METRIC INTERPRETATION AND GREATEST FIXPOINT SEMANTICS OF LOGIC PROGRAMS

by

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Abstract:

Using a compact metric space, we study the continuity properties of the transformation associated to a logic program. We show among other things that this transformation is weakly intersection-continuous on that metric space. We deduce from this result a greatest fixpoint semantics for logic programs computing on infinite trees.

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Introduction:

The intent of this paper is to give a greatest fixpoint semantics for infinite computations in Logic Programming.

In [6] it is shown that the least Herbrand model of a logic program P [5] is also the least fixpoint $\bigcup_{n \in N} T_P^n(\phi)$ of the set-theoretic transformation $T_P: \mathcal{P}(H_b) \to \mathcal{P}(H_b)$ associated to this program. This gives a semantics a la Scott-Stratchey for logic programs. In [1] the authors show that the complementary set of the intersection $\bigcap_n T_P^n(\overline{H}_b)$ in the Herbrand universe H_b is exactly the finite failure set of P, i.e., the set of all element $a \in H_b$ such that there exists a finite SLD tree for P together with query a which contains no success branches.

Using ideas from sort theory, we suggested in [7] how results of logic program computations could be obtained by starting from a large set of possible results, and shrinking this set step by step until the final result is obtained. In this paper, we explicit this approach and show how it gives a greatest fixpoint semantics for logic programs computing on infinite trees. Some tools we have (metric on trees, compact spaces, ...) are similar to those already used in [2] for giving a semantics to non deterministic recursive program schemes.

I. Definitions:

Let V be a set of variables, and $F = F_0 \cup F_1 \cup F_2 \ldots$ a set of functional letters, where $f \in F_i \Leftrightarrow$ arity (f) = i. We assume that F_0 is finite, i.e., we have a finite set of constant symbols.

Let $R = R_0 \cup R_1 \cup \ldots$ be a set of relation symbols with $r \in R_i \Leftrightarrow$ arity (r) = i.

Let H_u be the Herbrand universe generated by F (i.e., the set of all terms constructed from F), and H_b be the Herbrand base generated by F and R (i.e., the set of all formulas constructed from F and R). In fact H_b is the free R-magma generated by H_u : $H_b = M(R, H_u)$.

The set of trees H_u can be supplied with a distance d defined as follows:

$$d(t,t') = 0 \quad \text{if } t = t'$$
$$2^{-\inf\{n:\ \alpha_n(t) \neq \alpha_n(t')\}} \text{ otherwise}$$

where $\alpha_n(t)$ denotes the cut at height n of tree t. In the metric space H_u a sequence $(x_n)_{n \in N}$ converges iff it is stationary, i.e., $\exists a \in H_u\ \exists N \in N\ p \geq N \Rightarrow x_p = a$.

The completed metric space constructed from H_u will be denoted $\overline{H}_u$. Since F_0 is finite, $\overline{H}_u$ is a compact space [2].

The very same process may be applied to $H_b = M(R, H_u)$ and yields a complete metric space $\overline{H}_b$ which will be the completed Herbrand base. One easily verifies that $\overline{H}_b$ is the free R-magma generated by $\overline{H}_u$; i.e., $\overline{H}_b = M(R, \overline{H}_u) = \overline{M(R, H_u)}$.

The set $2^{\overline{H}_b}$ of closed subsets of $\overline{H}_b$ can be equipped with the Hausdorff distance:

$$d(A,B) = \inf\{\varepsilon: \ A \subseteq V_\varepsilon(B),\ B \subseteq V_\varepsilon(A)\}$$

where

$$V_\varepsilon(A) = \{y\in\overline{H}_b: \ \exists x\in A\ d(x,y)<\varepsilon\}.$$

The space $2^{\overline{H}_b}$ is a compact metric space (therefore a complete space) for this distance, if R_o is finite.

