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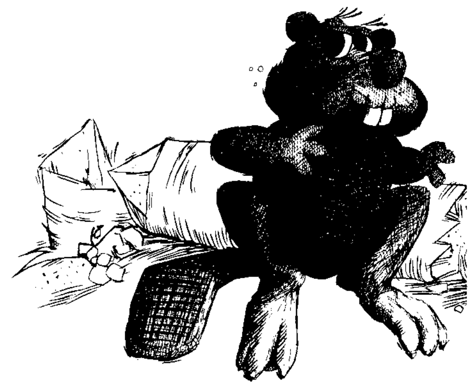

Restricted-Orientation Convexity

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Abstract

A *convex set* can be defined as a set of points whose intersection with any line is either empty or connected. The notion of a convex set is one of the most fruitful concepts in contemporary geometry and that of the *convex hull* of a set of points (the smallest convex set containing the points) is considered central to computational geometry. For practical reasons computational geometry has also studied *orthogonally convex sets* which are sets of points whose intersection with any horizontal or vertical line is empty or connected.

In this paper we generalize the notion of a convex set to include both definitions of convexity and derive several basic results based on this definition thereby unifying these two areas.

Keywords: Convex Sets, Convex Hulls, VLSI Design, Restricted-Orientation Convexity, Computational Geometry.

1 Introduction

In the fifteen years or so of its existence the field of computational geometry has bifurcated quite markedly into the study of algorithms for either orthogonal[1] or arbitrarily oriented objects. Possibly the main reason for this is that the major application areas of computational geometry, namely:

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[1]An *orthogonal* (also known as rectilinear, isothetic, iso-oriented, x-y or aligned) object is a planar figure composed solely of horizontal and vertical lines, segments and rays.

VLSI design; computer aided design; digital picture processing; computer vision and computer graphics, have traditionally placed heavy emphasis on orthogonally oriented objects. This in turn is due to technical restrictions, as for example, the fact that most input/output devices and layout schemes have been orthogonal. Recent technical advances in VLSI design however now allow objects to have more than the usual two orientations and as a result designers are now concerned with objects with horizontal, vertical and lines of 45° and 135° ([16]). Some companies now offer the capability of any finite number of orientations.

Another justification for the special study of orthogonal polygons might be that such polygons are very constrained and so it is a commonplace that algorithms for orthogonal polygons are very "tight" ([12]) since we can very often examine all possible cases. It is natural to speculate on whether we can increase the number of allowed orientations and still have fast algorithms.

Convex sets are a comparatively recent but very fruitful concept in geometry having applications in optimization, statistics, geometric number theory, functional analysis and combinatorics ([5,7]) and this is one of the reasons for the inordinate interest in convex sets in computational geometry. But its study is also practically motivated since the convex hull of an object typically has much less complexity than the object itself and so it is much used in testing for intersections among objects ([7,14]).[2] Convex polygons also crop up in decomposition results since, as is typical, there are very good algorithms for convex polygons and so polygons are decomposed into convex subparts to answer various queries ([7,14]). Finally, the convex hull was one of the first concepts studied in computational geometry ([13]) and so deserves especial attention.

In [9] we defined and gave optimal algorithms to construct various new versions of the convex hull of a *finitely-oriented* polygon (meaning a polygon whose edge orientations belong to only a fixed *finite* set of orientations). The new notion of convexity introduced in that paper was a natural generalization of the well-known concept of *orthogonal* convexity (see [6], for example) and new versions of the hull introduced there were meant to generalize the definitions used with respect to orthogonal polygons. As it turned out, this was an advantageous generalization since the new notion completely encompassed the old and there was no additional complexity, in fact the convex hull algorithms were slightly simplified. The reason for the simplification being that the generalization allowed the identification of inessential details that were specific only to orthogonal polygons.

In this paper we investigate the next most general concept of *restricted-orientation convexity* and apply it to *arbitrary* sets of points thereby gen-

[2]The same reason suffices to explain the great popularity of the "bounding box" of an object in computer graphics and computer vision.

eralizing all our previous results and, among other things, verifying a few otherwise unsupported observations in the literature.

Curiously, our investigation demonstrates that in the plane we may treat restricted-orientation convexity just as if it were orthogonal convexity. In other words, it is always possible to construct a case analysis which is only concerned with at most two orientations at a time. It seems natural to conjecture that this relationship also holds in three dimensions (that is, we only need results for three dihedral orientations) and so on to higher dimensions. This result is interesting on several levels, the most important of which is the purely practical one that, ***in terms of convexity, there is no loss in going from orthogonal to arbitrarily many orientations.***

Because we shall refer to some of them in the body of the paper we list here some of the most salient properties of planar convex sets ([3]). In the following $\mathbf{P}$ is a planar convex set:

1. $\mathbf{P}$ is simply connected.

2. The intersection of $\mathbf{P}$ and any line is either empty or a connected set.

3. $\mathbf{P}$ is the intersection of all convex sets which contain it.

4. If $p \notin \mathbf{P}$ then there exists a line separating p and $\mathbf{P}$.

5. $\mathbf{P}$ is the intersection of all halfplanes which contain it.

6. If $p, q \in \mathbf{P}$ then the line segment joining p and q is in $\mathbf{P}$.

Except for property (1), all of these are defining characteristics of convex sets. In the concluding section of this paper we list the corresponding properties of our more general "convex" sets which include these as special cases.

This paper is subdivided into the following sections: Section 2 establishes the conventions that we will be adhering to in this paper. Section 3 contains the definition of these new "convex" sets and several of their more elementary properties. Section 4 contains two theorems; the "Separation Theorem" which gives exact conditions on when a point can belong to the "convex hull" of a set and the "Decomposition Theorem" which establishes a kind of incremental property of these new "convex" sets. Section 5 introduces the notion of a "stairline" which serves as a suitable analogue of a straight line for these sets. Section 6 contains two theorems; the "Characterization Theorem" which characterizes these new sets completely in terms of their boundary (using stairlines) in much the same way that planar convex sets have been so characterized and the "Visibility Theorem" which characterizes these sets in terms of a generalization of visibility. Finally, in section 7, we summarize the properties established in this paper and point the way to further work in this area.

2 Agreements

All of our results are described in $\Re^2$ since planar relationships admit of easy visualization, but the results are easily generalizeable to $\Re^n$ (and in fact to any finite-dimensional normed linear space) in the usual way. We assume the reader's familiarity with such elementary topological concepts as (path-) connectedness, closure, simplicity, separability, support, interior and boundary of planar figures.

We shall denote subsets of $\Re^2$ by **bold face capital letters** (e.g., **P** and **Q**) and elements of such sets by *lower case italic letters* (e.g., p and q). We treat a subset of $\Re^2$ as a set of interior points together with its boundary.

We shall use the symbol $\mathcal{O}$, with or without subscripts, to refer to a set (possibly empty) of orientations. A collection of lines, segments and rays is said to be *$\mathcal{O}$-oriented* if the set of orientations[3] of the elements of the collection is a subset of $\mathcal{O}$. Thus we shall speak of "$\mathcal{O}$-lines", "$\mathcal{O}$-segments" and "$\mathcal{O}$-rays" to mean $\mathcal{O}$-oriented lines, segments and rays. By extension, we call a polygon an "$\mathcal{O}$-polygon" if its edges are $\mathcal{O}$-segments.

