METRIC INTERPRETATION AND GREATEST FIXPOINT SEMANTICS OF LOGIC PROGRAMS

by

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Abstract:

Using a compact metric space, we study the continuity properties of the transformation associated to a logic program. We show among other things that this transformation is weakly intersection-continuous on that metric space. We deduce from this result a greatest fixpoint semantics for logic programs computing on infinite trees.
METRIC INTERPRETATION AND GREATEST FIXPOINT
SEMANTICS OF LOGIC PROGRAM

Introduction:

The intent of this paper is to give a greatest fixpoint semantics for infinite computations in Logic Programming.

In [6] it is shown that the least Herbrand model of a logic program $P$ [5] is also the least fixpoint $\bigcup_{n \in \mathbb{N}} T^n_P (\emptyset)$ of the set-theoretic transformation $T_P: P(H_b) \rightarrow P(H_b)$ associated to this program. This gives a semantics a la Scott-Strachey for logic programs. In [1] the authors show that the complementary set of the intersection $\bigcap_{n \in \mathbb{N}} T^n_P (H_b)$ in the Herbrand universe $H_b$ is exactly the finite failure set of $P$, i.e., the set of all element $a \in H_b$ such that there exists a finite SLD tree for $P$ together with query $a$ which contains no success branches.

Using ideas from sort theory, we suggested in [7] how results of logic program computations could be obtained by starting from a large set of possible results, and shrinking this set step by step until the final result is obtained. In this paper, we explicit this approach and show how it gives a greatest fixpoint semantics for logic programs computing on infinite trees. Some tools we have (metric on trees, compact spaces, ...) are similar to those already used in [2] for giving a semantics to non deterministic recursive program schemes.
I. Definitions:

Let $V$ be a set of variables, and $F = F_0 \cup F_1 \cup F_2 \ldots$ a set of functional letters, where $f \in F_i \Rightarrow \text{arity } (f) = i$. We assume that $F_0$ is finite, i.e., we have a finite set of constant symbols.

Let $R = R_0 \cup R_1 \cup R_2 \ldots$ be a set of relation symbols with $r \in R_i \Rightarrow \text{arity } (r) = i$.

Let $H_u$ be the Herbrand universe generated by $F$ (i.e., the set of all terms constructed from $F$), and $H_b$ be the Herbrand base generated by $F$ and $R$ (i.e., the set of all formulas constructed from $F$ and $R$). In fact $H_b$ is the free $R$-magma generated by $H_u$: $H_b = M(R, H_u)$.

The set of trees $H_u$ can be supplied with a distance $d$ defined as follows:

$$d(t, t') = 0 \quad \text{if } t = t'$$
$$= 2^{-\inf\{n: \alpha_n(t) \neq \alpha_n(t')\}} \quad \text{otherwise}$$

where $\alpha_n(t)$ denotes the cut at height $n$ of tree $t$. In the metric space $H_u$, a sequence $(x_n)_{n \in \mathbb{N}}$ converges if it is stationary, i.e., $\exists a \in H_u \exists N \in \mathbb{N} \forall n \geq N \Rightarrow x_n = a$.

The completed metric space constructed from $H_u$ will be denoted $\overline{H_u}$. Since $F_0$ is finite, $\overline{H_u}$ is a compact space [2].

The very same process may be applied to $H_b = M(R, H_u)$ and yields a complete metric space $\overline{H_b}$ which will be the completed Herbrand base. One easily verifies that $\overline{H_b}$ is the free $R$-magma generated by $\overline{H_u}$; i.e., $\overline{H_b} = M(R, \overline{H_u}) = \overline{M(R, H_u)}$. 
The set \( \mathcal{H}_b \) of closed subsets of \( \mathcal{H}_b \) can be equipped with the Hausdorff distance:

\[
d(A, B) = \inf\{\varepsilon: A \subseteq \mathcal{V}_\varepsilon(B), B \subseteq \mathcal{V}_\varepsilon(A)\}
\]

where

\[
\mathcal{V}_\varepsilon(A) = \{y \in \mathcal{H}_b: \exists x \in A, d(x, y) < \varepsilon\}.
\]

The space \( \mathcal{H}_b \) is a compact metric space (therefore a complete space) for this distance, if \( R_0 \) is finite.

A (infinitary) substitution \( \theta \) is a function \( \theta: \mathcal{V} \to \mathcal{H}_u \) whose domain \( D(\theta) = \{x \in \mathcal{V}: \theta(x) \neq x\} \) is finite. We equip the set \( \Theta \) of all substitutions with the topology of simple convergence, i.e.,

\[
\forall \text{ sequence } (\theta_n)_{n \in \mathbb{N}} \text{ of } \Theta, \quad \lim_{n \to \infty} \theta_n(x) = \overline{\theta}(x).
\]

Let \( P \) be a logic program, i.e., a finite set of Horn clauses. We shall be concerned by the continuity of the following transformation \( T \):

\[
T: \mathcal{P}(\mathcal{H}_b) \to \mathcal{P}(\mathcal{H}_b)
\]

associated to program \( P \), when this transformation is extended to subsets of the completed Herbrand base constructed from the symbols of \( P \):

\[
S \to \{a\theta: (a \leftarrow b_1 \ldots b_m) \in \mathcal{P}, \theta \text{ substitution, } b_1 \theta, \ldots, b_m \theta \in S\}
\]

Lemma 1: Let \( \kappa: a \leftarrow b_1 \ldots b_m \) be a Horn clause, and define

\[
T_\kappa: \mathcal{P}(\mathcal{H}_b) \to \mathcal{P}(\mathcal{H}_b)
\]

\[
S \to \{a\theta \in \mathcal{H}_b: \theta \text{ substitution; } b_1 \theta, \ldots, b_m \theta \in S\}
\]
Then $T_k$ is closed in the following sense: if $S$ is a closed subset of $\mathbb{H}_b$, then $T_k(S)$ is also a closed subset of $\mathbb{H}_b$.