A (infinitary) substitution θ is a function $\theta: \ V \to \overline{H}_u$ whose domain $D(\theta) = \{x\in V : \theta(x) \neq x\}$ is finite. We equip the set Θ of all substitutions with the topology of simple convergence, i.e.,

$$\forall \text{ sequence } (\theta_n)_{n\in N} \text{ of } \Theta,\ \lim_n \theta_n = \overline{\theta} \Leftrightarrow \forall x\in V\ \lim_n \theta_n(x) = \overline{\theta}(x).$$

Let P be a logic program, i.e., a finite set of Horn clauses. We shall be concerned by the continuity of the following transformation T:

$$T: \ P(H_b) \to P(H_b)$$

associated to program P, when this transformation is extended to subsets of the completed Herbrand base constructed from the symbols of P:

$$S \to \{a\theta: \ (a \leftarrow b_1\wedge\ldots\wedge b_m)\in P,\ \theta \text{ substitution},\ b_1\theta,\ldots,b_m\theta\in S\}$$

Lemma 1: Let κ: $a \leftarrow b_1\wedge\ldots\wedge b_m$ be a Horn clause, and define

$$T_\kappa: \ P(\overline{H}_b) \to P(\overline{H}_b)$$

$$S \to \{a\theta\in\overline{H}_b: \ \theta \text{ substitution};\ b_1\theta,\ldots,b_m\theta\in S\}$$

Then T_κ is closed in the following sense: if S is a closed subset of $\overline{H}_b$, then $T_\kappa(S)$ is also a closed subset of $\overline{H}_b$.

Proof: Assume $S \subseteq \overline{H}_b$ is closed. Then either $T_\kappa(S) = \phi$ or $T_\kappa(S) \neq \phi$. In the second case let $(a\theta_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $T_\kappa(S)$ (all Cauchy sequences of $T_\kappa(S)$ are of this form). Is the limit $\lim_n a\theta_n$ an element of $T_\kappa(S)$?

By definition of T_κ, $(b_1\theta_n)_{n\in\mathbb{N}}, \ldots, (b_m\theta_n)_{n\in\mathbb{N}}$ are all in S and are all Cauchy sequences. Therefore $\lim_n(b_1\theta_n), \ldots, \lim_n(b_m\theta_n)$ are all in S since S is closed i.e., using the simple convergence topology of Θ, $b_1(\lim_n \theta_n), \ldots, b_m(\lim_n \theta_n) \in S$. Therefore $\lim_n a\theta_n = a\cdot(\lim_n \theta_n) \in T_\kappa(S)$ i.e., $T_\kappa(S)$ is closed and $T_\kappa(S) \in 2^{\overline{H}_b}$.

Hence transformation T_κ is defined from $2^{\overline{H}_b}$ into $2^{\overline{H}_b} \cup \{\phi\}$.

Corollary: For any finite program P, the transformation T_P: $\mathcal{P}(\overline{H}_b) \to \mathcal{P}(\overline{H}_b)$ is defined from $2^{\overline{H}_b}$ into $2^{\overline{H}_b} \cup \{\phi\}$.

Proof: If $P = \{\kappa_i\}_{i\in I}$, I finite, it is sufficient to notice that $T_P(S) = \bigcup_{i\in I} T_{\kappa_i}(S)$, and the finite union of closed sets is a closed set.

II. Some properties of Hausdorff distance

Lemma 2: (i) If $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $2^{\overline{H}_b}$ and if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $\overline{H}_b$ such that $\forall n\in\mathbb{N}$ $x_n \in S_n$, then $(\lim_n x_n) \in \lim_n S_n$.

(ii) If $(S_n)_{n\in\mathbb{N}}$ is a Cauchy sequence of $2^{\overline{H}_b}$, then $\forall x \in \lim\limits_n S_n$, there exists a Cauchy sequence $(x_p)_{p\in\mathbb{N}}$ such that $\forall p\ x_p \in S_{k_p}$ and $x = \lim\limits_p x_p$. □

<u>Proof</u>: (i) Let $S = \lim\limits_n S_n$. Then

$$S_n \to S \Leftrightarrow \forall\varepsilon>0\ \exists N\ \forall p \geq N \quad \inf\{\eta:\ S_p \subseteq V_\eta(S),\ S \subseteq V_\eta(S_p)\} < \varepsilon$$

$$\Leftrightarrow \quad \forall\varepsilon > 0\ \exists N\ \forall p \geq N \quad S_p \subseteq V_\varepsilon(S),\ S \subseteq V_\varepsilon(S_p)$$

$$\Leftrightarrow \quad \forall\varepsilon > 0 \quad \exists N\ \forall p \geq N \quad (\forall x\in S_p\ \exists y \in S\ d(x,y) < \varepsilon) \quad \text{and}$$

