Handling Degeneracy in a Nonlinear l_1 Algorithm

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ABSTRACT

This thesis is concerned with handling degeneracy in a nonlinear optimization algorithm based on an active-set strategy. Although the solution to the problem is given in the context of nonlinear l_1 optimization, the approach is more general and can be used to overcome the non-uniqueness of dual variables in methods that use optimality conditions constructively.

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CHAPTER 1

Introduction

This thesis presents a framework for the constrained nonlinear l_1 optimization problem:

$$\underset{x\varepsilon\mathbf{R}^n}{\textit{minimize}} \sum_{i=1}^{m} |f_i(x)| \qquad \text{(CNLL1)}$$

$$\textit{subject to } f_i(x) = 0\,,\ i\varepsilon\mathbf{I}_{EQ}$$

$$f_i(x) \geq 0\,,\ i\varepsilon\mathbf{I}_{IN}$$

where, the f_i's are twice continuously differentiable functions mapping $\mathbf{R}^n$ to $\mathbf{R}^1$.

The l_1 problem arises in the robust analysis of experimental data. In that context let f_i, $i\varepsilon\{1, ..., m\}$ be defined as

$$f_i(x) = h(t_i,x) - y_i \qquad (1.1)$$

Here the y_i's are considered to be observations at time t_i, and $h(t,x)$ is a model with the unknown parameter vector $x\varepsilon\mathbf{R}^n$.

When the data y_i are good, i.e. have normally distributed small errors, the traditional method of data fitting by minimizing

$$\frac{1}{2}\sum_{i=1}^{m}(h(t_i,x) - y_i)^2 \qquad (1.2)$$

is ideal. But when a small portion of data is contaminated by bad values the least-squares technique (1.2) can be quite inadequate. On the other hand the l_1 technique will often ignore that small portion of bad data.

A simple linear example from Bartels and Conn [1] will suffice to illustrate that fact. Let

$$h(t,x) = x_1 + tx_2\,,$$

and consider the data that roughly correspond to $h(t,x)$ with $x_1 = 0$ and $x_2 = 1$.

t	y
1	0.75
2	2.00
3	3.00
4	4.25
5	4.75
6	6.50
7	7.25
8	0.00

It is obvious that the y value for $t = 8$ is wild. The l_1 estimation of x for this data is $\bar{x}_1 = -0.1875$, $\bar{x}_2 = 1.0625$ which is quite reasonable. The least squares estimation of x for this data is $\hat{x}_1 = +1.848$, $\hat{x}_2 = +0.381$. The two lines are

$$y_1 = -0.1875 + 1.0625t$$

$$y_2 = 1.848 + 0.381t.$$

Figure 1.1 shows the l_1 line in solid and the l_2 line in dashes. Note that the l_1 line ignores the eighth value and reproduces a good model based on the first seven values. The least squares line tries to average the error in the model and gets influenced by the large error for the eighth value. So, the computed least squares model has no reasonable agreement with any portion of the data.

Recently, two algorithm for nonlinear l_1 optimization, based on an active set strategy, have appeared in the literature. Bartels and Conn [1] proposed extending their algorithm for the linear l_1 problem, Bartels and Conn [2] , to nonlinear problems. The exact penalty approach of Coleman and Conn [9,10] was used as a theoretical foundation. Independently, Murray and Overton [30] managed to transform the l_1 problem into a nonlinearly constrained one without nondifferentiable parts.

Both algorithms faired well on nondegenerate problems but had substantial difficulties with degenerate problems, i.e. the problems where so many $f_i's$ were close to zero (hence, initially regarded as active in the neighborhood of the solution) and where consequently their gradients were linearly dependent. Unfortunately, l_1 data fitting problems generally have the property that many functions are zero or nearly so at a solution. So, a good way of handling degeneracy is a necessary building block for l_1 algorithms.

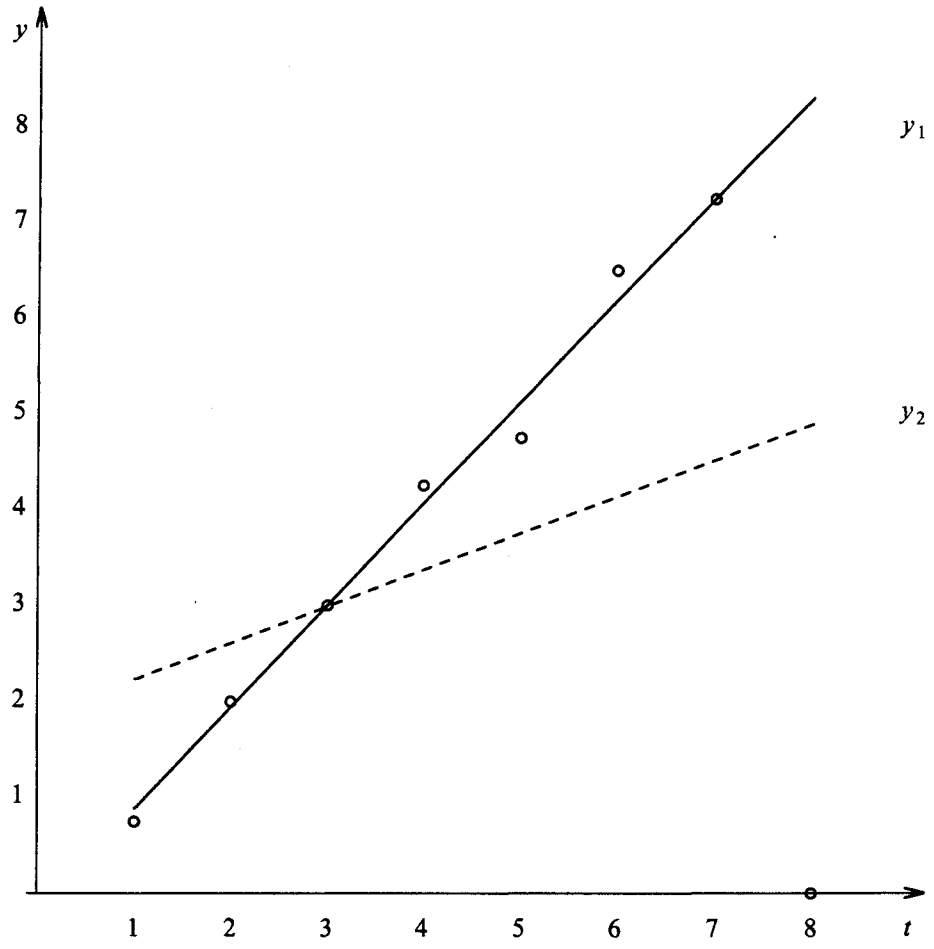

Figure 1.1

The purpose of this thesis is to propose a way to handle degeneracy in a nonlinear l_1 optimization algorithm. The nonlinear l_1 algorithm of Bartels and Conn [1] is modified to cope with degeneracy and the results of numerical tests are presented. The basic ideas appear to be applicable to Murray and Overton's algorithm [30]. Moreover, the same ideas could be applied in any method that uses optimality conditions constructively.

Chapter 2 introduces notation and mathematical background. A framework for the nonlinear l_1 optimization problem is presented in Chapter 3. This chapter is intended to be motivational rather than formal. The theoretical material resumes in Chapter 4 which deals with

degeneracy. In Chapter 5, the Bartels/Conn algorithm for nonlinear l_1 optimization is extended to cope with degeneracy. (The proofs of convergence are given.) Chapter 6 shows how the Murray/Overton algorithm for nonlinear l_1 optimization might be extended to cope with degeneracy; no convergence proofs are given. Numerical results for the extended Bartels/Conn algorithm are given in Chapter 7. Concluding remarks are given in Chapter 8. Appendices I and II suggest alternative ways of resolving degeneracy at a stationary point.

CHAPTER 2

Preliminaries

2.1. Notation and mathematical background

$\mathbf{R}^n$ will denote the n-dimensional real vector space. Vectors will be represented with lowercase letters and are assumed to be column vectors, unless superscripted with a transpose symbol T. (So, x is a column vector and x^T is a row vector.)

Matrices will be represented by capital letters.

A matrix $B_{n\times n}$ is *positive definite* if $x^TBx > 0$ for all $x \varepsilon \mathbf{R}^n$.

We will consider algorithms that produce a sequence of points in $\mathbf{R}^n$, $x^1, x^2, \cdots, x^i, \cdots$ converging to a point $x^* \varepsilon \mathbf{R}^n$. We will use definitions of Q-rate of convergence given in Ortega and Rheinboldt [32]. A sequence of points $\{x^k\}$ converging to x^* is said to converge

1. *linearly*, if there is a constant C, $0 < C < 1$, and an integer $k_0 \geq 1$, such that

$$\| x^{k+1} - x^* \| \leq C \| x^k - x^* \|, \text{ for all } k \geq k_0;$$

2. *superlinearly*, if

$$\lim_{k\to\infty} \frac{\| x^{k+1} - x^* \|}{\| x^k - x^* \|} \to 0,$$

3. *quadratically*, if there is a constant C, $C > 0$, and an integer $k_0 \geq 1$, such that

$$\| x^{k+1} - x^* \| \leq C \| x^k - x^* \|^2, \text{ for all } k \geq k_0;$$

4. *2−step superlinearly*, if

$$\lim_{k\to\infty} \frac{\| x^{k+1} - x^* \|}{\| x^{k-1} - x^* \|} \to 0,$$

5. *2−step quadratically*, if there is a constant C, $C > 0$, and an integer $k_0 \geq 1$, such that

$$\| x^{k+1} - x^* \| \leq C \| x^{k-1} - x^* \|^2, \text{ for all } k \geq k_0;$$

The following norms will be used:

$$\| x \|_2 = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}}, \quad (2-norm),$$

$$\| x \|_1 = \sum_{i=1}^{n} | x_i |, \quad (1-norm),$$

$$\| x \|_\infty = \max_{1 \le i \le n} \{ | x_i | \}, \quad (infinity-norm).$$

All functions used in the thesis are assumed to be *twice continuously differentiable.* If f is a function mapping $\mathbf{R}^n \rightarrow \mathbf{R}^1$, then the gradient of f at x is a column vector

$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \cdots, \frac{\partial f(x)}{\partial x_n} \right)^T$$

The Hessian of f at x is an $n \times n$ symmetric matrix

$$G(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \right)$$

(The argument x will be omitted whenever there is no ambiguity)

Through out this thesis the *little* o notation will be used:

$g(\alpha) = o(h(\alpha))$ means $\frac{g(\alpha)}{h(\alpha)} \rightarrow 0$ *as* $\alpha \rightarrow 0$.

The Taylor expansions of a twice continuously differentiable function will be used extensively in this work and with that in mind, we will state a few forms of the expansion.

The Taylor expansion of f up to the first-order term can be written as

$$f(x) = f(y) + \nabla f(\xi)^T (x-y),$$

where

$$\xi = y + \theta(x-y), \text{ for some } 0 \le \theta \le 1, \text{ and } x, y \in \mathbf{R}^n.$$

Alternatively, one may write

$$f(x) = f(y) + \nabla f(x)^T (x-y) + o(\| x-y \|_2).$$

Similarly, the Taylor expansion of f up to the second-order term is

$$f(x) = f(y) + \nabla f(\xi)^T (x-y) + \frac{1}{2}(x-y)^T G(\xi)(x-y),$$

where

$\xi = y + \theta(x-y)$, for some $0 \leq \theta \leq 1$, and $x, y \varepsilon \mathbf{R}^n$

We can also write

$$f(x) = f(y) + \nabla f(x)^T(x-y) + \frac{1}{2}(x-y)^T G(x)(x-y) + o(\| x-y \|_2^2).$$

2.2. Optimality conditions for the constrained nonlinear problem

Some basic definitions and results from optimization theory will be stated here.

Let us consider the constrained nonlinear minimization problem

$$\underset{x \varepsilon \mathbf{R}^n}{\textit{minimize}}\ f(x) \tag{CNP}$$

$$\textit{subject to}\ \ f_i(x) = 0,\ i \varepsilon I_{EQ} \tag{2.2.1}$$

$$f_i(x) \geq 0,\ i \varepsilon I_{IN} \tag{2.2.2}$$

where functions f_i are twice continuously differentiable mappings of $\mathbf{R}^n$ to $\mathbf{R}^1$, and the sets I_{EQ} and I_{IN} are mutually disjoint.

A *feasible point*, x, for the problem (CNP) is a point which satisfies (2.2.1) and (2.2.2).

The *active set* at any point x is

$$I_A(x) = \{i \mid f_i(x) = 0 \text{ and } i \varepsilon I_{EQ} \bigcup I_{IN}\},$$

and the *violated set* at any point x is

$$I_I(x) = \{i \mid f_i(x) \neq 0,\ i \varepsilon I_{EQ}\} \bigcup \{i \mid f_i(x) < 0,\ i \varepsilon I_{IN}\}.$$

Let us denote the *feasible region* of (CNP) by $\Omega = \{x \mid f_i(x) = 0,\ i \varepsilon I_{EQ} \text{ and } f_i(x) \geq 0,\ i \varepsilon I_{IN}\}$.

We say that x^* is a (*weak*) *global minimizer* of f on Ω if

$$f(x^*) \leq f(x) \text{ , for every } x \varepsilon \Omega,$$

and x^* is an *isolated global minimizer* of f on Ω if

$$f(x^*) < f(x) \text{ , for every } x \varepsilon \Omega, \text{ and } x \neq x^*.$$

A point x^* is said to be a (*weak*) *local minimizer* if there exists $\delta > 0$ such that

$$f(x^*) \leq f(x) \text{ , for all } x \varepsilon N(x^*, \delta) \bigcap \Omega,$$

where $N(x^*, \delta)$ is the $\delta-$*neighborhood* of x^*, i.e.

$$N(x^*,\delta) = \{x: \| x-x^* \| \leq \delta, x \neq x^*\}.$$

Similarly, x^* is an *isolated local minimizer* if

$$f(x^*) < f(x), \text{ for all } x \varepsilon N(x^*,\delta) \bigcap \Omega, \text{ and } x \neq x^*.$$

In general, algorithms that use just local information will be able to locate only local minimizers.

We require some results on optimality conditions given, for example, in Fiacco and McCormick [15].

Suppose that x is a feasible point of (CNP) and the problem functions, f and f_i's are differentiable. Let us define a set of directions that are first-order descent and feasible, i.e.

$$Z(x)=\{z \mid z^T \nabla f_i(x) \geq 0 \ (i \varepsilon I_{IN} \bigcap I_A), \ z^T \nabla f_i(x)=0 \ (i \varepsilon I_{EQ}), \ \textit{and} \ z^T \nabla f(x)<0 \}. \quad (2.2.3)$$

The most general *first−order* necessary conditions for a local minimizer have to insure that the set $Z(x^*)$ is empty. (The term first-order refers to the assumption of once continuous differentiability of the problem functions.)

In order to state the necessary conditions for (CNP), we have to define the Lagrange function, $L(x,\lambda,w)$, associated with (CNP):

$$L(x,\lambda,w) = f(x) - \sum_{i \varepsilon I_{IN}} \lambda_i f_i(x) + \sum_{i \varepsilon I_{EQ}} w_i f_i(x) \quad (2.2.4)$$

Theorem 2.2.1 If

(a) x^* satisfies the constraints of (CNP),

(b) the functions f and $f_i's$ are once differentiable, and

(c) at x^* the set $Z(x^*)$ is empty,

then it follows that there exist vectors λ^* and w^* such that (x^*,λ^*,w^*) satisfies

$$f_i(x) \geq 0, \ i \varepsilon I_{IN} \quad (2.2.5)$$

$$f_i(x) = 0, \ i \varepsilon I_{EQ} \quad (2.2.6)$$

$$\lambda_i f_i(x) = 0, \ i \varepsilon I_{IN} \quad (2.2.7)$$

$$\lambda_i \geq 0, \ i \varepsilon I_{IN} \quad (2.2.8)$$

$$\nabla L(x,\lambda,w) = 0. \quad (2.2.9)$$

■

In applying Theorem 2.2.1 one has to be able to determine if the set $Z(x^*)$ is empty, and that is a necessary and sufficient condition for the existence of the vectors (λ^*,w^*), called *Lagrange multipliers*.

Many conditions have been established as *constraint qualifications* that guarantee that the set $Z(x^*)$ is empty (see O.L. Mangasarian [25]) but only one among them, namely the linear independence of the gradients of the active functions, can be practically verified. Although all other ones are just of theoretical importance, we have to assume one of them, for example, the Kuhn-Tucker constraint qualification condition, in order to insure that the first-order necessary conditions hold at the minimizer.

Definition 2.2.2 [Kuhn-Tucker constraint qualification]

Let x^* be a point satisfying (2.2.1) and (2.2.2) and assume that the functions f_i are once continuously differentiable. Then a first-order constraint qualification holds at x^* if for any nonzero vector z, such that $z^T \nabla f_i(x^*) \geq 0$, $i \varepsilon I_{IN} \bigcap I_A$ and $z^T \nabla f_i(x^*) = 0$, $i \varepsilon I_{EQ}$, z is tangent to a once continuously differentiable arc emanating from x^* and contained in the constraint region Ω.

Now, the Kuhn-Tucker necessity theorem is a simple corollary of Theorem 2.2.1.

Theorem 2.2.2 [Kuhn-Tucker Necessity Theorem]

If the functions f and f_i, $i \varepsilon I_{EQ} \bigcup I_{IN}$, are once continuously differentiable at x^* and if the Kuhn-Tucker constraint qualification holds at x^*, then a necessary condition that x^* be a local minimizer of (CNP) is that there exist vectors λ^* and w^* such that (x^*,λ^*,w^*) satisfy (2.2.5)-(2.2.9). ■

Similarly, the second-order sufficiency conditions for a point to be a minimizer are based on the second-order constraint qualifications.

Definition 2.2.3 [Second-Order Constraint Qualification]

Let x^* be a feasible point of (CNP), and assume that the functions $\{f_i\}$, $i \varepsilon I_{EQ} \bigcup I_{IN}$ are twice continuously differentiable. A second-order constraint qualification holds at x^* if the following is true: Let y be any nonzero vector such that $y^T \nabla f_i(x^*) = 0$ for all $i \varepsilon I_A \bigcap I_{IN}$, and such that $y^T \nabla f_i(x^*) = 0$ for all $i \varepsilon I_{EQ}$. Then y is the tangent of an arc $\alpha(\theta)$, twice continuously differentiable, along which $f_i[\alpha(\theta)] = 0$, $i \varepsilon I_A \bigcap I_{IN}$ and $f_i[\alpha(\theta)] = 0$, $i \varepsilon I_{EQ}$, where $\theta \varepsilon [0,\varepsilon]$, $\varepsilon > 0$.

Theorem 2.2.4 [Second-Order Necessary Conditions]

If the functions f and $\{f_i\}$, $i \varepsilon I_{EQ} \bigcup I_{IN}$ are twice continuously differentiable and if the first- and second-order constraint qualifications hold at a point x^*, then necessary conditions that x^* be a local minimizer to (CNP) are that there exist vectors λ^* and w^* such that (2.2.5)-(2.2.9) hold and such that for every vector y, where $y^T \nabla f_i(x^*) = 0$ for all $i \varepsilon I_A \bigcap I_{IN}$ and $y^T \nabla f_i(x^*) = 0$, $i \varepsilon I_{EQ}$, it follows that

$$y^T \nabla^2 L(x,\lambda,w) y \geq 0. \tag{2.2.10}$$

■

Finally, second-order sufficiency conditions for an isolated local minimizer are given in the following theorem.

Theorem 2.2.4 [Second-Order Sufficiency Conditions]

Sufficient conditions that a point x^* be an isolated local minimizer of (CNP), where f and $\{f_i\}$ are twice continuously differentiable functions, are that there exist vectors λ^* and w^* such that the triple (x^*,λ^*,w^*) satisfies (2.2.5)-(2.2.9) and for every nonzero vector y satisfying $y^T \nabla f_i(x^*) \geq 0$, $i \varepsilon I_{IN}$; $y^T \nabla f_i(x^*) = 0$, for all $i \varepsilon \{i \mid \lambda_i^* > 0\} \bigcap I_{IN}$; and $y^T \nabla f_i(x^*) = 0$, $i \varepsilon I_{EQ}$; it follows that

$$y^T \nabla^2 L(x,\lambda,w) y > 0. \tag{2.2.11}$$

■

Further details are given in [15].

2.3. Optimality conditions for the l_1 problem

2.3.1. First-order necessary conditions for the l_1 problem

Let $f_i : R^n \rightarrow R$ be twice continuously differentiable; where $i \varepsilon \{1,...,m\}$. The unconstrained l_1 minimization problem is defined as

$$\underset{x \varepsilon \mathbf{R}^n}{minimize} \sum_{i=1}^{m} |f_i(x)| \tag{UNLL1}$$

This problem can be restated as

$$\underset{x \varepsilon \mathbf{R}^n,\, u \varepsilon \mathbf{R}^m,\, v \varepsilon \mathbf{R}^m}{minimize} \sum_{i=1}^{m} (u_i + v_i) \tag{P1}$$

$$\text{subject to } f_i(x) - u_i + v_i = 0 ; \quad i \varepsilon \{1,..., m\}$$

$$u_i \geq 0 , \quad i \varepsilon \{1,..., m\}$$

$$v_i \geq 0 , \quad i \varepsilon \{1,..., m\}.$$

The Lagrangian for (P1) can be written as:

$$L(x,u,v;\gamma,\mu,\lambda) = \sum_{i=1}^{m} (u_i + v_i) - \sum_{i=1}^{m} \gamma_i u_i - \sum_{i=1}^{m} \mu_i v_i + \sum_{i=1}^{m} \lambda_i (f_i(x) - u_i + v_i), \tag{2.3.1}$$

where,

$x \varepsilon R^n$; $\lambda,\mu,\gamma \varepsilon R^m$; and $u, v \varepsilon R^m$

The first-order necessary conditions for optimality of (P1) are:

1. $\nabla_{(x,u,v)} L(x,u,v;\gamma,\mu,\lambda) = 0$ (2.3.2)

2. $\gamma_i \geq 0$; $\mu_i \geq 0$; $i \varepsilon \{1,..., m\}$ (2.3.3)

3. $f_i(x) - u_i + v_i = 0$; $u_i \geq 0$; $v_i \geq 0$; $i \varepsilon \{1,..., m\}$ (2.3.4)

4. $\gamma_i u_i = 0$; $\mu_i v_i = 0$; $i \varepsilon \{1,..., m\}$. (2.3.5)

In our case, equation (2.3.2) gives:

$$\sum_{i=1}^{m} \lambda_i \nabla_x f_i(x) = 0 \tag{2.3.6}$$

$$1 - \gamma_i - \lambda_i = 0; \quad i \varepsilon \{1,..., m\} \tag{2.3.7}$$

$$1 - \mu_i + \lambda_i = 0; \quad i \varepsilon \{1,..., m\}. \tag{2.3.8}$$

Taking into account equalities (2.3.7) and (2.3.8, and the inequality (2.3.3), we get

$$-1 \leq \lambda_i \leq +1. \tag{2.3.9}$$

Let us assume that (x,u,v) is an optimal point for (P1). If for some i it is true that $u_i > 0$, then $v_i = 0$ and $f_i(x) = u_i$. Furthermore, condition (2.3.5) implies that $\gamma_i u_i = 0$, and that leads to $\gamma_i = 0$. Hence, equation (2.3.7) will give us $\lambda_i = 1$. Similarly, for $v_i > 0$, i.e. $f_i(x) < 0$ we get $\lambda_i = -1$.

Now, equality (2.3.6) can be written as:

$$\sum_{i \varepsilon I_I} \sigma_i \nabla_x f_i(x) + \sum_{i \varepsilon I_A} \lambda_i \nabla_x f_i(x) = 0 \tag{2.3.10}$$

where,

$$\sigma_i = sign(f_i(x)), I_I = \{i \,|\, f_i(x) \neq 0\}, \text{ and } I_A = \{i \,|\, f_i(x) = 0\}.$$

Finally, we can state the first-order necessary theorem for the l_1 nonlinear problem.

Theorem 2.3.1

Under a first-order constraint qualification, the first-order necessary conditions for optimality of (UNLL1) can be written as

$$\sum_{i \varepsilon I_I} \sigma_i \nabla_x f_i(x) + \sum_{i \varepsilon I_A} \lambda_i \nabla_x f_i(x) = 0 \tag{2.3.11}$$

and

$$-1 \leq \lambda_i \leq +1. \tag{2.3.12}$$

where,

$$\sigma_i = sign(f_i(x)), I_I = \{i \,|\, f_i(x) \neq 0\}, \text{ and } I_A = \{i \,|\, f_i(x) = 0\}.$$

This result follows immediately from the discussion above.

