it is cloudy]", we must further accept that "[it will rain] ≥ likely". Note that "[it will rain] = likely" is not a necessary conclusion given (H1) and (H2), since there may be other, more compelling, reasons why it may rain.

Since this form of implication is not truth-functional or compositional in FREGE's sense, the expression "[it will rain | it is cloudy]" may be assigned any probability as long as the constraint—

(H3) [it will rain & it is cloudy] = [it will rain | it is cloudy] [it is cloudy]

is satisfied. The fact that the weather is sunny does not force us to abandon equation (H1). For example, the sentence "If it were cloudy, it would likely rain" manages to encode two statements of probabilistic logic. The subjunctive mood of the verb suggests that "[it is cloudy]=0" while the conditional information is encoded by (H1) as before. Thus, using a single parsimonious notation, one can also represent the probabilistic counterfactuals found in English.

Representing counterfactuals, however, is only part of the problem of explaining how people reason with them; we will return to this point when we discuss synthetic reasoning. For now, as is appropriate to the study of analytic logic, we will assume that all of the relevant hypotheses have been given in advance. Once these are given, there is no distinction between counterfactuals and other conditionals in probabilistic logic.

In classical multi-valued logics [N. RESCHER 1969], which are invariably compositional in FREGE's sense, there are many examples wildly at odds with linguistic evidence. Some of these logics do not even recognise "P $\supset$ P" to be a tautology. Compositionality, however, has not been blamed as the source of these problems.

Compositionality is also inconsistent with the linguistic evidence for other connectives: "&" and "v". I direct my attention next to "fuzzy logic", but the analysis presented here applies with equal force to any compositional logic that satisifes the two simple constraints mentioned below.

L. ZADEH's "fuzzy set theory" has been used in various ways to devise "fuzzy propositional logics". ("Fuzzy set theory" should not be confused with "fuzzy logics.") "Fuzzy logics" are compositional in FREGE's sense since they assume there is some function, say f(p,q), such that

(F1) $[A\&B] = f([A], [B])$

where the symbols [A&B], [A] and [B] stand for "fuzzy truth-values". (The interpretation of term "fuzzy truth-value" is unimportant for my argument.) These logics also satisfy the further constraint —

(F2) if $p \neq 0$ and $q \neq 0$, then $f(p,q) \neq 0$

This condition states that a conjunction cannot be assigned the "truth-value" 0 if neither of its parts are assigned 0. ZADEH originally chose real numbers in the interval [0,1] for "truth-values" and let f(p,q) = min(p,q). J. GOGUEN generalised this to f(p,q) = glb(p,q) over some distributive lattice of "truth-values" with a 0 [GUPTA et al 1979, p. 53], but as long as (F2) holds the following counter-example obtains.

The counter-example involves the simple experiment of tossing a fair coin. Let A denote that it lands heads, and B denote its landing tails. Clearly there is no justification for assigning either [A]=0 or [B]=0, and therefore [A&B]≠0 if we accept both (F1) and (F2) above. But A&B is a logical absurdity and should be assigned "truth-value" 0; if this is done, however, then 0≠0, a contradiction. Hence, to accept "fuzzy logic" we must reject the Law of Contradiction (which says A&~A is always false) or else, equally absurdly, reject identifying 0 with impossibility.

No English speaker, on the other hand, after admitting that the coin may possibly land heads or tails ([A]≠0 and [~A]≠0) could ever be fooled into believing that both can be the outcome of one toss. Neither "fuzzy logic", nor any other logic obeying (F1) and (F2), is consistent with the interpretation of conjunction in English.

So how then, if FREGE's simple kind of compositionality is not at work in the English language, are propositions to be convincingly assigned probabilities based on their component parts? This can only be possible through a knowledge of the dependencies between the situations described by each proposition. In no situation when a coin lands heads does it also land tails, and this establishes a connection between the propositions A and B. These dependencies can be represented as statements about conditional probabilties. Independence statements are an important special class of such statements.

The notions of compositionality, probability and (statistical) independence are all intertwined. Compositionality, for example, can be "protected" as a scientific hypothesis by modifying other assumptions in our theory of reasoning. (I. LAKATOS has explained how such maneuvers are common in the history of science.) How then, if we wished to do so, could we make probabilistic logic compositional? One simple way is to define the probability of A to be "the set of possible worlds in which A holds". When "probabilities" are equal to such sets of possible worlds we observe that $[A\&B] = [A] \cap [B]$, which also denys (F2). We have solved one problem, but we have created others. It seems unintuitive that probabilities be so intimately connected with the propositions they are assigned to. In fact, for definitions of probability which deny (F2), the connection so intimate that statistically independent events cannot be assigned arbitrary probabilities by the builder of a statistical model. Definitions which deny (F2) end up denying that probabilities are a measure of belief that may be applied to arbitrary propositions.

English is equipped with expressions to denote (statistical) independence. The statement, "It would be just coincidence if John and Frank both went to Paris" is one example. This is easily represented in probabilistic logic by the pair of equations:

[John went to Paris | Frank went to Paris] = [John went to Paris | Frank didn't go to Paris],

[Frank went to Paris | John went to Paris] = [Frank went to Paris | John didn't go to Paris].

This pair of equations does not identify any specific probabilities for these events, and the English sentence doesn't either.

On the other hand, assertions of dependence and independence between propositions are not possible in the object language of classical logic.