Because we wish to preserve symmetry of direction in this paper we shall assume that the set $\mathcal{O}$ is symmetric about the horizontal, that is, if it contains an orientation $\theta < 180°$ it also contains an orientation $\theta + 180°$ and similarly for $\theta > 180°$. Hence we shall specify a set of orientations only by the set of orientations less than 180°, it being understood that all the complementary orientations are present. So, for example, if we say that the set of orientations $\mathcal{O}$ has two orientations we mean that it has four orientations two of which are complementary to the other two.

The notion of $\mathcal{O}$-orientation has been previously defined (for finite $\mathcal{O}$) in [4,9,16,17] and, in a slightly related form, in [2]. As mentioned in the Introduction there is a vast literature concerning the special case of $\mathcal{O} = \{0°, 90°\}$ (more exactly, $\mathcal{O} = \{0°, 90°, 180°, 270°\}$). $\{0°, 90°\}$-objects are more usually called orthogonal (also; rectilinear, isothetic, iso-oriented, x-y or aligned) objects (see [7,18] for further references).

We shall assume that $\mathcal{O}$ is representable as a list of disjoint *closed* ranges where some (or all) of the ranges may collapse to just single orientations. For example, $\mathcal{O}$ may be the set $\{\theta_1, [\theta_2, \theta_3], [\theta_4, \theta_5], \theta_6, \theta_7\}$ (all $\theta_i < 180°$). (Note that we do not list the orientations greater than 180°.)

We also assume that $\mathcal{O}$ is kept in sorted order so that given a range in $\mathcal{O}$ we can speak of the "next" range in $\mathcal{O}$ (the successor of the last range is the first range). In the above example, $\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_5 < \theta_6 < \theta_7$

[3]The orientation of a directed line is the counterclockwise angle made with the horizontal in a directed plane (in the goniometric sense). The orientation of an undirected line is the smaller of the two possible orientations. We will only discuss undirected lines in this paper.

and $[\theta_2, \theta_3]$ is the "next" range after the "range" $\theta_1 (= [\theta_1, \theta_1])$.

We call the open range (θ_1, θ_2) *$\mathcal{O}$-free* if there are no orientations in $\mathcal{O}$ in the range (θ_1, θ_2). We call the open range (θ_1, θ_2) a *maximal* $\mathcal{O}$-free range if (θ_1, θ_2) is $\mathcal{O}$-free and $\theta_1, \theta_2 \in \mathcal{O}$. If $\mathcal{O}$ has n ranges then $\mathcal{O}$ divides the set of all orientations, $[0°, 180°)$, into $2n$ maximal $\mathcal{O}$-free ranges. In the example the maximal $\mathcal{O}$-free ranges are $(\theta_1, \theta_2), (\theta_3, \theta_4), (\theta_5, \theta_6), (\theta_6, \theta_7), (\theta_7, \theta_1 + 180°)$, plus the five complementary ranges.

We shall use the notation $L(p, q)$ to mean the line passing through the points p and q and similarly, the notation $LS(p, q)$ to mean the line segment with endpoints p and q. We also use the notation $\Theta(L)$ (where L is a line, segment or ray) to mean the orientation of L. If L is a line, segment or ray and $\Theta(L) \notin \mathcal{O}$ then by *the* maximal $\mathcal{O}$-free range of L we mean the unique maximal $\mathcal{O}$-free range in which $\Theta(L)$ lies.

Any collection of lines, segments and rays having (one, two or) three orientations in the plane can be mapped onto another collection having the same incidence structure as the first but with (one, two or) three completely different orientations. For this reason we frequently, for ease of exposition, assume that when considering a particular L and $\mathcal{O}$ where $\Theta(L) \notin \mathcal{O}$ and $\mathcal{O}$ has two or more orientations that $(0°, 90°)$ is L's maximal $\mathcal{O}$-free range.

3 Restricted-Orientation Convexity

Property (2) of convex sets stated in the Introduction can be taken as a defining characteristic of convex sets (as are all the others except for property (1)). A set is said to *convex* if its intersection with *any* line is empty or connected. Here however we are only interested in intersections with a particular class of lines, namely those whose orientations belong in some restricted set of orientations. As a result we speak of ***restricted-orientation convexity.*** The phrasing is somewhat unhappy since it implies that it is a restriction on normal convexity when in fact the opposite is the case, restricted-orientation convexity includes (normal) convexity as a special case.

From this point on we assume that we have chosen some fixed set $\mathcal{O}$ of orientations (none of our results depend on the particular set chosen).

Definition: We say that $\mathbf{P}$ is *$\mathcal{O}$-convex* if the intersection of $\mathbf{P}$ and any $\mathcal{O}$-line is either empty or connected.

(Note that if $\mathbf{P}$ is a polygon then $\mathbf{P}$ is $\{\theta\}$-convex if and only if $\mathbf{P}$ is monotone in $\theta + 90°$.)

This is a natural generalization of the notion of orthogonal convexity — a set is orthogonally convex if its intersection with any horizontal or vertical line is either empty or connected. The practical motivation for such a definition is that in many instances (example, VLSI design) the possible

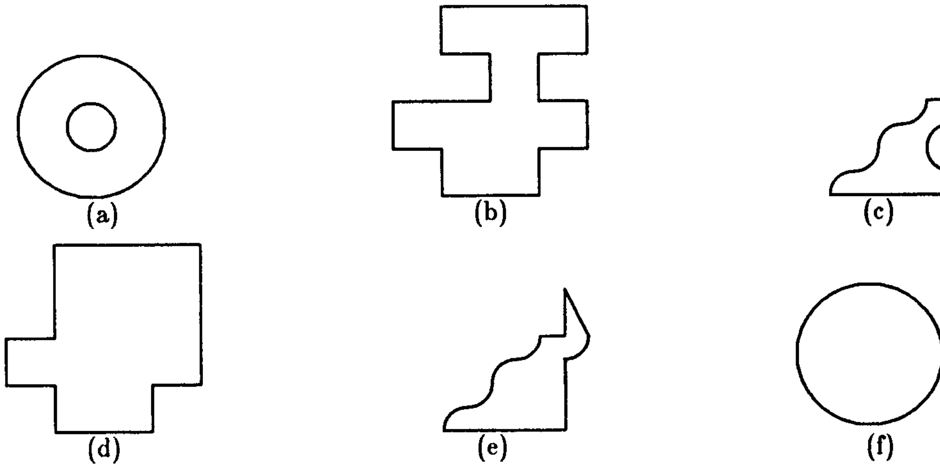

Figure 1: $\mathcal{O}$-convex figures.

lines are restricted to just horizontal and vertical ones and we are mostly concerned with the intersection of these lines and planar figures. Orthogonal convexity has been defined not only in computational geometry ([18]) but also in digital picture processing ([11]) and for polyominoes ([1]).

Figure 1 contains some example figures which are $\mathcal{O}$-convex for various $\mathcal{O}$. Figure 1 (a) is not $\mathcal{O}$-convex for any non-empty $\mathcal{O}$, but is $\mathcal{O}$-convex if $\mathcal{O} = \emptyset$, as are all the other figures. Figures 1 (b) and (c) are convex with respect to any horizontal line, as are (d), (e) and (f), so they are all $\{0°\}$-convex besides being $\emptyset$-convex. Note that (b) and (c) are not convex in any other direction. Figures 1 (d), (e) and (f) are convex with respect to any vertical line as well and so they are also $\{0°, 90°\}$-convex. Note that (d) is not convex in any other direction. Figures 1 (e) and (f) are convex with respect to any line with orientation in the range $\{[90°, 180°]\}$ and so they are also $\{[90°, 180°]\}$-convex. Note that (e) is not convex in any other direction. Figures 1 (f) is $\mathcal{O}$-convex for any $\mathcal{O}$.