It is a simple exercise to show that the constrained nonlinear l_1 minimization problem:

$$\underset{x\varepsilon\mathbf{R}^n}{minimize} \sum_{i=1}^{m} |f_i(x)| \tag{CNLL1}$$

$$subject\ to\ f_i(x) = 0\ ,\ i\varepsilon\mathbf{I}_{EQ}$$

$$f_i(x) \geq 0\ ,\ i\varepsilon\mathbf{I}_{IN}$$

has the following first-order optimality conditions at a local minimizer $x\varepsilon\mathbf{R}^n$

$$\sum_{i\varepsilon I_{IO}} \sigma_i \nabla_x f_i(x) + \sum_{i\varepsilon I_{AO}} \lambda_i \nabla_x f_i(x) = 0 \tag{2.3.13}$$

$$-1 \leq \lambda_i \leq +1\ ,\ i\varepsilon I_{AO} \bigcap \{1, \cdots, m\} \tag{2.3.14}$$

$$f_i(x) \geq 0\ ,\ i\varepsilon I_{IN} \tag{2.3.15}$$

$$f_i(x) = 0\ ,\ i\varepsilon I_{EQ} \tag{2.3.16}$$

$$\lambda_i f_i(x) = 0\ ,\ i\varepsilon I_{IN} \tag{2.3.17}$$

$$\lambda_i \geq 0\ ,\ i\varepsilon I_{IN} \tag{2.3.18}$$

where,

$$\sigma_i = sign(f_i(x)),\ i\varepsilon\{1, \cdots, m\}\ ,$$

$$I_{IO} = \left\{ i \mid f_i(x) \neq 0\ ,\ i\varepsilon\{1, \cdots, m\} \right\},$$

$$I_{AO} = \left\{ i \mid f_i(x) = 0\ ,\ i\varepsilon\{1, \cdots, m\} \bigcup I_{EQ} \bigcup I_{IN} \right\},\text{and}$$

the Kuhn−Tucker constraint qualification is assumed.

2.3.2. Second-order optimality conditions for the l_1 problem

Second-order sufficiency conditions for the (UNLL1) problem are given in Charalambous [6], but as pointed by Ben-Tal and Zowe [3] the proof is incorrect. Although Ben-Tal and Zowe [3] derived second-order sufficiency conditions for the (UNLL1) problem in a quite general form, they needed the linear independence of gradients of the active functions to obtain a useful statement. In order to resolve this problem we have derived a proof for the second-order sufficiency conditions

which does not depend on the linear independence of gradients of the active functions.

In section 2.3.1, we restated the unconstrained l_1 minimization problem (UNLL1) as a non-linear problem (P1) and found the corresponding Lagrangian (2.3.1). According to theorem 2.2.4, sufficient conditions that a point $(x^{*T},u^{*T},v^{*T})^T$ be an isolated minimizer of (P1), where f_i are twice continuously differentiable functions, are that there exist vectors γ^*, μ^* and λ^* such that the tuple $(x^*,u^*,v^*;\gamma^*,\mu^*,\lambda^*)$ satisfies

1. $\nabla_{(x,u,v)}L(x,u,v;\gamma,\mu,\lambda) = 0$ (2.3.19)

2. $\gamma_i \geq 0;\ \mu_i \geq 0;\quad i\varepsilon\{1,\ldots, m\}$ (2.3.20)

3. $f_i - u_i + v_i = 0;\ u_i \geq 0;\ v_i \geq 0\ ;\quad i\varepsilon\{1,\ldots, m\}$ (2.3.21)

4. $\gamma_i u_i = 0;\ \mu_i v_i = 0\ ;\quad i\varepsilon\{1,\ldots, m\}$ (2.3.22)

for every nonzero vector $y = (y_x{}^T, y_u{}^T, y_v{}^T)^T$ satisfying

$$y^T\nabla_{(x,u,v)}u_i = y_{u_i} = 0,\ i\varepsilon\{i \mid \gamma_i>0\}, \tag{2.3.23}$$

$$y^T\nabla_{(x,u,v)}v_i = y_{v_i} = 0,\ i\varepsilon\{i \mid \mu_i>0\}, \tag{2.3.24}$$

$$y^T\nabla_{(x,u,v)}u_i = y_{u_i} \geq 0,\ i\varepsilon\{1,\ldots, m\}, \tag{2.3.25}$$

$$y^T\nabla_{(x,u,v)}v_i = y_{v_i} \geq 0,\ i\varepsilon\{1,\ldots, m\}, \text{ and} \tag{2.3.26}$$

$$y^T\nabla_{(x,u,v)}(f_i - u_i + v_i)(x^*,u^*,v^*) = y_x{}^T\nabla f_i(x^*) - y_{u_i} + y_{v_i} = 0,\ i\varepsilon\{1,\ldots, m\} \tag{2.3.27}$$

it follows that

$$y^T\nabla^2_{(x,u,v)}L(x,u,v;\gamma,\mu,\lambda)y > 0. \tag{2.3.28}$$

In order to derive the second-order sufficiency conditions for the original l_1 problem (UNLL1), let us state some relations between γ_i's, μ_i's and λ_i's.

Notice that from (2.3.19) we have

$$1 - \gamma_i - \lambda_i = 0;\quad i\varepsilon\{1,\ldots, m\} \tag{2.3.29}$$

$$1 - \mu_i + \lambda_i = 0;\quad i\varepsilon\{1,\ldots, m\}. \tag{2.3.30}$$

If we consider the relations (2.3.29), (2.3.30) and (2.3.20) we obtain the following:

a) $-1 \leq \lambda_i \leq +1,$

b) $-1 < \lambda_i < +1 \iff \gamma_i > 0$ and $\mu_i > 0,$

c) $\lambda_i = +1 \iff \gamma_i = 0$ and $\mu_i = 2,$

d) $\lambda_i = -1 \iff \mu_i = 0$ and $\gamma_i = 2.$

So, we need to consider only three cases:

Case 1 $-1 < \lambda_i < +1$ ($\gamma_i > 0$ and $\mu_i > 0$)

From the relations (2.3.23), (2.3.24) and (2.3.27) we have

$$y_{u_i} = 0, \tag{2.3.31}$$

$$y_{v_i} = 0, \text{ and} \tag{2.3.32}$$

$$y_x^T \nabla_x f_i(x^*) = 0. \tag{2.3.33}$$

Case 2 $\lambda_i = +1$ ($\gamma_i = 0$ and $\mu_i = 2$)

From the relations (2.3.25), (2.3.24) and (2.3.27) we have

$$y_{u_i} \geq 0, \tag{2.3.34}$$

$$y_{v_i} = 0, \text{ and} \tag{2.3.35}$$

$$y_x^T \nabla_x f_i(x^*) = y_{u_i} \geq 0. \tag{2.3.36}$$

Case 3 $\lambda_i = -1$ ($\gamma_i = 2$ and $\mu_i = 0$)

From the relations (2.3.23), (2.3.26) and (2.3.27) we get

$$y_{u_i} = 0, \tag{2.3.37}$$

$$y_{v_i} \geq 0, \text{ and} \tag{2.3.38}$$

$$y_x^T \nabla_x f_i(x^*) = -y_{v_i} \leq 0. \tag{2.3.39}$$

The special structure of the Lagrangian (2.3.1) will reduce (2.3.28) to

$$y_x^T \Big(\sum_{i \in I_I} \sigma_i \nabla_x^2 f_i(x^*) + \sum_{i \in i_A} \lambda_i \nabla_x^2 f_i(x^*) \Big) y_x > 0. \tag{2.3.40}$$

Finally, we can state the second-order sufficiency theorem for the l_1 nonlinear problem.

Theorem 2.3.2 [Sufficiency Conditions]

Let the functions $f_i(x)$, $i \varepsilon \{1, \cdots, m\}$ be twice continuously differentiable. Then x^* is a strong

local minimizer of (UNLL1) if there exist λ_i^* , $i \varepsilon I_A$ such that

$$\sum_{i \varepsilon I_I} \sigma_i \nabla_x f_i(x^*) + \sum_{i \varepsilon I_A} \lambda_i \nabla_x f_i(x^*) = 0 \tag{2.3.41}$$

$$-1 \leq \lambda_i \leq +1$$

where,

$$\sigma_i = sign(f_i(x)),\ I_I = \{i \,|\, f_i(x) \neq 0\},\ \text{and}\ I_A = \{i \,|\, f_i(x) = 0\},$$

and for every nonzero vector d satisfying

$$d^T \nabla f_i(x^*) = 0 \ , i \varepsilon \{i : |\lambda_i| < 1\} \tag{2.3.42}$$

$$d^T \nabla f_i(x^*) \geq 0 \ , i \varepsilon \{i : \lambda_i = 1\} \tag{2.3.43}$$

$$d^T \nabla f_i(x^*) \leq 0 \ , i \varepsilon \{i : \lambda_i = -1\}, \tag{2.3.44}$$

it follows that

$$d^T \left[\sum_{i \varepsilon I_I} \sigma_i \nabla^2 f_i(x) + \sum_{i \varepsilon I_A} \lambda_i \nabla^2 f_i(x) \right] d > 0. \tag{2.3.45}$$

■

This result follows immediately from the discussion above.

Following the same idea it is straightforward to derive the second-order necessary conditions for (UNLL1).

Theorem 2.3.3 [Necessary Conditions]

If the functions f and $\{f_i\}$ are twice continuously differentiable and if the first- and second-order constraint qualifications hold at a point x^*, then necessary conditions that x^* be a local minimizer to (UNLL1) are that there exist λ_i^* , $i \varepsilon I_A$ such that

$$\sum_{i \varepsilon I_I} \sigma_i \nabla_x f_i + \sum_{i \varepsilon I_A} \lambda_i \nabla_x f_i = 0 \tag{2.3.46}$$

$$-1 \leq \lambda_i \leq +1 \tag{2.3.47}$$

where,

$$\sigma_i = sign(f_i(x)),\ I_I = \{i \,|\, f_i(x) \neq 0\},\ \text{and}\ I_A = \{i \,|\, f_i(x) = 0\}, \tag{2.3.48}$$

and for every nonzero vector d satisfying

$$d^T \nabla f_i(x^*) = 0 \ , i\varepsilon\{i: |\lambda_i| < 1\} \tag{2.3.49}$$

$$d^T \nabla f_i(x^*) \geq 0 \ , i\varepsilon\{i:\lambda_i = 1\} \tag{2.3.50}$$

$$d^T \nabla f_i(x^*) \leq 0 \ , i\varepsilon\{i:\lambda_i = -1\}, \tag{2.3.51}$$

it follows that

$$d^T \left[\sum_{i\varepsilon I_I} \sigma_i \nabla^2 f_i(x) + \sum_{i\varepsilon I_A} \lambda_i \nabla^2 f_i(x) \right] d \geq 0. \tag{2.3.52}$$

CHAPTER 3

Framework for the l_1 problem

3.1. Introduction

We have defined the constrained nonlinear l_1 minimization problem to be

$$\underset{x\varepsilon\mathbf{R}^n}{minimize} \sum_{i=1}^{m} |f_i(x)| \qquad \text{(CNLL1)}$$

$$subject\ to\ f_i(x) = 0\ ,\ i\varepsilon\mathbf{I}_{EQ}$$

$$f_i(x) \geq 0\ ,\ i\varepsilon\mathbf{I}_{IN}$$

where functions f_i are twice continuously differentiable mappings of $\mathbf{R}^n$ to $\mathbf{R}^1$, and the sets $\{1, \cdots, m\}$, I_{EQ} and I_{IN} are mutually disjoint.

The objective function of the (CNLL1) problem is not differentiable at a point where $f_i = 0$, for any $i\varepsilon\{1, \cdots, m\}$, and that fact prevents us from directly applying standard methods for nonlinear optimization. The problem (CNLL1) can be viewed as minimizing a piecewise-differentiable objective function subject to differentiable constraints.

Instead of reformulating (CNLL1) by introducing new variables, as is done in the work of Murray and Overton [30], we will note that by forming an exact penalty function

$$\Phi(x) = \sum_{i=1}^{m} \mu_i |f_i| + \sum_{i\varepsilon\mathbf{I}_{EQ}} |f_i| - \sum_{i\varepsilon\mathbf{I}_{IN}} min(0, f_i) \qquad (3.1.1)$$

$$\mu_i > 0$$

we will get a uniform treatment of all functions defining the problem. That idea was introduced by Bartels and Conn [2] in their algorithm for linear l_1 minimization.

From now on we proceed to solve the following problem

$$\underset{x\in\mathbf{R}^n}{\textit{minimize}}\ \Phi(x). \tag{EPP}$$

The exact penalty function $\Phi(x)$ will be our natural merit function, i.e. we will try to construct a sequence of points $\{x^k\}$ such that

$$\Phi(x^{k+1}) < \Phi(x^k) \tag{3.1.2}$$

and $x^k \rightarrow x^*$, $k \rightarrow \infty$, where x^* is a local minimum of (EPP). Also, our goal is to achieve at least superlinear convergence in the final phase of the algorithm.

Let us simplify notation by defining

$$\bar{f}_i(x) = \begin{cases} \mu_i f_i(x) & , i\in\{1,\cdots,m\} \\ f_i(x) & , i\in I_{EQ} \\ f_i(x) & , i\in I_{IN} \end{cases} \tag{3.1.3}$$

and

$$I_E = \{1,\cdots,m\} \bigcup I_{EQ}. \tag{3.1.4}$$

Then (EPP) becomes

$$\Phi(x) = \sum_{i\in I_E} |\bar{f}_i(x)| - \sum_{i\in I_{IN}} min(0,\bar{f}_i(x)) \tag{3.1.5}$$

As mentioned in Chapter 2, the set of $\bar{f}_i$'s which are zero at any given point, i.e. the *active set*, will be of special importance. For a given x let

$$I_A(x) = \{i \mid \bar{f}_i(x) = 0 \text{ and } i\in I_E \bigcup I_{IN}\} \tag{3.1.6}$$

denote the set of active indices at x, and let

$$I_I(x) = I_E \bigcup \{i \mid \bar{f}_i(x) < 0 \text{ and } i\in I_{IN}\} - I_A(x) \tag{3.1.7}$$

represent the set of inactive indices.

We will define a matrix formed from the gradients of the active functions:

$$A(x) = \left[\cdots \nabla\bar{f}_i(x) \cdots\right]_{i\in I_A(x)}.$$

Only in this chapter we will assume that $A(x)$ has full rank. An orthogonal decomposition of $A(x)$ will be represented by

$$A(x)=\begin{bmatrix} W(x) & Z(x) \end{bmatrix}\begin{bmatrix} R(x) \\ 0 \end{bmatrix}. \tag{3.1.8}$$

The columns of $W(x)$ span the range-space of $A(x)$ and the columns of $Z(x)$ span the null-space of $A(x)^T$, i.e

$$Z(x)^T A(x) = 0. \tag{3.1.9}$$

For brevity of notation will often suppress the explicit reference to x; that is we will write A for $A(x)$, I_A for $I_A(x)$, f_i for $f_i(x)$, etc.

3.2. First-order change in the penalty function

By expanding all functions in (3.1.5) to the first-order about x we get

$$\Phi(x+\alpha p) = \Phi(x) \tag{3.2.1}$$

$$+ \alpha g^T p \tag{3.2.2}$$

$$+ \alpha\left[\sum_{i\varepsilon I_{AE}} | \nabla \bar{f}_i{}^T p | - \sum_{i\varepsilon I_{AI}} min(0, \nabla \bar{f}_i{}^T p)\right] \tag{3.2.3}$$

$$+ o(\alpha)$$

where

p is an arbitrary direction,

$\alpha \geq 0$ and small enough,

$$I_{AE} = I_A \bigcap I_E \; ; \; I_{AI} = I_A \bigcap I_{IN}, \tag{3.2.4}$$

$$g = \sum_{i\varepsilon I_E} \sigma_i \nabla \bar{f}_i + \sum_{i\varepsilon I_{IN}} \sigma_i^- \nabla \bar{f}_i, \tag{3.2.5}$$

$$\sigma_i = \begin{cases} +1 & , \text{if } \bar{f}_i(x) > 0 \\ 0 & , \text{if } \bar{f}_i(x) = 0 \text{ , } i\varepsilon I_E \\ -1 & , \text{if } \bar{f}_i(x) < 0 \end{cases} \tag{3.2.6}$$

and

$$\sigma_i^- = \begin{cases} 0 & , \text{if } \bar{f}_i(x) \geq 0 \\ -1 & , \text{if } \bar{f}_i(x) < 0 \end{cases}, \; i \varepsilon I_{IN} \tag{3.2.7}$$

Formula (3.2.1) is similar to the first-order Taylor expansion, but in addition to the differentiable part (3.2.2) it also has the non-differentiable part (3.2.3). Bartels and Conn [2] pointed out that, in the case of linear functions, the differentiable part can be viewed as the gradient of Φ restricted to the null-space of A^T. The same remark is valid for the nonlinear case, and the vector g will be called the *restricted gradient* of Φ.

Instead of choosing an arbitrary vector p we would like to find a vector that will give us a direction of descent for Φ. The following discussion reflects that given in [2].

There are three cases to consider.

Case 1 the orthogonal projection of g onto the null-space of A^T is not zero, i.e.

$$ZZ^T g \neq 0 \tag{3.2.8}$$

Then the projected restricted gradient

$$p = -ZZ^T g \tag{3.2.9}$$

is a descent direction, but so is

$$p = -Z(Z^T BZ)^{-1} Z^T g \tag{3.2.10}$$

where

$$B = \sum_{i \varepsilon I_E} \sigma_i \nabla^2 \bar{f}_i + \sum_{i \varepsilon i_{IN}} \sigma_i^- \nabla^2 \bar{f}_i \tag{3.2.11}$$

and $Z^T BZ$ is assumed positive definite. The direction (3.2.10) can be derived from the local quadratic model of Φ restricted to the null-space of A^T.

Each of these p's satisfies

$$g^T p < 0, \tag{3.2.12}$$

and for $\alpha > 0$ and small enough we have

$$\Phi(x + \alpha p) = \Phi(x) + \alpha g^T p + o(\alpha) \tag{3.2.13}$$

$$< \Phi(x) \tag{3.2.14}$$

A suitable α can be found by a line search algorithm for non-differentiable functions. See, for example, the work of Murray and Overton [28].

Case 2 The orthogonal projection of g onto the null-space of A^T is zero. Thus, the vector g is in the range-space of A, i.e.

$$g = A\lambda = \sum_{i \varepsilon I_{AE}} \lambda_i \nabla \bar{f}_i + \sum_{i \varepsilon I_{AI}} \lambda_i \nabla \bar{f}_i \tag{3.2.15}$$

for some coefficients λ_i. Clearly, λ is the vector of Lagrange multipliers, and the λ values can be obtained by solving

$$\underset{\lambda}{minimize} \quad \| A\lambda - g \|_2^2. \tag{3.2.16}$$

In order to handle this case it is convenient to substitute (3.2.15) into (3.2.2). The first-order change to $\Phi(x)$ is now given by

$$\alpha \left[\sum_{i \varepsilon I_{AE}} \lambda_i \nabla \bar{f}_i{}^T p + \sum_{i \varepsilon I_{AI}} \lambda_i \nabla \bar{f}_i{}^T p \right] + \alpha \left[\sum_{i \varepsilon I_{AE}} | \nabla \bar{f}_i{}^T p | - \sum_{i \varepsilon I_{AI}} \min(0, \nabla \bar{f}_i{}^T p) \right] \tag{3.2.17}$$

By combining those two terms, we can write the change to $\Phi(x)$ as

$$\alpha \left[\sum_{i \varepsilon I_{AE}} \left\{ \lambda_i \pi_i + 1 \right\} | \nabla \bar{f}_i{}^T p | + \sum_{i \varepsilon I_{AI}} \left\{ \lambda_i + \pi_i^- \right\} \nabla \bar{f}_i{}^T p \right] \tag{3.2.18}$$

where

$$\pi_i = \begin{cases} +1 & , \text{if } \nabla \bar{f}_i(x)^T p > 0 \\ 0 & , \text{if } \nabla \bar{f}_i(x)^T p = 0 , \\ -1 & , \text{if } \nabla \bar{f}_i(x)^T p < 0 \end{cases} \quad i \varepsilon I_E \tag{3.2.19}$$

and

$$\pi_i^- = \begin{cases} 0 & \text{if } \nabla \bar{f}_i(x)^T p \geq 0 \\ -1 & \text{if } \nabla \bar{f}_i(x)^T p < 0 \end{cases}, \; i \varepsilon I_{IN}. \tag{3.2.20}$$

Now, it is easy to verify that the first-order change to $\Phi(x)$ will be nonnegative for all $\alpha \geq 0$ and all choices of $p \neq 0$ if the following conditions hold

$$-1 \leq \lambda_i \leq +1 \quad \text{for} \quad i \varepsilon I_{AE} \tag{3.2.21}$$

$$0 \leq \lambda_i \leq +1 \quad \text{for} \quad i \varepsilon I_{AI}. \tag{3.2.22}$$

That means that our model cannot provide us any more with a first-order descent direction. Second-order information should be used to verify whether the point x is an optimizer.

Note that we have just derived the first-order optimality conditions for (3.1.1):

$$\begin{aligned} A(x)\lambda - g(x) &= 0 \\ \bar{f}_i(x) &= 0 \,, \; i \varepsilon I_A \\ -1 \leq \lambda_i \leq +1 \quad &\text{for} \quad i \varepsilon I_{AE} \\ 0 \leq \lambda_i \leq +1 \quad &\text{for} \quad i \varepsilon I_{AI}. \end{aligned} \tag{3.2.23}$$

Case 3 If the conditions (3.2.21) or (3.2.22) are violated, let us say for some λ_i, then we will refer to the current x as a *non-optimal stationary point.* We would like to find a direction p that will allow us to descend from x and at the same time change the active set. The direction obtained as a solution to

$$A^T p = -sign(\lambda_i) e_i \tag{3.2.24}$$

will remove the i^{th} function from the active set and will also give us a descent direction. (e_i is the vector having zero components everywhere except one at the i^{th} position.)

It is a straightforward exercise to show that the direction p from (3.2.24) is a descent direction for Φ, for $\alpha > 0$, i.e.

$$\begin{aligned} \Phi(x + \alpha p) &= \Phi(x) + \alpha \omega_i + o(\alpha) \\ &< \Phi(x), \end{aligned} \tag{3.2.25}$$

where

$$\omega_i = \begin{cases} (\lambda_i \pi_i + 1) \,, & \text{if} \quad i \varepsilon I_{AE} \\ (\lambda_i + \pi_i^-)(-sign(\lambda_i)) \,, & \text{if} \quad i \varepsilon I_{AI}. \end{cases} \tag{3.2.26}$$

Again, a suitable α for (3.2.25) to hold will be found by a line search in the direction p.

3.3. Local change in a neighborhood of the optimal point

So far we know how to find a direction of descent for $\Phi(x)$ far away from a local minimizer and at a stationary point which is not a minimizer. The only outstanding problem is how to proceed when the current point x is close to a local minimizer. We would like to solve the system of nonlinear equations (3.2.23) to obtain a local minimizer, using a solution process which has a higher than linear rate of convergence.

For this reason, an algorithm for nonlinear optimization based on the idea of solving the nonlinear equations (3.2.23) by the Newton method will be discussed in the next section. Then, some well-known, and successfully implemented, algorithms will be related to that *Newton−like* method.