So far, we have used counterexamples to eliminate all theories which are based on FREGE's simple notion of compositionality. This also includes R. MONTAGUE's semantic analysis of English [DOWTY et al 1985]. MONTAGUE semantics, in its current form, can only give a truth-functional account of the sentence "When ~P, then P", and therefore cannot classify this sentence as an absurdity. It is also not clear to me how the idea of (statistical) independence is to be captured in MONTAGUE semantics. This is not to say, however, that some other intensional logic cannot be used to accomplish these aims, but instead that MONTAGUE semantics gives too simple an account of English propositional logic.

The psychologist P. N. JOHNSON-LAIRD has proposed a far-reaching performance theory of how people reason [GARDNER 1985, pp.361-370]. He explains that people test the validity of conclusions by creating and examining "mental models" (or "situations" or "possible worlds") in which the hypotheses hold. If the conclusion is not rejected in any situation that is examined, it is taken to be valid. According to this view, the assertion "If ~P then P" prompts us to imagine only situations where ~P holds, and to test to see if P holds in them also; we do not examine any other situations. Thus, in every situation we examine in this case, we see that P cannot hold, and we therefore conclude the assertion is absurd.

This process for testing validity is easy to understand and it is certainly logical. Mathematical probability theory, which originated well before mathematical logic, has implicitly accepted this view, and goes on to define the probability of A to be some measure of the relative size of the set of possible worlds in which A holds. JOHNSON-LAIRD's process uses ~P as a selection rule, and not as a truth-value—simple compositionality, therefore, cannot be used to account for how meaning is assigned to "If ~P, then P". [BARWISE & PERRY 1983] have formulated, apparently independently of JOHNSON-LAIRD, a related theory of "situation semantics" for English; their theory, however, is not as concerned with the mechanics of reasoning as JOHNSON-LAIRD's is.

I was originally dissatisfied by the requirement the Q cannot be an absurdity in the conditional [P|Q] and I sought ways to eliminate it. But without it, it was impossible to find a reasonable axiomatisation of probabilistic logic that avoided the following inconsistency: assume Q=**0**, then $1=[Q|Q]=[\mathbf{0}\&Q|Q]=[\mathbf{0}|Q]=0$. Therefore this requirement was a mathematical necessity. It turns out, in fact, to be a necessity sanctioned by the JOHNSON-LAIRD theory. If the antecedent is absurd, $Q=\mathbf{0}$, then both $[P|Q]=1$ and $[P|Q]=0$ are accepted as valid assertions, since in every situation that Q holds (namely none), "P is true" is never rejected, and neither is "P is false"—you can say anything you like about non-existent situations. Notice, however, that if the antecedent Q is merely false, then there are still situations in which it is true and therefore false propositions remain meaningful and acceptable as antecedents of conditionals.

Consider again, for example, the sentence "If $1\neq1$, then $1=1$." If you can imagine worlds in which $1\neq1$ (which is what I assumed earlier), then the consequent is false in each one of them and you judge this conditional to be absurd. If, however, you cannot imagine any possible world at all in which $1\neq1$, then you cannot form any opinion whatsoever about $1=1$ according to the JOHNSON-LAIRD theory.

Though we have rejected many traditional theories as inadequate accounts of the logic of English, we have not yet eliminated *theories of comparative probabilities.* These theories use only statements of the form "$[A|B]\geq[P|Q]$" and avoid all mention of explicit probability values. But such theories are inconsistent with the simple fact that English has many words and phrases that denote probabilities: "likely", "very unlikely", "almost certain", "negligible", "impossible", and so on. Moreover, it is conceivable for any of these probabilities to be assigned to any (non-trivial) proposition by a speaker of English.

Furthermore speakers of English recognize a crude algebra involving these probabilties: if P is "likely" then ~P is "unlikely"; if P is "possible" (i.e. $[P]\neq0$) then ~P is "impossible" (i.e. $[\sim P]=0$); if P and Q are independent, P is "true", and Q is "almost impossible", then P&Q is "almost impossible" = "true"·"almost impossible". There is some fragmentary recognition by English speakers of the inverse of a probability and of the product of two probabilities. The algebra of probabilities also respects condition (F2).

Furthermore, the work of KAHNEMAN & TVERSKY can be used to support this claim that reasoning involves some sort of system of probabilities that can be arbitrarily assigned to propositions. They have shown that people make decisions based on the shapes of probability distributions. Given two similar lotteries, people will regularly choose the one which maximises their guaranteed winnings instead of their mathematically expected winnings. But how can this be if there are no notions of probabilities as autonomous elements of reasoning? This concludes our list of arguments against a theory of purely comparative probability.

JOHNSON-LAIRD went on to show how his performance theory predicted where human subjects were most likely to make mistakes in syllogistic reasoning. Such mistakes, when they are pointed out, however, are quickly recognised as such. Thus, there is an intuitive grasp of the logical standards of validity (the competence theory) which is different from how these standards are tested (the performance theory).