Proof: Assume $S \subseteq \mathbb{H}_b$ is closed. Then either $T_k(S) = \emptyset$ or $T_k(S) \neq \emptyset$.

In the second case let $(a^{\theta_n})_{n \in \mathbb{N}}$ be a Cauchy sequence of $T_k(S)$ (all Cauchy sequences of $T_k(S)$ are of this form). Is the limit $\lim_{n} a^{\theta_n}$ an element of $T_k(S)$?

By definition of $T_k$, $(b^l_{\theta_n})_{n \in \mathbb{N}}, \ldots, (b^m_{\theta_n})_{n \in \mathbb{N}}$ are all in $S$ and are all Cauchy sequences. Therefore $\lim_{n} (b^l_{\theta_n}), \ldots, \lim_{n} (b^m_{\theta_n})$ are all in $S$ since $S$ is closed i.e., using the simple convergence topology of $\emptyset$,

$b^i_{\lim n} \subseteq S$. Therefore $\lim_{n} a^{\theta_n} = a^{\lim (\theta_n)} \in T_k(S)$ i.e., $T_k(S)$ is closed and $T_k(S) \subseteq \mathbb{H}_b$.

Hence transformation $T_k$ is defined from $\mathbb{H}_b$ into $\mathbb{H}_b \cup \{\emptyset\}$.

Corollary: For any finite program $P$, the transformation $T_P: P(\mathbb{H}_b) \rightarrow P(\mathbb{H}_b)$ is defined from $\mathbb{H}_b$ into $\mathbb{H}_b \cup \{\emptyset\}$.

Proof: If $P = \{ \langle k_i \rangle \}_{i \in I}$ is finite, it is sufficient to notice that

$T_P(S) = \bigcup_{i \in I} T_{k_i}(S)$, and the finite union of closed sets is a closed set.

II. Some properties of Hausdorff distance

Lemma 2: (i) If $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence of $\mathbb{H}_b$ and if $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence of $\mathbb{H}_b$ such that $\forall n \in \mathbb{N} x_n \subseteq S_n$, then $(\lim_{n} x_n) \subseteq \lim_{n} S_n$. 


(ii) If \( (S_n)_{n \in \mathbb{N}} \) is a Cauchy sequence of \( 2^\mathbb{N}_b \), then \( \forall x \in \lim S_n \), there exists a Cauchy sequence \( (x_p)_{p \in \mathbb{N}} \) such that \( \forall p, x \in S_{k_p} \) and \( x = \lim x_p \). 

Proof: (i) Let \( S = \lim S_n \). Then

\[
S_n \to S = \forall \varepsilon > 0 \ \exists N \ \forall p \geq N \ \inf \{ n : S_p \subseteq V_n(S), S \subseteq V_n(S_p) \} < \varepsilon
\]

\[
\Rightarrow \ \forall \varepsilon > 0 \ \exists N \ \forall p \geq N \ S_p \subseteq V_\varepsilon(S), S \subseteq V_\varepsilon(S_p)
\]

\[
\Rightarrow \ \forall \varepsilon > 0 \ \exists N \ \forall p \geq N \left( \forall x \in S_p \ \exists y \in S \ d(x, y) < \varepsilon \right) \text{ and }
\]

\[
\left( \forall y \in S \ \exists x \in S_p \ d(x, y) < \varepsilon \right).
\]

Let \( (x_n)_{n \in \mathbb{N}} \) be a Cauchy sequence with \( x_n \in S_n \). Then

\[
\forall \varepsilon > 0 \ \exists N \ \forall p \geq N \ \exists y \ d(x_p, y) < \varepsilon \quad (*), \text{ and since } (x_n) \text{ is Cauchy, for the same } \varepsilon
\]

\[
\exists N' \ \forall p, q \geq N' \ d(x_p, x_q) < \varepsilon.
\]

Call \( x = \lim x_p \). Condition (*) implies \( \forall p \exists \varepsilon > 0 \ \forall p \geq N \ d(x_p, S) < \varepsilon \) if we define \( d(x_p, S) = \inf_{y \in S} d(x_p, y) \).

Now if \( y \in S \) we have \( d(x, y) \leq d(x, x_p) + d(x_p, y) \). Taking the inf over \( S \) we have
\[ \inf_{y \in S} d(x,y) = d(x,S) \leq d(x,x_p) + \inf_{y \in S} d(x_p,y) \]

i.e., \( d(x,S) \leq d(x,x_p) + d(x_p, S) \).

Now \( d(x,x_p) \to 0 \) and \( d(x_p, S) \to 0 \) when \( p \to \infty \). Therefore \( d(x, S) = 0 \);

since \( S \) is closed, this implies \( x \in S \), i.e., \( \lim_{p} x_p \in S \).

(ii) Let \( (S_n)_{n \in \mathbb{N}} \) be a Cauchy sequence of \( \overline{2^H} \) and \( x = \lim_{p} S_p = S \). We have

\[ S = \lim_{p} S_p = \]

\[ \forall \varepsilon > 0 \ \exists N \ \forall p \geq N \ \inf\{ \eta : S_p \subseteq V_{\eta}(S), S \subseteq V_{\eta}(S_p) \} < \varepsilon \).

Now \( S \subseteq V_{\eta}(S_p) \Rightarrow \forall \varepsilon \in S \exists z \in S_p \ d(z,y) < \varepsilon \). Thus

\[ \forall \varepsilon > 0 \ \exists N_\varepsilon \ \forall p \geq N_\varepsilon \ \forall \varepsilon \in S \exists z \in S_p \ d(z,y) < \varepsilon . \]

Take a sequence \( \varepsilon_q = (\frac{1}{q}), q > 0 \), fix \( y = x \in S \) and define \( \forall q > 0 \ x_q = \text{some } z \in S_{N_{\varepsilon_q}} \) such that \( d(x,z) < \varepsilon_q \). Then \( (x_n) \) is Cauchy, \( \lim_{n} x_n = x \) and \( S_{k_p} = S_{N_{\varepsilon_p}} \).