3.3.1. Newton-like algorithm for nonlinear optimization

Suppose that we have a pair (x,λ) that is close to a solution (x^*,λ^*) of

$$g(x) - A(x)\lambda = 0 \tag{3.3.1}$$

$$\bar{f}_A(x) = 0, \tag{3.3.2}$$

where

$$\bar{f}_A(x) = \left[\cdots \bar{f}_i(x) \cdots \right]^T_{i\varepsilon I_A}, \tag{3.3.3}$$

and we want to solve (3.3.1)-(3.3.2) by holding λ fixed, making a small step p from x in order to satisfy (3.3.1)-(3.3.2), and then estimating λ at $x+p$. Let $p=v+h$, where $v \varepsilon$ *Span*$(A(x))$ and $h\varepsilon$ *Span*$(Z(x))$ = *Null Space of* $A(x)^T$. Then we have

$$g(x+v+h) - A(x+v+h)\lambda = 0 \tag{3.3.4}$$

$$\bar{f}_A(x+v+h) = 0 \tag{3.3.5}$$

If we use the Taylor expansion of (3.3.4)-(3.3.5) we get

$$g(x)+B[v+h] - A(x)\lambda - \left[\sum_{i\varepsilon I_A} \lambda_i \nabla^2 \bar{f}_i(x)\right][v+h] = 0 \tag{3.3.6}$$

$$\bar{f}_A(x) + A(x)^T[v+h] = 0,$$

which is satisfied to $o(||v+h||)$.

By rearranging (3.3.6) and taking in account that $A(x)^T h=0$ we get

$$g(x)-A(x)\lambda+[B-\sum_{i\varepsilon I_A}\lambda_i\nabla^2 f_i(x)][v+h] = 0 \tag{3.3.7}$$

$$\bar{f}_A(x)+A(x)^T v = 0. \tag{3.3.8}$$

One way of solving the system (3.3.7)-(3.3.8) is to assume that the matrix $A(x)$ is full rank and obtain v according to

$$v = -A(x)[A(x)^T A(x)]^{-1}\bar{f}_A(x). \tag{3.3.9}$$

Vector h can be expressed as a linear combination of columns of Z, i.e. $h=Zh_Z$.

Assuming that the projected Hessian

$$Z^T[B-\sum_{i\varepsilon I_A}\lambda_i\nabla^2 f_i(x)]Z$$

is positive definite we get

$$h = -Z[Z^T(B-\sum_{i\varepsilon I_A}\lambda_i\nabla^2 f_i(x))Z]^{-1}Z^T\{g(x)+(B-\sum_{i\varepsilon I_A}\lambda_i\nabla^2 f_i(x))v\} \tag{3.3.10}$$

Then, the new x, x^+, is obtained as

$$x^+=x+v+h, \tag{3.3.11}$$

and the new λ is obtained by solving

$$\underset{\lambda}{\textit{minimize}} \quad ||A(x+v+h)\lambda - g(x+v+h)||_2^2. \tag{3.3.12}$$

In summary, the method can be described as

1. Compute v by (3.3.9)
2. Compute h by (3.3.10)
3. Update x by (3.3.11)
4. Compute λ by (3.3.12).

Although the way of computing a new λ (3.3.12) made the overall algorithm something less than the Newton method for solving (3.3.1)-(3.3.2), the quadratic convergence is retained if the initial point is sufficiently close to the solution. M. H. Wright [35] established that the quadratic con-

vergence of the Newton method in (x,λ) will not be impaired if λ is computed by (3.3.12). R. A. Tapia [34] showed that using (3.3.12) we get a method that converges Q-quadratically in x, according to Ortega and Rheinboldt's [32] definition. (See the work by Nocedal and Overton [31] for a detailed discussion on Newton-like methods.)

3.3.2. Some algorithms that are superlinearly convergent

Murray and Overton [30] use (3.3.9), (3.3.10) (with $B = \sum_{i \varepsilon I_E} \sigma_i \nabla^2 \bar{f}_i$) and (3.3.12). (Notice that they consider only the unconstrained l_1 problem.) Conn and Coleman [10] justify a slightly different v and h, i.e.

$$h = -Z[Z^T(B - \sum_{i \varepsilon I_A} \lambda_i \nabla^2 f_i(x))Z]^{-1} Z^T g(x) \tag{3.3.13}$$

$$v = -A(A^TA)^{-1} \bar{f}_A(x+h). \tag{3.3.14}$$

They prove the 2-step superlinear convergence of their method.

3.4. Framework

The discussion in the two preceding sections suggests a framework for solving the nonliner l_1 optimization problem (CNLL1). The crucial question is how well satisfied at x are the first-order necessary conditions for (3.1.5):

$$A(x)\lambda - g(x) = 0 \tag{3.4.1}$$

$$\bar{f}_i(x) = 0\ ,\ i \varepsilon I_A \tag{3.4.2}$$

$$-1 \leq \lambda_i \leq +1 \quad \text{for } i \varepsilon I_{AE} \tag{3.4.3}$$

$$0 \leq \lambda_i \leq +1 \quad \text{for } i \varepsilon I_{AI}. \tag{3.4.4}$$

1) If $A\lambda - g \neq 0$, or equivalently $\| ZZ^Tg \|$ is large, the current point x is far from a stationary point, and we would like to attempt so-called *normal descent*. A descent direction like

$$p = -Z(Z^TBZ)^{-1}Z^Tg \tag{3.4.5}$$

where

$$B = \sum_{i \varepsilon I_E} \sigma_i \nabla^2 \bar{f}_i + \sum_{i \varepsilon i_{IN}} \sigma_i^- \nabla^2 \bar{f}_i, \tag{3.4.6}$$

derived from a local quadratic model and used in the Bartels and Conn algorithm [1] would be possible. Another choice would be the the direction

$$p = -Z(Z^T \tilde{B} Z)^{-1} Z^T g \tag{3.4.7}$$

where $\tilde{B}$ is given by

$$\tilde{B} = \sum_{i \varepsilon \{1, \cdots, m\}} \sigma_i \nabla^2 f_i - \sum_{i \varepsilon i_A} \lambda_i \nabla^2 f_i.$$

This direction is used by Murray and Overton [30], and is obtained from a local quadratic model, too. (Notice that they consider only the unconstrained l_1 problem (UNLL1).)

2) If $A(x)\lambda - g(x) \approx 0$, $\bar{f}_i(x) \approx 0$, $i \varepsilon I_A$ and some multiplier λ_i violates (3.4.3) or (3.4.4), the current point x is a stationary point, but it is not optimal. The active set is incorrectly chosen, and we would like to attempt so-called *dropping descent*. The direction p obtained as a solution to

$$A^T p = -sign(\lambda_i) e_i \tag{3.4.8}$$

will provide descent as well as removing the i^{th} function from the active set.

In order to find an α such that

$$\Phi(x + \alpha p) < \Phi(x) \tag{3.4.9}$$

a line search algorithm has to be used in 1) and 2). That could lead to linear convergence, but sufficient decrease obtained on the merit function Φ for every line search will force the algorithm eventually to reach a neighborhood of the local minimum x^*. Following the terminology of Coleman and Conn [9,10], and others we call 1) and 2) *the global phase* of the framework.

3) If all first-order necessary conditions (3.4.1)-(3.4.4) are approximately satisfied we should solve (3.4.1) and (3.4.2), i.e.

$$A(x)\lambda - g(x) = 0,$$

$$\bar{f}_i(x) = 0 \text{ , } i \varepsilon I_A.$$

In order to achieve a fast asymptotic convergence, a superlinearly convergent algorithm might be used. Following general terminology, again, we call part 3) *the asymptotic phase* of the framework.

l_1 framework

{Select any $x \varepsilon \mathbf{R}^n$. Determine tolerances}

repeat

{Identify I_A}

{Form g; compute the multipliers λ_i's}

if {(3.4.1)-(3.4.4) are approximately satisfied }

then

/* Asymptotic phase */

{Find p }

if {sufficient decrease in $\Phi(x)$ in moving from x to $x+p$}

then

$x \leftarrow x + p$

go to OUT

else

{Modify tolerances}

/* Global phase */

if {x is close to a non-optimal stationary point}

then

{Find a descent direction p that moves away from the stationary point}

go to ALPHA

{Find the normal descent direction p}

ALPHA:

{Find α}

$x \leftarrow x + \alpha p$

OUT:

until {termination test satisfied or $\Phi(x)$ does not decrease}

The two previously mentioned algorithms for solving (CNLL1), [1] and [30], can easily fit in this framework, but those are not the only possibilities. In the asymptotic phase, we can use any other superlinearly convergent algorithm, and moreover we could use an algorithm based on updating the projected Hessian

$$Z^T\left[\sum_{i\varepsilon I_E}\sigma_i\nabla^2\bar{f}_i + \sum_{i\varepsilon I_{IN}}\sigma_i^-\nabla^2\bar{f}_i - \sum_{i\varepsilon I_A}\lambda_i\nabla^2\bar{f}_i\right]Z. \tag{3.4.10}$$

See a paper by Nocedal and Overton [31] in this regard.

3.5. Reducing the penalty parameters

If the selected direction p fails to decrease the penalty function or if a reasonable termination criterion is satisfied we conclude that the minimum of Φ has been found. Now, we test whether the computed x^* is feasible for (CNLL1). If it is, we take x^* as a solution to (CNLL1). In the case that x^* is infeasible for (CNLL1) we have three possibilities:

a) the penalty constants μ_i, $i\varepsilon\{1, ..., m\}$ are too large,

b) the original problem (CNLL1) is infeasible, or

c) a minimizer of the penalty function that is not a minimizer of the original problem (CNLL1) has been found.

To distinguish case a) from cases b) and c) we need to reduce μ_i, $i\varepsilon\{1, ..., m\}$. If we have to do it repeatedly and μ_i, $i\varepsilon\{1, ..., m\}$ becomes close to the machine precision we can only conclude that the original problem is infeasible or a minimizer of the penalty function that is not a minimizer of the original problem (CNLL1) has been found. (See Coleman and Conn [8] for an example of such a situation.)

The above discussion suggests the following algorithm for reducing penalty parameters:

Reducing μ

{Select any $\mu_i > 0$, $i \varepsilon \{1, ..., m\}$}

repeat

{Apply l_1 framework}

{Reduce μ_i, $i \varepsilon \{1, ..., m\}$}

until {feasible or μ_i, $i \varepsilon \{1, ..., m\}$ too small}

CHAPTER 4

Degeneracy

4.1. Introduction

The preceding chapter provided us with some terminology and a framework upon which to base further discussions but with no rigorous results.

In this chapter we will remove the assumption that the gradients of the active functions are linearly independent and attempt to develop out framework into something more carefully constructed that will admit dependencies in these gradients. A point where the matrix, A, formed from the gradients of active functions, has linearly dependent columns is called a *degenerate point*. A degenerate point can be stationary or nonstationary. The optimality conditions are the same as for a nondegenerate problem but much of the discussion in Chapter 3 breaks down.

More specifically:

(1) We can no longer determine the vector of Lagrange multipliers uniquely from

$$\underset{\lambda}{minimize} \quad \| A\lambda - g \|_2^2.$$

(2) If a stationary point is degenerate we can no longer use

$$A^T p = -sign(\lambda_i) e_i$$

as a dropping descent.

(3) If a degenerate point is encountered in the asymptotic phase we will have problems computing both null-space and range-space components h and v.

(4) The Hessian used in computing the null-space component by (3.3.13) or (3.3.10) is defined in terms of Lagrange multipliers, and as already mentioned they are not uniquely determined.

In the context of the nonlinear programming, we know of almost no work done to address these questions directly. The practical resolution of degeneracy appears to have been attempted up to now

in the following ways:

1. ignore the degeneracy, i.e. assume it will never happen but implement some sort of safeguard to recognize (and possibly cope with) attempts to solve a singular system of equations; (done in many nonlinear optimization algorithms);
2. try to select a working set, i.e. a largest set of active constraints whose gradients are linearly independent and work just with it; (done in the algorithm of Murray and Overton [30]);
3. perturb the active constraints randomly and then find a descent direction for the new problem. (done in the algorithm of Bartels and Conn [1]).

None of those techniques proves to be robust for the nonlinear l_1 problems for the following reasons:

1. The problem of degeneracy in an l_1 code cannot be ignored. It happens too often in the context of data fitting. For example, the well known test problem given by Wood [26], has seven problem functions in four variables and all seven of them are zero at the solution. Moreover, even if a large number of exactly zero functions are not encountered at an optimal point, good data-fitting models will often yield problem-function values (residuals) very close to zero at optimality, so that large numbers of problem functions will be interpreted as active under any reasonable computational tolerance.
2. So far there is no way to select a correct working set without trying all possible combinations, and that can be exceedingly expensive. If the working set is chosen incorrectly the algorithm can stall by trying to move in the direction that is descent for the working set but ascent for the original problem. Large numbers of cycles can be wasted before a direction determined from a working set proves to be an actual descent direction or before an indication of optimality is attained. Furthermore, some record-keeping mechanism for selecting subsets should be included in the algorithm to prevent the repetition of working sets, which increases the programming complexity of the algorithm.
3. The perturbation technique is quite successful in resolving the degeneracy for mildly nonlinear constrained problems. In the case of highly nonlinear constraints the degeneracy still

can be resolved but at possible cost of such outcomes as:

a) the optimizer of the perturbed problem being arbitrarily far away from the current point,

b) the final active set of the perturbed problem being totally different from the current one,

c) the optimizer of the perturbed problem being strongly infeasible for the original problem,

d) the perturbed problem yielding an optimizer although the current point of the original problem had not been optimal.

The extent to which the solution to the perturbed problem can be reinterpreted, possibly by some form of continuation method, to yield information about the current point of the original problem deserves further study.

It has been recognized that a sound way of handling degeneracy in a nonlinear l_1 programming is a necessary building block for a robust l_1 algorithm. We will propose an alternative way of resolving degeneracy in the context of the nonlinear l_1 framework, but we believe the proposed resolution is more general. For example,

a) the resolution of degeneracy at a stationary point could be considered for use in any algorithm for continuous optimization that uses optimality conditions constructively.

b) the resolution of degeneracy in the asymptotic phase could be considered for use in any Newton-like algorithm.

4.2. Resolving of degeneracy at a stationary point

Suppose that at a point x the matrix A has linearly dependent columns. Then a vector of Lagrange multipliers, λ, cannot be uniquely determined from:

$$\underset{\lambda}{minimize} \quad \| A\lambda - g \|_2^2. \tag{4.2.1}$$

But, the necessary conditions for the optimality of a nonlinear l_1 minimization problem state that there should exist a vector of Lagrange multipliers, λ, that satisfies:

$$-1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE} \tag{4.2.2}$$

$$0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI}$$

At a degenerate point we can combine (4.2.1) and (4.2.2) and solve the following constrained minimization problem:

$$\underset{\lambda}{minimize} \quad \| A\lambda - g \|_2^2 \tag{4.2.3}$$

$$subject\ to \quad -1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI}$$

If the solution to (4.2.3) has zero residual we know from Section 3.2 that we have satisfied the first-order necessary conditions for optimality. Otherwise, we can use the following theorem to find a descent direction for $\Phi(x)$.

Theorem 4.1 Let λ^* be a solution to (4.2.3). If

$$\| A\lambda^* - g \|_2^2 \neq 0$$

then

$$p = A\lambda^* - g \tag{4.2.4}$$

is a descent direction for the exact penalty function $\Phi(x)$.

Proof: Let us recall that the first-order expansion of the exact penalty function $\Phi(x)$ is given by

$$\Phi(x+\alpha p) = \Phi(x) + \alpha \Big[g^T p + \sum_{i \varepsilon I_{AE}} | \nabla \bar{f}_i{}^T p | - \sum_{i \varepsilon I_{AI}} min(0, \nabla \bar{f}_i{}^T p) \Big] + o(\alpha).$$

We will also need the following well-known lemma from convex analysis (see, for example Luenberger [24], page 69).

Lemma 4.1 Let y be a vector in a finite-dimensional vector space H and let K be a closed convex subset of H. Then there is a unique vector $k_0 \varepsilon K$ such that

$$\| y-k_0 \|_2 \leq \| y-k \|_2$$

for all $k \varepsilon K$. Furthermore, a necessary and sufficient condition that k_0 be the unique minimizing vector is that

$$(y-k_0)^T(k-k_0) \leq 0 \quad \text{for all} \quad k \varepsilon K. \tag{4.2.5}$$

■

Note that the set

$$\{A\lambda \mid (-1 \leq \lambda_i \leq +1, i \varepsilon I_{AE}) \text{ and } (0 \leq \lambda_i \leq +1, i \varepsilon I_{AI}\} \tag{4.2.6}$$

is convex and closed.

We can choose the following vectors:

$$\begin{aligned} y &= g, \\ k_0 &= A\lambda^*, \text{ and} \\ k &= A\lambda, \ (-1 \leq \lambda_i \leq +1, \ i \varepsilon I_{AE}) \text{ and } (0 \leq \lambda_i \leq +1, \ i \varepsilon I_{AI}) \end{aligned} \tag{4.2.7}$$

Then, according to Lemma 4.1,

$$(y-k_0)^T(k-k_0) \leq 0. \tag{4.2.8}$$

or

$$(g-A\lambda^*)^T(A\lambda-A\lambda^*) \leq 0 \tag{4.2.9}$$

or

$$(g-A\lambda^*)^T(A\lambda-g-(A\lambda^*-g)) \leq 0 \tag{4.2.10}$$

or

$$(-p)^T(A\lambda-g-p) \leq 0 \tag{4.2.11}$$

or

$$-p^T(A\lambda-g)+p^Tp \leq 0. \tag{4.2.12}$$

But, by the assumption that the residual is nonzero, $p \neq 0$. That implies $p^Tp > 0$.

So, (4.2.12) will give us

$$p^T(-A\lambda + g) < 0. \tag{4.2.13}$$

If we choose

$$\lambda_i = \begin{cases} -sign(a_i^T p) & , i \varepsilon I_{AE} \\ sign(min(0, a_i^T p)) & , i \varepsilon I_{AI} \end{cases}, \tag{4.2.14}$$

where a_i is the i^{th} column of A, it follows that

$$g^T p + \sum_{i \varepsilon I_{AE}} |a_i^T p| - \sum_{i \varepsilon I_{AI}} \min(0, a_i^T p) < 0, \tag{4.2.15}$$

and therefore p is a descent direction for the exact penalty function $\Phi(x)$. ■

It is important to notice that the problem (4.2.3) is nondegenerate, and that $p = A\lambda^* - g$ is unique, although the vector λ^* is not.

Other methods can be used in solving

$$A\lambda - g = 0 \tag{4.2.16}$$

$$subject\ to \quad -1 \leq \lambda_i \leq +1, \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i \leq +1, \quad i \varepsilon I_{AI}.$$

If the residual is zero, we know that we are at an optimal point for our model and the asymptotic phase should be entered. But if the residual is nonzero, then finding a descent direction could be based on some other idea. Using l_1 and l_∞ norms in solving (4.2.16) has been explored in Appendices I and II.

An algorithm for solving (4.2.3) is given in the next section. The algorithm is a modification of the nonnegative least-squares algorithm from Lawson and Hanson [23].

4.3. Bounded least-squares problem

The bounded least-squares problem

$$\underset{\lambda}{\textit{minimize}} \quad \| A\lambda - g \|_2^2 \tag{BLS}$$

$$\textit{subject to} \quad -1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI}$$

can be solved efficiently by taking into account the very special structure of the constraints.

Let us assume that λ is a feasible point for (BLS), and define a function Ψ as

$$\Psi(\lambda)=(A\lambda - g)^T(A\lambda - g) = \lambda^T A^T A\lambda - 2\lambda^T A^T g + g^T g \tag{4.3.1}$$

Now, partition the coordinates of λ in the following way:

$$B = \{i \mid -1 < \lambda_i < +1 , i \varepsilon I_{AE}\} \bigcup \{i \mid 0 < \lambda_i < +1 , i \varepsilon I_{AI}\} \tag{4.3.2}$$

$$L = \{i \mid \lambda_i = -1 , i \varepsilon I_{AE}\} \bigcup \{i \mid \lambda_i = 0 , i \varepsilon I_{AI}\}, \text{ and}$$

$$U = \{i \mid \lambda_i = +1 , i \varepsilon I_A\}$$

Our task is to find a step s s.t. $\Psi(\lambda) > \Psi(\lambda+s)$ and that a point $\lambda+s$ is feasible.

$$\Psi(\lambda+s) = \Psi(\lambda)+s^T \nabla\Psi(\lambda)+\frac{1}{2} s^T \nabla^2\Psi(\lambda)s$$

where,

$$\nabla\Psi(\lambda)=s^T[2A^T A\lambda-2A^T g], \text{ and}$$

$$\nabla^2\Psi(\lambda)=s^T A^T As.$$

We would like to have

$$\Psi(\lambda+s)-\Psi(\lambda)<0$$

at the point $\lambda+s$. One way of finding such s is to consider the following subproblem:

$$\underset{s}{\textit{minimize}} \quad \frac{1}{2} s^T A^T As+s^T A^T(A\lambda-g) \tag{4.3.3}$$

$$\textit{subject to} \quad -1 \leq \lambda_i+s_i \leq +1 , \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i+s_i \leq +1 , \quad i \varepsilon I_{AI}$$

The feasibility of $\lambda+\alpha s$ is guaranteed (for α small enough) if

$s_i \geq 0$ for all $i \varepsilon L$, and

$s_i \leq 0$ for all $i \varepsilon U$, and

otherwise, no restriction on s_i for all $i \varepsilon B$.

So, we can attempt to find s that solves

$$\underset{s}{minimize} \ \frac{1}{2} s^T A^T A s + s^T A^T (A\lambda - g) \tag{4.3.4}$$

$$subject\ to \ s_i = 0, \quad i \varepsilon L \bigcup U$$

The above constrained minimization problem can be reformulated as an unconstrained one if we define s as $s = I_B b$ where, I_B is obtained from the identity matrix by substituting zero for all the ones on the diagonal not corresponding to the indices from the set B. With this formulation the problem (4.3.4) becomes

$$\underset{b}{minimize} \ \frac{1}{2} b^T I_B^T A^T A I_B b + b^T I_B^T A^T (A\lambda - g) \tag{4.3.5}$$

This minimizer will be found among the vectors b satisfying

$$I_B^T A^T A I_B b + I_B^T A^T (A\lambda - g) = 0 \tag{4.3.6}$$

We can define a matrix $I_{L \bigcup U}$ in a way similar to defining I_B. Taking into account that $I = I_B + I_{L \bigcup U}$, we can rearrange (4.3.6) into

$$I_B^T A^T A I_B b + I_B^T A^T A I_B^T = -I_B^T A^T A I_{L \bigcup U} \lambda + I_B^T A^T g. \tag{4.3.7}$$

Finally, we can write (4.3.7) as

$$I_B^T A^T A I_{L \bigcup U} (b + \lambda) = I_B^T A^T (g - A I_{L \bigcup U} \lambda). \tag{4.3.8}$$

The equations (4.3.8) can be recognized as the normal equations for the least-squares solution of

$$A I_B z \approx g - A I_{L \bigcup U} \lambda \tag{4.3.9}$$

where, $z = b + \lambda$.

So, after solving the least-squares problem (4.3.9) we can get the following two outcomes:

Case 1 $z \neq 0$

There is one component of z for each $j \varepsilon B$. We have to look at each z_j in order to see whether it is inside the interval

$[-1\ ,\ +1]$, for $i \varepsilon I_{AE}$ or $[0\ ,\ +1]$, for $i \varepsilon I_{AI}$.

If it is not we save

$$\alpha_j = |z_j - \lambda_j|$$

and the corresponding j.

If any α_j exists, then we find q s.t. α_q has minimal value and let $\alpha = \alpha_q$. Otherwise let $\alpha = 1$.

The new λ is obtained in the following way:

$$\lambda_j \leftarrow \lambda_j + \alpha(z_j - \lambda_j) \quad \text{for all } j \varepsilon B$$

$$\lambda_j \leftarrow \lambda_j \quad \text{for all } j \varepsilon L \bigcup U$$

We redefine B and $L \bigcup U$ as

$$L \bigcup U \leftarrow L \bigcup U \bigcup \{j\}$$

$$B \leftarrow B - \{j\}$$

for all λ_j at a bound.

Case 2 $z = 0$

We are at a global minimum on the active set $L \bigcup U$.

To see if we are at a global minimum with respect to all activities we need to compute a vector of Lagrange multipliers u, i.e. to find a solution to the problem

$$\underset{u}{\textit{minimize}} \quad \| I_{L \bigcup U} u - (2A^T A\lambda - 2A^T g) \|_2 \tag{4.3.10}$$

The solution to this problem is given by

$$u_j = [2A^T(A\lambda - g)]_j \quad \text{for all } j \varepsilon L \bigcup U$$

So, if $u_j \geq 0$ for all $j \varepsilon L$ and

$u_j \leq 0$ for all $j \varepsilon U$ we are at an optimal point.