Likewise KAHNEMAN & TVERSKY have pointed out many predictable patterns of mistakes when people do probabilistic reasoning. Most of these counter-examples to rationality, however, are flaws in synthetic reasoning or decision-making. These include the use of inappropriate prior probabilities, of "irrational" utility functions, and other deviations from ordinary Bayesian decision theory. There are, also, genuine lapses from the standards of analytic reasoning. For example, KAHNEMAN & TVERSKY's observations suggest that patients are more likely to choose surgery if it offered a 90% chance of living than if it offered a 10% chance of dying. The predictable mistakes of analytic probabilistic reasoning, however, share the same characteristic as the deductive errors studied by JOHNSON-LAIRD: namely, though people regularly make these mistakes (performance), they can immediately recognise their mistake once it is pointed out (competence). I do not believe, therefore, that KAHNEMAN & TVERSKY's observations threaten our thesis that probabilistic logic is a *competence theory* of analytic reasoning.

III AXIOMS OF ANALYTIC REASONING

III.1 Axioms for a Probabilistic Logic

This subsection is concerned with axiomatizing the inference rules and consistency properties of probabilistic logic in a way that takes into account the foregoing discussion. Probabilities surely exist in English, but what the precise set is—if it ever can be made precise—is a question we will leave unanswered. Therefore the axiomatization presented here is that of a "generic" probabilistic logic, one which is silent on the specific choice of probabilities. Probabilities are just uninterpreted formal marks in the theory. Notice that this axiomatization *defines* what the necessary consequences obtainable from a set of hypotheses are—if this set is incomplete, or inconsistent with linguistic evidence, then this axiomatization will have to be modified. I am not concerned here with devising an immutable normative theory of logic.

Probabilistic logic is not about synthetic reasoning, instead it sets the standards of inference for analytic reasoning. From just the hypothesis that ten coin tosses all landed heads, the outcome of the eleventh toss cannot be determined by probability theory alone. The "facts" do not speak for themselves. They must be interpreted and synthesized into a collection of hypotheses (a hypothetical model) before an inference about the eleventh toss can be made.

Probabilistic logic deals with a set, **T**, of subjective truth-values or probabilities, and, **B**, a collection of propositions (or events). Following the example of [R.T. COX 1946], we assume probabilistic logics posses two functions, **h** and **i**, named respectively *product* and *inversion.* We write **h**(p,q)=p·q.

T is a partially ordered set (a poset) of values, ordered by the relation "≤". Furthermore T is equipped with an identity relation, "=", that is related to "≤" in the following way—

For any p,q in **T** $\quad p = q \quad$ if $\quad p \leq q$ and $q \leq p$.

Contrary to what may be familiar to many, we take a boolean algebra, **B**, to be merely a set of expressions generated by finite combinations of the symbols "&", "~", "v", and a countably infinite set of propositional letters (called generators): P, Q, R, S, ... These expressions are partitioned into equivalence classes according an equivalence relation, "≈". The boolean equivalence relation is characterised by a finite set of equations [BURRIS et al 1981] of which "~(P & Q) ≈ (~P) v (~Q)" is one example. An algebraist will recognise this as the description of the *free boolean algebra on a countable set of generators.*

The first ten axioms below are first order axioms that capture various rules of inference in probabilistic logic. The remaining three axioms are not first order; they assert that certain kinds of statistical models (collections of hypotheses framed in the language of probabilistic logic) are guaranteed to be consistent. Axioms 5, 6, 8, and 9 are taken from [COX 1946].

AXIOMS

(1) The set **T** of probabilities is **partially ordered**, with relations ≤, =. The set **B** of propositions is a **boolean algebra**, with a relation ≈, and **B** is closed under finite combinations of the operations &, v, ~.

Conventions: Capital letters below stand for **arbitrary** propositions in **B**, with the **proviso** that no antecedent part of any conditional probability expression may be a logical absurdity. Greek letters stand for **arbitrary** probabilities in **T**. [P] = [P|**1**], 1=[**1**|**1**] and 0=[**0**|**1**] are abbreviations.

(2) $[P_1 | Q_1] = [P_2 | Q_2]$ if $P_1 \approx P_2$ and $Q_1 \approx Q_2$

(3) $[P|X] \leq [X|X]$

(4) $[P\&X|X] = [P|X]$

(5) $[P\&Q] = \mathbf{h}([P|Q], [Q])$

(6) $\mathbf{h}(\alpha,\beta) \leq \mathbf{h}(\xi,\psi)$ if $\alpha \leq \xi$ and $\beta \leq \psi$ (order preserving)

(7) $[Q] = 0$ or $[P|Q] = 0$ if $[P\&Q]=0$

(8) $[\sim P|X] = \mathbf{i}([P|X])$

(9) $\mathbf{i}(\psi) \leq \mathbf{i}(\xi)$ if $\xi \leq \psi$ (order inverting)

(10) $[P|X] \leq [P] \leq [P|\sim X]$ if $[P|X] \leq [P|\sim X]$ (sandwich)

For any X, Y, and Z that are **distinct generators** of the boolean algebra, **B**, and arbitrary probabilities ξ, ψ, ζ, the following sets of hypotheses are always algebraically **consistent** (meaning that it is not possible to derive two different probabilities for the same conditional expression and that it is possible to assign the remaining probabilities in a consistent manner). These are not, therefore, first order axioms.