**Definition:** An element \( a \) is a point of accumulation of a set \( U \) if and only if \( \forall \varepsilon > 0 (B_\varepsilon(a) - \{a\}) \cap U \neq \emptyset \), where \( B_\varepsilon(a) \) is the open ball of radius \( \varepsilon \) centered in \( a \).

**Lemma 3:** If \( (A_n)_{n \in \mathbb{N}} \) is a decreasing sequence of non-empty closed subsets in \( \overline{2^H} \), then its limit for the Hausdorff distance \( \lim_{n} A_n \) is the intersection \( \cap_{n} A_n \).
Proof: (See [4]) First notice that we have the equivalences

\[
x \in \bigcap_{n} A_n \iff (\exists \text{ sequence } \{a_n\}_{n \in \mathbb{N}} (a_n \in A_n) \text{ s.t. } \lim_{n} a_n = x)
\]

\[
\iff (\exists \text{ sequence } \{a_n\}_{n \in \mathbb{N}} (a_n \in A_n) \text{ such that } x \text{ is a point of accumulation of } \{a_n\})
\]

Indeed if we assume that we have a sequence \( \{a_n\}, a_n \in A_n \), with a point of accumulation \( x \) and \( x \notin \bigcap_{n} A_n \), then \( \exists p x \notin A_p \). Since \( A_n \) is decreasing, \( \forall q \geq p x \notin A_q \). But \( A_p \) is closed and subsequence \( (a_p, a_{p+1}, a_{p+2}, \ldots) \) has the same points of accumulation as \( \{a_n\} \), therefore \( x \in A_p \). Contradiction.

This shows in particular that \( \bigcap_{n} A_n \) is non-empty, since every sequence \( \{a_n\}_{n \in \mathbb{N}} \) has at least one point of accumulation \( x \in \bigcap_{n} A_n \).

Similarly assume \( \exists (a_n), a_n \in A_n, \lim_{n} (a_n) = x \). Then necessarily \( x \in \bigcap_{n} A_n \) because every limit is a point of accumulation. Now we show that \( d(\bigcap_{n} A_n, A_n) \to 0 \).

Assume we do not have \( d(\bigcap_{n} A_n, A_n) \to 0 \), then since \( u_n \to 0, v_n \to 0 \) in \( \mathbb{R} = \max(u_n, v_n) \to 0 \). i.e., \( \max(u_n, v_n) \to 0 \) \( \Rightarrow \) \{not \( u_n \to 0 \) or not \( v_n \to 0 \)\}

and since

\[
d(\bigcap_{n} A_n, A_p) = \max(\rho(\bigcap_{n} A_n, A_p), \rho(A_p, \bigcap_{n} A_n))
\]

with

\[
\rho(A,B) = \sup_{b \in B} \delta(A,b) = \inf \{\varepsilon: A \subseteq V_{\varepsilon}(B)\}
\]

\[
\delta(A,b) = \inf \{d(a,b): a \in A\}
\]
We have two cases to consider:

1. If we do not have $\rho(\bigcap_{n} A_n, A_n) \to 0$ then for some subsequence $\rho(\bigcap_{n} A_n, A_p)$, we have $\rho(\bigcap_{n} A_n, A_p) > \delta > 0$. Thus there would be a sequence of points $a_p \in A_p$ for which $\delta(\bigcap_{n} A_n, a_p) > \delta$. But this contradicts the fact that the $a_p$ must have a point of accumulation $x \in \bigcap_{n} A_n$.

2. If we do not have $\rho(\bigcap_{n} A_n, A_n) \to 0$ then for some subsequence $\rho(A_p, \bigcap_{n} A_n)$, we have $\rho(A_p, \bigcap_{n} A_n) > \delta > 0$. Thus there would be a sequence of points $a_p \in \bigcap_{n} A_n$ for which $\delta(A_p, a_p) > \delta$. Because of the compactness, the sequence $(a_p)$ has a convergent subsequence $a_q \to x$, where $x \in \bigcap_{n} A_n$ and $\delta(A_q, x) \to 0$.

   From $|\delta(A_q, a_q) - \delta(A_q, x)| \leq d(x, a_q)$, it follows that $\delta(A_q, x_q) \to 0$ contradicting $\delta(A_p, a_p) > \delta > 0$.

   Whence the lemma.

III. Some properties of unification:

   The reason why the theory of program schemes [2] cannot be applied here at once is that unification is not continuous in general. Indeed let us consider the following function, which is defined from the cartesian product of the complete Herbrand base and the set of atoms into the set of truth values $\{tt, ff\}$ supplied with the discrete topology:
unif: $\overline{H_b} \times M(R, M(F, V \cup H_u)) \to \{tt, ff\}$

$(u, a) \to tt$ if $\exists$ substitution $\theta \ u = a\theta$

$ff$ otherwise.

This function is not continuous in general. As an example consider the Cauchy sequence of atoms

$$\{a_n\}_{n \in \mathbb{N}} = \{s^n((x, y))\}_{n \in \mathbb{N}}.$$

Its limit is the infinite tree $s^\omega \in \overline{H_b}$. Then we have

$$\forall n \in \mathbb{N} \ \ unif (s^\omega, a_n) = ff$$

whereas $unif (s^\omega, \lim_{n \to \infty} a_n) = tt$. Notice that this argument applies only when $\overline{H_b}$ contains infinite trees, i.e., $\overline{H_b} \neq H_b$.

The first question is: under which conditions can we make unification continuous?

**Definition:** An atom $a \in M(R, M(F, V))$ is normal iff $a \notin \overline{H_b}$ and no variable $x$ has more than one occurrence in $a$.

**Lemma 4:** (i) Assume $\overline{H_b}$ contains at least one infinite tree. Then the function:

$$\lambda u. \ unif (u, a): \ u \to tt \ if \ \exists \ \theta \ u = a\theta$$

$ff$ otherwise
is continuous if and only if a is a normal atom.