Otherwise, we choose j_0 where u_{j_0} is out of the bounds and consider the following problem:

$$\underset{s}{\textit{minimize}}\ \frac{1}{2}s^TA^TAs+s^TA^T(A\lambda-g) \qquad (4.3.11)$$

$$\textit{subject to}\ s_j=0\ ,\ j\varepsilon L\bigcup U-\{j_0\}$$

That is, we redefine B and $L\bigcup U$ as

$$L\bigcup U \leftarrow L\bigcup U-\{j_0\}$$

$$B \leftarrow B\bigcup\{j_0\}$$

and then we minimize as above.

The finite-step convergence of the Lawson and Hanson algorithm (NNLS) [23] is not impaired by these modifications. A similar algorithm, with more flexibility in simple bounds, was developed by Hanson [22].

Both algorithms are quite inefficient in the context in which the (BLS) problem is used. They do not use the information from the previous point x_k, i.e. λ_k and the row and column ordering of A_k. It is worth noticing that based on the work of Stoer [33], Gill et al. [20] have implemented an algorithm for the linearly constrained least squares problem in which the information from the previous stage can be used. When dealing with a rank deficient matrix A_k, an extension of this method (under preparation [21]) uses the full orthogonal decomposition of A_k to get the final λ_k.

4.4. Resolving degeneracy far from a stationary point

In order to find a descent direction from a point far from a stationary one we need a matrix Z, such that

$$Z^TA = 0.$$

Whether A has full column rank or not, the matrix Z is readily available from the *full orthogonal decomposition* of A (see Lawson and Hanson [23]). That is, if A is $n\times k$ and has rank $r \leq min(n,k)$,

$$A = Q\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}P^T \tag{4.4.1}$$

where, Q is an orthogonal $n \times n$ matrix, P is an orthogonal $k \times k$ matrix and R is a nonsingular $r \times r$ matrix; and

$$Q = [W\ Z]_{n \times (r \mid n-r)}. \tag{4.4.2}$$

If the Lagrange multipliers are necessary in forming the normal direction, as in Section 3.2, we suggest using the estimates for the λ's obtained from (BLS).

4.5. Resolving degeneracy in the asymptotic phase

We will assume here, for the sake of discussion, that there is a neighborhood of a degenerate local minimum x^* where the rank of A will not change and will be the same as the rank of $A(x^*)$. Also, we will assume that the global phase will be able to reach any neighborhood of x^*, possibly at a linear rate of convergence.

4.5.1. Degenerate range-space component

Suppose that we have a full orthogonal decomposition of the matrix A.

$$A_{n \times k} = Q\begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}P^T \tag{4.5.1}$$

where, Q is an orthogonal $n \times n$ matrix, P is an orthogonal $k \times k$ matrix and R is a nonsingular $r \times r$ matrix.

Let us partition P and Q in the following way:

$$P = [V\ S]_{k \times (r \mid k-r)} \tag{4.5.2}$$

$$Q = [W\ Z]_{n \times (r \mid n-r)} \tag{4.5.3}$$

Note that $Span(W) = Span(A)$ and $Span(Z) =$ *null-space of* A^T.

A step p is formed from the null-space component h and the range-space component v, as $p = h + v$.

As derived in Section 3.3, the range-space component can be obtained from the formula

$$A^T v = -\bar{f}_A \tag{4.5.4}$$

Taking the orthogonal decomposition of A into account we can write (4.5.4) as

$$P\begin{bmatrix} R^T & 0 \\ 0 & 0 \end{bmatrix} Q^T v = -\bar{f}_A \tag{4.5.5}$$

$$[V\ S]\begin{bmatrix} R^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W^T \\ Z^T \end{bmatrix} v = -\bar{f}_A \tag{4.5.6}$$

$$VR^T W^T v = -\bar{f}_A \tag{4.5.7}$$

$$R^T W^T v = -V^T \bar{f}_A \tag{4.5.8}$$

So, v can be viewed as a direction obtained by the Gauss-Newton method on the zero-residual problem.

Note that in the context of l_1 programming the two practically-encountered cases of (4.5.1) are

$$A = \begin{bmatrix} W\ Z \end{bmatrix}\begin{bmatrix} R \\ 0 \end{bmatrix} \tag{4.5.9}$$

(i.e. columns of A are linearly independent) and

$$A = \begin{bmatrix} W \end{bmatrix}\begin{bmatrix} R\ 0 \end{bmatrix}\begin{bmatrix} V^T \\ S^T \end{bmatrix} \tag{4.5.10}$$

(i.e. too many functions are active, but their gradients span the n-dimensional space R^n).

The general case (4.5.1) could happen if, for instance, the active functions $\bar{f}_i$, $i \varepsilon I_A$ are themselves linearly dependent.

Note that for the complete generality it would be necessary to consider an expansion of the form

$$A(x) = \begin{bmatrix} W(x)\ Z(x) \end{bmatrix}\begin{bmatrix} R(x) & 0 \\ 0 & M(x) \end{bmatrix}\begin{bmatrix} V(x)^T \\ S(x)^T \end{bmatrix}, \tag{4.5.11}$$

for some $M(x) \rightarrow 0$ as $x \rightarrow x^*$. This is a far more difficult case to handle, and we will not deal with it in this thesis.

4.5.1.1. Descent property of the degenerate range-space component

The following lemma shows that v, computed from (4.5.8), is a correct direction in an iterative procedure to solve

$$\bar{f}_A = 0. \tag{4.5.12}$$

Lemma 4.2 The range-space component obtained from

$$R^T W^T v = -V^T \bar{f}_A \tag{4.5.13}$$

is a descent direction for the least-squares merit function

$$\Psi(x) = \frac{1}{2} \bar{f}_A^T \bar{f}_A. \tag{4.5.14}$$

Proof: The direction v will be descent for $\Psi(x)$ if

$$\nabla \Psi(x)^T v < 0. \tag{4.5.15}$$

$$\nabla \Psi(x)^T v = (A \bar{f}_A)^T v \tag{4.5.16}$$

$$= \bar{f}_A^T [V \; S] \begin{bmatrix} R^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} W^T \\ Z^T \end{bmatrix} v \tag{4.5.17}$$

$$= \begin{bmatrix} \bar{f}_A^T V & \bar{f}_A^T S \end{bmatrix} \begin{bmatrix} R^T W^T v \\ 0 \end{bmatrix} \tag{4.5.18}$$

$$= \bar{f}_A^T V (R^T W^T v) \tag{4.5.19}$$

$$= \bar{f}_A^T V (-V^T \bar{f}_A) \tag{4.5.20}$$

$$< 0 \tag{4.5.21}$$

■

4.5.2. Degenerate null-space component

For the null-space component h in the Newton-like algorithm we need the matrix Z such that $Z^T A = 0$, and we need estimates of the Lagrange multipliers λ's. As pointed out in Section 4.5.1, the matrix Z is obtained from the full orthogonal decomposition of A. The estimate of the Lagrange multipliers is obtained from the (BLS) problem. Note that those Lagrange multipliers satisfy their respective bounds, i.e.

$$-1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI}.$$

4.6. Applicability to Existing Algorithms

We will end this chapter by pointing out to what extent the foregoing discussion is consistent with the practice carried on in the Bartels/Conn [1] and Murray/Overton [30] algorithms for nondegenerate (CNLL1) problems.

4.6.1. Degeneracy far from a stationary point

The normal descent direction (3.4.5), used by Bartels and Conn [1], need not be changed at all. On the other hand, if we use the direction (3.4.7), given by Murray and Overton [30], we also need an estimate of λ. It is not clear what the best estimate would be. If we solve (BLS) the residual vector will give us the descent direction immediately, so we do not need (3.4.7). If we solve

$$\underset{\lambda}{minimize} \quad \| A\lambda - g \|_2$$

there is no guarantee that λ's are between the corresponding bounds.

It seems that the direction (3.4.5) should be preferred in the degenerate case.

4.6.2. The degenerate range-space component

Bartels and Conn [1] obtain the nondegenerate range-space component v from the formula

$$A^T v = -\bar{f}_A(x+h).$$

So, by using the full orthogonal decomposition of A (4.5.1) the degenerate range-space component v could be obtained from

$$R^T W^T v = -V^T \bar{f}_A(x+h).$$

Murray and Overton [30] could use (4.5.8).

4.6.3. The degenerate null-space components

The null-space component can be obtained either from the formula

$$h = -Z(Z^T BZ)^{-1} Z^T (g + Bv) \tag{4.6.1}$$

where,

$$B = \sum_{i \in \{1, \cdots, m\}} \sigma_i \nabla^2 f_i - \sum_{i \in I_A} \lambda_i \nabla^2 f_i, \tag{4.6.2}$$

or from

$$h = -Z(Z^T BZ)^{-1} Z^T g \tag{4.6.3}$$

where,

$$B = \sum_{i \in I_E} \sigma_i \nabla^2 \bar{f}_i + \sum_{i \in I_{IN}} \sigma_i^- \nabla^2 \bar{f}_i - \sum_{i \in I_A} \lambda_i \nabla^2 \bar{f}_i, \tag{4.6.4}$$

depending on whether we are using the Murray/Overton [30] or the Bartels/Conn [1] asymptotic step.

CHAPTER 5

Handling degeneracy in the Bartels/Conn algorithm

As an example of a nonlinear l_1 algorithm that can fit in our framework we will extend the Bartels and Conn [1] algorithm to handle degeneracy. Proofs of global and asymptotically superlinear convergence under certain assumptions will be given.

For a given x and $\varepsilon \geq 0$ let

$$I_A(x,\varepsilon) = \{i: |\bar{f}_i(x)| \leq \varepsilon \text{ and } i \varepsilon I_E \bigcup I_{IN}\} \tag{5.0.1}$$

denote the set of ε-active indices at x, and let

$$I_I(x,\varepsilon) = (I_E \bigcup \{i: \bar{f}_i(x) < -\varepsilon \text{ and } i \varepsilon I_{IN}\}) - I_A(x,\varepsilon) \tag{5.0.2}$$

represent the set of ε-inactive indices.

Although the theoretical discussion was given for the activity tolerance $\varepsilon = 0$ we do not want to follow the active functions so closely. There is a danger that an active function can be repeatedly dropped from and added to the active set (*zig−zagging*). Furthermore, the problem functions are nonlinear and there is a possibility of making too many small steps just in order to stay on the active set manifold. So, we would like to use an ε considerably greater than the machine precision.

If we use the expansion (3.2.1)-(3.2.3) with $\varepsilon > 0$, the first-order change in an ε-active function could predict an increase in the penalty function $\Phi(x,\varepsilon)$, although for small $\alpha > 0$ we would expect a decrease in $\Phi(x,0)$. For example, if $\bar{f}_i(x) \neq 0$, $i \varepsilon I_E$, is considered to be an ε-active function, $\varepsilon > 0$, then the first-order change in $\Phi(x,\varepsilon)$ in a direction p, for $\alpha > 0$, will contain the contribution $|\nabla \bar{f}_i{}^T p|$. But, in the 0-active model, the same function $\bar{f}_i(x) \neq 0$, $i \varepsilon I_E$, will be considered inactive and the first-order change in $\Phi(x,0)$ in the direction p will contain the contribution $\nabla \bar{f}_i{}^T p$.

Similarly, if $\bar{f}_i(x) \neq 0$, $i \varepsilon I_{IN}$, is considered to be an ε-active function, $\varepsilon > 0$, then the first-order change in $\Phi(x,\varepsilon)$ in a direction p will contain the contribution $-min(0,\nabla \bar{f}_i{}^T p)$. But, in the 0-active model the same function $\bar{f}_i(x) \neq 0$, $i \varepsilon I_{IN}$ will be considered inactive and the first-order change in $\Phi(x,0)$ in the direction p will contain the contribution $\frac{(sign(\bar{f}_i(x))-1)}{2} \nabla \bar{f}_i{}^T p$.

So, for all $\alpha > 0$ small enough

$$\Delta\Phi(x) \leq \Delta\Phi(x,\varepsilon), \tag{5.0.3}$$

where $\Delta\Phi(x,\varepsilon)$ represents the first-order change in $\Phi(x,\varepsilon)$ in a direction p, i.e.

$$\Delta\Phi(x,\varepsilon) = \alpha \left[g^T p + \sum_{i \varepsilon I_{AE}(x,\varepsilon)} | \nabla \bar{f}_i{}^T p | - \sum_{i \varepsilon I_{AI}(x,\varepsilon)} min(0, \nabla \bar{f}_i{}^T p) \right] \tag{5.0.4}$$

and $\Delta\Phi(x)$ is (3.2.2)-(3.2.3).

As a result of (5.0.3) there is an $\alpha > 0$ such that if a direction p is a descent direction for $\Phi(x,\varepsilon)$ it is a descent direction for $\Phi(x)$, too.

Now, the normal descent direction, $p(x,\varepsilon)$, will still give the first-order descent on $\Phi(x,\varepsilon)$ due to the fact that $p(x,\varepsilon)$ is in the null-space of $A(x,\varepsilon)^T$.

For dropping descent a similar observation holds. If $g(x,\varepsilon)$ is in the range-space of $A(x,\varepsilon)$ and if we combine the terms in (5.0.4) in the way that was done in Section 3.2 we obtain

$$\alpha \left[\sum_{i \varepsilon I_{AE}(x,\varepsilon)} \left\{ \lambda_i \pi_i + 1 \right\} | \nabla \bar{f}_i{}^T p | + \sum_{i \varepsilon I_{AI}(x,\varepsilon)} \left\{ \lambda_i + \pi_i^- \right\} \nabla \bar{f}_i{}^T p \right] \tag{5.0.5}$$

where

$$\pi_i = \begin{cases} +1 & , \text{if } \nabla \bar{f}_i(x)^T p > 0 \\ 0 & , \text{if } \nabla \bar{f}_i(x)^T p = 0 \\ -1 & , \text{if } \nabla \bar{f}_i(x)^T p < 0 \end{cases}, \quad i \varepsilon I_E \tag{5.0.6}$$

and

$$\pi_i^- = \begin{cases} 0 & \text{if } \nabla \bar{f}_i(x)^T p \geq 0 \\ -1 & \text{if } \nabla \bar{f}_i(x)^T p < 0 \end{cases}, \; i \varepsilon I_{IN}. \tag{5.0.7}$$

Again, when the columns of $A(x,\varepsilon)$ are linearly independent, it is easy to verify that the first-order change to $\Phi(x,\varepsilon)$ will be nonnegative for all $\alpha \geq 0$ and all choices of $p \neq 0$ if the following conditions hold

$$-1 \leq \lambda_i \leq +1 \quad \text{for} \quad i \varepsilon I_{AE} \tag{5.0.8}$$

$$0 \leq \lambda_i \leq +1 \quad \text{for} \quad i \varepsilon I_{AI}. \tag{5.0.9}$$

That means that our ε-activity model cannot always provide us with a first-order descent direction. There are two actions we might take if we perceive that $\Phi(x,\varepsilon)$ does not predict the local behavior of $\Phi(x,0)$.

(a) We could assume that the point x is close to a local minimum x^*, and that the active functions have been correctly chosen. In order to find a point where the first order optimality conditions are satisfied exactly, we need to solve the following system of equations:

$$A(x)\lambda - g(x) = 0 \tag{5.0.10}$$

$$\bar{f}_i(x) = 0 \, , \; i \varepsilon I_A(x,\varepsilon).$$

In other words, we would like to make the ε-active functions zero (to machine precision) and at the same time to retain the relation (5.0.10) and to decrease the penalty function. If this attempt fails we could assume that our ε-activity model is too crude and that we should reduce ε.

(b) If the conditions (5.0.8) or (5.0.9) are violated, let us say for some λ_i, then we consider the current x to be a non-optimal stationary point. We would like to find a direction p that will allow us to descend from x and at the same time change the active set. The direction obtained as a solution to

$$A(x,\varepsilon)^T p = -sign(\lambda_i) e_i \tag{5.0.11}$$

will remove the i^{th} function from the active set and will also give us a descent direction. If this does not provide sufficient decrease for $\Phi(x,0)$, we could assume that ε-activity model is too crude and that we should reduce ε.

If the columns of $A(x,\varepsilon)$ are linearly dependent then the residual vector, $p = A(x,\varepsilon)\lambda - g$, obtained from (BLS) will tell us which of the above cases hold. If $\| p \| < \delta_d$, $\delta_d > 0$, then we could assume the case (a). Otherwise, we will assume the case (b) and Theorem 4.1 tells us that p is a descent direction.

5.1. Extended Bartels/Conn algorithm

Bartels and Conn [1] motivate the task of solving (CNLL1) problem

$$\underset{x\varepsilon\mathbf{R}^n}{\textit{minimize}} \sum_{i=1}^{m} | f_i(x) | \qquad \text{(CNLL1)}$$

$$\textit{subject to } f_i(x) = 0\,, \quad i\varepsilon\mathbf{I}_{EQ}$$

$$f_i(x) \geq 0\,, \quad i\varepsilon\mathbf{I}_{IN}$$

by selecting a trial active index set I_A and the violated index set I_I suggested locally by x and some activity tolerance ε. Then, with I_A chosen, they propose solving

$$\underset{x}{\textit{minimize}} \ \Phi(x) = \sum_{i\varepsilon I_E} \sigma_i \bar{f}_i(x) + \sum_{i\varepsilon I_{IN}} \sigma_i^{-} \bar{f}_i(x) \qquad (5.1.1)$$

$$\textit{subject to } \bar{f}_i(x) = 0,\ i\varepsilon I_A(x,\varepsilon)$$

until they are sufficiently close to a minimum of Φ to satisfy a convergence criterion, or until it is evident that they are dealing with the wrong selection of I_A, in which case (5.1.1) is redefined (by redefining I_A) for further steps of the algorithm. The problems of the form (5.1.1) would be solved using exact penalty method of Coleman and Conn [9,10].

This way of solving (CNLL1) could hardly be incorporated in our framework. For one reason, the set of functions that form the objective function of (5.1.1) would change from step to step, depending which functions are currently active, and that means that at each step different sets of functions would be multiplied by a penalty parameter μ. Fortunately, just the motivation for solving (CNLL1) is wrong. Bartels and Conn implemented their algorithm exactly as described in our framework, i.e. they minimize

$$\Phi(x)=\mu\sum_{i=1}^{m}|f_i|+\sum_{i\varepsilon I_{EQ}}|f_i|-\sum_{i\varepsilon I_{IN}}min(0,f_i) \tag{5.1.2}$$

where $\mu > 0$.

The algorithm of Conn and Coleman [9,10] is used to find the minimum of the exact penalty function (5.1.2).

The estimates of Lagrange multipliers play an important role in the assessment of nearness to the optimal point, and in the decision about which phase the algorithm is considered to be in. In the degenerate case we will use our bounded least squares problem (BLS) to decide about the phase.

Algorithm 1

{Select any $x\varepsilon\mathbf{R}^n$, ε, δ, $\delta_d=\delta_0$, Λ and set $k=0$}

repeat

{Identify $I_A(x,\varepsilon)$; obtain the full orthogonal decomposition of $A(x,\varepsilon)$ }

{Form $g(x,\varepsilon)$; compute the multipliers, λ's, from (BLS)}

if {(3.4.1)-(3.4.4) are approximately satisfied and $\|A\lambda-g\|_2<\delta_d$}

then

/* Asymptotic phase */

{Find p's components h (see Section 4.6.3) and v (see Section 4.6.2)}

if $\{\Phi(x+h+v) < \Phi(x) - \delta(\|Z^Tg\|_2^2+\|f_A\|_1\}$

then

$x \leftarrow x + h + v$

go to OUT

else

repeat

if {either ε or Λ is too small} **then** STOP

$\varepsilon \leftarrow \frac{\varepsilon}{2}$

$\Lambda \leftarrow \frac{\Lambda}{2}$

$k \leftarrow k+1$

$\delta_d \leftarrow \dfrac{\delta_0}{k}$

{Identify $I_A(x,\varepsilon)$; obtain the full orthogonal decomposition of $A(x,\varepsilon)$ }

{Form $g(x,\varepsilon)$; compute the multipliers, λ's, from (BLS)}

until $\{ \| Z^T g \| \neq 0\}$

/* Global phase */

if $\{ \| Z^T g \| \leq \Lambda\}$

then

$\{p = A\lambda - g\}$

go to ALPHA

{Find the normal descent direction p (see Section 4.6.1)}

ALPHA:

{Find α}

$x \leftarrow x + \alpha p$

OUT:

until {termination test satisfied or $\Phi(x)$ does not decrease}

Note that the algorithm Algorithm 1 lays out the logical steps in the method for solving the l_1 nonlinear optimization problem. Computationally, it would be less expensive to test conditions for the global phase first.

Naturally, whenever we are dealing with a nondegenerate point we can use an appropriate step from the Bartels/Conn algorithm [1]. That could save some computational effort.

Algorithm 1 is designed to minimize the exact penalty function $\Phi(x)$ for a fixed $\mu>0$. The user must also supply an initial guess for x^0. In addition, the activity tolerance $\varepsilon>0$ and the projected gradient tolerance Λ are supplied by the user. Following the approach of Coleman and Conn [9,10] we will show that the global convergence properties are unaffected by the initial assignment

for x_0, ε, and Λ. However, the efficiency of the algorithm can be greatly influenced by the choice.

In the proofs, the following notational rule is used: if a quantity is sub- or superscripted, then it may change from iteration to iteration, otherwise it remains constant. As in Coleman and Conn [9,10], the termination criterion is not shown, but in practice the algorithm stops when at least all of the following conditions hold:

(1) $-1 \leq \lambda_i^k \leq +1\ ,\ i \varepsilon I_{AE}$

$0 \leq \lambda_i^k \leq +1\ ,\ i \varepsilon I_{AI}$, and

$\| A\lambda_i^k - g \|_2^2 \leq TOL0,$

(2) $\| Z_k^T g(x^k) \| \leq TOL1,$

(3) $\| \bar{f}_A(x^k) \| \leq TOL2.$

Further conditions based upon the history by which (2) and (3) become satisfied over the last several iterations may also be useful to make a computer implementation more robust.(See a discussion of terminating iterative processes in, for example, Dennis and Schnabel [13].)

The algorithm minimizes the exact penalty function $\Phi(x)$ for a given and fixed μ. If μ is less than a threshold value then the obtained minimizer will usually also solve the (CNLL1) problem. But, the threshold value of μ is a function of the multipliers at the solution of the original problem, see Charalambous [5], and therefore this value is unknown a priori. Consequently, an initial choice of μ that is too large can lead to a minimum of $\Phi(x)$ that is infeasible. So far there is no proven successful strategy for choosing $\mu's$ a priori, but we adopted the usual practice, see Bartels and Conn [2], of reducing μ whenever an infeasible optimum for (CNLL1) is found.

5.2. Implementation techniques

Numerical techniques described in Gill et al. [19] are used here. Just the basic ideas will be mentioned and the reader is referred to the works by Gill and Murray [17], and Murray and Wright [27] for more details.

5.2.1. The range step and estimating multipliers

When $A_{n\times t}$, (the matrix formed from the gradients of the active functions has linearly independent columns), we obtain an orthogonal decomposition of A such that

$$A_{n\times t} = Q_{n\times n}\tilde{R}_{n\times t} = \begin{bmatrix} W_{n\times t} & Z_{n\times(n-t)} \end{bmatrix} \begin{bmatrix} R_{t\times t} \\ 0 \end{bmatrix}, \quad t \leq n, \tag{5.2.1}$$

where R is upper triangular. Then the vector of multiplier estimates, λ, can be computed by solving

$$R\lambda = W^T g. \tag{5.2.2}$$

Similarly, the range-space component can be computed by solving

$$R^T\bar{v} = -\bar{f}_A(x^k + h^k) \tag{5.2.3}$$

and then setting

$$v^k = Q\bar{v}. \tag{5.2.4}$$

When $A_{n\times t}$ has linearly dependent columns, we compute a complete orthogonal decomposition of A, i.e.

$$A_{n\times t} = Q_{n\times n}\tilde{R}_{n\times t}P^T_{t\times t} = \begin{bmatrix} W_{n\times r} & Z_{n\times(n-r)} \end{bmatrix} \begin{bmatrix} R_{r\times r} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V^T_{t\times r} \\ S^T_{t\times(t-r)} \end{bmatrix}, \tag{5.2.5}$$

where the rank of A is r. (The lower two zeros and $Z_{n\times(n-r)}$ are often vacuous on practical problems; see (4.5.10).)