Lemma 3.1 *All planar convex sets are $\mathcal{O}$-convex.*

Proof: If $\mathcal{O}$ is empty then, vacuously, all sets are $\mathcal{O}$-convex. Suppose that $\mathcal{O}$ is non-empty. Since convex sets are by definition sets whose intersection with any line is either empty of a connected set then they are *a fortiori* $\mathcal{O}$-convex for any $\mathcal{O}$. ◇

Indeed, we can go further and completely characterize convex sets as a sub-class of $\mathcal{O}$-convex sets:

Observation: A planar set is convex if and only if it is $\{[0°, 180°)\}$-convex.

In fact it is easy to construct examples to show that this is true for no smaller set of orientations. For example if we delete just *one* orientation (say θ_1) then any set consisting of just two distinct points on a $\{\theta_1\}$-line is $\{\theta\}$-convex for all $\theta \neq \theta_1$ but is, of course, not convex. In fact examples like these establish that the statement "for all **P**, **P** is connected if **P** is $\mathcal{O}$-convex" holds if and only if $\mathcal{O} = \{[0°, 180°)\}$.

Note that the following sets are convex, and hence $\mathcal{O}$-convex for any $\mathcal{O}$: the empty set, $\Re^2$, and, any point, line, segment, ray or halfplane in $\Re^2$.

There is a straightforward extension of $\mathcal{O}$-convexity to $\Re^n$ which singles out a subclass, $\mathcal{O}$, of $(n-1)$-dimensional hyperplanes and defines a set to be $\mathcal{O}$-convex if its intersection with any translate of such a hyperplane is empty or connected. Any k-flat in $\Re^n$ is $\mathcal{O}$-convex in that sense. For $n = 2$ the flats are: any point (a 0-dimensional hyperplane), any line (a 1-dimensional hyperplane) and the whole plane (a 2-dimensional hyperplane).

Lemma 3.2 *If $\mathcal{C}$ is a non-empty collection of $\mathcal{O}$-convex sets, then $\bigcap \mathcal{C}$ is $\mathcal{O}$-convex.*

Proof: The result is vacuously true if $\mathcal{O}$ is empty since all sets are $\emptyset$-convex. If $\mathcal{O}$ is non-empty but there are no two points in $\bigcap \mathcal{C}$ which lie on an $\mathcal{O}$-line then the intersection of any $\mathcal{O}$-line and $\bigcap \mathcal{C}$ is either empty or a single point.

Suppose then that $\mathcal{O}$ is non-empty and that there exists at least two points in $\bigcap \mathcal{C}$ which lie on an $\mathcal{O}$-line. If such a pair exist then they belong to each member of $\mathcal{C}$. Since each member is $\mathcal{O}$-convex, the segment joining such a pair is in each member of $\mathcal{C}$ and so is in $\bigcap \mathcal{C}$. Hence $\bigcap \mathcal{C}$ is $\mathcal{O}$-convex. $\diamondsuit$

Definition: We call the intersection of all $\mathcal{O}$-convex sets containing **P** the *$\mathcal{O}$-hull* of **P**, and write $\mathcal{O}$-*hull*(**P**).

Observe that $\forall \mathcal{O}, \mathbf{P}$; $\mathbf{P} \subseteq \mathcal{O}\text{-}hull(\mathbf{P})$ even when $\mathcal{O} = \emptyset$ or $\mathbf{P} = \emptyset$ (or both).

Lemma 3.3 $\forall \mathcal{O}, \mathbf{P}$; *$\mathcal{O}$-hull(**P**) exists, is unique and is the smallest $\mathcal{O}$-convex set which contains **P**.*

Proof: Since the hull is defined as the intersection of a non-empty collection of sets (there exists at least one set which contains **P** and is $\mathcal{O}$-convex, namely the entire plane $\Re^2$) it always exists and is unique.

$\mathcal{O}$-*hull*(**P**) contains **P** and it must be $\mathcal{O}$-convex by Lemma 3.2. Also, no $\mathcal{O}$-convex set containing **P** can be smaller than $\mathcal{O}$-*hull*(**P**) since $\mathcal{O}$-*hull*(**P**) is contained in all such sets. Since $\mathcal{O}$-*hull*(**P**) is unique it is then the smallest such set. $\diamondsuit$

This means that we can speak of ***the* $\mathcal{O}$*-hull*** of any set and be assured of its existence and uniqueness. If $\mathcal{O} = \emptyset$ then $\mathcal{O}$-***hull***$(\mathbf{P}) = \mathbf{P}$, for all $\mathbf{P}$, since $\mathbf{P}$ is the smallest set containing $\mathbf{P}$ which is not required to be convex in any direction. Similarly, if $\mathbf{P} = \emptyset$ then $\mathcal{O}$-***hull***$(\mathbf{P}) = \mathbf{P}$, for all $\mathcal{O}$, since the intersection of every $\mathcal{O}$-line and $\mathbf{P}$ is empty. When $\mathcal{O} = \{\theta\}$ and $\mathbf{P}$ is a polygon then the $\mathcal{O}$-***hull*** of $\mathbf{P}$ has been called the "θ-visibility hull" of $\mathbf{P}$ ([12,15]).

Note that in Figure 1, (f) is the $\mathcal{O}$-***hull*** of (a) for any non-empty $\mathcal{O}$ and (d) and (e) are the $\{90°\}$-***hulls*** of (b) and (c) respectively.

Lemma 3.4 $\forall \mathcal{O}, \mathbf{P}$; $\mathbf{P}$ is $\mathcal{O}$-convex if and only if $\mathcal{O}$-***hull***$(\mathbf{P}) = \mathbf{P}$

Proof: $\mathcal{O}$-***hull***$(\mathbf{P}) = \bigcap\{\mathbf{Q} \mid \mathbf{P} \subseteq \mathbf{Q} \wedge \mathbf{Q}$ is $\mathcal{O}$-convex$\}$. If $\mathbf{P}$ is $\mathcal{O}$-convex then $\mathbf{P}$ must be a member of the intersected family, and so $\mathcal{O}$-***hull***$(\mathbf{P}) = \bigcap\{\mathbf{Q} \mid \mathbf{P} \subseteq \mathbf{Q} \wedge \mathbf{Q}$ is $\mathcal{O}$-convex$\} \cap \mathbf{P}$. Hence, $\mathcal{O}$-***hull***$(\mathbf{P}) \subseteq \mathbf{P}$. But $\mathbf{P} \subseteq \mathcal{O}$-***hull***$(\mathbf{P})$, hence, $\mathcal{O}$-***hull***$(\mathbf{P}) = \mathbf{P}$.

Conversely, if $\mathcal{O}$-***hull***$(\mathbf{P}) = \mathbf{P}$ then, since $\mathcal{O}$-***hull***$(\mathbf{P})$ is $\mathcal{O}$-convex, $\mathbf{P}$ is $\mathcal{O}$-convex. $\diamond$

Note that when $\mathcal{O}$ is the set of all orientations then this lemma reduces to property 3 stated in the Introduction.