(ii) For every Cauchy sequence of atoms \((a_n)_{n \in \mathbb{N}}\), if \(\lim_{n} \text{unif}(u, a_n) = \text{tt}\) then \(\text{unif}(u, \lim_{n} a_n) = \text{tt}\).

**Proof:** (i) It is enough to show that the inverse image

\[(\lambda u. \text{unif}(a, a))^{-1}(\text{tt}) = \{a \in H_{\mathcal{B}}: \exists \theta \ u = a\theta\} = U(a)\]

is both open and closed if and only if a is normal. We first remark that \(U(a)\) is always closed.

Let a be non-normal; we show that \(U(a)\) is not open. Since a is not normal, then \(a = p(\ldots, x, \ldots, x, \ldots)\) for some \(p \in R\) and \(x \in V\). Let \(v\) be an infinite tree in \(H_{\mathcal{B}}\) and \(a_n(v)\) be the cut at height \(n\) of tree \(v\). Then if \(\theta\) is some ground closure substitution for \(p(\ldots, a_n(v), \ldots, a_{n+1}(v), \ldots)\), \(\forall n \ p(\ldots, a_n(v), \ldots, a_{n+1}(v), \ldots) \theta \not\in U(a)\), but

\[\lim_{n} p(\ldots, a_n(v), \ldots, a_{n+1}(v), \ldots) \theta = p(\ldots, v, \ldots, v, \ldots) \theta \in U(a).\]

Therefore \(U(a)\) is not open. Conversely, let a be normal. Let \((u_n)\) be a Cauchy sequence such that \(\lim_{n} u_n = u \in U(a)\). Let \(a^{-1}\) be the tree obtained from a by cutting off all the variable leaves of a. Then since \(\forall \varepsilon > 0 \ \exists N \ \forall p \geq N \ d(u_p, \lim_{n} u_n) < \varepsilon\), by taking \(\varepsilon = 2^{-\text{height}(a)}\), we have that \(\forall p \geq N, a^{-1}\) is a subtree of \(u_p\). Therefore since all the variable leaves of a are distinct variables, for every \(p \geq N\) we can directly and unambiguously
read a substitution \( \theta_p \) such that \( u_p = a \theta_p \), i.e., \( u_p \in U(a) \). Thus
\[
\exists N \forall p \geq N u_p \in U(a).
\]
Therefore \( U(a) \) is open.

(ii) Let \( a = \lim a_n \). This atom \( a \) has a finite number of variables
\( \{x_0, x_1, ..., x_p\} \) and each one of these variables occurs at a finite height.

1st case: \( \{x_0, x_1, ..., x_p\} = \emptyset \)

We then have the following situation: \( \lim a_n = a \in \bar{H} \) and
\[
\exists N \forall p \geq N \exists \theta \in \bar{H} \mid a \theta_p = u, \text{ since } \lim \text{unif} \ (u, a_n) = \text{tt}. \text{ Now we claim } a = u.
\]
Indeed:
\[
a = \lim a_n = \forall \varepsilon > 0 \exists N \forall p \geq N \ d(a_p, a) < \varepsilon.
\]
Since \( a \) is constant, in the sequence \( a_p \) the variables are as high as we want them to be (they are "pushed at infinity"). Therefore the distance
\( d(a_p, u) \to 0 \). Now \( d(a, u) = d(a, a_p) + d(a_p, u) \) and both \( d(a, a_p) \) and
\( d(a_p, u) \) go to 0. Thus \( d(a, u) = 0 \) i.e., \( a = u \).

2nd case: \( \{x_0, ..., x_p\} \neq \emptyset \)

Let \( k \) be the maximum occurrence height of all variables \( x_0, ..., x_p \),
and let \( \varepsilon = 2^{-k} \). Then since \( (a_n)_{n \in \mathbb{N}} \) is Cauchy, \( \exists N \in \mathbb{N} \forall p, q \geq N \ d(a_p, a_q) < \varepsilon \),
thus all trees \( a_p \), for \( p \geq N \varepsilon \), coincide up to height \( k \) included. Now
\[ \lim_{n \to \infty} (u, a_n) = \text{tt} \iff (\exists N \forall q \geq N \exists \theta_q \ u = a_q \theta_q) \]

By taking \( q \geq \max(N, N) \), we have that for each given \( x \in \{x_0, \ldots, x_p\} \) and \( \forall q \geq \max(N, N) \) all the \( \theta_q(x) \) are equal between themselves. These common values of \( \theta_q \), \( q \geq \max(N, N) \) over the set \( \{x_0, \ldots, x_p\} \) give a substitution \( \theta \). By using the first case of this proof for the other branches of \( u_q \), we have that substitution \( \theta \) verifies \( u = a\theta \).  

III. Continuous transformations

3.1. Clauses of the form \( a + b \) (\( a, b \) atoms): We have the following lemma.

**Lemma 5:** Let \( \kappa : a + b \) be a Horn clause, and assume \( \overline{H_b} \) contains an infinite tree. Then the associated function \( t_\kappa : \overline{H_b} \to 2^{\overline{H_b} \cup \{\phi\}} \)

\[ t_\kappa : u \to \{a\theta \in \overline{H_b} : u = b\theta, \ \theta \text{ ground substitution}\} \]

is continuous iff the atom \( b \) is normal.

**Proof:** (i) Assume \( b \) is not normal, i.e., \( b = p(..., x, \ldots, x, \ldots) \) for some \( p \in R \) and \( x \in V \). Then for some infinite tree \( v \) and ground substitution \( \theta \), the sequence \( u_n = p(..., \alpha_n(v), \ldots, \alpha_{n+1}(v), \ldots)\theta \) is Cauchy and

\[ \lim_{n \to \infty} u_n = p(..., v, \ldots, v, \ldots)\theta \]. But \( \forall n \in \mathbb{N} \ t_\kappa(u_n) = \phi \) and

\[ t_\kappa(\lim_{n \to \infty} u_n) \not\in a\theta \cdot x^<v \), where \( \theta \cdot x^<v \) is the ground substitution obtained from \( \theta \).
by modifying it at variable $x$ by $v$. Hence $t_k$ is discontinuous.