We compute the multiplier estimates as a solution to the (BLS) problem, i.e. we solve

$$\begin{aligned} &\underset{\lambda}{\textit{minimize}} \quad \| A\lambda - g \|^2 \\ &\textit{subject to} \quad -1 \leq \lambda_i \leq +1, \quad i\varepsilon I_{AE} \\ &\qquad\qquad 0 \leq \lambda_i \leq +1, \quad i\varepsilon I_{AI}. \end{aligned} \tag{BLS}$$

The range-space component is found by solving

$$R^T\bar{v} = -V^T\bar{f}_A(x^k + h^k) \tag{5.2.6}$$

and then setting

$$v^k = W\bar{v} \tag{5.2.7}$$

5.2.2. Null-space component

The null-space component is obtained by solving the system

$$(Z_k^T B_k Z_k)\bar{h} = -Z_k^T g \tag{5.2.8}$$

and then setting

$$h^k = Z\bar{h}. \tag{5.2.9}$$

Z^TBZ is assumed to be positive definite, so the LDL^T decomposition exists and if

$$Z^TBZ = LDL^T \tag{5.2.10}$$

the solution to (5.2.8) is obtained as

$$L\tilde{h} = -Z^Tg \tag{5.2.11}$$

$$L^T\bar{h} = D^{-1}\tilde{h}. \tag{5.2.12}$$

The asymptotic convergence results will assume that the projected Hessian approximation, Z^TBZ, approach the true projected Hessian in the limit.

5.3. Convergence results

Convergence results from Coleman and Conn [9,10] will be closely followed in this chapter and just the proofs that are modified in order to handle degeneracy will be shown.

5.3.1. Definitions and assumptions

Here we use the convention that $\|\cdot\|$ denotes the 2-norm unless indicated otherwise.

(i) The reduced function of $\Phi(x,\varepsilon)$, i.e. the differentiable part of $\Phi(x,\varepsilon)$, is defined to be

$$\Phi_1(x,\varepsilon) = \sum_{i\in I_E \cap I_I(x,\varepsilon)} \sigma_i \bar{f}_i + \sum_{i\in I_{IN} \cap I_I(x,\varepsilon)} \sigma_i^- \bar{f}_i.$$

The reduced gradient of $\Phi(x,\varepsilon)$ is

$$\nabla\Phi_1(x,\varepsilon) = \sum_{i\in I_E \cap I_I(x,\varepsilon)} \sigma_i \nabla\bar{f}_i + \sum_{i\in I_{IN} \cap I_I(x,\varepsilon)} \sigma_i^- \nabla\bar{f}_i \quad (= g). \tag{5.3.1}$$

(ii) Let A_k denote an $n \times t_k$ matrix whose columns are the gradients of the $\varepsilon-$active functions. We

will not assume A_k to be of full rank. The matrix Z_k satisfies $Z_k^T Z_k = I_{(n-r)\times(n-r)}$, (where $I_{(n-r)\times(n-r)}$ is an $(n-r)\times(n-r)$ identity matrix), and $A_k^T Z_k = 0$; where $r = rank(A_k)$. If A_k is full rank, i.e. $r = t_k$, the matrix Z_{k-j} satisfies $Z_{k-j}^T Z_{k-j} = I_{n-r+1\times n-r+1}$, and $\nabla \bar{f}_i(x^k) Z_{k-j} = 0$ if $i \varepsilon I_A(x^k,\varepsilon) - \{j\}$, for some $j \varepsilon I_A(x^k,\varepsilon)$.

Define

$$G_L(x^k) = \nabla^2 \Phi_1(x^k) - \sum_{i \varepsilon I_A(x^k,\varepsilon)} \lambda_i^k \nabla^2 \bar{f}_i(x^k)$$

where λ^k is the least-squares solution to $A_k \lambda = g$ when A_k is full rank, and λ^k is the minimum-l_2-norm solution to

$$\underset{\lambda}{\textit{minimize}} \quad \| A\lambda - g \|^2$$

$$\textit{subject to} \quad -1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI}$$

when A_k is not full rank.

At times, an arbitrary but fixed point $\bar{x}$ will be considered. At such a point, $\bar{A}$ will denote a matrix whose columns are the gradients of the precisely active functions. That is, the columns of $\bar{A}$ belong to $\{\nabla \bar{f}_i(\bar{x}) \mid i \varepsilon I_A(\bar{x},0)\}$. The matrix $\bar{Z}$ satisfies $\bar{A}^T \bar{Z} = 0$, $\bar{Z}^T \bar{Z} = I_{(n-r)\times(n-r)}$, where A is $n \times t$ and $r = rank(\bar{A})$. If $\bar{x}$ is a stationary point, then $\bar{\lambda}$ satisfies

$$\bar{A}\bar{\lambda} = g(\bar{x})$$

(iv) A vector $\bar{x}$ is termed *a stationary point* of Φ if there exists a vector λ satisfying

$$A(\bar{x},0)\lambda = g(\bar{x}).$$

If λ satisfies the above equations and

$$-1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE}$$

$$0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI}$$

then we call $\bar{x}$ *a first-order point* of Φ.

If

$$-1 < \lambda_i < +1 , \quad i \varepsilon I_{AE} \qquad (5.3.2)$$

$$0 < \lambda_i < +1 , \quad i \varepsilon I_{AI}$$

then $\bar{x}$ is *a strict first-order point.* If inequalities (5.3.2) hold and

$$y^T(\sum_{i \varepsilon I_E(\bar{x},0)} \sigma_i \nabla^2 \bar{f}_i(\bar{x}) + \sum_{i \varepsilon I_{IN}(\bar{x},0)} \sigma_i^- \nabla^2 \bar{f}_i(\bar{x}) - \sum_{i \varepsilon I_A(\bar{x},0)} \lambda_i \nabla^2 \bar{f}_i(\bar{x}))y > 0$$

for all y satisfying $\bar{A}^T y = 0$, $y \neq 0$, then we term $\bar{x}$ *a strict second-order point of* Φ.

(v) We make the following line search assumptions:

(a) If h^k is a normal descent direction for Φ at x^k, then α^k is determined so that

$$\Phi(x^k) - \Phi(x^k + \alpha^k h^k) \geq \gamma_1((h^k)^T g^k)^2, \gamma_1 > 0.$$

where

$$g^k = \sum_{i \varepsilon I_E \bigcap I_I} \sigma_i \nabla \bar{f}_i + \sum_{i \varepsilon \tilde{I}_E(x^k,\varepsilon)} \sigma_i \nabla \bar{f}_i$$

$$+ \sum_{i \varepsilon I_{IN} \bigcap I_I} \sigma_i^- \nabla \bar{f}_i + \sum_{i \varepsilon \tilde{I}_{IN}(x^k,\varepsilon)} \sigma_i^- \nabla \bar{f}_i$$

and

$$\tilde{I}_{IN}(x^k,\varepsilon) = \{i \varepsilon I_A(x^k,\varepsilon)\} \bigcap \{i \varepsilon I_{IN} : \nabla \bar{f}_i{}^T h^k < 0\},$$

$$\tilde{I}_E(x^k,\varepsilon) = \{i \varepsilon I_A(x^k,\varepsilon)\} \bigcap \{i \varepsilon I_E : | \nabla \bar{f}_i{}^T h^k | > 0\}.$$

(It is a direct extension of a result of Conn and Pietryzkowski (Proposition 1, [12]) to show that the above condition can be satisfied.)

(b) If $p^k = A_k \lambda^k - g(x^k))$ is a dropping descent direction (4.2.4) for Φ at x^k, then α^k is determined so that

$$\Phi(x^k) - \Phi(x^k + \alpha^k p^k) \geq \gamma_1 (p^k)^T p^k, \gamma_1 > 0.$$

(The first-order expansion of $\Phi(x,\varepsilon)$:

$$\Phi(x^k + \alpha p^k, \varepsilon) \leq \Phi(x^k, \varepsilon)$$

$$+ \alpha\left[g^T p^k + \sum_{i\varepsilon I_{AE}(x^k,\varepsilon)} | \nabla \bar{f}_i{}^T p^k | - \sum_{i\varepsilon I_{AI}(x^k,\varepsilon)} min(0, \nabla \bar{f}_i{}^T p^k)\right]$$

$$+ o(\alpha)$$

and the relation

$$g^T p^k + \sum_{i\varepsilon I_{AE}(x^k,\varepsilon)} | \nabla \bar{f}_i{}^T p^k | - \sum_{i\varepsilon I_{AI}(x^k,\varepsilon)} min(0, \nabla \bar{f}_i{}^T p^k) \leq -(p^k)^T p^k,$$

which is derived in the proof for Theorem 4.1, show that the above condition can be satisfied).

(vi) Let Ω denote a compact set, where $\{x^k\}\varepsilon\Omega$.

(vii) Let S_0, $\bar{S}$, $\hat{S}$ denote the set of stationary points of Φ in Ω, the set of stationary but not first-order points, and the set of first-order points respectively.

5.3.2. Global convergence

Lemma 5.1 We assume that

(i) f_i, $i\varepsilon I_E \bigcup I_{IN}$ are twice continuously differentiable on a compact set Ω, where $\{x^k\}\varepsilon\Omega$.

(ii) $\bar{x}$ is any strict-second order point of Φ on Ω.

(iii) We have a complete orthogonal decomposition of $A(x^k)$, i.e.

$$A(x^k) = \begin{bmatrix} W(x^k) & Z(x^k) \end{bmatrix} \begin{bmatrix} R(x^k) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V(x^k)^T \\ S(x^k)^T \end{bmatrix}.$$

There is a neighborhood of $\bar{x}$ such that A does not change its rank and there is a positive constant b_0 such that $\| R^{-1}(x^k) \| \leq b_0$ for all $k > K_0$.

(Note this is certainly true when there are more active functions than the dimensionality of the space, n, and the rank of A is equal to n. Also, it is true when the degeneracy is caused by the presence of the linearly dependent functions. It is also true in the standard nondegenerate case.)

$f_A \varepsilon$ *Range space*(A^T).

Also, $x^k \neq \bar{x}$ for all k, and W, Z, R, V and S are computed as continuous functions of x.

(iv) $I_A(x^k,\varepsilon) = I_A(\bar{x},0)$, for all $k > K_0$.

(v) There exist positive constants b_1, b_2 such that for all vectors $w \varepsilon R^{n-r}$, $w \neq 0$

$$b_1 \| w \|^2 \leq w^T(Z_k^T B_k Z_k)w \leq b_2 \| w \|^2.$$

Then there exist positive constants Δ_1, Δ_2, and δ such that

(1) $\| x^k - \bar{x} \| \leq \Delta_1$

(2) $\| Z_k^T B Z_k - Z_k^T G_L(x^k) Z_k \| \leq \Delta_2$

implies that

$$\Phi(x^k + h^k + v^k) - \Phi(x^k) \leq -\delta(\| Z_k^T g(x^k) \|_2^2 + \| \bar{f}_A(x^k) \|_1),$$

where

$$h^k = -Z_k(Z_k^T B_k Z_k)^{-1} Z_k^T g(x^k),$$

$$v^k = -WR^{-T} V^T \bar{f}_A(x^k + h^k).$$

Proof: (In order to simplify notation, the "k" superscripts and subscripts will be dropped. If an argument is unspecified, it is assumed to be x. In addition, I_A and I_I denote $I_A(\bar{x},0)$ and $I_I(\bar{x},0)$, respectively.)

(a) Consider changes in Φ_1.

By Taylor's theorem

$$\Phi_1(x+h+v) = \Phi_1(x) + g^T[h+v] + \frac{1}{2}[h+v]\nabla^2\Phi_1[h+v] + o(\| h+v \|^2). \tag{5.3.3}$$

Recall

$$G_L(x) = \nabla^2\Phi_1(x) - \sum_{i \varepsilon I_A} \lambda_i \nabla^2 \bar{f}_i(x)$$

Thus

$$h = -Z(Z^T G_L Z)^{-1} Z^T g + ZEZ^T g \tag{5.3.4}$$

where

$$E = (Z^T G_L Z)^{-1} - (Z^T B Z)^{-1}.$$

Notice that

$$g = A\lambda + Zw, \tag{5.3.5}$$

for some $w \varepsilon R^{n-r}$. Hence by (5.3.3)-(5.3.5)

$$\Phi_1(x+h+v) - \Phi_1(x) = -g^T Z(Z^T G_L Z)^{-1} Z^T g + \lambda A^T v \tag{5.3.6}$$

$$+ \frac{1}{2} h^T \nabla^2 \Phi_1 h + \frac{1}{2} v^T \nabla^2 \Phi_1 v + h^T \nabla^2 \Phi_1 v$$

$$+ o(\| h+v \|^2) + g^T Z E Z^T g$$

But

$$A^T v = -\bar{f}_A(x+h), \tag{5.3.7}$$

where $\bar{f}_A = (\cdots \bar{f}_i \cdots)^T_{i \varepsilon I_A}$. Since $\nabla \bar{f}_i^T h = 0$, $i \varepsilon I_A$, it follows that

$$\bar{f}_i(x+h) = \bar{f}_i(x) + \frac{1}{2} h^T \nabla^2 \bar{f}_i h + o(\| h \|^2) \tag{5.3.8}$$

Therefore, by (5.3.7) and (5.3.8)

$$\lambda^T A^T v = -\sum_{i \varepsilon I_A} \lambda_i (\bar{f}_i + \frac{1}{2} h^T \nabla^2 \bar{f}_i h) + o(\| h \|^2)$$

Hence (5.3.6) can be written

$$\Phi_1(x+h+v) - \Phi_1(x) = -g^T Z(Z^T G_L Z)^{-1} Z^T g + \frac{1}{2} h^T (\nabla^2 \Phi_1 - \sum_{i \varepsilon I_A} \lambda_i \nabla^2 \bar{f}_i) h \tag{5.3.9}$$

$$- \sum_{i \varepsilon I_A} \lambda_i \bar{f}_i + \frac{1}{2} v^T \nabla^2 \Phi_1 v + h^T \nabla^2 \Phi_1 v$$

$$+ o(\| h+v \|^2) + o(\| h \|^2) + g^T Z E Z^T g.$$

But, by the definition of G_L

$$h^T (\nabla^2 \Phi_1 - \sum_{i \varepsilon I_A} \lambda_i \nabla^2 \bar{f}_i) h = h^T G_L h$$

$$= \left[-g^T Z (Z^T B Z)^{-1} Z^T + g^T Z E^T Z^T \right] G_L \left[-Z(Z^T B Z)^{-1} Z^T g + Z E Z^T g \right]$$

$$= g^T Z (Z^T G_L Z)^{-1} Z^T g - 2 g^T Z E Z^T g$$

$$+ g^T Z E Z^T G_L Z E Z^T g$$

using (5.3.4). Substituting this expression into (5.3.9) results in

$$\Phi_1(x+h+v) - \Phi_1(x) = -\frac{1}{2}g^T Z(Z^T G_L Z)^{-1} Z^T g \qquad (5.3.10)$$

$$- \sum_{i \varepsilon I_A} \lambda_i \bar{f}_i + \frac{1}{2} v^T \nabla^2 \Phi_1 v + h^T \nabla^2 \Phi_1 v$$

$$+ e(x),$$

where

$$e(x) = o(\| h+v \|^2) + o(\| h \|^2) - g^T ZEZ^T g + \frac{1}{2} g^T ZEZ^T G_L ZEZ^T g.$$

(b) Consider changes in $\bar{f}_i$, $i \varepsilon I_A$.

By Taylor's theorem

$$\bar{f}_i(x+h+v) - \bar{f}_i(x) = \nabla \bar{f}_i{}^T v + \frac{1}{2} h^T \nabla^2 \bar{f}_i h + \frac{1}{2} v^T \nabla^2 \bar{f}_i v \qquad (5.3.11)$$

$$+ h^T \nabla^2 \bar{f}_i v + o(\| h+v \|^2).$$

But $A^T v = -\bar{f}_A(x+h)$, and for each $i \varepsilon I_A$,

$$\bar{f}_i(x+h) = \bar{f}_i(x) + \frac{1}{2} h^T \nabla^2 \bar{f}_i h + o(\| h \|^2);$$

$$\nabla \bar{f}_i{}^T v = -\bar{f}_i(x) - \frac{1}{2} h^T \nabla^2 \bar{f}_i h + o(\| h \|^2),$$

and thus using (5.3.11),

$$\bar{f}_i(x+h+v) - \bar{f}_i(x) = -\bar{f}_i(x) + \frac{1}{2} v^T \nabla^2 \bar{f}_i v + h^T \nabla^2 \bar{f}_i v \qquad (5.3.12)$$

$$+ o(\| h+v \|^2).$$

Case 1. $(\bar{f}_i(x+h+v) \geq 0, \bar{f}_i(x) < 0)$

Then,

$$min(0, \bar{f}_i(x)) - min(0, \bar{f}_i(x+h+v)) = \bar{f}_i(x), \; i \varepsilon I_{AI}$$

$$|\bar{f}_i(x+h+v)| - |\bar{f}_i(x)| = \bar{f}_i(x+h+v) - (-\bar{f}_i(x))$$

$$= \frac{1}{2} v^T \nabla^2 \bar{f}_i v + h^T \nabla^2 \bar{f}_i v + \bar{f}_i(x)$$

$$+ o(\| h+v \|^2), \quad i \varepsilon I_{AE}.$$

Case 2. $(\bar{f}_i(x+h+v) < 0, \bar{f}_i(x) < 0)$

Then, using (5.3.12),

$$min(0,\bar{f}_i(x)) - min(0,\bar{f}_i(x+h+v)) = \bar{f}_i(x) - \frac{1}{2}v^T \nabla^2 \bar{f}_i v - h^T \nabla^2 \bar{f}_i v \tag{5.3.13}$$

$$+ o(\| h+v \|^2), \quad i \varepsilon I_{AI}$$

$$|\bar{f}_i(x+h+v)| - |\bar{f}_i(x)| = -\bar{f}_i(x+h+v) - (-\bar{f}_i(x))$$

$$= -\frac{1}{2}v^T \nabla^2 \bar{f}_i v - h^T \nabla^2 \bar{f}_i v + \bar{f}_i(x)$$

$$+ o(\| h+v \|^2), \quad i \varepsilon I_{AE}.$$

Case 3. $(\bar{f}_i(x+h+v) \geq 0, \bar{f}_i(x) \geq 0)$

Then,

$$min(0,\bar{f}_i(x)) - min(0,\bar{f}_i(x+h+v)) = 0, \; i \varepsilon I_{AI}$$

$$|\bar{f}_i(x+h+v)| - |\bar{f}_i(x)| = \bar{f}_i(x+h+v) - \bar{f}_i(x)$$

$$= +\frac{1}{2}v^T \nabla^2 \bar{f}_i v + h^T \nabla^2 \bar{f}_i v - \bar{f}_i(x)$$

$$+ o(\| h+v \|^2), \quad i \varepsilon I_{AE}.$$

Case 4. $(\bar{f}_i(x+h+v) < 0, \bar{f}_i(x) \geq 0)$

Then, using (5.3.12),

$$min(0,\bar{f}_i(x)) - min(0,\bar{f}_i(x+h+v)) = -\frac{1}{2}v^T \nabla^2 \bar{f}_i v - h^T \nabla^2 \bar{f}_i v$$

$$+ o(\| h+v \|^2), \quad i \varepsilon I_{AI}.$$

$$|\bar{f}_i(x+h+v)| - |\bar{f}_i(x)| = -\bar{f}_i(x+h+v) - \bar{f}_i(x)$$

$$= -\frac{1}{2}v^T \nabla^2 \bar{f}_i v - h^T \nabla^2 \bar{f}_i v - \bar{f}_i(x)$$

$$+ o(\| h+v \|^2), \quad i \varepsilon I_{AE}.$$

Using cases 1-4

$$\sum_{i\varepsilon I_{AE}} |\bar{f}_i(x+h+v)| - \sum_{i\varepsilon I_{AE}} |\bar{f}_i(x)| + \sum_{i\varepsilon I_{AI}} min(0,\bar{f}_i(x)) - \sum_{i\varepsilon I_{AI}} min(0,\bar{f}_i(x+h+v)) \tag{5.3.14}$$

$$\leq \eta^T \bar{f}_A + \frac{1}{2}\sum_{i\varepsilon I_A} |v^T \nabla^2 \bar{f}_i v| + \sum_{i\varepsilon I_A} |h^T \nabla^2 \bar{f}_i v| + o(\|h+v\|^2),$$

where

$$\eta_i = \begin{cases} 1 & \text{, if } \bar{f}_i(x) < 0 \text{ and } i\varepsilon I_{AE} \bigcup I_{AI} = I_A \\ -1 & \text{, if } \bar{f}_i(x) \geq 0 \text{ and } i\varepsilon I_{AE} \\ 0 & \text{, if } \bar{f}_i(x) \geq 0 \text{ and } i\varepsilon I_{AI} \end{cases}$$

(c) Consider changes in Φ.

We now combine parts (a) and (b) to obtain

$$\Phi(x+h+v) - \Phi(x) \leq -\frac{1}{2} g^T Z (Z^T G_L Z)^{-1} Z^T g + (\eta-\lambda)^T \bar{f}_A \tag{5.3.15}$$

$$+ \frac{1}{2}|v^T \nabla^2 \Phi_1 v| + |h^T \nabla^2 \Phi_1 v| + \frac{1}{2}\sum_{i\varepsilon I_A} |v^T \nabla^2 \bar{f}_i v|$$

$$+ \sum_{i\varepsilon I_A} |h^T \nabla^2 \bar{f}_i v| + e(x).$$

But

$$v = -WR^{-T}V^T \bar{f}_A(x+h)$$

$$= -WR^{-T}V^T\left[\bar{f}_A + \frac{1}{2}q\right] + o(\|h\|^2),$$

where $q = (\cdots h^T \nabla^2 \bar{f}_i h \cdots)_i \varepsilon I_A$. Define

$$H_1 = \nabla^2 \Phi_1 WR^{-T}V^T, \; H_2 = VR^{-1}W^T H_1,$$

$$\overline{H}_i = \nabla^2 \bar{f}_i WR^{-T}V^T \text{ and}$$

$$\tilde{H}_i = VR^{-1}W^T \overline{H}_i, \; i\varepsilon I_A.$$

Thus

$$v^T \nabla^2 \Phi_1 v = \bar{f}_A(x)^T w(x) + \frac{1}{4} q^T H_2 q + o(\|h\|^2) \tag{5.3.16}$$

where $w(x) = H_2\left[\bar{f}_A + q(x)\right]$. But $q^T H_2 q = O(\|h^4\|)$, and hence by (5.3.16)

$$\frac{1}{2}\,|\,v^T\nabla^2\Phi_1 v\,|\,\leq \frac{1}{2}\sum_{i\varepsilon I_A}|\,\bar{f}_i(x)\,|\,|\,w_i(x)\,|\,+\,o(\|\,h\,\|^2). \tag{5.3.17}$$

Similarly, we can define vectors

$u^j(x) = \tilde{H}_j\left[\bar{f}_A + q(x)\right]$, $j\varepsilon I_A$. If we let $\bar{u}_i = \sum_{j\varepsilon I_A}|\,u_i^j(x)\,|$, for $i\varepsilon I_A$, we obtain

$$\frac{1}{2}\sum_{i\varepsilon I_A}|\,v^T\nabla^2\bar{f}_i v\,|\,\leq \frac{1}{2}\sum_{i\varepsilon I_A}|\,\bar{f}_i(x)\,|\,|\,\bar{u}_i(x)\,|\,+\,o(\|\,h\,\|^2). \tag{5.3.18}$$

Consider now the terms $h^T\nabla^2\Phi_1 v$, $h^T\nabla^2\bar{f}_i v$, $i\varepsilon I_A$. Following lines similar to that used above, it can be shown

$$|\,h^T\nabla^2\Phi_1 v\,|\,\leq \sum_{i\varepsilon I_A}|\,\bar{f}_i(x)\,|\,|\,y_i(x)\,|\,+\,o(\|\,h\,\|^2) \tag{5.3.19}$$

and

$$\sum_{i\varepsilon I_A}|\,h^T\nabla^2\bar{f}_i v\,|\,\leq \sum_{i\varepsilon I_A}|\,\bar{f}_i(x)\,|\,|\,\bar{s}_i(x)\,|\,+\,o(\|\,h\,\|^2). \tag{5.3.20}$$

where $y^T = h^T H_1$, $s^j = h^T\bar{H}_j$, $\bar{s}_j = \sum_{i\varepsilon I_A}|\,s_i^j(x)\,|$, for $j\varepsilon I_A$. Now, define

$$c_i(x) = \frac{1}{2}\,|\,w_i(x)\,|\,+\,\frac{1}{2}\,|\,\bar{u}_i(x)\,|\,+\,|\,\bar{s}_i(x)\,|\,+\,|\,y_i(x)\,|.$$

By (5.3.14), (5.3.17)-(5.3.20) (and noting that $\bar{x}$ is a strict second-order point of Φ, so for Δ_1 sufficiently small $(\eta_i-\lambda_i)\bar{f}_i = -\,|\,\lambda_i-\eta_i\,|\,|\,\bar{f}_i\,|\,$)

$$\Phi(x+h+v)-\Phi(x)\leq -\frac{1}{2}g^T Z(Z^T G_L Z)^{-1}Z^T g \tag{5.3.21}$$

$$+\sum_{i\varepsilon I_A}(-\,|\,\lambda_i-\eta_i\,|+c_i(x))\,|\,\bar{f}_i(x)\,|\,+\,e(x)$$

By assumption (ii), for Δ_1 sufficiently small there exists a positive constant $\bar{b}_2$ such that

$$g^T Z(Z^T G_L Z)^{-1}Z^T g \geq \frac{2}{\bar{b}_2}\,\|\,Z^T g\,\|^2. \tag{5.3.22}$$

Moreover, by assumption (ii), $\lambda_i-\eta_i\neq 0$ and $c_i(x)\to 0$ as $x\to\bar{x}$ (by assumption (iv) and continuity of c_i). Therefore for Δ_1 sufficiently small, and for some $\delta_1>0$,

$$\Phi(x+h+v)-\Phi(x)\leq -\delta_1(\|\,Z^T g\,\|_2^2+\|\,\bar{f}_A\,\|_1)+e(x). \tag{5.3.23}$$

Consider,

$$e(x) = o(\| h+v \|^2) + o(\| h \|^2) - g^T ZEZ^T g$$

$$+ g^T ZEZ^T G_L ZEZ^T g.$$

But

$$\| h+v \|_2^2 = \| h \|_2^2 + \| v \|_2^2$$

$$\leq L_1 \| Z^T g \|_2^2 + L_2 \| \bar{f}_A \|_2^2, L_1 > 0, L_2 > 0.$$

(Since R^{-1}, $(Z^T BZ)^{-1}$, and Z are bounded on Ω.)