(11) $\{ [X]=\xi, [Y|X]=\psi, [Z|X\&Y]=\zeta \}$ (chains)

(12) $\{ [X|Y]=[X|\sim Y]=\xi, [Y|X]=[Y|\sim X]=\psi \}$ (independence)

(13) $\{ [X\&Y]=\xi, [Y]=\psi \}$ if $\xi \leq \psi$

Before explaining what the axioms do say, I will first point out what they do not say. The product, **h**, is not assumed to be associative or commutative: this turns out to be a logical consequence of these axioms. The product need not be cancellative (i.e. $p \cdot q = p \cdot r$ and $p \neq 0$ together do not imply $q = r$). Neither product nor inversion are assumed to be continuous in some topology. The probabilities 0 and 1 are, for the moment, merely special abbreviations. That [P|P] =1 independently of the choice of P, is another consequence of the axioms. I do not assume the existence of a function **g** such that [A v (~A)&B] = **g**([A], [(~A)&B]); thus, *finite additivity is not assumed.* (The possibility of additivity, continuity, cancellativity or even divisibility for probabilities is not, however, precluded.)

Axiom 2 states that, if two propositions are boolean equivalent, one may be substituted for the other. Thus [P| Q&R] = [P| R&Q] is justified because of the boolean equivalence Q&R ≈ R&Q. Mere equality of truth-values, however, is not sufficient to justify such a substitution. For example

[bees get around|bees fly] = [bees get around|3>2]

is not true in general even when [bees fly] = [3>2] = 1. From axioms 3 and 4 we can establish that [P|P] = [Q|Q] independent of the choice of P and Q. Axioms 5 and 6 are the product rule for probabilities. They posit the existence of an order preserving function for calculating [P&Q] from [P|Q] and [Q], a calculation which permits a generalisation of *modus ponens.* From the first six axioms, we can derive the BAYES-LAPLACE RULE:

$$[A|B][B] = [B|A][A].$$

Axiom 7, which is also condition (F2), says that if $p \cdot q = 0$ then $p = 0$ or $q = 0$. Suppose the contrary, namely that $p \cdot q = 0$ for some $p \neq 0$ and $q \neq 0$. Now consider a pair of (statistically) independent events assigned the probabilities p and q. The probability of the joint occurrence of these two independent events is therefore $p \cdot q = 0$, an absurd and unintuitive result.

Axioms 8 and 9 introduce a method for calculating [~P|X] from [P|X]. This makes **i** an anti-isomorphism of the poset **T**. (**T** is therefore self-dual.)

Axiom 10 deals with "~" symbols in the antecedent part of a conditional probability. I call it the "sandwich rule". It specializes to the cut rule of ordinary logic, and is also a theorem of ordinary numerical probability theory since [P] is a convex linear combination of [P|X] and [P|~X], which entails—

$$[P|X] \leq [P] = [P|X][X] + [P|{\sim}X][{\sim}X] \leq [P|{\sim}X].$$

Axiom 11 guarantees that assigning arbitrary probabilities to chains of three events will not run afoul of the algebraic machinery of probability theory. So, for example, given one had to choose probabilities p, q, and r for—

[march winds] = p

[april showers | march winds] = q

[may flowers | march winds & april showers] = r,

there is nothing inherent in the mathematics of probabilities that prevents one from making arbitrary choices for p, q, and r as long as they are elements of **T**. This is as it should be since probabilistic logic must be capable of respresenting arbitrary situations.

Axiom 12 similarly guarantees that assigning arbitrary probabilities to two statistically independent events will not run afoul of the algebraic machinery. Axiom 13 says for any $a \leq b$, we are assured that there is a c such that $a=bc$. This property is called *natural ordering* in [FUCHS 1963]. This axiom reflects the belief that any proposition or conditional, must in principle, have a probability, given a reasonable partial assignment of probabilities. If we knew, for example, that

[march winds] = p

$$[\text{march winds \& april showers}] = r \quad \text{where} \quad r \leq p$$

then we all believe that there exists some probability, q, for [april showers | march winds], so that $r = q \cdot p$, even though we may not know what it is.

Axioms 11, 12 and 13 are logically independent of each other and of the previous axioms. From axioms 11 and 12 we can establish that the product of probabilities is associative and commutative.

The logical independence of the axioms and their mathematical consequences are more fully studied in [ALELIUNAS 1986, 1987], which also contains detailed proofs of the assertions made above.

III.2 Relationship to Classical Propositional Logic and Numerical Probability

These axioms are logically **consistent** because both classical propositional logic and numerical probability theory are special instances of this theory.

Classical Propositional Logic (with conditionals)

$\alpha \in \{ \text{true, false} \}$, $\mathbf{h}(\alpha,\beta) = \alpha \,\&\, \beta$, $\mathbf{i}(\alpha) = {\sim}\alpha$.

This system is identical to propositional logic except that we permit conditional expressions, such as [P|Q]=1, as hypotheses. (Material implication can of course be represented using expressions like [Pv(~Q)]=1.)

Modus ponens does not require material implication for its existence; this deduction rule works just as well with conditionals. For example, from the two hypotheses {[P|Q]=1, [Q]=1} one can derive that [P&Q]=1, and consequently that [P]≥[P&Q]=1, from which it is obvious that [P]=1. Similarly one can prove that when all probabilities are restricted to being either 0 or 1, that the probabilities of conjunctions and disjunctions satisfy the simple formulas: [P&Q]=min([P], [Q]), and [PvQ]=max([P], [Q]). Compositionality therefore arises naturally in this situation; it is a combinatorial accident of two-valued logic, not an *a priori* requirement of logic.