(ii) Assume $b$ is normal, i.e., $U(b)$ is both open and closed. Then we have the equivalence:

$$\exists \theta \left( \lim_{n} u_n \right) = b \theta \iff (\exists N \forall p \leq N \exists \theta_p \ u_p = b \theta_p \text{ and } \lim_{p} \theta_p = \theta)$$

for every Cauchy sequence $(u_n)_{n \in \mathbb{N}}$ of $\overline{H}_b$, where the implication $\iff$ is obtained because $U(b)$ is open and the implication $\Rightarrow$ because $U(b)$ is closed. This equivalence implies:

$$a \theta \in t_k \left( \lim_{n} u_n \right) \Rightarrow$$

$$a \theta = a \lim_{n} \theta_n = \lim_{n} a \theta_n \in \lim_{n} t_k (u_n).$$

Whence the equality $t_k (\lim_{n} u_n) = \lim_{n} t_k (u_n)$. \hfill \Box

**Lemma 6:** If $b$ is a normal atom, then the transformation:

$$T_k : 2^{\overline{H}_b} \to 2^{\overline{H}_b} U(\phi), \ S \mapsto \{a \theta \in \overline{H}_b : b \theta \in S\}$$

associated with the clause $\kappa : a \leftarrow b$ is uniformly continuous for the Hausdorff distance.

**Proof:** (i) Assume that for $S, S' \in 2^{\overline{H}_b}$, $T_k (S), T_k (S') \neq \phi$. Then we must show:

$$\forall \varepsilon > 0 \ \exists \eta > 0 \ \forall (S, S') \in d(S, S') < \eta \Rightarrow d(T_k (S), T_k (S')) < \varepsilon.$$

We have
\[ d(T_k(S), T_k(S')) < \varepsilon \Rightarrow \]

\[ \forall u \in T_k(S), \exists v \in T_k(S') \quad d(u, v) < \varepsilon, \text{ and} \]

\[ \forall v \in T_k(S') \exists u \in T_k(S) \quad d(u, v) < \varepsilon. \]

Now \( d(u, v) < \varepsilon \Rightarrow u \) and \( v \) coincide up to height \( -1g_2(\varepsilon) \); let \( u = a_0, v = a_0 \). Then \( b \in S, b_0 \in S' \) and \( b_0 \) and \( b_0 \) coincide up to height:

\[ -1g_2(\varepsilon) - \text{height } (a) + m(b) \]

where \( m(b) \) is the minimal height of occurrence of a variable in atom \( b \). Then for \( \varepsilon > 0 \) given, it is enough to take \( \eta = \varepsilon \). \( 2^{\text{height}(a)} - m(b) \) in order to have \( \forall \varepsilon > 0 \exists \eta > 0 \quad d(S, S') < \eta = d(T_k(S), T_k(S')) < \varepsilon. \)

(ii) Now let \( S \in 2^b \) and assume \( T_k(S) = \emptyset \in 2^b \). Then \( S \cap U(b) = \emptyset \) where \( U(b) = (\lambda u \cdot \text{unif } (u, b))^{-1} (tt) \). Let \( (S_n)_{n \in \mathbb{N}} \) be any Cauchy sequence such that \( \lim_{n} S_n = S. \) Since \( U(b) \) is closed,

\[ \min_{x \in S} d(x, y) = \alpha \neq 0 \]

\[ x \in S, y \in U(b) \]

If we take \( \varepsilon = \frac{\alpha}{2} \), then since \( S_n \to S, \exists N \forall p \geq N \quad d(S_p, S) < \varepsilon. \) In particular, \( \forall p \geq N \quad S_p \cap U(b) = \emptyset \) i.e., \( T_k(S_p) = \emptyset. \) Thus we have shown:

\[ T_k(S) = \emptyset \quad \text{and } S_n \to S \text{ implies} \]

\[ \exists N \forall p \geq N \quad T_k(S_p) = \emptyset. \]
(iii) Let \((S_n)_{n \in \mathbb{N}}\) be any Cauchy sequence that \(S_n \to S\) and assume
\[\forall n \in \mathbb{N}, \exists S_n \cap U(b) = \phi.\] Then since \(b\) is normal, \(U(b)\) is open and \(S \cap U(b) = \phi.\) Therefore \(T_\kappa(S) = \phi.\)

Which completes the proof of the lemma.

3.2. General case:

The results of the preceding paragraph can be immediately extended to clauses of the general form \(\kappa: a + b_1 \ldots b_m\) only at the cost of having the right-hand side normal in a certain sense i.e., no variable occurs more than once in \(b_1 \ldots b_m.\) However, a weaker form of continuity is true for these clauses.

Lemma 7: Let \(\kappa: a + b_1 \ldots b_m, m \geq 1\) be a Horn clause, and let \((S_n)_{n \in \mathbb{N}}\) be a Cauchy sequence of \(2^U_b.\) Then \(\lim_{n \to \infty} T_\kappa(S_n) \subseteq \lim_{n \to \infty} (T_\kappa)_n.\)

Proof: \(\lim_{n \to \infty} T_\kappa(S_n)\) exists by construction (Lemma 1). Let \(x \in \lim_{n \to \infty} T_\kappa(S_n).\)

By Lemma 2 this implies \(\exists\) Cauchy sequence \((x_p), \lim_p x_p = x\) and \(\forall p x_p \in T_\kappa(S_k).\)

Now since \(\forall p x_p \in T_\kappa(S_k),\), we have \(x_p = a \theta_p\) and \(b_1 \theta_p \ldots b_m \theta_p \in S_{k_p}\) for some substitution \(\theta_p.\) Now \((x_p)\) is a Cauchy sequence implies \((b_1 \theta_p)_{p \in \mathbb{N}}, \ldots, (b_m \theta_p)_{p \in \mathbb{N}}\) are all Cauchy sequences and \(\forall p b_i \theta_p \in S_{k_p}.\)