Therefore by (2), for Δ_1, Δ_2 sufficiently small,

$$e(x) \leq \frac{1}{2}\delta_1(\| Z^T g \|_2^2 + \| \bar{f}_A \|_1). \tag{5.3.24}$$

Hence, if $\delta = \frac{1}{2}\delta_1$, by (5.3.23) and (5.3.24),

$$\Phi(x+h+v) - \Phi(x) \leq -\delta(\| Z^T g \|_2^2 + \| \bar{f}_A \|_1).$$

■

Theorem 5.1 We assume that

(1) the functions $\bar{f}_i$, $i \varepsilon I_E \bigcup I_{IN}$ are twice continuously differentiable

(2) $\{x^k\}$ is generated by the Algorithm 1 starting from an arbitrary initial point, and $\{x^k\} \varepsilon \Omega$, Ω is compact,

(3) the number of stationary points of Φ in Ω is finite,

(4) all first-order points of Φ in Ω are strict second-order points of Φ,

(5) if the vectors $\nabla \bar{f}_i(x^k)$, $i \varepsilon I_A(x^k, \varepsilon_k)$ are linearly dependent in a neighborhood of a degenerate minimum then there is $k > K_0$ such that

(a) active functions are identified,

(b) rank of A_k does not change and

$rank(A(x^k)) = rank(A(\bar{x}))$,

(c) $W(x)$, $Z(x)$, $R(x)$,$V(x)$ and $S(x)$ are continuous functions of x,

(d) there is a positive constant b_0 such that $\| R^{-1} \| \leq b_0$,

(e) $f_A \varepsilon$ *Range space*(A^T),

(6) the line search condition (v) is satisfied,

(7) if $\{x^{k_i}\}$ is a subsequence and $\bar{x}$ is a first-order point such that

$$x^{k_i} \rightarrow \bar{x}, \text{ and } I_A(x^{k_i},\varepsilon_k) = I_A(\bar{x},0),$$

then the second-order information is approximated so that

$$Z_{k_i}^T B_{k_i} Z_{k_i} \rightarrow \bar{Z}^T\Big(\sum_{i\varepsilon I_E(\bar{x},0)} \sigma_i \nabla^2 \bar{f}_i(\bar{x}) + \sum_{i\varepsilon I_{IN}(\bar{x},0)} \sigma_i^- \nabla^2 \bar{f}_i(\bar{x}) - \sum_{i\varepsilon I_A(\bar{x},0)} \lambda_i \nabla^2 \bar{f}_i(\bar{x})\Big)\bar{Z}.$$

Then, for all δ sufficiently small,

(i) $\varepsilon_k \not\rightarrow 0$,

(ii) $x^k \rightarrow \bar{x}\varepsilon\hat{S}$,

(iii) for k sufficiently large, the asymptotic (Newton) step is executed.

Proof: Part 1 [If $Z_k^T g(x^k) \neq 0$ then $-Z_k^T(Z_k^T B_k Z_k)^{-1} g(x^k)$ $(=h^k)$ is a descent direction for Φ, at x^k.] The first-order change in $\Phi(x,\varepsilon)$ in an arbitrary direction p is

$$\Delta\Phi(x,\varepsilon) = \alpha\left[g^T p + \sum_{i\varepsilon I_{AE}(x,\varepsilon)} |\nabla \bar{f}_i{}^T p| - \sum_{i\varepsilon I_{AI}(x,\varepsilon)} min(0,\nabla \bar{f}_i{}^T p)\right].$$

But when $p = h^k$ the first-order change is just

$$\Delta\Phi(x,\varepsilon) = -\alpha g(x^k)^T Z_k^T(Z_k^T B_k Z_k)^{-1} g(x^k) < 0,$$

and moreover

$$\Phi(x+\alpha h^k,\varepsilon) = \Phi(x,\varepsilon) - \alpha g(x^k)^T Z_k^T(Z_k^T B_k Z_k)^{-1} g(x^k) + o(\alpha).$$

Hence for all α sufficiently small

$$\Phi(x+\alpha h^k,\varepsilon) < \Phi(x,\varepsilon).$$

Part 2 [If $\| A\lambda - g \| \neq 0$, where λ is obtained from the (BLS) subproblem, then $p = A\lambda - g$ is a descent direction for Φ.]

The above result is established in Theorem 4.1.

Part 3 [There exist positive scalars Λ, δ such that $\| x^k - \bar{x} \| \leq \Lambda$, $\bar{x}\varepsilon\bar{S}$, and $I_A(x^k,\varepsilon) = I_A(\bar{x},0)$ implies that

(i) $\| \bar{p} \| = \| A(\bar{x},0)\bar{\lambda} - g(\bar{x}) \| \neq 0$, and

(ii) $(A(x^k,\varepsilon_k)\lambda - g(x^k))^T(A(x^k,\varepsilon_k)\lambda - g(x^k)) = (p^k)^T p^k \geq \delta_d$.]

Since $|\bar{S}|$ is finite it is clear that there exists a positive scalar $\bar{\delta}$ such that if $\bar{x}$ is any member of $\bar{S}$ then $\| A(\bar{x},0)\bar{\lambda} - g(\bar{x}) \| = \| \bar{p} \| \geq \bar{\delta}$

Let us consider the following problem

$$\underset{\lambda}{minimize}\ (\bar{A}\lambda - \bar{g})^T(\bar{A}\lambda - \bar{g})$$

on a compact set

$$\Omega(x,\lambda) = \mathbf{R}^n \times [-1\ ,\ +1]^l \times [0\ ,\ +1]^s,$$

where $l = |I_A(\bar{x},0) \cap I_E|$ and $s = |I_A(\bar{x},0) \cap I_{IN}|$.

By Theorem 2.3.2 in Fiacco [16], pp. 25-26, there exists a positive constant Λ such that for every x satisfying $\| x - \bar{x} \| \leq \Lambda$ the solution to the above problem changes differentiably.

Now, it follows that if $\| x^k - \bar{x} \| \leq \Lambda$, $\bar{x} \varepsilon \bar{S}$, and $I_A(x^k,e_k) = I_A(\bar{x},0)$ then

$$(A(x^k,\varepsilon_k)\lambda - g(x^k))^T(A(x^k,\varepsilon_k)\lambda - g(x^k)) = (p^k)^T(p^k) > \frac{1}{2}\bar{\delta} = \delta_d.$$

Part 4 [For all k sufficiently large the degenerate dropping step is not executed.]

Rather then considering a subsequence let us assume (without loss of generality) that the dropping step is executed for all k. But Algorithm 1 requires that $\| p^k \| = (p^k)^T p^k \geq \delta_d$, $\delta_d > 0$, and therefore by the line search assumption

$$\Phi(x^k,\varepsilon) - \Phi(x^k + \alpha p^k,\varepsilon) \geq \gamma_1 \delta_d.$$

Now, we get that $\Phi(x^k) \rightarrow -\infty$, which is contradictory.

Part 5 [If $x^{k_i} \rightarrow \bar{x}$ then for k_i sufficiently large, $I_A(x^{k_i},\varepsilon_{k_i}) \subseteq I_A(\bar{x},0)$.]

Suppose that $I_A(x^{k_i},\varepsilon_{k_i}) \supset I_A(\bar{x},0)$. We can assume (without loss of generality) that $I_A(x^{k_i},\varepsilon_{k_i}) = I_A(x^{k_i+1},\varepsilon_{k_i+1}), i = 1,\ldots,\infty$.

(i) Suppose $Z_{k_i}^T g(x^{k_i}) \not\rightarrow 0$. Then we can assume that $h^{k_i} \rightarrow \bar{h} \neq 0$. Clearly $-\bar{g}^T\bar{h} > 0$, where $\bar{g} = g(\bar{x})$.

Let $\bar{\beta}=-\bar{g}^T\bar{h}$. By continuity, $g^{k_i}\to\bar{g}$, where $g^{k_i}=g(x^{k_i})$.

Hence for k_i sufficiently large $-(h^{k_i})^T g^{k_i}\geq\frac{1}{2}\bar{\beta}$. By Part 4, h^{k_i} is a descent direction for Φ and therefore, applying the line search condition,

$$\Phi(x^k,\varepsilon)-\Phi(x^k+\alpha p^k,\varepsilon)\geq\gamma_1(\bar{\beta}/2)^2.$$

It follows that $\Phi(x^k)\to-\infty$, a contradiction.

(ii) Suppose $Z_{k_i}^T g(x^{k_i})\to 0$. Since $I_A(x^{k_i},\varepsilon_{k_i})\subseteq I_A(\bar{x},0)$, it follows that for some j, $c_j(x^{k_i})\not\to 0$. A consequence of this is that $\Lambda_k\to 0$. (Suppose $\Lambda_k\not\to 0$. By Part 4, the asymptotic step is attempted for all k sufficiently large. Since Φ is bounded below and $c_j(x^{k_i})\not\to 0$ it follows that for k sufficiently large the asymptotic step is unsuccessfully attempted. This implies $\Lambda_k\to 0$ and $\varepsilon_k\to 0$.) But $\varepsilon_k\to 0$, and $x^{k_i}\to\bar{x}$ implies $I_A(x^{k_i},\varepsilon_{k_i})\subseteq I_A(\bar{x},0)$.

Part 6 [The asymptotic step is successful for all k sufficiently large, $\varepsilon_k\not\to 0$, and $x^k\to\bar{x}\varepsilon\hat{S}$.]

(i) Suppose $\varepsilon_k\to 0$. Then $\Lambda_k\to 0$ and therefore $Z_{k_i}^T g(x^{k_i})\to 0$ for some subsequence $\{x^{k_i}\}$. Then, for all k sufficiently large $\|Z_k^T g(x^k)\|>\Lambda_k$. By Algorithm 1, it follows that Λ_k is not reduced, for all k sufficiently large, and thus $\Lambda_k\not\to 0$. Therefore we have a convergent subsequence $x^k\to\bar{x}$ and $Z_{k_i}^T g(x^{k_i})\to 0$. By Part 5, for k sufficiently large, $I_A(x^{k_i},\varepsilon_{k_i})\subseteq I_A(\bar{x},0)$. However, the assumptions 5(a)-5(e) and $Z_{k_i}^T g(x^{k_i})\to 0$, forces $I_A(x^{k_i},\varepsilon_{k_i})=I_A(\bar{x},0)$. Considering Parts 3 and 4, it follows that for at least one subsequence $\{x^{k_i}\}$, $x^{k_i}\to\bar{x}\varepsilon\hat{S}$, and $I_A(x^{k_i},\varepsilon_{k_i})=I_A(\bar{x},0)$. By Lemma 5.1, for k sufficiently large, iterations k_i+1, k_i+2, ... will be asymptotic steps. It follows that $\varepsilon_k\not\to 0$.

(ii) Suppose $\varepsilon_k\not\to 0$. Thus $\varepsilon_k=\varepsilon>0$, for k sufficiently large . By Part 5 it follows that there exists a subsequence $\{x^{k_i}\}$, such that $x^{k_i}\to\bar{x}$, and $I_A(x^{k_i},\varepsilon_{k_i})=I_A(\bar{x},0)$. Using an argument identical to that used in Part 5(i), we can establish that $\bar{x}\varepsilon S$. By Parts 3 and 4, $\bar{x}\varepsilon\hat{S}$. By Lemma 5.1 and the boundedness of Φ, for k_i sufficiently large, iterations $k=k_i+1$, k_i+2, ... are asymptotic steps and $x^k\to\bar{x}\varepsilon\hat{S}$. ■

5.3.3. Asymptotic 2-step superlinear convergence

As pointed out in Section 3.3, the asymptotic phase of Coleman and Conn's algorithm can be described as

Algorithm 2 (Asymptotic)

(0) Select an x^0 sufficiently close to x^* and set $k \leftarrow 1$.

(1) Determine the Lagrange multipliers $\{\lambda_i^k\}$.

(2) Update $Z_k^T B_k Z_k$ maintaining positive definiteness.

(3) Determine h^k from

$$h^k = -Z_k(Z_k^T B_k Z_k)^{-1} g(x^k).$$

(4) Determine v^k

$$A_k^T v^k = f_A(x^k + h^k).$$

(5) Update:

$$x^{k+1} \leftarrow x^k + h^k + v^k$$

go to (1).

Theoretically, it is not important how step (1) is performed as long as $\| \lambda^k - \lambda^* \| = O(\| x^k - x^* \|)$, $k \rightarrow \infty$, where $g(x^*) = \sum_{i \in I_A(x^*,0)} \lambda_i^* \nabla \bar{f}_i(x^*)$. If A is full rank, the least-squares solution to

$$A(x^k)\lambda = g(x^k)$$

is used. Otherwise, the solution to the bounded least squares problem

$$A(x^k)\lambda = g(x^k)$$

$$\textit{subject to} \quad -1 \leq \lambda_i \leq +1, \quad i \in I_{AE}$$

$$0 \leq \lambda_i \leq +1, \quad i \in I_{AI}$$

is used.

The result that the Algorithm 2 generates a sequence $\{x^k\}$, which satisfies

$$\frac{\| x^{k+1}-x^* \|}{\| x^{k-1}-x^* \|} \rightarrow 0$$

will be established. First, the assumptions and the preliminary results will be given.

Assumptions

(A) $\bar{f}_i$, $i \varepsilon I_E \bigcup I_{IN}$ are twice continuously differentiable;

(B) the second-order sufficiency conditions are satisfied at x^*;

(C) $\{x^k\}$ is generated by Algorithm 2, and $\{x^k\} \varepsilon \Omega$, where Ω is a compact set;

(D) if the columns of $A(x) = \left[\cdots \nabla \bar{f}_i \cdots \right]_{i \varepsilon I_A}$ are linearly dependent at x^* then the following assumptions hold for $\{x^k\} \varepsilon \Omega$:

(a) active functions are identified,

(b) rank of A_k does not change and $rank(A(x^k)) = rank(A(x^*))$,

(c) $W(x)$, $Z(x)$, $R(x)$, $V(x)$ and $S(x)$ are Lipschitz continuous functions of x (recall that $A(x) = \begin{bmatrix} W(x) & Z(x) \end{bmatrix} \begin{bmatrix} R(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V(x)^T \\ S(x)^T \end{bmatrix}$);

(d) there is a positive constant b_0 such that $\| R(x)^{-1} \| \leq b_0$,

(e) $f_A \varepsilon$ *Range space*(A^T).

Following the work by Coleman and Conn [10], we will first establish that the null-space step h^k, is bounded by the distance between x^k and x^*.

Lemma 5.2 Under assumptions (A)-(D) and assuming that there exist scalars b_1, b_2 $(0 < b_1 \leq b_2)$ such that

$$b_1 \| y \|^2 \leq y^T (Z_k^T B_k Z_k) y \leq b_2 \| y \|^2 \text{ all } k; \text{ all } y \varepsilon Span(Z_k) \tag{5.4.1}$$

then there exists an $L_1 > 0$ such that

$$\| h^k \| \leq L_1 \| x^k - x^* \|.$$

Proof: See [10]. ■

A similar bound exists for the range-space component v^k.

Lemma 5.3 Under assumptions (A)-(D) and (5.4.1), there exists $L_2 > 0$ such that

$$\| v^k \| \leq L_2 \| x^k - x^* \|.$$

Proof: For the nondegenerate case see [10]. In the degenerate case, we compute v^k as

$$v^k = -W_k R_k^{-T} V_k^T f_A \left(x^k + h^k \right). \tag{5.4.2}$$

But

$$\bar{f}_i(x^k + h^k) = \bar{f}_i(x^k) + O(\| h^k \|^2), \; i \varepsilon I_A \tag{5.4.3}$$

and thus

$$\| v^k \| \leq \| W_k \| \, \| R_k^{-T} \| \, \| V_k^T \| \, \| f_A(x^k) \| + O(\| h^k \|^2). \tag{5.4.4}$$

Using Lipschitz continuity of $\bar{f}_i$ and the boundedness of $\| R_k^{-1} \|$, the result follows. ■

Lemma 5.4 Under the assumptions of Lemmas 5.2 and 5.3, there exists an $L_3 > 0$ such that

$$\| x^{k+1} - x^* \| \leq L_3 \| x^k - x^* \|.$$

Proof: Recalling that $x^{k+1} = x^k + h^k + v^k$, the proof follows directly from Lemmas 5.3 and 5.4. ■

The columns of (A_k, Z_k) span $\mathbf{R}^n$, and therefore we can write

$$x^k - x^* = A_k w^k + Z_k u^k. \tag{5.4.5}$$

When the columns of A_k are linearly dependent w^k is not unique and in that case we think of it as the minimum-l_2-norm least-squares solution to

$$A_k w^k = x^k - x^*.$$

Using this representation of $x^k - x^*$, the following lemma can be easily established.

Lemma 5.5 Under assumptions (A)-(D) and (5.4.1), if

$$\frac{\| w^{k+1} \|}{\| x^k - x^* \|} \to 0 \quad \text{and} \quad \frac{\| u^{k+1} \|}{\| x^{k-1} - x^* \|} \to 0$$

$$\text{then} \quad \frac{\| x^{k+1} - x^* \|}{\| x^{k-1} - x^* \|} \to 0.$$

Proof: From the representation for $x^k - x^*$,

$$x^k - x^* = A_k w^k + Z_k u^k, \tag{5.4.6}$$

we get

$$\frac{\| x^{k+1}-x^* \|}{\| x^{k-1}-x^* \|} = \frac{\| A_{k+1}w^{k+1} - Z_{k+1}u^{k+1} \|}{\| x^{k-1}-x^* \|} \tag{5.4.7}$$

$$= \frac{\| W_{k+1}R_{k+1}V_{k+1}^T w^{k+1} - Z_{k+1}u^{k+1} \|}{\| x^{k-1}-x^* \|}$$

$$\leq \| W_{k+1} \| \, \| R_{k+1} \| \, \| V_{k+1}^T \| \frac{\| w^{k+1} \|}{\| x^{k-1}-x^* \|} + \| Z_{k+1} \| \frac{\| u^{k+1} \|}{\| x^{k-1}-x^* \|}$$

But

$$\| x^k - x^* \| \leq L_3 \| x^{k-1} - x^* \|, \; L_3 > 0 \tag{5.4.8}$$

or

$$\frac{1}{\| x^{k-1}-x^* \|} \leq L_3 \frac{1}{\| x^k - x^* \|}, \tag{5.4.9}$$

and that implies

$$\frac{\| x^{k+1}-x^* \|}{\| x^{k-1}-x^* \|} \leq \| W_{k+1} \| \, \| R_{k+1} \| \, \| V_{k+1}^T \| \, L_3 \frac{\| w^{k+1} \|}{\| x^k - x^* \|} \tag{5.4.10}$$

$$+ \| Z_{k+1} \| \frac{\| u^{k+1} \|}{\| x^{k-1}-x^* \|}.$$

From the assumption (D) and Lemma 5.4

$$\| W_{k+1} \| \, \| R_{k+1} \| \, \| V_{k+1}^T \| \, L_3 \leq L_4$$

and

$$\| Z_{k+1} \| \leq L_5,$$

where, L_4 and L_5 are positive constants. So

$$\frac{\| x^{k+1}-x^* \|}{\| x^{k-1}-x^* \|} \leq L_4 \frac{\| w^{k+1} \|}{\| x^k - x^* \|} + L_5 \frac{\| u^{k+1} \|}{\| x^{k-1}-x^* \|} \tag{5.4.11}$$

and consequently

$$\frac{\| x^{k+1}-x^* \|}{\| x^{k-1}-x^* \|} \to 0.$$

■

Lemma 5.5 allows us to prove 2-step superlinear convergence by showing separately that

$$\frac{\| w^{k+1} \|}{\| x^k - x^* \|} \to 0 \quad \text{and} \quad \frac{\| u^{k+1} \|}{\| x^{k-1} - x^* \|} \to 0. \tag{5.4.12}$$

Theorem 5.2 Under the assumptions of Lemmas 5.2 and 5.3, and assuming that $x^k \to x^*$, $k \to \infty$, then

$$\frac{\| w^{k+1} \|}{\| x^k - x^* \|} \to 0$$

Proof: If A is full rank see [10] *pp.* 129-130. Otherwise, from the Algorithm 2

$$x^{k+1} = x^k + v^k + h^k \tag{5.4.13}$$

$$= x^k - W_k R_k^{-T} V_k^T f_A(x^k + h^k) + h^k \tag{5.4.14}$$

$$= x^k - W_k R_k^{-T} V_k^T f_A(x^k) + h^k + y^k, \tag{5.4.15}$$

where $\| y^k \| = O(\| h^k \|^2)$. But for each $i \varepsilon I_A$

$$\bar{f}_i(x^k) = \nabla \bar{f}_i(\xi_i^k)^T (x^k - x^*), \tag{5.4.16}$$

where $\xi_i^k = x^k + \theta_i^k (x^k - x^*)$, $0 \leq \theta_i^k \leq 1$.