Every probabilistic logic that satisifies the above set of thirteen axioms, therefore, retains the capabilities of classical binary logic. Thus, when hypotheses are known with certainty, they can be used to obtain conclusions consistent with classical propositional logic. English logic doesn't lack the precision of mathematical logic, mathematical logic, instead, lacks the richness of English.

Simple Numerical Probability Theory (excludes infinite events)

$\alpha \in [0,1]$, $\mathbf{h}(\alpha,\beta) = \alpha \times \beta$, $\mathbf{i}(\alpha) = 1 - \alpha$.

This simple real-valued system does not mention any method for adding probabilities—this feature can be added later.

The probability that an infinite sequence of tosses of a fair coin all land heads is 0 according to the KOLMOGOROV probability calculus, a conclusion which conflicts with my insistence that probability 0 is the same as logical falsehood. The acceptance of only real-numbers as probabilities, of infinite events as meaningful, and of the importance of topological continuity for the product operation conspires to produce this conflict. There are therefore several ways to resolve it. One way, suggested by R. VON MISES and based on his analysis of the foundations of probability, is to eliminate such inifinite events from consideration entirely because they cannot be the subject of practical experiments [FINE 1973, p.64]. Another way to resolve this conflict is to enrich the set of real numbers with infinitesimals so that the probability of an infinite series of heads being observed is no longer 0, but an infinitesimally small value instead.

III.3 Examples of Probability Algebras

The formal marks that stand for probabilities in the above set of thirteen axioms admit many concrete interpretations. Each interpretation of these formal marks will be called a *probability algebra.* Two familiar ones have already been given in the previous section; several novel ones are given below. In most cases the choice for an inversion function is obvious, so that only the table of products is given.

My aim here is to illustrate the wide variety of systems of probability that can appear in a probabilistic logic; no claims for the superiority of any of them are advanced.

Example 1. T={0, m, 1}, ordered as in Figure 1(a), with mm=m. (In all probability algebras we have 1·x=x·1=x and 0·x=x·0=0 for any x.) This is the only 3-valued probability algebra.

Example 2. T={0,0',1',1}, ordered as in Figure 1(b). There are two possible probability algebras based on the total ordering of four probabilities. The two multiplication tables are—

	0	**0'**	**1'**	**1**
0	0	0	0	0
0'	0	0'	0'	0'
1'	0	0'	0'	1'
1	0	0'	1'	1

	0	**0'**	**1'**	**1**
0	0	0	0	0
0'	0	0'	0'	0'
1'	0	0'	1'	1'
1	0	0'	1'	1

(There are 3 distinct algebras possible on total orderings of 5 probabilities, and 7 distinct algebras for total orderings of 6 probabilities.)

Example 3. T={0,0',a,b,A,B,1',1}, ordered as in Figure 1(c). We let AB=a, AA=BB=b, Aa=Ab=Ba=Bb=aa=ab=bb=0', x1'=x if x≠1, and x0'=0' if x≠0. This is not a lattice and yet it is the basis of a simple probabilistic logic.

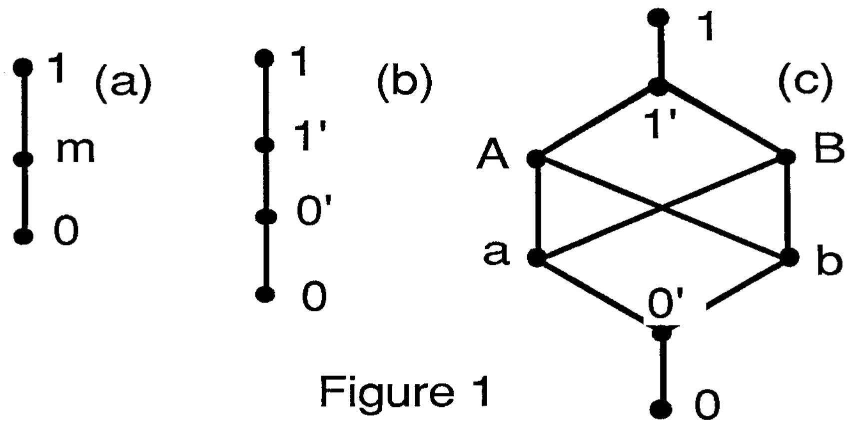

Figure 1

Example 4. This is an infinite algebra generated by one element, L. The probabilities in this algebra are **T** = { L^k | k is an integer, k=+∞, or k=-∞}, and they are totally ordered so that $0 < L^{k+1} < L^k < 1$, for any integer k. We define inversion by $\mathbf{i}(L^k) = L^{-k}$, and product is simply $L^j L^k = \min(L^j, L^k)$.

Example 5. Let **T** be the set of *nonstandard reals* in the interval [0,1]; these include infinitesimals. We use the extension of real number multiplication as the definition of the product **h**(p,q).

Example 6. Let **T** be the probability algebra generated by the formal symbols likely and unlikely, subject to the following additional constraints:

(1) $0 <$ unlikely $<$ likely < 1,

(2) **i**(likely) = unlikely.

Two randomly chosen terms are unlikely to be comparable —

(likely^3)·**i**(unlikely^2) is not comparable to (likely^3)·(**i**(unlikely))^2.