Therefore
\[\forall i \lim_{p \to \infty} (b_i \theta_p) = b_i \lim_{p \to \infty} \theta_p \in \lim_{p \to \infty} S_{k_p} = \lim_{n \to \infty} S_n.\]

Since \(\forall i b_i (\lim \theta_p) \in \lim_{n \to \infty} S_n,\) we have \(a (\lim \theta_p) = \lim_{p \to \infty} x_p = x \in T_\kappa(\lim_{n \to \infty} S_n).\)

Hence \(\forall x \in \lim_{n \to \infty} T_\kappa(S_n), x \in T_\kappa(\lim_{n \to \infty} S_n).\) Hence the lemma.
Corollary: If $\kappa$ is a definite Horn clause, and if $(S_n)_{n \in \mathbb{N}}$ is a decreasing sequence in $H^b$, then $\cap_{n \in \mathbb{N}} T_{\kappa} (S_n) = T_{\kappa} (\cap_{n \in \mathbb{N}} S_n)$. 

Proof: Because $T_{\kappa}$ is monotonic, we have $T_{\kappa} (\cap_{n \in \mathbb{N}} S_n) \subseteq \cap_{n \in \mathbb{N}} T_{\kappa} (S_n)$. Now by the preceding lemma $\lim_{n \to \infty} T_{\kappa} (S_n) \subseteq T_{\kappa} (\lim_{n \to \infty} S_n)$ and by Lemma 3, $\cap_{n \in \mathbb{N}} T_{\kappa} (S_n) = T_{\kappa} (\cap_{n \in \mathbb{N}} S_n)$. $\square$

Since $H^b$ is a compact metric space, the notion of a limit of closed subsets in the Hausdorff distance sense coincides with the notion of a limit in the Kuratowski-Painlevé sense, i.e., for any sequence $(S_n)_{n \in \mathbb{N}}$ of $H^b$, $\lim S_n$ exists in the Hausdorff distance sense if and only if $\lim_{n \to \infty} S_n = L(S_n) = LI(S_n) = L(S_n)$ where LS (resp. LI, resp. L) denotes the limit sup. (resp. the limit inf, resp. the limit in the Kuratowski-Painlevé sense) of sequence $(S_n)$. This result is given in Hausdorff [4] pp. 170-172.

We recall the definition of these limits.

**Limit Inf:**

$x \in LI (S_n) = \forall$ open neighbourhood $V(x)$ of $x$

$\exists N > 0 \ \forall p \geq N \ V(x) \cap S_p \neq \phi$

**Limit Sup:**

$x \in LS (S_n) = \forall$ open neighbourhood $V(x)$ of $x$

$\forall N > 0 \ \exists p \geq N \ V(x) \cap S_p \neq \phi$
Kuratowski-Painlevé limit:

If \( \text{LIS}_n(S_n) = \text{LS}_n(S_n) \) then by definition \( \text{L}_n(S_n) = \text{LI}_n(S_n) = \text{LS}_n(S_n) \).

Lemma 8: Let \( P \) be a logic program.

(i) For any sequence \( (S_n)_{n \in \mathbb{N}} \) of \( 2^\mathbb{H}_b \), \( \text{LS}_n T_p(S_n) \subseteq T_p(\text{LS}_n(S_n)) \)

(ii) For any Cauchy sequence \( (S_n)_{n \in \mathbb{N}} \) of \( 2^\mathbb{H}_b \), \( \text{LI}_n T_p(S_n) \subseteq T_p(\text{LI}_n(S_n)) \)

Proof: (i) Let \( x \in \text{LS}_n T_p(S_n) \). Then \( x \in \text{LS}_n T_p(S_n) = \)

\[ \forall \varepsilon > 0 \ \forall N > 0 \ \exists p > N \ \exists y \in T_p(S_p) \ d(x,y) < \varepsilon. \]

Take \( \varepsilon < 2^{-h(p)} \), where \( h(p) \) is the maximum height of the left-hand sides of \( P \). Then \( d(x,y) < \varepsilon = \exists \) clause \( k: (a + b_1\ldots b_m) \epsilon P \) such that \( x = a \sigma, y = a \theta \) where \( \sigma \) and \( \theta \) are substitutions and furthermore, since \( y \in T_p(S_p) \), \( b_1 \theta, \ldots, b_m \theta \in S_p \).

Let us take a decreasing sequence of \( \varepsilon \)'s such that \( \varepsilon < 2^{-h(p)} \). Each \( \varepsilon \) will give some \( y \) such that \( d(x,y) < \varepsilon \). Since \( P \) is finite, infinitely many such \( y \)'s will be such that \( y = a \theta \) and \( b_1 \theta, \ldots, b_m \theta \in S_p \). Now

\[ \lim_{\varepsilon \to 0} y = x = \lim_{\varepsilon \to 0} a \theta = a \lim_{\varepsilon \to 0} \theta = a \sigma. \]

Thus the sequence \( \theta \) converges in \( \Theta \) towards \( \sigma \), and \( a \sigma + b_1 \sigma \ldots b_m \sigma \) is an instantiation of \( k \).
Now we have $b_1 \sigma, \ldots, b_m \sigma \in \text{LS} \left( S_n \right)$ by construction of these limits. 

$b_i \sigma = b_i \left( \lim_{n \to \infty} \theta \right)$. Thus $x = a \sigma \in T_p \left( \text{LS} \left( S_n \right) \right) \subseteq T_p \left( \text{LS} \left( S_n \right) \right)$. Therefore

$\text{LS} \left( T_p \left( S_n \right) \right) \subseteq T_p \left( \text{LS} \left( S_n \right) \right)$.

(ii) By definition of these limits, we have $\text{LI} \left( S_n \right) \subseteq \text{LS} \left( S_n \right)$.