Thus if we define matrices $\tilde{A}_k = (\cdots \nabla \bar{f}_i(\xi_i^k) \cdots)_{i \varepsilon I_A}$, and $E_k = (\tilde{A}_k - A_k)$, then (5.4.15) becomes

$$x^{k+1} = x^k - W_k R_k^{-T} V_k^T A_k^T (x^k - x^*) - W_k R_k^{-T} V_k^T E_k^T (x^k - x^*) + h^k + y^k, \tag{5.4.17}$$

but $A_k = W_k R_k V_k^T$, so (5.4.17) becomes

$$x^{k+1} = x^k - W_k W_k^T (x^k - x^*) - W_k R_k^{-T} V_k^T E_k^T (x^k - x^*) + h^k + y^k, \tag{5.4.18}$$

Using (5.4.18) and then multiplying by W_k^T, we obtain

$$W_k^T (x^{k+1} - x^*) = -R_k^{-T} V_k^T E_k^T (x^k - x^*) + W_k^T y^k. \tag{5.4.19}$$

Adding $W_{k+1}^T (x^{k+1} - x^*)$ to both sides of (5.4.19) yields

$$W_{k+1}^T (x^{k+1} - x^*) = -R_k^{-T} V_k^T E_k^T (x^k - x^*) + (W_{k+1} - W_k)^T (x^{k+1} - x^*) + W_k^T y^k, \tag{5.4.20}$$

and thus, using (5.4.5), we obtain

$$R_{k+1} V_{k+1}^T w^{k+1} = -R_k^{-T} V_k^T E_k^T (x^k - x^*) + (W_{k+1} - W_k)^T (x^{k+1} - x^*) + W_k^T y^k, \tag{5.4.21}$$

or

$$w^{k+1} = V_{k+1}R_{k+1}^{-1}\left[-R_k^{-T}V_k^TE_k^T(x^k-x^*)+(W_{k+1}-W_k)^T(x^{k+1}-x^*)+W_k^Ty^k\right]. \quad (5.4.22)$$

But $\| y^k \| = O(\| h^k \|^2)$, and therefore using Lemma 5.2, assumption (D) and compactness, there exists an $L_6 > 0$ such that

$$\| V_{k+1}R_{k+1}^{-1}W_k^Ty^k \| \leq L_6 \| x^k-x^* \|^2. \quad (5.4.23)$$

Using Lipschitz continuity of $\nabla\bar{f}_i$, $i\varepsilon I_A$ and assumption (D) we know that there exists $\gamma > 0$ such that

$$\| W_{k+1}-W_k \| \leq \gamma \| x^{k+1}-x^k \|$$

$$\leq \gamma \| x^{k+1}-x^* \| + \gamma \| x^k-x^* \|.$$

Now, from Lemmas 5.2, 5.3 and 5.4 it follows that

$$\| V_{k+1}R_{k+1}^{-1} \| \| (W_{k+1}-W_k)^T(x^{k+1}-x^*) \| \leq L_7 \| x^k-x^* \|^2, \quad (5.4.24)$$

for some $L_7 > 0$. Therefore

$$\| w^{k+1} \| \leq \| V_{k+1} \| \| R_{k+1}^{-1} \| \| R_k^{-T} \| \| V_k^T \| \| E_k^T \| \| x^k-x^* \| \quad (5.4.25)$$

$$+ (L_6 + L_7) \| x^k-x^* \|^2.$$

But by the definition of E_k and the convergence assumption $\| E_k \| \to 0$, $k \to \infty$, and therefore the result follows. ■

Theorem 5.3 Under the assumptions of Theorem 5.2, and assuming that

$$Z(x^k)^TB(x^k)Z(x^k) \to Z(x^*)^TG_L(x^*,\lambda^*)Z(x^*),$$

then

$$\frac{\| u^{k+1} \|}{\| x^{k-1}-x^* \|} \to 0.$$

Proof: See [10] *pp.* 130-132. ■

Theorem 5.4 Under the assumptions (A)-(D) and

(i) $x^k \to x^*$, $k \to \infty$,

(ii) there exist scalars b_1, b_2 $(0<b_1\leq b_2)$ such that

$$b_1 \| y \|^2 \leq y^T(Z_k^TB_kZ_k)y \leq b_2 \| y \|^2 \text{ all } k; \text{ all } y\varepsilon Span(Z_k)$$

then

$$\frac{\| x^{k+1}-x^* \|}{\| x^{k-1}-x^* \|} \to 0.$$

Proof: It follows directly from Lemma 5.5 and Theorems 5.2 and 5.3. ■

Finally, the local convergence of Algorithm 2 can be established. That is, provided x^* is sufficiently close to x^*, then $x^k \to x^*$.

Lemma 5.6 Suppose that x^1 and x^2 are generated by Algorithm 2 with starting vector x^0. Under assumptions (A)-(D), if x^0 is sufficiently close to x^*, it follows that

$$\| A_2 w^2 \| \leq \frac{1}{4} \| x^0 - x^* \|.$$

Proof: If A is full rank see [10] *p.* 132. Otherwise, by (5.4.25)

$$\| A_2 w^2 \| \leq \| W_2 \| \| R_2 \| \| V_2^T \| \Big[\| V_2 \| \| R_2^{-1} \| \| R_1^{-T} \| \| V_1^T \| \| E_1^T \| \| x^1 - x^* \| + (L_4 + L_5) \| x^1 - x^* \|^2 \Big]. \tag{5.4.26}$$

Now, using Lemma 5.4, we have , for x^0 sufficiently close to x^*

$$\| A_2 w^2 \| \leq \frac{1}{4} \| x^0 - x^* \|.$$

■

Lemma 5.7 Under assumptions (A)-(D) and assuming that

$$Z(x^k)^T B(x^k) Z(x^k) \to Z(x^*)^T G_L(x^*,\lambda^*) Z(x^*), \text{ as } x^k \to x^*,$$

then for x^0, $Z(x^0)^T B(x^0) Z(x^0)$ sufficiently close to x^*, $Z(x^*)^T G_L(x^*,\lambda^*) Z(x^*)$, respectively,

$$\| u^2 \| \leq \frac{1}{4} \| x^0 - x^* \|.$$

Proof: See [10] p. 133. ■

Theorem 5.5 Under the assumptions of Lemma 5.6 and 5.7, then for x^0, $Z(x^0)^T B(x^0) Z(x^0)$ sufficiently close to x^*, $Z(x^*)^T G_L(x^*,\lambda^*) Z(x^*)$, respectively,

$$x^k \to x^*,$$

where $\{x^k\}$ is generated by Algorithm 2.

Proof: By Lemmas 5.6 and 5.7 and (5.4.5), $\| x^2 - x^* \| \leq \frac{1}{2} \| x^0 - x^* \|$. It follows that $x^{2k} \rightarrow x^*$. But, by Lemma 5.4 $x^{2k+1} \rightarrow x^*$, and therefore $x^k \rightarrow x^*$. ■

This completes the proof of the 2-step superlinear convergence of the extended Bartels/Conn algorithm for nonlinear l_1 optimization.

CHAPTER 6

Degeneracy and the Murray/Overton algorithm

Murray and Overton [30] consider the following unconstrained nonlinear l_1 problem:

$$\underset{x\varepsilon\mathbf{R}^n}{\textit{minimize}}\ \Phi_1(x) = \sum_{i=1}^{m} |f_i(x)|, \tag{UNLL1)}$$

where $f_i:\mathbf{R}^n \rightarrow \mathbf{R}^1$ are twice continuously differentiable.

They transform (UNLL1) into the nonlinearly constrained optimization problem

$$\underset{x\varepsilon\mathbf{R}^n,\ u\varepsilon\mathbf{R}^m}{\textit{minimize}}\ \sum_{i=1}^{m} u_i \tag{ELP}$$

$$\textit{subject to}\ \ c_i^{(\sigma)}(x,u) \geq 0\ ,\ i\varepsilon\{1, \cdots, m\};\ \sigma\varepsilon\{-1, +1\}$$

where $c_i^{(\sigma)}(x,u) = u_i - \sigma f_i(x)$.

Notice that the new problem has m more variables. For that reason they do not use the general nonlinear optimization algorithm to solve (ELP) but exploit its special structure and reduce it to a problem with n variables. Also, they point out that an important feature of transforming (UNLL1) into (ELP) is the fact that the l_1 function Φ_1 is a natural merit function for (ELP). Notice that in this unconstrained l_1 case, the exact penalty function Φ used by Bartels and Conn [1] is just $\mu\Phi_1$ (see (5.1.2)).

Basically, the method of Murray and Overton [30] for solving (UNLL1) consists of

1) obtaining a direction of descent from a local quadratic model based on a projected Lagrangian algorithm for (ELP), and

2) taking a step along the given direction which reduces the natural merit function Φ_1. The multiplier estimates, used to form the Lagrangian function, are obtained as the solution to

$$\underset{\lambda_C}{\textit{minimize}}\ \ \| A\lambda_C + g \|_2^2. \tag{6.1}$$

It is interesting to point out that Murray and Overton also make use of the second-order multiplier estimates:

$$\underset{\lambda_W}{minimize} \quad \| A\lambda_W - (g + B(h+v)) \|_2^2. \tag{6.2}$$

The second-order multiplier estimates are used to ensure that the accuracy with which the minimum on the manifold has been approximated, and the accuracy of the multiplier estimates, are sufficiently high compared to the uncertainty of the signs of multipliers.

The first-order estimates are used to define the Hessian of the Lagrangian function in a rather unusual way. Based on the fact that an active function for which the corresponding multiplier is out of bounds will not necessarily be deleted from the active set, Murray and Overton decide to use the following multipliers

$$\lambda_i = \begin{cases} -1 & \text{, if } (\lambda_C)_i < -1 \\ +1 & \text{, if } (\lambda_C)_i > +1 \\ (\lambda_C)_i & \text{, otherwise.} \end{cases} \tag{6.3}$$

This chopped off version of the Lagrange multipliers is used to form an approximation to the Hessian of Lagrangian. It is worth noting that our method of handling degeneracy at a stationary point by solving

$$\begin{aligned} &\underset{\lambda}{minimize} \quad \| A\lambda - g \|_2^2 \\ &subject\ to \quad -1 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AE} \\ &\qquad\qquad 0 \leq \lambda_i \leq +1 , \quad i \varepsilon I_{AI} \end{aligned} \tag{6.4}$$

obtains the Lagrange multipliers λ_i's in corresponding bounds, too, but now we have a guaranteed direction of descent if the residual is nonzero. We have not been able to see how to generalize the above-mentioned change in the local quadratic model in order to cope with degeneracy.

Another interesting feature of the Murray/Overton algorithm is the fact that the range-space component of the asymptotic phase is used to assess whether too many functions are active. The theorem that they state in this regard is given here according to our notation:

Theorem 6.1 Assume A has full rank and let v be defined as the solution of minimum-l_2-norm to

$$A^T v = -f_A. \tag{6.1}$$

If $g^T v - \sum_{i \varepsilon I_A} |f_i| > 0$, then for some i either $f_i > 0$ and $\lambda_i > +1$, or $f_i < 0$ and $\lambda_i < -1, i \varepsilon I_A$.

Their interpretation of the above situation is that too many functions have been selected to be active and are being forced to be approximately zero, thus forcing the inactive functions to increase in modulus more than the active ones are decreasing. We have tried a similar idea, but the results have been unimpressive. It seems, by observation, that the conditions of the theorem 6.1 are only satisfied far from a stationary point, so the test does not help us very often.

Although an active function, let us say the i^{th}, corresponding to a multiplier out of bounds can be removed from the set of active functions in the same way as in the Bartels/Conn algorithm, Murray and Overton try to use second order information to delete the active function. To explain their approach, let us define $\tilde{A}$ with A the column a_i deleted, and $\tilde{Z}$ by

$$\tilde{A}^T \tilde{Z} = 0, \tilde{Z}^T \tilde{Z} = I, \tilde{Z} = [Z\ z]. \tag{6.6}$$

They first try computing the Newton step, defined by

$$p_1 = -\tilde{Z}(\tilde{Z}^T B \tilde{Z})^{-1} \tilde{Z}^T (g + sign(\lambda_C)_i f_i), \tag{6.7}$$

where the i^{th} active function will be deleted. If the direction p_1 yields a descent, it will be taken as a search direction. Otherwise, they use p_2, where

$$p_2 = \tilde{Z} \begin{bmatrix} -(Z^T B Z)^{-1} Z^T (g + sign(\lambda_C)_i f_i) \\ -z^T (g + sign(\lambda_C)_i f_i) \end{bmatrix}. \tag{6.8}$$

They motivate the use of p_2 by pointing out that it combines the Newton step in the null space of the previous active function Jacobian with the steepest descent step in the new direction made available by moving off an active function. The steepest descent component is justified by necessity to ensure the first order feasibility for (ELP), and use is made of a Newton component in hopes of giving a larger reduction in the merit function Φ_1.

The k^{th} step of Murray/Overton algorithm is summarized as follows:

Algorithm (Murray/Overton)

{Select any $x \varepsilon \mathbf{R}^n$, $\varepsilon_{activity}$ and other tolerances}

repeat

{Identify I_A}

{Form g; compute the Lagrange multipliers}

{Obtain the decomposition $A = [W\ Z]\begin{bmatrix} R \\ 0 \end{bmatrix}$}

{Form $Z^T B Z$}

{Obtain the decomposition $Z^T B Z + E = LDL^T$ for a "small" diagonal matrix E}

if {(2.3.11) and (2.3.12) are approximately satisfied }

then

/* Asymptotic phase */

{$R^T v_W = -f_A$, and set $v = W v_W$}

{$(LDL^T) h_Z = -Z^T(g + Bv)$, and set $h = Z h_Z$}

if {sufficient decrease in Φ_1 expected by $g^T h < 0$}

then

go to ALPHA

else

go to QP2

/* Global phase */

DROP:

{Find p that deletes a term from the active set}

go to ALPHA

{Find the normal descent direction p}

QP2:

$p = -Z(LDL^T)^{-1}Z^T g$

ALPHA:

{Find α}

$x \leftarrow x + \alpha p$

until {termination test satisfied or $\Phi_1(x)$ does not decrease}

The matrix E in the above algorithm is zero if the LDL^T (Cholesky) decomposition of Z^TBZ exists, i.e. if Z^TBZ is numerically positive definitive. If this is not the case, E is chosen to be a diagonal matrix with diagonal elements large enough to ensure that L and D will exist. See [19] for further discussion of this method of modified *Cholesky decomposition.*

Although it looks as if the Murray/Overton algorithm deviates from our framework by performing a line search after the asymptotic phase, their line search algorithm always attempts a step size of one (i.e. $\alpha = 1$), and consequently we get the same outcome. The algorithm has an elaborate section to test whether the first-order optimality conditions are approximately satisfied. For ease of discussion we shall state the complete Murray/Overton algorithm in a step-by-step fashion (as done in the original paper) and then point out what might be done in order to handle the degeneracy.

Algorithm (Murray/Overton) Step by step

1. [Select active set.] Form f_A, A. Let $g = \sum_{i \in I_A} \sigma_i \nabla f_i$, where

$$\sigma_i = \begin{cases} -1 & \text{, if } f_i(x) < 0 \\ +1 & \text{, if } f_i(x) \geq 0 \end{cases} \text{, } i \in I_A.$$

2. [QR decomposition.] Obtain $A = [W\ Z]\begin{bmatrix} R \\ 0 \end{bmatrix}$.

3. [First-order multiplier estimate and one direction of search direction.]

Solve

$R\lambda_C = -W^T g$ and $R^T v_W = -f_A$,

and set $v = Wv_W$.

4. [Projected Hessian.] Form $Z^T BZ$, where B is given by

$$B = \sum_{i \in I_A} \lambda_i \nabla^2 f_i + \sum_{i \notin I_A} \sigma_i \nabla^2 f_i,$$

where

$$\lambda_i = \begin{cases} -1 & \text{, if } (\lambda_C)_i < -1 \\ +1 & \text{, if } (\lambda_C)_i > +1 \\ (\lambda_C)_i & \text{, otherwise.} \end{cases}$$

5. [Modified Cholesky decomposition.] Obtain $Z^T BZ + E = LDL^T$ for a "small" diagonal matrix E.

6. [Termination criteria.] If $\| f_A \|_2$ and $\| Z^T g \|_2$ are greater than prescribed tolerances then go to Step 7.

Otherwise:

- if $|\lambda_C| < 1$ and $E = 0$ ($Z^T BZ$ numerically positive definite) then STOP – x satisfies the convergence criteria.

- if $\max |(\lambda_C)_i| > 1$ then go to Step 13.

- if $\max |(\lambda_C)_i| = 1$ then optionally try the zero-multiplier procedure (see [18]).

- if $E \neq 0$ then optionally try the saddle-point procedure (see [29] and [17]).

7. [Other component of the search direction.] If $E \neq 0$ then go to Step 12. Otherwise solve $(LDL^T)h_Z = -Z^T(g + Bv)$, and set $h = Zh_Z$

8. [Second-order multiplier estimates.] Solve $R\lambda_W = -W^T(g + B(h+v))$.

9. [Check whether too many functions are active.] If $g^T v - \sum_{i \in I_A} |f_i| > 0$ then go to Step 13.

10. [Check multiplier estimates.] Let $|(\lambda_C)_j| = \max_{i \in I_A} |(\lambda_C)_i|$ and let $\rho_1 = max(|(\lambda_C)_j|, |(\lambda_W)_i|) - 1$ and $\rho_2 = (\| Z^T g \|_2 + \| f_A \|_2)/(1 + \Phi_1/m)$. If $\rho_2 < min(1,\rho_1)$ and $2|(\lambda_C)_j - (\lambda_W)_i| < \rho_1$ then go to Step 13.

11. [Direction of search.] If $g^T h < 0$ then set $p = v + h$ and go to Step 14.

12. [Alternative direction of search.] Solve $(LDL^T)h_1 = -Z^T g$ and set $p = Zh_1$. Go to Step 14.

13. [Delete a term from the active set.] Delete f_j from the active set, where $|(\lambda_C)_j| = \max_{i \in I_A} |(\lambda_C)_i|$, and compute the direction of search p by (6.7) or (6.8).

14. [Line search.] Replace x by $x + \alpha p$, where α is obtained from the line-search algorithm [28].

6.1. Extended Murray/Overton algorithm

In order to handle degeneracy in the Murray/Overton algorithm we would propose trying the same modifications that we applied to the Bartels/Conn algorithm. The following steps would be changed in the presence of degeneracy:

2'. [QRP^T decomposition.] Obtain

$$A(x^k) = \begin{bmatrix} W(x^k) & Z(x^k) \end{bmatrix} \begin{bmatrix} R(x^k) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V(x^k)^T \\ S(x^k)^T \end{bmatrix}.$$

3'. [First-order multiplier estimate and one direction of the search direction.] Solve

$$\begin{aligned} &\underset{\lambda_C}{\text{minimize}} \quad \| A\lambda_C - g \|_2^2 \\ &\text{subject to} \quad -1 \leq (\lambda_C)_i \leq +1, \quad i \in I_A, \end{aligned} \tag{6.9}$$

and,

$$R^T v_W = -V^T f_A(x^k),$$

and set $v = W v_W$.

4'. [Projected Hessian.] Form $Z^T B Z$, where B is given by

$$B = \sum_{i \in i_A} \lambda_i \nabla^2 f_i + \sum_{i \in / I_A} \sigma_i \nabla^2 f_i,$$

and where the λ_i's are obtained from (6.9).

6'. [Termination criteria.] If $\| f_A \|_2$ and $\| Z^T g \|_2$ are greater than prescribed tolerances, then go to Step 7.

Otherwise:

– if $|\lambda_C| < 1$ and $E = 0$ ($Z^T BZ$ is numerically positive definite) and

$\| A\lambda_C - g \|_2^2 = 0$

then STOP – x satisfies the convergence criteria.

– if $\| A\lambda_C - g \|_2^2 \neq 0$ then go to Step 13.

8., 9., 10.

Should be dispensed with.

13'. [Descent from a degenerate stationary point.] $p = A\lambda_C - g$ is a descent direction.

We have not tested these proposals computationally, nor have we investigated them theoretically.

CHAPTER 7

Numerical results

An experimental code, called CNLL1, has been implemented based on the ideas given in Chapter 5. The main body of the code resembles the code for nondegenerate l_1 problems by Bartels and Conn [1]. As pointed out in the Chapter 4, the (BLS) subproblem is currently solved by a modified *NNLS* algorithm (see Lawson and Hanson [23]). The code for the complete orthogonal decomposition is taken from the same source. Finally, the line search algorithm is obtained from Murray and Overton [28].

So far, not all of the requirements necessary for the convergence proofs have been implemented. The most important missing ingredients are the continuously varying Z , R and V matrices from the orthogonal decomposition of A

$$A(x) = \begin{bmatrix} W(x) & Z(x) \end{bmatrix} \begin{bmatrix} R(x) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V(x)^T \\ S(x)^T \end{bmatrix}.$$

The code from Lawson and Hanson [23] uses the standard Housholder transformations and, as shown in the work by Coleman and Sorenson [7], those orthogonal transformations do not necessarily give continuous changes even in the case of full-rank A. Some ways to make an orthogonal decomposition a continuous function of x were suggested by Coleman and Sorenson [7]. It is important to point out that, based on the following assumptions:

(a) f_i's are twice continuously differentiable, and

(b) A does not change rank in some neighborhood of x^*,

we can expect that there is a neighborhood of x^* such that the column ordering of A, induced by the column norms of A as used in Lawson and Hanson [23] will not change. So we can apply the results from Coleman and Sorenson [7] for the transformations comprising W and Z to get a continuous orthogonal decomposition

$$PAQ = \begin{bmatrix} W & Z \end{bmatrix} \begin{bmatrix} T \\ 0 \end{bmatrix},$$

where T is upper trapezoidal. The continuity of these transformations should imply an unchanging row ordering of T, from which the suggestions of Coleman and Sorenson [7] are applicable again to the transformations which produce V and S.

Also, as shown in section 4.3, the minimum-l_2-norm λ is not found. The code uses modified Lawson and Hanson [23] NNLS (nonnegative least squares) subroutine which tries to keep as many λ_i's at their bounds as it can.

In spite of not enforcing the above theoretical assumptions, we have not encountered any troubles as a result in solving test problems. For that reason the implementation of a complete orthogonal decomposition which maintains continuity of Z, R and V and finding the minimum-l_2-norm λ have been left for some future date.

The code has been tested on some well known problems compiled in Moré et al. [26], and given by El-Attar et al. [14]. Although the problems from Moré et al. [26] have originally been used mainly for testing l_2 methods, they are quite interesting for testing l_1 problems, too. Note, however, that the l_2 minimizer and minimum values reported in the literature will often differ from the l_1 minimizer and minimum.

For each of the test problems the name, the initial point, the minimizer obtained by our code, and the minimum value of the l_1 function are listed below.

EL-ATTAR 5.1

$x^0 = (1, 1)$

$x^* = (2.8425033e00, 1.9201751e00)$

$\sum_{i=1}^{m} |f_i(x^*)| = 4.7042427e-01$

EL-ATTAR 5.2

$x^0 = (1, 1, 1)$

$x^* = (5.3606100e-1, 0.0, 3.1929055e-02)$

$$\sum_{i=1}^{m} |f_i(x^*)| = 7.8942270e00$$

EL-ATTAR 6.1

$x^0 =$ (2, 2, 7, 0, -2, 1)

$x^* =$ (2.2407476e00, 1.8576930e00, 6.7700487e00 -1.6448982e00, 1.6589113e-01, 7.4228294e-01)

$$\sum_{i=1}^{m} |f_i(x^*)| = 5.5981421e-01$$

EL-ATTAR 6.2(I)

$x^0 =$ (1.7060000e-01, 1.7578000e00, 0.0, 9.5370000e-01, 0.0)

$x^* =$ (0.0, 8.5628785e00, 2.9312344e01, 2.4737458e01, 1.2228476e01)

$$\sum_{i=1}^{m} |f_i(x^*)| = 7.0718168e-02$$

MURRAY/OVERTON 1 (BARD)

$x^0 =$ (1, 1, 1)

$x^* =$ (1.0093749e-01, 1.5251270e00, 1.9721407e00)

$$\sum_{i=1}^{m} |f_i(x^*)| = 1.2434061e-01$$

MURRAY/OVERTON 3 (MADSEN)

$x^0 =$ (3, 1)

$x^* =$ (0.0, -2.0497181e-03)

$$\sum_{i=1}^{m} |f_i(x^*)| = 1.0000021e00$$

WOOD

$x^0 =$ (-3, -1, -3, -1)

$x^* =$ (1, 1, 1, 1)

$$\sum_{i=1}^{m} |f_i(x^*)| = 0.0$$

POWELL

$x^0 =$ (3, -1, 0, 1)

$x^* =$ (5.5879354e-09, -3.7252844e-10, 1.2501352e-09, 1.7157965e-09)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 2.9039014e-09

ROSENBROCK

$x^0 =$ (-1.2, 1.0)

$x^* =$ (9.9999949e-01, 9.9999899e-01)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 5.0664205e-07

WATSON

$x^0 =$ (0, 0, 0, 0)

$x^* =$ (-4.4271325e-01, 1.1932080e00, -4.7676029e-01, 3.8448579e-01)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 6.0185842e-01

HELIX

$x^0 =$ (-1, 0, 0)

$x^* =$ (1.0000001e+00, 2.9331204e-11, 4.2632564e-14)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 4.6643683e-10

MARTENSON

$x^0 =$ (-1.2, 1.0, 1.0)

$x^* =$ (-1.3629154e00, 2.8358438e00, 3.8859299e00)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 2.0052319e01

LRNKLS

$x^0 =$ (-1.2, 1.0, 1.0)

$x^* =$ (-3.2099085e-1, 2.6749238e-01, 1.0837855e00)

$$\sum_{i=1}^{m} |f_i(x^*)| = 3.6899270e\text{-}06$$

BIGGS-EXP5

$x^0 =$ (1, 1, 1, 1, 1)

$x^* =$ (6.1882386e-01, 1.0397100e01, 5.8688609e-01, 4.6106889e00, 3.3088040e00)

$$\sum_{i=1}^{m} |f_i(x^*)| = 3.1531560e\text{-}04$$

BIGGS-EXP6

$x^0 =$ (1, 1, 1, 1, 1)

Failure.