Corollary 3.1 $\forall \mathcal{O}, \mathbf{P}$; $\mathcal{O}$-***hull***$(\mathcal{O}$-***hull***$(\mathbf{P})) = \mathcal{O}$-***hull***$(\mathbf{P})$

Lemma 3.5 $\forall \mathcal{O}, \mathbf{P}, \mathbf{Q}$; $\mathbf{P} \subseteq \mathbf{Q} \Longrightarrow \mathcal{O}$-***hull***$(\mathbf{P}) \subseteq \mathcal{O}$-***hull***$(\mathbf{Q})$

Proof: If $\mathbf{P} \subseteq \mathbf{Q}$ then $\mathcal{O}$-***hull***$(\mathbf{Q})$ is an $\mathcal{O}$-convex set containing $\mathbf{P}$. But $\mathcal{O}$-***hull***$(\mathbf{P})$ is contained in all such sets. $\diamond$

Lemma 3.6 *If $\mathcal{O}$ is non-empty and $\mathbf{P}$ is connected, then $\mathcal{O}$-**hull**$(\mathbf{P})$ is simply connected.*

Proof: If $\mathbf{P}$ is empty we have nothing to prove, so suppose $\mathbf{P}$ is non-empty.

Suppose that $\mathcal{O}$-***hull***$(\mathbf{P})$ is disconnected. Since $\mathbf{P}$ is connected it can only belong to one of the connected components of $\mathcal{O}$-***hull***$(\mathbf{P})$ (it must belong to at least one otherwise $\mathcal{O}$-***hull***$(\mathbf{P})$ does not contain $\mathbf{P}$). This component must be $\mathcal{O}$-convex, otherwise the entire hull is not $\mathcal{O}$-convex. Hence we may discard all of the other components of $\mathcal{O}$-***hull***$(\mathbf{P})$ and have a smaller $\mathcal{O}$-convex set which contains $\mathbf{P}$. But $\mathcal{O}$-***hull***$(\mathbf{P})$ is the smallest such set. Therefore $\mathcal{O}$-***hull***$(\mathbf{P})$ must be connected if $\mathbf{P}$ is connected.

Suppose that $\mathcal{O}$-***hull***$(\mathbf{P})$ is connected but contains a hole. Since $\mathcal{O}$ is non-empty there must exist at least one $\mathcal{O}$-line which cuts this hole. Hence there exists an $\mathcal{O}$-line whose intersection with $\mathcal{O}$-***hull***$(\mathbf{P})$ is neither empty nor connected. But this implies that $\mathcal{O}$-***hull***$(\mathbf{P})$ is not $\mathcal{O}$-convex hence $\mathcal{O}$-***hull***$(\mathbf{P})$ must be simply connected. $\diamond$

Compare this lemma with property (1) stated in the Introduction.

Corollary 3.2 *If $\mathcal{O}$ is non-empty and* **P** *is connected and $\mathcal{O}$-convex, then* **P** *is simply connected.*

Proof: If **P** is $\mathcal{O}$-convex then it is its own hull. ◇

Lemma 3.7 *A set is $\mathcal{O}$-convex if and only if it consists of a set of disjoint connected components such that each component is $\mathcal{O}$-convex and no $\mathcal{O}$-line intersects any pair of components.*

Proof: Let **P** consist of a set of disjoint connected components such that each component is a connected $\mathcal{O}$-convex set and no $\mathcal{O}$-line intersects any pair of components. Since no $\mathcal{O}$-line can intersect any two of them simultaneously and each component is separately $\mathcal{O}$-convex, the entire collection is $\mathcal{O}$-convex.

Conversely, let **P** be disconnected and $\mathcal{O}$-convex. If one of its components is not $\mathcal{O}$-convex then **P** cannot be $\mathcal{O}$-convex. Similarly, if there exists an $\mathcal{O}$-line which intersects any two components then **P** cannot be $\mathcal{O}$-convex. ◇

Observe that if $\mathcal{O}$ is the set of all orientations then for each pair of connected components there exists at least one $\mathcal{O}$-line which intersects them. Hence, all $\{[0°, 180°)\}$-convex sets are connected.

4 The Decomposition Theorem

Intuitively, we think of the action of forming the *$\mathcal{O}$-hull* of a set **P** as sweeping a line of each orientation in $\mathcal{O}$ across **P** and adding suitable line segments to the hull formed so far so that it is convex in each direction in $\mathcal{O}$. (Note that if $\mathcal{O}$ is empty then we do not add anything to **P**.) Thinking of it this way it does not seem sensible that the hull we eventually produce is changed if we decide to change the order of orientations in which we sweep. As we shall prove in Theorem 4.2 this is, in fact, the case but only for *connected* sets. For disconnected sets Lemma 4.2 is the strongest possible result.

Lemma 4.1 *If* **P** *is connected and $p \in \mathcal{O}$-hull(**P**), then each $\mathcal{O}$-line through p intersects* **P**.

Proof: If either $\mathcal{O}$ or **P** is empty then the lemma is vacuously true since then $\mathcal{O}$-*hull*(**P**) = **P**. Further, if $p \in \mathbf{P}$ we have nothing to prove. So suppose that both $\mathcal{O}$ and **P** are non-empty and that $p \notin \mathbf{P}$.

Suppose that there exists a $\theta \in \mathcal{O}$ such that the $\{\theta\}$-line through p does not intersect **P**. Then, by the continuity of $\Re^2$ and the fact that **P** is connected, there exists a convex set (and hence an $\mathcal{O}$-convex set) which contains $\mathbf{P}$ and does not contain p, namely, any halfplane bounded by a $\{\theta\}$-line separating p and $\mathbf{P}$. Hence p cannot be in the intersection of all $\mathcal{O}$-convex sets which contain $\mathbf{P}$ and so cannot be in the hull. $\diamond$

Theorem 4.1 (The Separation Theorem) *Let* $\mathbf{P}$ *be connected and* $p \notin \mathbf{P}$. $p \in \mathcal{O}\text{-}hull(\mathbf{P})$ *if and only if there exists a* $\theta \in \mathcal{O}$ *such that the* $\{\theta\}$*-line through* p *intersects* $\mathbf{P}$ *in, at least, two points on either side of* p.

Proof: If either $\mathcal{O}$ or $\mathbf{P}$ is empty then the lemma is vacuously true since then $\mathcal{O}\text{-}hull(\mathbf{P}) = \mathbf{P}$. So suppose that both $\mathcal{O}$ and $\mathbf{P}$ are non-empty.

If $p \notin \mathbf{P}$ and there exists an $\mathcal{O}$-line which intersects $\mathbf{P}$ at two points which bracket p then p must be in the $\mathcal{O}$-*hull* of $\mathbf{P}$ (else the $\mathcal{O}$-*hull* would not be $\mathcal{O}$-convex).

Conversely, if $\mathbf{P}$ is connected and $p \in \mathcal{O}\text{-}hull(\mathbf{P}) \setminus \mathbf{P}$ then all $\mathcal{O}$-lines through p must intersect $\mathbf{P}$ (Lemma 4.1).

We shall prove the claim for the three cases in which we have either exactly one orientation in $\mathcal{O}$, two or more with at least one $\mathcal{O}$-free range and finally if $\mathcal{O}$ is all orientations (that is, there are no $\mathcal{O}$-free ranges).