Axiom 13 guarantees that this algebra contains a solution, x, to the following equation, even though x cannot be expressed as a combination of products and inversions of the generators of this probability algebra—

$\text{unlikely}^2 = \text{x}\cdot\text{likely}^2.$

It is tempting to use a system like this one to model the probabilities that appear in English. But even if we choose this system for example, there are still questions such as whether "P is likely" is to be represented as $[P]\geq\text{likely}$, or as $[P]=\text{likely}$.

Example 7. Let **T** be the set of real numbers in the interval [0,1] with the product defined by p·q=min(p, q). This system, which contains Example 4 as a subalgebra, is reminiscent of the fuzzy logics derived from ZADEH's "fuzzy set theory", except, in this case the connectives are treated in a logically sound way.

The probability algebra with three values is equivalent to a congruence algebra of the real numbers in [0,1] under multiplication. The three congruence classes are {0}, the open interval (0,1), and {1}. This, however, is the only probability algebra that is a nontrivial homomorphic image of the simple real-valued probability calculus.

In general, there is no translation of the probabilities of one probability algebra into those of another. There should be no surprise that "likely" has no interpretation in terms of numerical probabilities—this is typical.

This suggests why it is difficult to translate the knowledge in, say, **Harrison's Principles of Internal Medicine** into statements involving only real-valued probabilities [PETERSDORF et al 1983]. Physicians, like everyone else, use probabilities such as "likely" to their express their beliefs. Representing this sort of knowledge using a non-standard probabilistic logic can be more natural and therefore less likely to cause misunderstandings between humans and machines. (Certainly MYCIN has already demonstrated that medical knowledge can be profitably coerced into a very *ad hoc* form, showing us that even an inconsistent normative theory can enjoy some practical success.)

A catalog of probability algebras, and of their algebraic properties is given in [ALELIUNAS 1986, 1987].

III.3 The Special Place of Ordinary Numerical Probability Theory

Of course this is far from being the first attempt to study probability theory abstractly. The following theorem shows why many of these attempts are doomed to re-invent nothing more than simple real-valued probability theory. By "simple real probability theory" I mean the system without infinite events or addition that was described in section III.2.

A CHARACTERIZATION THEOREM for SIMPLE REAL PROBABILITY THEORY
Under the conditions established by Axioms 1 through 13 (actually, Axioms 7 and 12 may be omitted), the following two statements are equivalent:

(S1) T is a totally ordered set of probabilities and it is archimedean ordered with respect to the product. (Archimedean ordering means that a^n becomes smaller than b for some n, for any $a \neq 1$ and $b \neq 0$.)

(S2) T is isomorphic to a subalgebra of simple real-valued probability theory. In other words, besides **T** =[0,1], we could also have **T** = all the rational numbers in [0,1] for instance.

PROOF SKETCH: The full proof is given in [ALELIUNAS 1986, 1987]. The only interesting part is that (S1) implies (S2). This relies on the theorem that states there are only three Archimedean totally ordered semigroups that are also naturally ordered, a result is due to O. HOLDER at the turn of the century [see FUCHS 1963, p.165]. Two of these semigroups can be ruled out because they are not self-dual, leaving only the system isomorphic to the real numbers in the interval [0,1]. QED

This result substantially extends the theorem obtained by R. T. COX in 1946. COX's initial hypotheses included: probabilities are real numbers, the product of probabilities is continuous, and finite additivity holds for the probabilities of disjoint events. None of these assumptions appear in this version of the theorem. Moreover, COX was only able to provide sufficient conditions for (S2) to hold, even with his stronger assumptions. The present theorem establishes both necessary and sufficient conditions under weaker hypotheses. Some improvements to COX's original theorem are reported in [ACZEL 1966], but again assumptions similar to COX's were made.

It is shown further in [ALELIUNAS 1986, 1987] that the ubiquitous assumption that probabilities are finitely additive for disjoint events precludes all the finite probability algebras. This dramatically increases the chances that some additional assumption about probabilities, apparently minor, will lead inevitably back to ordinary real-valued probability again.

It is obvious now, however, that COX's theorem cannot be interpreted as a definitive argument for the inevitability of simple real-valued probability, as has been done by some.

In light of this theorem, it is also not surprising that simple real-valued probability theory continues to be unwittingly re-invented by writers wishing to develop novel theories for reasoning with "probabilities". Consider, for example, the MYCIN system and the DEMPSTER-SHAFER theory. Both, unfortunately, refuse to distinguish between analytic and synthetic reasoning—the authors bury general statistical hypotheses into the inference rules. This has delayed a satisfactory understanding of the properties of these systems.

[D. HECKERMAN 1986] shows that the inference rules of MYCIN can be translated into the language of ordinary probability theory. Once this is done, however, the hidden statistical assumptions become evident. Under this translation, HECKERMAN shows that MYCIN has shortcomings as a modelling language for diagnostic situations in which there are three or more alternative diagnoses.

Recently the DEMPSTER-SHAFER theory of evidential reasoning has been shown by [E. KYBURG 1985] to be mathematically isomorphic to a fragment of ordinary probability theory, though this is certainly not evident in its original form [BUCHANAN and SHORTLIFFE 1984, pp.272-294]. (More precisely, KYBURG shows that the DEMPSTER-SHAFER theory can be simulated by the algebra of convex real-valued probability assignments.)

IV SYNTHETIC REASONING: JUDGING COMPETING HYPOTHESES

One form of synthetic reasoning is the process of inventing, and judging *hypothetical models* against empirical fact. (Each model is, itself, a collection of individual hypotheses.) Each hypothetical model must be consistent with the facts and other firmly held beliefs. The set of possible hypothetical models is, however, invariably a competitive one—they are inconsistent among themselves.