Now if $\left( S_n \right)_{n \in \mathbb{N}}$ is Cauchy, then by Hausdorff's theorem we have

$\text{LI} \left( S_n \right) = \text{LS} \left( S_n \right) = \lim_{n \to \infty} \left( S_n \right)$. Now we have by (i):

$\text{LI} \left( T_p \left( S_n \right) \right) \subseteq \text{LS} \left( T_p \left( S_n \right) \right) \subseteq T_p \left( \text{LS} \left( S_n \right) \right) = T_p \left( \text{LI} \left( S_n \right) \right)$

which completes the proof.

Remark 1: If the sequence $\left( S_n \right)$ is not Cauchy, then $\text{LI} \left( T_p \left( S_n \right) \right) \subseteq T_p \left( \text{LI} \left( S_n \right) \right)$ does not hold, as shown by the following example.

Take $P: \{p(b) + y(f(x), f^2(x))\}$

$p(b) + r(f(x), f^3(x))$}

and the sequence of sets $\left( S_n \right)_{n \in \mathbb{N}}$ defined by:

$S_{2n} = \{q(f^n(b), f^{n+1}(b))\}$

$S_{2n+1} = \{r(f^n(b), f^{n+2}(b))\}$
Then $\text{LI}(S_n) = \emptyset$ thus $T_p(\text{LI}(S_n)) = \emptyset$, and $\text{LS}(S_n) = \{q(f^{(n)}, f^{(n)})\}$

thus $T_p(\text{LS}(S_n)) = \{p(b)\}$.

On the other hand:

$$T_p(S_{2n}) = \{p(b)\} = T_p(S_{2n+1}).$$

Thus:

$$\text{LI}(T_p(S_n)) = \text{LS}(T_p(S_n)) = \{p(b)\}.$$

Therefore

$$\{p(b)\} = \text{LI}(T_p(S_n)) \neq T_p(\text{LI}(S_n)) = \emptyset.$$

**Remark 2:** If $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, then $(T_p(S_n))_{n \in \mathbb{N}}$ is not necessarily a Cauchy sequence. Consider the following example:

**P:**

$$\{p(a) \leftarrow q(u, v)$$

$$p(b) \leftarrow q(f(x), f^2(x)) \}$$

$S_{2n} = \{q(f^n(a), f^{n+1}(a))\}$

$S_{2n+1} = \{q(f^n(a), f^{n+2}(a))\}$
Then $\lim_{n} L(S_n) = L(S) = \{q(f^\omega, \rho^\omega)\} = \lim_{n} S_n$

$$T_p(LI(S_n)) = T_p(LS(S_n)) = \{p(a), p(b)\}$$

But we also have

$$T_p(S_{2n}) = \{p(a), p(b)\}$$

$$T_p(S_{2n+1}) = \{p(a)\}$$

Therefore

$$LI(T_p(S_n)) = \{p(a)\}$$

$$LS(T_p(S_n)) = \{p(a), p(b)\}$$

and the Hausdorff limit of $(T_p(S_n))$ does not exist.

**Theorem 1:** Let $P = \{\kappa_i : i \in I\}$ be a logic program. Then:

(i) For every Cauchy sequence $(S_n)_{n \in \mathbb{N}}$ of $\mathbb{H}^b$, if $\lim_{n} T_p(S_n)$ exists, then

$$\lim_{n} T_p(S_n) \subseteq T_p(\lim_{n} S_n).$$

(ii) $T_p$ is $\mathbb{N}$-continuous i.e., for every decreasing sequence $(S_n)_{n \in \mathbb{N}}$ of $\mathbb{H}^b$,

$$T_p(\bigcap_{n} S_n) = \bigcap_{n} T_p(S_n)$$
Proof: (i) Follows immediately from the above theorem and Hausdorff's theorem. A more direct proof is as follows:

Let \( x \in \lim_{n} T_{p}(S_{n}) = \lim_{n} (U_{i} \kappa_{i}(S_{n})). \)

Then by lemma 2(ii), there exists a Cauchy sequence \((x_{p})_{p \in \mathbb{N}}\) such that \( (\lim_{p} x_{p}) = x \), and \( \forall p \, x_{p} \in U_{i} \kappa_{i}(S_{k_{p}}) \). Let us take \( \varepsilon > 0 \) such that:

\[-1g_{2}(\varepsilon) = (\text{maximal height of left-hand sides of } P) + 1.\]

Then we get

\[ \exists N \, \forall p \geq N \, x_{p} = a \theta_{p} \text{ for some fixed left-hand side } a \text{ of } P. \text{ Let } \kappa: a + b_{1} \ldots + b_{m} \text{ be the clause containing this left-hand side. Then:} \]

\[ \forall p \geq N \, x_{p} = a \theta_{p} \in T_{\kappa}(S_{k_{p}}) \text{ and } x = \lim_{p} a \theta_{p} = a \theta \]

Hence \( \forall p \geq N \, \exists b_{1} \theta_{p}, \ldots, b_{m} \theta_{p} \in S_{k_{p}} \) and the sequences \((b_{i} \theta_{p})_{p \in \mathbb{N}}, \ldots, (b_{m} \theta_{p})_{p \in \mathbb{N}}\)

are all Cauchy sequences. Their limits \( b_{1} \theta, \ldots, b_{m} \theta \) are all in \( \lim_{k_{p}} S_{k_{p}} = \lim_{n} S_{n} \) according to Lemma 2(i). Hence:

\[ a \theta + b_{1} \theta + \ldots + b_{m} \theta \]

is an instantiation of \( \kappa \), therefore:

\[ x = a \theta \in T_{\kappa}(\lim_{n} S_{n}) \subseteq T_{p}(\lim_{n} S_{n}) \]
i.e., \( x \in T_P(\lim S_n) \). Whence \( \lim_{n} (T_P(S_n)) \subseteq T_P(\lim_{n} S_n) \) whenever the first limit exists.