$$(\forall y \in S\ \exists x\in S_p\ d(x,y) < \varepsilon).$$

Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with $x_n \in S_n$. Then

$\forall\varepsilon > 0\ \exists N\ \forall p \geq N\ \ \exists y\ d(x_p,y) < \varepsilon$ (*) , and since (x_n) is Cauchy, for the same ε

$$\exists N'\ \forall p,q \geq N' \quad d(x_p, x_q) < \varepsilon.$$

Call $x = \lim\limits_p x_p$. Condition (*) implies $\forall\varepsilon > 0\ \exists N\ \forall p \geq N\ d(x_p, S) < \varepsilon$ if we define $d(x_p, S) = \inf\limits_{y\in S} d(x_p, y)$.

Now if $y\in S$ we have $d(x,y) \leq d(x,x_p) + d(x_p, y)$. Taking the inf over S we have

$$\inf_{y\in S} d(x,y) = d(x,S) \leq d(x,x_p) + \inf_{y\in S} d(x_p,y)$$

i.e., $d(x,S) \leq d(x,x_p) + d(x_p, S)$.

Now $d(x, x_p) \to 0$ and $d(x_p, S) \to 0$ when $p \to \infty$. Therefore $d(x, S) = 0$; since S is closed, this implies $x\in S$, i.e., $\lim_p x_p \in S$.

(ii) Let $(S_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of $2^{\overline{H}b}$ and $x \in \lim_p S_p = S$. We have $S = \lim_p S_p \Leftrightarrow$

$$\forall\varepsilon > 0 \quad \exists N \; \forall p \geq N \quad \inf\{\eta: \; S_p \subseteq V_\eta(S), \; S \subseteq V_\eta(S_p)\} < \varepsilon \quad .$$

Now $S \subseteq V_\eta(S_p) \Leftrightarrow \forall y\in S \; \exists z \in S_p \;\; d(z,y) < \varepsilon$. Thus

$$\forall\varepsilon > 0 \quad \exists N_\varepsilon \; \forall_p \geq N_\varepsilon \quad \forall \; y\in S \; \exists z\in S_p \quad d(z,y) < \varepsilon.$$

Take a sequence $\varepsilon_q = (\frac{1}{q}), q > 0$, fix $y = x\in S$ and define $\forall \; q > 0 \; x_q =$ some $z\in S_{N_{\varepsilon_q}}$ such that $d(x,z) < \varepsilon_q$. Then (x_n) is Cauchy, $\lim_n x_n = x$ and $S_{k_p} = S_{N_{\varepsilon_p}}$. □

<u>Definition</u>: An element a is a point of accumulation of a set U if and only if $\forall\varepsilon > 0 \; (B_\varepsilon(a) - \{a\}) \cap U \neq \phi$, where $B_\varepsilon(a)$ is the open ball of radius ε centered in a.

<u>Lemma 3</u>: If $(A_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty closed subsets in $2^{\overline{H}b}$, then its limit for the Hausdorff distance $\lim_n A_n$ is the intersection $\bigcap_n A_n$.

Proof: (See [4]) First notice that we have the equivalences

$$x \in \bigcap_n A_n \Leftrightarrow (\exists \text{ sequence } \{a_n\}_{n\in N}\ (a_n \in A_n) \text{ s.t. } \lim_n a_n = x)$$

$$\Leftrightarrow (\exists \text{ sequence } \{a_n\}_{n\in N}\ (a_n \in A_n) \text{ such that } x \text{ is a point of accumulation of } \{a_n\})$$

Indeed if we assume that we have a sequence (a_n), $a_n \in A_n$, with a point of accumulation x and $x \notin \bigcap_n A_n$, then $\exists p$ $x \notin A_p$. Since A_n is decreasing, $\forall q \geq p$ $x \notin A_q$. But A_p is closed and subsequence $(a_p, a_{p+1}, a_{p+2}, \ldots)$ has the same points of accumulation as (a_n), therefore $x \in A_p$. Contradiction. This shows in particular that $\bigcap_n A_n$ is non-empty, since every sequence $(a_n)_{n\in N}$ has at least one point of accumulation $x \in \bigcap_n A_n$.