Case 1: $\mathcal{O} = \{\theta\}$

The $\{\theta\}$-line through p must cut $\mathbf{P}$. Suppose that it only cuts it on one side of p (say to the right of p). Then we may delete p and all other points in $\{\theta\}\text{-}hull(\mathbf{P})$ on the left θ-ray from p and so obtain a smaller $\{\theta\}$-convex set which contains $\mathbf{P}$. But $\{\theta\}\text{-}hull(\mathbf{P})$ is the smallest such set. Hence p cannot be in $\{\theta\}\text{-}hull(\mathbf{P})$. Hence the $\{\theta\}$-line through p must cut $\mathbf{P}$ on both sides of p.

Case 2: $\mathcal{O}$ contains two or more orientations but not all.

Every $\mathcal{O}$-line through p must cut $\mathbf{P}$. Suppose that none of them cut $\mathbf{P}$ both to the left and to the right of p. Since $\mathbf{P}$ is connected this means that there exists at least one $\mathcal{O}$-convex halfplane containing $\mathbf{P}$ and not p. The simplest such halfplane is bounded by the first $\{\theta\}$-line through p which does not cut $\mathbf{P}$ to the left of p and the first $\{\theta\}$-line through p which does not cut $\mathbf{P}$ to the right of p (see Figure 2 for a simple example with $\mathcal{O} = \{0°, 90°, 135°\}$). This halfplane must be $\mathcal{O}$-convex as no $\mathcal{O}$-line can intersect both of the boundary $\mathcal{O}$-lines since the entire range is $\mathcal{O}$-free.

Hence p cannot be in $\mathcal{O}\text{-}hull(\mathbf{P})$ for it would not be contained in the intersection of all $\mathcal{O}$-convex sets which contain $\mathbf{P}$. Hence at least one of the $\mathcal{O}$-lines through p must cut $\mathbf{P}$ to the left and to the right of p.

Case 3: $\mathcal{O} = \{[0°, 180°)\}$.

Here $\mathcal{O}\text{-}hull(\mathbf{P})$ is the normal convex hull of $\mathbf{P}$. Since $\mathcal{O}$ contains all possible orientations and, by Lemma 4.1 each of them must intersect $\mathbf{P}$

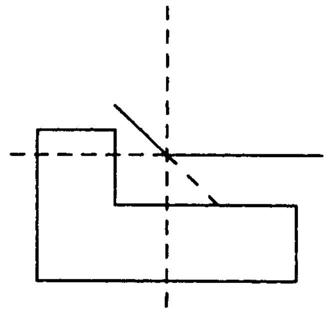

Figure 2: A halfplane containing **P** and not p.

then there must exist an infinity of $\mathcal{O}$-lines which intersect **P** at points bracketting p. $\diamond$

Observe that if $\mathcal{O}_1 \subseteq \mathcal{O}_2$ then $\mathcal{O}_1$-*hull*(**P**) $\subseteq \mathcal{O}_2$-*hull*(**P**) for all **P** since $\mathcal{O}_2$-*hull*(**P**) contains **P** and is $\mathcal{O}_1$-convex. In some sense as a set of orientations $\mathcal{O}$ "grows" to include all possible orientations, the set $\mathcal{O}$-*hull*(**P**) "grows" to the (normal) convex hull of **P**.

Lemma 4.2 $\forall \mathcal{O}_1, \mathcal{O}_2, \mathbf{P}$; $\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P})) \subseteq (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})$

Proof: $\forall \mathcal{O}_2, \mathbf{P}$; $\mathbf{P} \subseteq \mathcal{O}_2\text{-}hull(\mathbf{P})$ and so, from Lemma 3.5, $\mathcal{O}_1\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P}))$. Also, $\mathcal{O}_2\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P}))$. Hence, $(\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P})) \subseteq \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P}))$.

Since $\mathcal{O}_2 \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ an $(\mathcal{O}_1 \cup \mathcal{O}_2)$-convex set is $\mathcal{O}_2$-convex. Hence, $\forall \mathbf{P}$; $\mathcal{O}_2\text{-}hull(\mathbf{P}) \subseteq (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})$. And so, by Lemma 3.5 we have that, $\mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P})) \subseteq \mathcal{O}_1\text{-}hull((\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})) = (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})$ (since an $(\mathcal{O}_1 \cup \mathcal{O}_2)$-convex set is $\mathcal{O}_1$-convex). $\diamond$

This result also holds if we replace $\mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P}))$ by $\mathcal{O}_2\text{-}hull(\mathcal{O}_1\text{-}hull(\mathbf{P}))$.

Simple counter-examples show that all these results are best possible, in that, there exists sets for which the respective converses are false. However, we can strengthen Lemma 4.2 considerably by restricting **P** to be connected.

Theorem 4.2 (The Decomposition Theorem) *If* **P** *is connected then* $\forall \mathcal{O}_1, \mathcal{O}_2$

$$
\begin{aligned}
(\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P}) &= \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P})) \\
&= \mathcal{O}_2\text{-}hull(\mathcal{O}_1\text{-}hull(\mathbf{P})) \\
&= \mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P})
\end{aligned}
$$

Proof: First, from Lemma 4.2 we have that,

$\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}) \subseteq \mathcal{O}_1\text{-}hull(\mathcal{O}_2\text{-}hull(\mathbf{P})) \subseteq (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})$.

Therefore all we need establish is that if $\mathbf{P}$ is connected then $(\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P}) \subseteq (\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}))$.

Observe that if $\mathbf{P}$ is empty or if both $\mathcal{O}_1$ and $\mathcal{O}_2$ are empty then the theorem is true. So assume that $\mathbf{P}$ is non-empty and $\mathcal{O}_1 \cup \mathcal{O}_2$ is non-empty.

Let $p \in (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P})$.

If $p \in \mathbf{P}$ then p is in both the $\mathcal{O}_1$-*hull* and the $\mathcal{O}_2$-*hull* of $\mathbf{P}$. Hence $p \in (\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}))$.

If $p \in (\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P}) \setminus \mathbf{P}$ then, from Theorem 4.1, we know that there must exist a $\theta \in \mathcal{O}_1 \cup \mathcal{O}_2$ such that the $\{\theta\}$-line through p cuts $\mathbf{P}$ to the left and right of p. But this means that p must be in either $\mathcal{O}_1\text{-}hull(\mathbf{P})$ or in $\mathcal{O}_2\text{-}hull(\mathbf{P})$ for θ belongs to one of $\mathcal{O}_1$ or $\mathcal{O}_2$. Hence, $p \in (\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}))$.

Hence, $(\mathcal{O}_1 \cup \mathcal{O}_2)\text{-}hull(\mathbf{P}) \subseteq (\mathcal{O}_1\text{-}hull(\mathbf{P}) \cup \mathcal{O}_2\text{-}hull(\mathbf{P}))$. ◇

Corollary 4.1 *If* $\mathbf{P}$ *is connected and* $\mathcal{O} = \bigcup \mathcal{O}_i$ *then* $(\bigcup \mathcal{O}_i)\text{-}hull(\mathbf{P}) = \bigcup(\mathcal{O}_i\text{-}hull(\mathbf{P}))$

This corollary verifies Toussaint and Sack's observation ([15]) that the (normal) convex hull is the union of the "visibility hulls" over all directions of visibility.[4]

Sack ([12]) showed, in the orthogonal case, that the horizontal hull of the vertical hull of an orthogonal polygon (or alternately the vertical hull of the horizontal hull) is equivalent to the union of both hulls. It was taken as self-evident that the union is the smallest horizontally and vertically convex polygon enclosing the orthogonal polygon. Corollary 4.1 validates that assumption.