Any hypothetical model is unavoidably the child of two parents: an *a priori* set of modelling assumptions, and the relevant empirical evidence. Synthetic reasoning, more so than analysis, is sensitive to a reasoner's goals, and values. Some of these *a priori* components of this kind of synthetic reasoning are—

(a) A language for analytic reasoning, a logic.

(b) A set of competitive models (perhaps with initial preferences).

(c) A method of using empirical evidence and *a priori* rules to assign preferences.

(d) A decision rule that turns preferences into commitments: either actions or decisions to accept certain hypotheses ([J. W. TUKEY 1960] is very clear about this distinction.)

Several normative recipes exist that flesh out this skeleton in very convincing ways. Here are two broadly sketched recipes—

	A Typical Hypothetico-Deductive Recipe	**A Typical Bayesian Recipe**
(a) Logic	Classical Logic	Real-valued Probability
(b) Prior Probabilities	None, other than admitting a hypothetical model as worthy of consideration.	Use the Maximum Entropy Principle, if possible.
(c) Evaluation	Reject models whose hypotheses or predictions disagree with the data. Rank the remaining ones using Occam's Razor, symmetry, or some other rule-of-thumb—each one is invariably imperfect.	Apply Bayes-Laplace Rule to obtain posterior probabilities.
(d) Decisions	Often avoided, unless one model has clearly ranked very highly according to some heuristic.	Choose a numerical subjective utility function. Make the decision that maximises subjective expected utility.

The Bayesian recipe is normative, but it has been in vogue with economists, for instance, as a description of individual economic behaviour in society. Psychologists, however, now seriously question the wisdom of adopting untested normative theories as scientific models. [KAHNEMAN & TVERSKY 1982], in particular, show that people do not maximise a simple subjective expected utility; that they often ignore prior distributions; that if a utility function exists it probably isn't representable as a single number; that people seek to maximize certain gain and not long-run mathematical expectation; and so on.

By why stop here at dismantling the normative theory of Bayesian statistics? The existence of mathematically consistent non-numerical probabilistic logics challenges even the choice of logic. In fact once you strip away the *a priori* conventions about logic there is little left to distinguish Bayesian statistics from the hypothetico-deductive method, as the example below will demonstrate.

The remaining skeleton offers no normative guidance, nor does it have any testable empirical content. But it is for this very reason that I believe this skeleton suggests a *disciplined and unprejudiced* way of presenting any competence theory of synthetic reasoning for judging competing hypothetical models.

It is easy to see how different ways of fleshing it out can be used to reduce both modelling effort and computational costs, at the expense of reduced precision. The Bayesian apparatus can be still be operated, in the main, with a foreign probabilistic logic. Consider, for example modelling coin tossing as a Bernoulli process with the non-numerical probabilities defined in Example 4, namely: 0, ..., L^2, L^1, L^0, L^{-1}, ..., 1. Assuming equal prior probabilities for the hypotheses (let's say they all have prior probability p), we calculate the joint probability that "the probability of heads is L^k" & "we see m heads and n tails" to be—

$$(L^k)^m \cdot (L^{-k})^n \cdot p = \min(L^k, L^{-k}) \cdot p \quad \text{for } m \geq 1 \text{ and } n \geq 1.$$

These results are (only infinitesimally) better than would be obtained by the hypothetico-deductive method—there we could only rule out the hypothesis that heads were impossible as soon as we saw a head, and the hypothesis that tails were impossible as soon as we saw a tail. In exchange for the crudeness, however, we find nothing to stop us from assigning equal prior probabilities (since they do not add)—a common stumbling block in the traditional Bayesian approach. And we need not be very concerned about reconciling observed numerical frequencies with these probabilities since there is no correspondence; these probabilities reflect a reasoner's propositional attitudes and not objectively measurable quantities.

V SYNTHETIC REASONING: PLAUSIBLE INFERENCE

The approach to synthetic reasoning that I have just described respects the conventional division between analysis and synthesis established by philosophers. There is another kind of synthetic reasoning, I will call it synthetic inference, that shares a lot in common with analytic inference; both are concerned with making and justifying inferences from a fixed set of beliefs (hypotheses). Unlike the previous kind of synthetic reasoning, this kind is characterised by the fact that all inferences are consistent with each other and with the fixed set of beliefs. We begin this discussion by reviewing JOHNSON-LAIRD's theory of inference.

JOHNSON-LAIRD's work suggests that there is only one psychological mechanism for doing inference, and that this mechanism doesn't use symbolic-mechanical rules of inference such as *modus ponens.* In short it is a theory of "inference without proof theory". We must therefore also explain, what impact, if any, this has on our claims concerning probabilistic logics.

JOHNSON-LAIRD's performance theory, you will recall, is a description of how people convince themselves to accept a conclusion. Though this theory was initially developed to account for the mistakes people regularly make when evaluating syllogisms (an analytic form of reasoning), there is nothing to stop us, and in fact the law of parsimony encourages us, to hypothesize that the same mechanism is used for evaluating all kinds of inferences, both synthetic and analytic without distinction.