(ii) It is enough to show that for every decreasing sequence \((S_n)_{n \in \mathbb{N}}\) of \( \overline{H_b} \), \( T_P(\bigcap_{n} S_n) \subseteq \bigcap_{n} T_P(S_n) \), but this is given by the monotonicity of \( T_P \). From (i) and Lemma 3 we get \( \bigcap_{n} T_P(S_n) \subseteq T_P(\bigcap_{n} S_n) \). Whence the theorem. \( \Box \)

IV. Greatest fixpoint theorem:

**Theorem 2:** Let \( P \) be a logic program; then \( \bigcap_{n} T_P(\overline{H_b}) \) is the greatest fixpoint of \( T_P \).

**Proof:** The sequence \((T_P^n(\overline{H_b}))_{n \in \mathbb{N}}\) is decreasing and \( T_P \) is \( \omega \)-continuous; whence:

\[
T_P(\bigcap_{n} T_P^n(\overline{H_b})) = \bigcap_{n} T_P^{n+1}(\overline{H_b}) = \bigcap_{n} T_P^n(\overline{H_b})
\]

The fact that this fixpoint is the greatest one is obvious. \( \Box \)

V. An example

We consider "Hamming's problem" as discussed in Dijkstra [3]: construct the sorted list of all natural numbers \( \neq 0 \) containing no prime factors other than 2, 3 or 5. This can be formalized by using a logic program. We first give our notation:
\[ u \cdot x \] is the list resulting from inserting atom \( u \) at the beginning of list \( x \).

\[ <, > \] are relation symbols with their usual meaning (infix notation is used here).

\[ \text{Prod}(a,b,c) \] means \( c = a \cdot b \)

\[ F(x,n,y) = \text{list } y \text{ is obtained by multiplying all the elements of list } x \text{ by } n \text{ (scalar multiplication)} \]

\[ M(x,y,z) = \text{sorted list } z \text{ is obtained by merging without repetition sorted lists } x \text{ and } y. \]

From this follows the following logic program \( P \) with query \( \text{Eq}(x,y) \):

\[
[ 0. \ F(\text{nil}, n, \text{nil}) + ]
1. \ F(a \cdot x, n, b \cdot y) + \text{Prod}(a, n, b) \cdot F(x, n, y)
2. \ M(u \cdot x, u \cdot y, u \cdot z) + M(x, y, z)
3. \ M(x_1 \cdot x, y_1 \cdot y, x_1 \cdot z) + (x_1 < y_1) \cdot M(x_1, y_1, y, z)
4. \ M(x_1 \cdot x, y_1 \cdot y, y_1 \cdot z) + (y_1 < x_1) \cdot M(x_1 \cdot x, y, z)
5. \ \text{Eq}(x, y) + F(y, 3, u) \cdot F(y, 5, v) \cdot M(u, v, w) \cdot F(y, 2, z) \cdot M(w, z, x)
6. \ + \text{Eq}(x, 1 \cdot x)
\]

The desired infinite list, which is computed by fair derivations

\[ \lambda = 2.3.5.6.8.9.10.12.15.16.18.20.24. \ldots \]

verifies \( \text{Eq}(\lambda, 1 \cdot \lambda) \in \bigcap_{n \in \mathbb{N}} T^n_p(H_b) \) i.e., \( \forall n \in \mathbb{N} \ \text{Eq}(\lambda, 1 \cdot \lambda) \in T^n_p(H_b) \),

where \( H_b \) is the completed Herbrand base associated with the above program.
The proof is as follows: For \( n = 0 \), it is obvious. Let \( n \neq 0 \). To make things simpler let us assume that \(<, >\) and Prod produce their results "instantaneously". We may remark that:

\[
\forall q > 0, q \in \mathbb{N} \left\{ F(\tilde{\alpha}, \tilde{u}, m, \tilde{\beta}, \tilde{v}) : \tilde{\alpha}, \tilde{u}, \tilde{\beta}, \tilde{v} \text{ lists}, |\tilde{\alpha}| = |\tilde{\beta}| = q, \tilde{\alpha} = m \ast \tilde{\beta} \right\} \subseteq T_p^q(\overline{H}_b)
\]

From this we deduce, if \((\ell/n+1)\) denote the initial segment of length \(n+1\) of list \(\ell\), that

\[
\{ \text{Eq}((\ell/n+1), \tilde{u}, 1.(\ell/n+1), \tilde{v}) : \tilde{u}, \tilde{v} \text{ lists} \} \subseteq \mathcal{T}_p^{n+1}(\overline{H}_b)
\]

by applying the fifth clause once.

Whence \( \text{Eq}(\ell, 1.\ell) \in \mathcal{T}_p^{n+1}(\overline{H}_b) \).

In fact we even have:

\[
[\text{Eq}(x, 1.x)] \cap (\cap_{n} \mathcal{T}_p^n(\overline{H}_b)) = \{ \text{Eq}(\ell, 1.\ell) \} \quad (\ast)
\]

where \([\text{Eq}(x, 1.x)]\) denotes the subset of \(\overline{H}_b\) obtained by replacing every occurrence of a free variable in \(\text{Eq}(x, 1.x)\) by every possible element of the Herbrand universe.

The property \((\ast)\) above is just an occurrence of a more general result shown in \([8]\), which can be stated as follows. Let \( P \) be a finite logic program, and \( t \) a query; assume \( \Delta \) is a generic name for finite derivations which are fair up to \( q > 0 \), and where \( \theta_1, \ldots, \theta_n \) is the sequence of most general unifiers associated with \( \Delta \).
Then:

\[
\bigcup [t\theta_1 \ldots \theta_n] = [t] \cap T_p^q(H_b)
\]

\[\Delta \text{ finite derivation}
\]

\[\text{fair up to } q\]

When \(q\) goes to \(+\infty\), this becomes

\[
\bigcup [t\theta_1 \ldots \theta_n \ldots] = [t] \cap (\bigcap_{q} T_p^q(H_b))
\]

\[\Delta \text{ fair derivation}\]

which is a generalization of (\(\ast\)).
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