This decomposition result immediately yields an algorithm to find the hull of any connected set given that we can find the hull in one direction. It turns out though that connected $\mathcal{O}$-convex sets have considerably more structure than this and we can exploit this structure to construct optimal algorithms to find the hull of any connected set (see [9] for the special case of finite $\mathcal{O}$, see [8] for the general case).

5 The Notion of a Stairline

To characterize $\mathcal{O}$-convex sets we need a new definition of "line" more appropriate to $\mathcal{O}$-convex sets. We call these generalized lines "stairlines" and

[4]Interestingly, the Decomposition Theorem bears a strong resemblance to the double integration rule where *if* $f(x,y)$ *is continuous* then $\int\int f(x,y)dxdy = \int\int f(x,y)dydx$.

we define and investigate them in this section. First though we need the concept of the ***span*** of a continuous curve in the plane.

Definition: We say that the continuous plane curve S has ***span*** $[\theta_1, \theta_2]$ $(\theta_1 \leq \theta_2)$ if for any two distinct points $p, q \in S$ $\Theta(L(p,q)) \in [\theta_1, \theta_2]$.

(Of course, $\theta_1 = \theta_2$ if and only if the curve is a line, segment or ray with orientation θ_1.)

As an illustration: if S is a continuous curve with span $[0°, 90°]$ and $(x_1, y_1), (x_2, y_2)$ are any two points on S then either $(x_1 \leq x_2$ and $y_1 \leq y_2)$ or $(x_1 \geq x_2$ and $y_1 \geq y_2)$.

Definition: We say that a continuous curve in the plane with span $[\theta_1, \theta_2]$ is an ***$\mathcal{O}$-stairline*** if (θ_1, θ_2) is $\mathcal{O}$-free.

(Note that if $\theta_1 = \theta_2$ then (θ_1, θ_2) is vacuously $\mathcal{O}$-free since there are ***no*** orientations in the range and so any line, segment or ray is an $\mathcal{O}$-stairline.)

We chose the name "stairline" as a merger of (orthogonal) ***staircases*** ([18]) and (straight) ***lines***. By analogy with lines, segments and rays we also use the terms "$\mathcal{O}$-stairsegment" and "$\mathcal{O}$-stairray" with the obvious meanings. Note that a line, segment or ray of ***any*** orientation is an $\mathcal{O}$-stairline, $\mathcal{O}$-stairsegment or $\mathcal{O}$-stairray.

Remark: To avoid excessive terminology we shall assume for the rest of his paper that $\mathcal{O}$ is understood and we shall just refer to "stairlines" ("stairsegments" or "stairrays"). Also, if a result is stated for stairlines we do not add the cumbersome qualifications that it also holds for stairsegments and stairrays.

Lemma 5.1 *If S is a stairline, then S is $\mathcal{O}$-convex.*

Proof: Suppose S is an stairline with span $[\theta_1, \theta_2]$. If $\theta_1 = \theta_2$ then S is a straight line and is hence $\mathcal{O}$-convex. Suppose then that $\theta_1 \neq \theta_2$. Suppose that there exists an $\mathcal{O}$-line L which cuts S at two distinct points p and q. Since S has span $[\theta_1, \theta_2]$ then $\Theta(L) = \Theta(L(p,q)) \in [\theta_1, \theta_2]$. Since (θ_1, θ_2) is $\mathcal{O}$-free then $\Theta(L)$ can only be θ_1 or θ_2.

Suppose $\Theta(L) = \theta_1$ and $\theta_1 \in \mathcal{O}$. Without loss of generality assume that $[\theta_1, \theta_2] = [0°, 90°]$ and that p is to the left of q (that is, p and q lie on a horizontal line). Consider any point r on S in between p and q. r must be ***on or above*** the horizontal line segment $LS(p,q)$ otherwise $\Theta(L(p,r)) \notin [\theta_1, \theta_2]$. Similarly, r must be ***on or below*** the horizontal line segment $LS(p,q)$ otherwise $\Theta(L(q,r)) \notin [\theta_1, \theta_2]$. Hence $r \in LS(p,q)$ for all r in S in between p and q. That is, between p and q, S is a line segment. Hence, even if θ_1 (or θ_2) is an orientation in $\mathcal{O}$ then S is $\mathcal{O}$-convex. $\diamond$

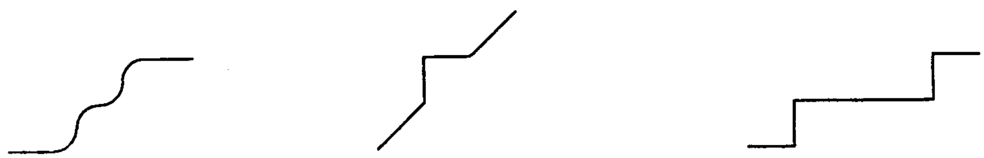

Figure 3: A variety of stairsegments.

If S divides the plane into two halfspaces we call them both "stair-halfplanes" for obvious reasons. Beware! It is easy to fall into the habit of thinking of "stair-halfplanes" as just halfplanes with "wavy line boundaries". This is only true if $\mathcal{O} \neq \emptyset$.

Corollary 5.1 *All stair-halfplanes are $\mathcal{O}$-convex.*

Corollary 5.2 *If* $\mathbf{P}$ *is connected and there exists a stair-halfplane which contains* $\mathbf{P}$ *and not the point* p*, then* $p \notin \mathcal{O}$*-hull*$(\mathbf{P})$.

Definition: We say that a stairline composed of a sequence of connected line segments is a *polygonal* stairline.

It is easy to show that if a connected sequence of segments $l_1, l_2, ..., l_m$ forms a stairline, stairsegment or stairray with span $[\theta_1, \theta_2]$ then

(1) $\forall 1 \leq i \leq m\,;\quad \Theta(l_i) \in [\theta_1, \theta_2]$.

(2) $\forall 2 \leq i \leq m-1\,;\quad l_i$ meets l_{i-1} and l_{i+1} only at its endpoints.

Polygonal stairlines have been previously defined for the special case of orthogonal objects (see [18] for references) in that special case they are known as "staircases". See Figure 3 for examples of a stairsegment, a polygonal stairsegment, and an $\mathcal{O}$-oriented polygonal stairsegment for $\mathcal{O}$ any subset of $\{[90°, 180°]\}$.

Definition: We call the set of all stairsegments joining p and q the *$\mathcal{O}$-region* of p and q and write $\mathcal{O}$*-region*(p, q).

Note that if $\Theta(LS(p,q)) \in \mathcal{O}$ then $\mathcal{O}$*-region*$(p,q) = LS(p,q)$. Of course, if $\mathcal{O}$ consists of all orientations then, for all p and q, $\mathcal{O}$*-region*$(p,q) = LS(p,q)$. On the other hand if $\mathcal{O}$ is empty then every range is $\mathcal{O}$-free and so *any* continuous curve connecting p and q for any p and q is a "stairsegment", hence $\emptyset$*-region*$(p,q) = \Re^2$.

Definition: If $\mathcal{O}$ has at least two orientations then we say that *the parallelogram induced by p and q*, $\|pq$, is $LS(p,q)$ if $\Theta(LS(p,q)) \in \mathcal{O}$. Else it is the parallelogram with diagonal endpoints p and q and with sides of orientations θ_1 and θ_2, where (θ_1, θ_2) is $LS(p,q)$'s maximal $\mathcal{O}$-free range.