Consider the counterfactual "If today were Sunday, then I couldn't go to the bank". JOHNSON-LAIRD's theory explains how people form an opinion about this statement—the process is the same as the one people use to understand the more straighforward sentence: "If today were Sunday, then it wouldn't be a weekday." Recall that the first step is to consider "mental models" or situations in which "today is not Sunday", and the next step is to check to see if "the bank is open" in any of these situations. But of course *the opinion formed as a result of these tests depends critically on the set of situations which were considered.*

If a *mathematically exhaustive set of situations* is examined then one reaches an opinion about "If today were Sunday then I couldn't go to the bank" that is sound according the standards of analytic logic. If, for instance, I happened to be alert and in a contrary mood, it may cross my mind to consider a situation where I am in China (where I assume some banks are open on Sundays). The conclusion "I can't go to the bank" is, therefore, not an incontestable consequence of the antecedent "If today were Sunday."

If, however, only a *set of familiar situations* is examined then one reaches an *opinion that may not be incontestable. This opinion may, however, be very useful.* Suppose I only considered situations where I found myself in Canada; I may have excluded situations involving other countries as irrelevant, or I may have not even thought of such situations in the first place. In this case I would conclude that the counterfactual "If today were Sunday then I couldn't go to the bank" makes a lot of sense.

To exclude situations consciously I must have had in mind an additional unstated premise that I did not make explicit—so this is a case of reasoning from further known, but unstated, premises. If challenged I would mention them to justify my inference. On the other hand it is also very likely that I unconsciously ignored some potentially relevant situations.

Thus, according to the JOHNSON-LAIRD theory, the difference between synthetic inference and analytic inference depends entirely on the set of situations that are used to form an opinion about the conclusion. A person's confidence in her opinion depends on her confidence that she has thoroughly searched the appropriate set of possible situations. (Of course, in everyday life, there is little demand for doing meticulous searches.) Thus, according to the JOHNSON-LAIRD theory, there is only one way that all inferences get judged, and only the collection of situations that were used varies in each instance. Under this view there is no difference between analytic and synthetic inference, only a difference in the care with which conclusions are tested.

Logicians call this a "model-theoretic" approach to inference. It can dispense with a rigid formal calculus for expressing hypotheses, conclusions, and the proof theory that connects them. All that is required is a mechanism for generating imaginary situations, and a procedure that tests whether various "statements" hold in these situations. These "statements" need not have a meticulously regulated internal structure; all that is important is having the ability to judge if a "statement" holds in a situation. This is "inference without proof theory".

"Inference without proof theory" is a very attractive psychological theory of reasoning. This theory does not require a commitment to any single formal symbolic calculus of logic—in fact it can test the inferences of any formal calculus for which the collection of situations constitutes an appropriate set of "models" for this calculus (in the logician's sense). The proof theoretic approach, on the other hand, demands careful attention to specific rules of well-formedness for expressions (since their internal structure will be examined and manipulated), and to specific mechanical inference rules. This, compared to the requirements of "inference without proof theory", seems to call for a lot of additional mental machinery—and we still require all the machinery of the model-theoretic approach in order to realise the semantic associations between our symbols and our experiences of reality.

How does this affect the status of the claim that probabilistic logic is a competence theory of analytic reasoning? First, observe that the set of situations *actually used* to test an inference may be characterised by a set of logical sentences; these sentences are may be called "implicit" hypotheses. Hence, for the purposes of specifying a *competence theory* of synthetic inference we can choose either the model-theoretic description that uses a restricted set of models (JOHNSON-LAIRD), or a proof-theoretic description that invokes "implicit" hypotheses. If all situations are tested (no restrictions), then this is equivalent to no "implicit" hypotheses, both of which represent the special case of analytic inference. Thus there is no inconsistency between the two approaches as long as we restrict ourselves to competence theory. We have, instead, two different and valuable ways of looking at the same thing—one way provides more details about the implementation in people (a performance theory), the other way can help us understand the general principles used to generate the test situations.

VI CONCLUSIONS

This approach to probability theory is mathematically unusual because it is non-additive. Probabilities are not merely numbers. These appear to be necessary properties of any theory that attempts to account for the probabilities that appear in English.

Is a knowledge of probabilities prewired? Or, if it is learned, are there significant variations between people or cultures?

But merely observing uniformity in the use of probabilities by different people still does not give conclusive evidence of their psychological "reality" as part of a deeper language of thought. Though probabilities are real features of English, perhaps they are just surface reports on the results of some more quantitative estimates of the relative sizes of sets of imaginary situations, estimates that are carried out in some speechless corner of the mind.

Obviously an important motivation for describing people's competence at analytic reasoning, aside from the scientific one, is to use this understanding to build better, more natural, formal representations of people's knowledge. If we do not try to build machines that reason like us, aiming instead, I suppose, for "better" artificial intelligences, I do not see how we can trust ourselves to understand what these machines will be doing except in an idealistic sense.

Declarative descriptions of machines are usually better than non-declarative ones. But what's the good of a declarative description (of some machine) that is phrased in an unintuitive logic? Our natural habits of mind determine how we will understand these descriptions. It is cold comfort if they are only understood by logicians and philosophers—and even they cannot avoid slipping into natural ways of reasoning and thinking. Logics that do not conform to our natural mental habits make the task of understanding and verifying declarative descriptions based on them all that much harder and error-prone.

I hope, finally, that I have succeeded in showing that even the study of logic can benefit from empirical input sometimes.

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