POLYNOMIAL CURVE FITTING

$x^0 =$ (-0.1, 0.1, 0.0, 0.0, 0.1)

$x^* =$ (1.0000000e00, -2.0833333e00, 1.4583333e00, -4.1666665e-01, 4.1666664e-01)

$$\sum_{i=1}^{m} |f_i(x^*)| = 2.3376197e\text{-}07$$

BOX-3

$x^0 =$ (-0.1, 0.1, 1.0)

$x^* =$ (-2.6958381e-01, -2.6958383e-01, -2.4883775e-08)

$$\sum_{i=1}^{m} |f_i(x^*)| = 1.3479876e\text{-}07$$

BOX-2

$x^0 =$ (-0.1, 0.1)

$x^* =$ (1.0000000e00, 9.9999754e00)

$$\sum_{i=1}^{m} |f_i(x^*)| = 1.4104855e\text{-}09$$

BROWN & DENNIS

$x^0 =$ (25, 5, -5, -1)

$x^* =$ (-1.0223572e+01, 1.1908427e+01, -4.5804122e-01, 5.8031970e-01)

$$\sum_{i=1}^{m} |f_i(x^*)| = 9.0323431e+02$$

CRAGG & LEVY

$x^0 = (1, 2, 2, 2)$

$x^* = (-6.5421164e-03, 9.9357426e-01, 1.0021038e+01, 1.0000000e00)$

$$\sum_{i=1}^{m} |f_i(x^*)| = 1.0642390e-05$$

BEALE

$x^0 = (1, 1)$

$x^* = (2.9999955e+00, 4.9999931e-01)$

$$\sum_{i=1}^{m} |f_i(x^*)| = 3.6954880e-06$$

BRANIN

$x^0 = (2, 0)$

$x^* = (-6.9075031e-07, 6.9080852e-07)$

$$\sum_{i=1}^{m} |f_i(x^*)| = 4.1446806e-06$$

FREUDENSTEIN & ROTH

$x^0 = (0.5, -2)$

$x^* = (6.4427691e+00, -8.9837897e-01)$

$$\sum_{i=1}^{m} |f_i(x^*)| = 9.8979273e+00$$

JENNRICH & SIMPSON

$x^0 = (0.3, 0.4)$

$x^* = (2.5584278e-01, 2.5584280e-01)$

$$\sum_{i=1}^{m} |f_i(x^*)| = 3.2091946e+01$$

KOWALIK & OSBORN

$x^0 =$ (0.25, 0.39, 0.415, 0.39)

$x^* =$ (-3.8207465e-01, -2.9640751e-02, 3.3936000e-01, -1.6528348e+00)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 6.7670248e-02

OSBORN (FIRST)

$x^0 =$ (0.5, 1.5, -1.0, 0.01, 0.02)

$x^* =$ (1.0671568e00, 1.8025689e00, -1.8073120e00, 3.4459896e-03, 1.0885496e-03)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 8.2037266e-01

OSBORN (SECOND)

$x^0 =$ (1.3, 0.65, 0.65, 0.7, 0.6, 3.0, 5.0, 7.0, 2.0, 4.5, 5.5)

Failure.

ENGVALL

$x^0 =$ (1, 2, 0)

$x^* =$ (2.9426883e-07, -2.8494105e-07, 1.0000000e+00)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 3.8743019e-06

ZANGWILL (SECOND)

$x^0 =$ (100, -1, 2.5)

$x^* =$ (-4.4703484e-08, 4.2319298e-06, 4.2027968e-06)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 8.5688371e-06

MARKET RESPONSE 1

$x^0 =$ (-1.2, 1.0, 1.0)

$x^* =$ (-2.2306226e+02, -7.9779944e00, -1.8429138e-01)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 4.3779707e+01

MARKET RESPONSE 2

$x^0 =$ (-1.2, 1.0, 1.0)

Failure.

CHEMICAL REACTION

$x^0 =$ (-1.2, 1.0, 1.0, 1.0)

$x^* =$ (-7.8844978e00, 7.9498363e01, -9.8983908e00, 8.8180143e00)

$\sum_{i=1}^{m} |f_i(x^*)| =$ 5.2873616e+01

The numerical results are presented in Table 7.1. This table shows the problem name, number of functions (m), number of variables (n), number of iterations (iter), number of function evaluations (fun), number of gradient evaluations (grad), the success (S) or the failure (F) and the column dimension of Z at the solution (dim(Z)). The letter D next to S or F indicates that the degeneracy has been encountered. More specifically, DP indicates that a degenerate non-optimal point has been encountered, and DO that a degenerate local optimizer has been found.

The computation was carried out on the Honeywell 6660 computer at the University of Waterloo's Mathematics Faculty Computing Facility. The algorithm stops when all of the following conditions hold:

(1) $-1 \leq \lambda_i^k \leq +1$, $i \varepsilon I_{AE}$

$0 \leq \lambda_i^k \leq +1$, $i \varepsilon I_{AI}$, and

$\| A\lambda_i^k - g \|_2^2 \leq 8\varepsilon_{machine}(1 + \| \bar{f}_{iA} \|)$ $(= TOL0)$,

(2) $\| Z_k^T g(x^k) \| \leq 10^{-3} \| g \|$ $(= TOL1)$,

(3) $\| \bar{f}_A(x^k) \| \leq 10^{-5}$ $(= TOL2)$.

The tolerances have been the following: $\varepsilon_{activity} = 0.1$ and $\Lambda = 0.1 \| f_A \|$. No attempts have been made to optimize the value of different tolerances in order to get a better performance. The goal has been to explore how the (BBC) algorithm copes with degeneracy.

Name	m	n	iter	fun	grad	S/F	dim(Z)
EL-ATTAR 5.1	3	2	8	21	19	S	0
EL-ATTAR 5.2	6	3	5	10	6	S	2
EL-ATTAR 6.1	51	6	28	88	85	SDP	0
EL-ATTAR 6.2(I)	51	5	17	58	55	SDP	0
M/O 1 (BARD)	15	3	13	29	26	SDO	0
M/O 3 (MADSEN)	3	2	13	20	15	S	0
WOOD	7	4	25	45	45	SDO	0
POWELL	4	4	3	9	9	SDO	2
ROSENBROCK	2	2	51	103	103	S	0
WATSON	31	4	24	62	61	SDP	0
HELIX	3	3	11	31	31	S	0
MARTENSON	3	3	8	16	13	S	1
LRNKLS	3	3	5	13	12	SDO	2
BIGGS-EXP5	5	5	21	67	63	S	1
BIGGS-EXP6	6	6				F	
POLYNOMIAL CURVE FITTING	5	5	5	11	10	S	0
BOX-3	3	3	9	19	18	SDO	1
BOX-2	2	2	17	41	35	S	0
BROWN & DENNIS	20	4	2	9	9	S	4
CRAGG & LEVY	5	4	15	36	36	SDO	0
BEALE	3	2	8	17	17	SDO	0
BRANIN	2	2	7	11	11	S	0
FREUDENSTEIN & ROTH	2	2	8	12	12	S	1
JENNRICH & SIMPSON	10	2	12	33	32	S	1
KOWALIK & OSBORN	11	4	26	67	67	SDP	1
OSBORN (FIRST)	33	5	55	241	239	SDP	3
OSBORN (SECOND)	65	11	72	257	255	F	
ENGVALL	5	3	10	19	19	SDO	0
ZANGWILL (SECOND)	3	3	15	35	33	S	0
MARKET RESPONSE 1	8	3	40	96	59	S	2
MARKET RESPONSE 2	8	3				F	
CHEMICAL REACTION	37	4	79	208	198	SDP	1

Table 7.1 Numerical results

The results shows that the main goal has been achieved, i.e. the CNLL1 code has been able to solve the degenerate problems from the set of the well known test problems. It is worth noticing that the Bartels/Conn and Murray/Overton algorithms could not deal successfully with many of these problems. Although three problems ended with failure, i.e they only exhibited a linear rate of

convergence, a closer investigation showed that the projected Hessians for all three had negative eigenvalues along the paths taken. Moreover, none of these problems was degenerate. The current implementation of the (BBC) algorithm uses exact Hessians and makes no provision for indefiniteness.

In conclusion, we can say that the proposed way of handling degeneracy enabled us to design and implement a more reliable algorithm for nonlinear l_1 optimization. Since a considerable number of l_1 optimization problems exhibit some sort of degeneracy, as it can be seen even from the above well-known test problems, it is the author's belief that the proposed methods for handling degeneracy should be investigated for use in any nonlinear l_1 algorithm in which our basic l_1 framework has been followed.

CHAPTER 8

Concluding remarks

8.1. Contributions

In this thesis we have studied the problem of handling degeneracy in a constrained nonlinear l_1 algorithm. In summary, the major contributions in this thesis can be stated as follows:

(1) A proof of the second-order optimality conditions for the nonlinear l_1 problem that does not assume the linear independence of the gradients of the active functions.

(2) A direction of descent from a degenerate non-optimal stationary point for constrained nonlinear optimization algorithms based on an exact l_1 penalty function.

(3) A direction of descent in a neighborhood of a degenerate optimal point in Newton-like optimization algorithms that explicitly computes the step from the null space and range space components.

(4) A general outline for exact-penalty approaches for solving the constrained nonlinear l_1 optimization.

(5) An extension of the Bartels/Conn [1] algorithm which handles degeneracy.

8.2. Further work and open questions

Some of the important open problems are listed below.

(1) Near-degeneracy is a harder problem than the true degeneracy. For instance, if we are attempting the simple dropping step ($A^T p = -sign(\lambda_i)e_i$) and if the columns of A are almost linearly dependent than the computed "descent" direction p could be poorly determined. In that case, it would be interesting to explore whether the way in which we find a direction of descent from a nonoptimal stationary point will give any improvement.

(2) The current implementation of the BBC algorithm uses the exact projected Hessian and that can be quite expensive for some practical problems. It is necessary to explore how well the schemes for updating the projected Hessian, proposed by Coleman and Conn [11] and Nocedal and Overton [31], fare in the context of nonlinear l_1 optimization.

(3) The Coleman/Conn algorithm for nonlinear optimization [9,10] assumes that the gradients of the active functions will be linearly independent. That is not always a realistic assumption. It would be useful to apply our results for handling degeneracy to the Coleman/Conn algorithm. The convergence proofs for the (BBC) algorithm can be used for the Coleman/Conn algorithm extended to handle degeneracy.

(4) In general, when we are dealing with a degenerate optimizer, the full orthogonal decomposition of A will be

$$A(x) = \begin{bmatrix} W(x) & Z(x) \end{bmatrix} \begin{bmatrix} R(x) & 0 \\ 0 & M(x) \end{bmatrix} \begin{bmatrix} V(x)^T \\ S(x)^T \end{bmatrix},$$

for some $M(x) \rightarrow 0$ as $x \rightarrow x^*$. This is a far more difficult obstacle to overcome and bears an obvious relationship with Newton's method applied to singular problems.

8.3. Historical note

An independent suggestion of solving a constrained norm problem to estimate Lagrange multipliers in the presence of degeneracy for linear l_1 problems was made during the development of this work by G.A. Watson of Dundee, Scotland.

Appendix I

Using the l_1 norm to resolve degeneracy

As pointed out in Chapter 2, the first-order optimality conditions for the (UNLL1) problem

$$\underset{x\varepsilon\mathbf{R}^n}{minimize} \sum_{i=1}^{m} |f_i(x)| \qquad \text{(UNLL1)}$$

are

$$A\lambda - g = 0 \qquad \text{(A1.1)}$$

$$-1 \leq \lambda_i \leq +1\,, \quad i\varepsilon I_{AE}.$$

If we solve

$$\underset{\lambda}{minimize} \quad \| A\lambda - g \|_1 \qquad \text{(BL1)}$$

$$subject\ to \quad -1 \leq \lambda_i \leq +1\,, \quad i\varepsilon I_{AE}$$

and the residual ($\| A\lambda - g \|_1$) is zero, we know that the current point, x, is an optimal point to first order. Otherwise, the direction of descent can be found from reformulating (BL1) as a linear programming problem and then using duality.

Let us restate (BL1) as

$$\underset{u\varepsilon\mathbf{R}^m,\, v\varepsilon\mathbf{R}^m}{minimize} \sum_{i=1}^{m}(u_i + v_i) \qquad \text{(LPR)}$$

$$subject\ to \quad A\lambda - g = u - v\,;$$

$$-1 \leq \lambda \leq +1$$

$$u \geq 0\,,$$

$$v \geq 0.$$

The (LPR) problem can be written in the canonical form as

$$\textit{minimize}\,[0^T\ 1^T\ 1^T]\begin{bmatrix}\lambda\\ u\\ v\end{bmatrix} \tag{CLPR}$$

$$\textit{subject to}\ \begin{bmatrix}-A & +I & -I\\ A & -I & +I\\ +I & 0 & 0\\ -I & 0 & 0\\ 0 & +I & 0\\ 0 & 0 & +I\end{bmatrix}\begin{bmatrix}\lambda\\ u\\ v\end{bmatrix} \geq \begin{bmatrix}-g\\ g\\ -1\\ -1\\ 0\\ 0\end{bmatrix}$$

The dual problem for (CLPR) is

$$\textit{minimize}\,[+g^T\ -g^T\ +1^T\ +1^T\ 0^T\ 0^T]\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\end{bmatrix} \tag{DCLPR}$$

$$\textit{subject to}\ \begin{bmatrix}-A^T & +A^T & +I & -I & -I & 0 & 0\\ +I & -I & 0 & 0 & 0 & +I & 0\\ -I & +I & 0 & 0 & 0 & 0 & +I\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\\ y_4\\ y_5\\ y_6\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ y_4 \geq 0,\ y_5 \geq 0,\ y_6 \geq 0.$$

Taking in account the very simple structure of (DCLPR) we can rewrite it as

$$\underset{y_1,\cdots,y_6}{\textit{minimize}}\ \ g^T(y_1 - y_2) + \sum_i((y_3)_i + (y_4)_i) \tag{A1.2}$$

$$\textit{subject to}\ \ A^T(y_1 - y_2) = (y_3 - y_4)$$

$$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, y_4 \geq 0, y_5 \geq 0, y_6 \geq 0.$$

Now, if we rename $y_1 - y_2$ as p , y_3 as r and y_4 as s we get

$$\textit{minimize}\ \ g^T p + \sum_i(r_i + s_i) \tag{A1.3}$$

subject to $a_i^T p = r_i - s_i$

$$r_i \geq 0 \, , \, s_i \geq 0.$$

and that can be restated as:

$$\textit{minimize} \quad g^T p + \sum_i | \, a_i^T p \, | \, . \tag{A1.4}$$

Finally, we recognize that p gives us locally the best direction of descent for the (UNLL1) problem, provided that the residual $|| A\lambda - g ||_1$ is nonzero.

This way of resolving degeneracy at a stationary point could be useful in dealing with problems where the dimensionality is large and the matrix A has a sparse structure. A good code for linear programming should be able to cope successfully with those problems; note: it may have to deal with degeneracy.

We have not explored this approach any further.

Appendix II

Resolving degeneracy by using norm duality

Another way of testing whether the first-order optimality conditions

$$A\lambda - g = 0 \tag{A2.1}$$

$$-1 \leq \lambda_i \leq +1\,, \quad i \varepsilon I_{AE}$$

for the (UNLL1) problem

$$\underset{x \varepsilon \mathbf{R}^n}{\textit{minimize}} \sum_{i=1}^{m} |f_i(x)| \tag{UNLL1}$$

are satisfied at a current point x is to solve

$$\underset{\lambda}{\textit{minimize}} \quad \|\lambda\|_\infty \tag{A2.2}$$

$$\textit{subject to} \quad A\lambda = g,$$

provided that the system $A\lambda = g$ is consistent.

If λ_i's are within the bounds (A2.1) then we know that we are dealing with a first order point. Otherwise, we have to rely on the duality theorem, see Luenberger [24], to obtain a descent direction.

Definition A2.1

Vectors $w \varepsilon \mathbf{R}^n$ (with norm $\|\cdot\|_r$) and $z \varepsilon \mathbf{R}^n$ (with norm $\|\cdot\|_q$), where $\frac{1}{r} + \frac{1}{q} = 1$, are said to be *aligned* if

$$w^T z = \sum_{i=1}^{n} w_i z_i = \|w\|_r \|z\|_q$$

with $1 \leq r \leq \infty$.

Theorem A2.1 (Duality Theorem)

Given a system of m consistent linear equations of n unknowns

$$A\lambda = g$$

then

$$\underset{A\lambda=g}{\text{minimize}} \ \| \lambda \|_r = \underset{\| A^T d \|_q \leq 1}{\text{maximize}} \ g^T d$$

where $\frac{1}{r}+\frac{1}{q}=1$. Furthermore, optimal λ and $A^T d$ are aligned.

Now, returning to our original problem (A2.2) we see that the dual problem is

$$\underset{d}{\text{maximize}} \ g^T d \qquad \text{(A2.3)}$$

$$\text{subject to} \quad \| A^T d \|_1 \leq 1.$$

Suppose that we have d^*, a solution to (A2.3). Then $p = -d^*$ gives us a direction of descent if

$$g^T d^* > 1.$$

In that case

$$g^T p + \sum_i | a_i^T p | < 0,$$

i.e. p is a descent direction for the problem (UCLL1).

In the case of $g^T d^* < 1$, we can use the alignment of $A^T d^*$ and λ^* to find λ^*, see Cadzow [4].

We have not explored this approach any further.

References

[1] Bartels, R.H and Conn, A.R., "An Approach to Nonlinear l_1 Data Fitting," pp. 48-58 in *Numerical Analysis; Lecture Notes in Mathematics*, ed. J.P. Hennart,Springer-Verlag (1981).

[2] Bartels, R.H. and Conn, A.R., "Linearly constrained discrete l_1 problems.," *ACM Trans. on Math. Software* **6** pp. 594-608 (1980).

[3] Ben-Tal, A. and Zowe, J., "Necessary and sufficient optimality conditions for a class of nonsmooth minimization problems," *Mathematical Programming* **24** pp. 70-91 North-Holland Publishing Company, (1982).

[4] Cadzow, J.A., "Functional analysis and the optimal control of linear discrete systems," *Int. J. Control* **17** pp. 481-495 (1973).

[5] Charalambous, C., "A lower bound for the controlling parameter of the exact penalty function," *Mathematical Programming* **15** pp. 278-290 (1978).

[6] Charalambous, C., "On the conditions for optimality for the nonlinear l_1 problem," *Mathematical Programming* **17** pp. 123-135 (1979).

[7] Coleman, T.F and Sorensen, D.C., "A note on the computation of an orthonormal basis for the null space of a matrix," *Mathematical Programming* **29** pp. 234-242 (1984).

[8] Coleman, T. F. and Conn, A. R., "Second-Order Conditions for an Exact Penalty Function," *Mathematical Programming* **19** pp. 178-185 (1980).

[9] Coleman, T.F. and Conn, A.R., "Nonlinear Programming via an Exact Penalty Function: Asymptotic Analysis," *Mathematical Programming* **24** pp. 123-136 (1982).

[10] Coleman, T.F. and Conn, A.R., "Nonlinear Programming via an Exact Penalty Function: Global Analysis," *Mathematical Programming* **24** pp. 137-161 (1982).

[11] Coleman, T.F. and Conn, A.R., "On the Local Convergence of a Quasi-Newton Method for the Nonlinear Programming Problem," *SIAM J. Numer. Anal.* **24** pp. 755-769 (1984).

[12] Conn, A. R. and Pietrzykowski, T., "A penalty function method converging directly to a constrained minimum," *SIAM J. Numer. Analysis* **14** pp. 348-375 (1977).

[13] Dennis, J. E. and Schnabel, R. B., *Numerical Methods for Unconstrained Optimization and Nonlinear Equations,* Prentice-Hall (1983). Prentice-Hall Series in Computational Mathematics; Cleve Moler, Advisor

[14] El-Attar, R.A., Vidyasagar, M., and Dutta, S.R.K., "An Algorithm for l_1-norm Minimization with Application to Nonlinear l_1-approximation," *SIAM J. Numer. Anal.* **16** pp. 70-86 (1979).

[15] Fiacco, A.V. and McCormick, G.P., *Nonlinear Programming: Sequential Unconstrained Minimization Techniques,* John Wiley, New York (1968).

[16] Fiacco, A. V., *Introduction to Sensitivity and Stability Analysis in Nonlinear Programming,* Academic Press (1983). Mathematics in Sciences and Engineering; A Series of Monographs and Textbooks

[17] Gill, P.E. and Murray, W., "Newton-type methods for unconstrained and linearly constrained optimization," *Mathematical Programming* **7** pp. 311-350 (1974).

[18] Gill, P.E. and Murray, W., "The computation of Lagrange multiplier estimates for constrained optimization," *Mathematical Programming* **17** pp. 234-242 (1979).

[19] Gill, P.E., Murray, W., and Wright, M.H., *Practical Optimization,* Academic Press (1981).

[20] Gill, P. E., Murray, W., Saunders, M. A., and Wright, M. H., "Procedures for Optimization Problems with a Mixture of Bounds and General Linear Constraints," *ACM Trans. on Math. Software* **10** pp. 282-298 (1984).

[21] Gill, P. E., 1984.

[22] Hanson, R.J., *Linear Least Squares with Bounds and Linear Constraints,* Sandia Laboratories Report SAND82-1517, Albuquerque, New Mexico (1982). Submitted to SIAM J. Sci. and Stat. Comp.

[23] Lawson, C.L. and Hanson, R.J., *Solving Least Squares Problems,* Prentice-Hall (1974).

[24] Luenberger, D.G., *Optimization by Vector Space Methods,* John Wiley & Sons, Inc. (1969).

[25] Mangasarian, O.L., *Nonlinear Programming,* McGraw-Hill Book Co., New York (1969).

[26] More', J.J, Garbow, B.S., and Hillstrom, K.H., "Testing Unconstrained Optimization Software," *ACM Trans. on Math. Software* **7** pp. 17-41 (1981).

[27] Murray, W. and Wright, M.H., *Projected Lagrangian methods based on the trajectories of penalty and barrier functions,* Department of Operational Research, Stanford University, Stanford, CA (1978). Technical Report SOL 78-23

[28] Murray, W. and Overton, M., "Steplength algorithms for minimizing a class of non-differentiable functions," *Computing* **23** pp. 309-331 (1979).

[29] Murray, W. and Overton, M., "A projected Lagrangian algorithm for nonlinear minimax optimization," *SIAM J. Sci. and Stat. Comp.* **1** pp. 345-370 (1980).

[30] Murray, W.A. and Overton, M.L., "A Projected Lagrangian Algorithm for Nonlinear l_1 Optimization," *SIAM J. Sci. and Stat. Comp.* **2** pp. 207-224 (1981).

[31] Nocedal, J. and Overton, M., *Projected Hessian Updating Algorithms for Nonlinearly Constrained Optimization,*.ie !"Submitted to SIAM J. Numer. Anal."" Submitted to SIAM J. Numer. Anal. 1985.

[32] Ortega, J.M. and Rheinboldt, W.C., *Iterative Solution of Nonlinear Equations in Several Variables,* Academic Press, New York (1970).

[33] Stoer, J., "On the numerical solution of constrained least-squares problems," *SIAM J. Numer. Anal.* **8** pp. 382-411 (1971).

[34] Tapia, R.A., "Diagonalized Multiplier Methods and Quasi-Newton Methods for Constrained Optimization," *J. Opt. Theory Appl.* **22** pp. 135-194 (1977).

[35] Wright, M.H., "Numerical Methods for Nonlinearly Constrained Optimization," Ph.D. thesis, Report CS-76-566, Computer Science Department, Stanford University, Stanford, California (1976).