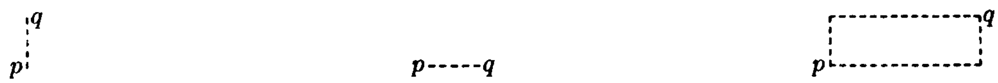

Figure 4: Orthogonal induced parallelograms.

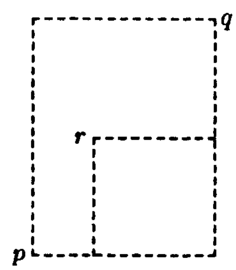

Figure 5: A stairsegment from p to q through r.

Note that if $\mathcal{O} = \{[0°, 180°)\}$ then, for all distinct p and q, $\|pq = LS(p, q)$. See Figure 4 for examples of $\|pq$ for $\mathcal{O} = \{0°, 90°\}$.

If $\Theta(LS(p, q)) \notin \mathcal{O}$ then we call the two sets of segments connecting p and q the *arms* of $\|pq$.

Lemma 5.2 *If p and q are two points in the plane and $\mathcal{O}$ is a set of at least two orientations, then any stairsegment joining p and q must lie wholly in $\|pq$. Further, all points in $\|pq$ lie on some stairsegment joining p and q.*

Proof: If $\Theta(LS(p, q)) \in \mathcal{O}$ then the lemma is true, so suppose otherwise. Without loss of generality, let $(0°, 90°)$ be $LS(p, q)$'s maximal $\mathcal{O}$-free range.

If any continuous path from p to q leaves the parallelogram $\|pq$ then it can only be monotone in either the horizontal or vertical direction but not both and so cannot be a stairsegment. Hence when $\mathcal{O}$ contains two or more orientations then all stairsegments must lie in $\|pq$.

If $r \in \|pq$ the we can easily construct a stairsegment joining p and q passing through r (see Figure 5 for a simple example stairsegment for $\mathcal{O} = \{0°, 90°\}$). $\diamond$

Hence, when $\mathcal{O}$ has two or more orientations, $\mathcal{O}$-*region*$(p, q) = \|pq$.

With respect to $\mathcal{O}$-convex sets stairlines are the most natural analogues of straight lines with respect to convex sets, in that: there exists a stairsegment which realises the shortest distance between any two points; an $\mathcal{O}$-line meets a stairline at at most one point (unless collinear with some part of the stairline); and two stairlines with disjoint spans can only intersect at at most one point. However, the intersection of two stairlines with non-disjoint spans is either empty, connected or disconnected — unlike the simpler case for straight lines. Further, stairlines can be non-intersecting without being parallel (in the conventional sense). Also two *points* may define exactly one

$\mathcal{O}$-line or infinitely many stairlines (that is, all stairlines passing through their *$\mathcal{O}$-region*). Perhaps a closer analogy would be to say that two stairlines with non-disjoint spans are *parallel* and that they are *collinear* if they intersect anywhere. If two stairlines *have disjoint spans* then they behave just like normal straight lines (i.e., intersect exactly once etc.).

With stairlines standing for lines we can generalize convexity in other ways than the one we investigate in this paper. For example, we call a set $\mathbf{P}$ "strongly $\mathcal{O}$-convex" if for every pair of points p and q in $\mathbf{P}$ *all* stairsegments with endpoints p and q lie in $\mathbf{P}$. It is possible to prove that this definition of convexity always produces *convex* (in the normal sense) *$\mathcal{O}$-oriented* sets. Indeed, when $\mathcal{O} = \{0°, 90°\}$ then the strong $\mathcal{O}$-convex hull of $\mathbf{P}$ is just the *bounding box* of $\mathbf{P}$. We investigated the notion of strong $\mathcal{O}$-convexity in a previous paper ([9]) and we show in [10] that both $\mathcal{O}$-convexity and strong $\mathcal{O}$-convexity along with many other natural definitions of convexity are essentially the same.

6 Other Characterizations of $\mathcal{O}$-Convex Sets

In this section we characterize *connected* $\mathcal{O}$-convex sets by deriving conditions on the form their boundary must take and proving a generalized version of property 6 (see the Introduction).

Definition: We say that p is an *$\mathcal{O}$-extremal* of $\mathbf{P}$ if p is a point of support of $\mathbf{P}$ with respect to an $\mathcal{O}$-line.

We now show that the boundary of a closed connected $\mathcal{O}$-convex set may be completely characterized in terms of stairsegments.

Definition: We say that a portion of a continuous curve in the plane is a *maximal stairsegment* in the curve if it is a stairsegment and it is not a proper subset of any other stairsegment in the curve.

Theorem 6.1 The Characterization Theorem *A simply connected closed set is $\mathcal{O}$-convex if and only if the portions of its boundary in between any two consecutive $\mathcal{O}$-extremal points are maximal stairsegments.*

Proof: If $\mathbf{P}$ is closed and simply connected and its boundary is made up only of stairsegments meeting at $\mathcal{O}$-extremal points in $\mathbf{P}$ then the only way in which $\mathbf{P}$ could fail to be $\mathcal{O}$-convex is if some $\mathcal{O}$-line intersects some one of the stairsegments more than once, since no $\mathcal{O}$-line can intersect such a set more than twice. But this is impossible, since any $\mathcal{O}$-line can only intersect a stairline at most once (unless it is collinear with some part of the stairline). Hence such a set must be $\mathcal{O}$-convex.

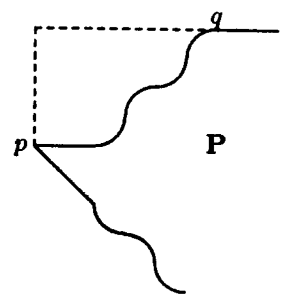

Figure 6: $S(p,q)$ is a maximal stairsegment.

Suppose now that **P** is a connected closed $\mathcal{O}$-convex set. Consider any pair of distinct consecutive $\mathcal{O}$-extremal points p and q of **P**. If $LS(p,q) \subseteq \mathbf{P}$ then $\Theta(LS(p,q)) \in \mathcal{O}$ and hence $LS(p,q)$ is a stairsegment joining p and q. So let $S(p,q)$ be the portion of **P**'s boundary connecting p and q where $S(p,q)$ is not a line segment. Since p and q are distinct consecutive extremal points of **P** then $\Theta(LS(p,q)) \notin \mathcal{O}$. Without loss of generality, assume that $LS(p,q)$'s maximal $\mathcal{O}$-free range is $(0°, 90°)$ and that p is below and to the left of q (see Figure 6).

Now sweep a horizontal line from q down to p. If at any time in this sweep this line intersects $S(p,q)$ more than once then **P** cannot be $\mathcal{O}$-convex, and similarly for a vertical line sweeping from p to q. Hence $S(p,q)$ is a stairsegment connecting p and q. Trivially, it is maximal since it's endpoints are $\mathcal{O}$-extremal in **P**. $\diamond$

Observe that in the normal convex hull (that is, $\mathcal{O} = \{[0°, 180°)\}$) *all* points are $\mathcal{O}$-extremal and so the maximal stairsegments in the boundary shrink to points.

Corollary 6.1 *A polygon is $\mathcal{O}$-convex if and only if its boundary consists of a sequence of polygonal stairsegments meeting at convex interior angles.*

Corollary 6.2 *An $\mathcal{O}$-polygon is $\mathcal{O}$-convex if and only if its boundary consists of a sequence of $\mathcal{O}$-oriented polygonal stairsegments meeting at convex interior angles.*

For the special case of finite $\mathcal{O}$ Corollary 6.1 has been stated without proof in [17] and it was proved in a different, more direct, way in [9].

Note that the characterization of the boundary of $\mathcal{O}$-convex polygons as a sequence of polygonal stairsegments is a direct generalization of the case for orthogonal polygons ([18]).

In the theory of (normal) convex sets two points are said to be ***visible*** from each other in a set if the line segment joining them lies wholly in the set. Thinking of stairlines as the analogues of straight lines we are led to define a generalized form of visibility in which two points in a set are visible from each other if there exists *at least one stairsegment* joining them which lies wholly in the set. This leads to the next characterization of $\mathcal{O}$-convex sets and againg it only applies to *connected* $\mathcal{O}$-convex sets.

Theorem 6.2 (The Visibility Theorem) *If* $\mathbf{P}$ *is connected, then* $\mathbf{P}$ *is* $\mathcal{O}$*-convex if and only if for all* p *and* q *in* $\mathbf{P}$ *there exists a stairsegment in* $\mathbf{P}$ *with endpoints* p *and* q.

Proof: Suppose that $\mathbf{P}$ is connected and for all $p, q \in \mathbf{P}$ there exists a stairsegment joining them lying in $\mathbf{P}$. If $\Theta(LS(p,q)) \in \mathcal{O}$ then $\|pq$ collapses to $LS(p,q)$. Hence there is only one stairsegment joining p and q and so it must lie in $\mathbf{P}$. Hence $\mathbf{P}$ is $\mathcal{O}$-convex.

Conversely, suppose that $\mathbf{P}$ is connected and $\mathcal{O}$-convex. If $p, q \in \mathbf{P}$ and $\Theta(LS(p,q)) \in \mathcal{O}$ then there exists a stairsegment lying in $\mathbf{P}$ joining p and q — namely, $LS(p,q)$ (else $\mathbf{P}$ is not $\mathcal{O}$-convex). Suppose then that $\Theta(LS(p,q)) \notin \mathcal{O}$. Consider $\|pq$. If either arm of $\|pq$ lies in $\mathbf{P}$ then there exists a stairsegment lying in $\mathbf{P}$ joining p and q since either arm of $\|pq$ is a stairsegment. Assume then that neither arm lies wholly in $\mathbf{P}$. Since the lower arm (say) consists of two $\mathcal{O}$-segments and it does not lie wholly in $\mathbf{P}$ then it must intersect the boundary of $\mathbf{P}$ exactly twice (else $\mathbf{P}$ would not be $\mathcal{O}$-convex).

Both of these intersection points must belong to one maximal stairsegment since if they belonged to separate maximal stairsegments than there must be at least one $\mathcal{O}$-extremal point on $\mathbf{P}$'s boundary between the two intersection points. This means that there must exist at least one $\mathcal{O}$-orientation in $LS(p,q)$'s $\mathcal{O}$-free range. But this is impossible.

Now we can construct a stairsegment lying in $\mathbf{P}$ connecting p and q by starting at p and following the lower arm until we encounter $\mathbf{P}$'s boundary then follow the boundary until we intersect the arm again, then follow the arm to q. $\diamond$

Note that in normal convexity this theorem collapses to property 6 stated in the Introduction, since all (normal) convex sets are connected.

7 Conclusions

We have shown that $\mathcal{O}$-convex sets contain both convex sets and orthogonally convex sets as sub-classes and that the properties of both can be explained as special cases of the properties of $\mathcal{O}$-convex sets. The main characteristic of convex sets that we have lost in generalizing to $\mathcal{O}$-convex sets is *connectivity*. A convex set is always connected.

Connected $\mathcal{O}$-convex sets have all of the properties of convex sets listed at the beginning of the paper if we interpret a "line" as a stairline and generalize the betweenness relation to reflect the fact that the "line segment" joining two points is no longer necessarily unique and can be any stairsegment connecting them. And so, *any* point in *$\mathcal{O}$-region(p, q)* is "between" p and q. In the following we assume that $\mathbf{P}$ is a *connected* $\mathcal{O}$-convex set.

(1) If $\mathcal{O}$ is non-empty then $\mathbf{P}$ is simply connected (Lemma 3.6). Indeed, the connected components of any $\mathcal{O}$-convex set are simply connected once $\mathcal{O}$ is non-empty (Lemma 3.7 together with Lemma 3.6).

(2) The intersection of $\mathbf{P}$ and any $\mathcal{O}$-line is either empty or a connected set (by Definition). This is true even if $\mathbf{P}$ is allowed to be disconnected. One of the points of this property for convex sets is that lines are themselves convex. We could obtain the needed analogy by saying that the intersection of any two $\mathcal{O}$-convex sets is again $\mathcal{O}$-convex (Lemma 3.2) (although observe that the intersection of two connected $\mathcal{O}$-convex sets may be disconnected).

(3) $\mathbf{P}$ is the intersection of all $\mathcal{O}$-convex sets which contain it (Lemma 3.4). This is true even if $\mathbf{P}$ is allowed to be disconnected.

(4) If $p \notin \mathbf{P}$ then there exists a stairline separating p and $\mathbf{P}$ (Theorem 4.1 and Corollary 5.2).

(5) $\mathbf{P}$ is the intersection of all stair-halfplanes which contain it (Corollary 5.2).

(6) If $p, q \in \mathbf{P}$ then there exists a stairsegment in $\mathbf{P}$ connecting p and q (Theorem 6.2).

Restricted-orientation convexity is a generalization of orthogonal convexity which has itself been separately defined in computational geometry, digital picture processing, VLSI design and combinatorics. Restricted-orientation convexity serves as a useful vantage point to survey and unify many scattered results and observations in the literature of computational geometry. We have shown that restricted-orientation convexity is a reasonable generalization of convexity since properties analogous to those of normal convex sets hold for these more general "convex" sets.

It may be argued that since computational geometry concerns itself with figures in $\Re^n$ that it is not necessary to develop the theory of $\mathcal{O}$-convex sets in as general a setting as is possible. There are two telling rejoinders to this point of view, the first being a purely practical one. To take but one pertinent example, the history of algorithms for finding the convex hull of a simple polygon illustrates that unaided *geometric intuition* is not sufficiently powerful to avoid egregious errors[5] Any theoretical machinery that may

[5]There have been several algorithms proposed over time (and accepted as correct) which were later shown to be incorrect.

aid insight is desirable. Secondly, there is a well-demonstrated synergism between theoretical investigations and practical problems, in that practice suggests new areas for theory and in turn a developing theory suggests a broadening and sharpening of practique. Finally, if any further justification were needed, we submit that the study of restricted-orientation convexity is of sufficient interest and importance in its own right.

Besides the above justifications we believe that this material will be beneficial in at least two practical areas (restricted-orientation VLSI design and restricted-orientation robotic path problems) and that it is of continuing theoretical interest as evidenced by further work in "starshapedness", "visibility", the computation of nearness of "convex" polygons, etc. ([8]).

It is our opinion that, while the practical concerns from which computational geometry grew will continue to change and expand, the broad outlines of computational geometry that serve to delienate it from classical geometry and combinatorial geometry are now sufficiently well defined that it can now, in its turn, give impetus to the development of new directions of geometry.

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