Similarly assume $\exists (a_n)$, $a_n \in A_n$, $\lim_n (a_n) = x$. Then necessarily $x \in \bigcap_n A_n$ because every limit is a point of accumulation. Now we show that $d(\bigcap_n A_n, A_n) \to 0$. Assume we do not have $d(\bigcap_n A_n, A_n) \to 0$, then since $u_n \to 0$, $v_n \to 0$ in $\mathbb{R} \Leftrightarrow$ $\max(u_n, v_n) \to 0$. i.e., Not $(\max(u_n, v_n) \to 0) \Leftrightarrow \{\text{not } (u_n \to 0) \text{ or not } v_n \to 0\}$ and since

$$d(\bigcap_n A_n, A_p) = \max\left(\rho(\bigcap_n A_n, A_p)\ ,\ \rho(A_p, \bigcap_n A_n)\right)$$

with

$$\rho(A,B) = \sup_{b\in B} \delta(A,b) = \inf\{\varepsilon :\ A \subseteq V_\varepsilon(B)\}$$

$$\delta(A,b) = \inf\{d(a,b) : a \in A\}$$

We have two cases to consider:

1. If we do not have $\rho(\bigcap_n A_n, A_n) \to 0$ then for some subsequence $\rho(\bigcap_n A_n, A_p)$, we have $\rho(\bigcap_n A_n, A_p) > \delta > 0$. Thus there would be a sequence of points $a_p \in A_p$ for which $\delta(\bigcap_n A_n, a_p) > \delta$. But this contradicts the fact that the a_p must have a point of accumulation $x \in \bigcap_n A_n$.

2. If we do not have $\rho(A_n, \bigcap_n A_n) \to 0$ then for some subsequence $\rho(A_p, \bigcap_n A_n)$, we have $\rho(A_p, \bigcap_n A_n) > \delta > 0$. Thus there would be a sequence of points $a_p \in \bigcap_n A_n$ for which $\delta(A_p, a_p) > \delta$. Because of the compactness, the sequence (a_p) has a convergent subsequence $a_q \to x$, where $x \in \bigcap_n A_n$ and $\delta(A_q, x) \to 0$.

From $|\delta(A_q, a_q) - \delta(A_q, x)| \leq d(x, a_q)$, it follows that $\delta(A_q, x_q) \to 0$ contradicting $\delta(A_p, a_p) > \delta > 0$.

Whence the lemma.

III. Some properties of unification:

The reason why the theory of program schemes [2] cannot be applied here at once is that unification is not continuous in general. Indeed let us consider the following function, which is defined from the cartesian product of the complete Herbrand base and the set of atoms into the set of truth values {tt, ff} supplied with the discrete topology:

$$\text{unif}: \ \overline{H}_b \times M(R, M(F, V \cup \overline{H}_u)) \to \{tt, ff\}$$

$$(u,a) \to tt \text{ if } \exists \text{ substitution } \theta \quad u = a\theta$$
$$ff \text{ otherwise.}$$

This function is not continuous in general. As an example consider the Cauchy sequence of atoms

$$\{a_n\}_{n \in \mathbb{N}} = \{s^n(+ (x,y))\}_{n \in \mathbb{N}} .$$

Its limit is the infinite tree $s^\omega \in \overline{H}_b$. Then we have

$$\forall\, n \in \mathbb{N} \quad \text{unif}\ (s^\omega, a_n) = ff$$

whereas unif $(s^\omega, \lim_n a_n) = tt$. Notice that this argument applies only when $\overline{H}_b$ contains infinite trees, i.e., $\overline{H}_b \neq H_b$.

The first question is: under which conditions can we make unification continuous?

<u>Definition</u>: An atom $a \in M(R, M(F,V))$ is <u>normal</u> iff $a \notin \overline{H}_b$ and no variable x has more than one occurrence in a.

<u>Lemma 4</u>: (i) Assume $\overline{H}_b$ contains at least one infinite tree. Then the function:

$$\lambda u.\,\text{unif}\ (u,a): \ u \to tt \text{ if } \exists\theta \quad u = a\theta$$
$$ff \text{ otherwise}$$

is continuous if and only if a is a normal atom.

(ii) For every Cauchy sequence of atoms $(a_n)_{n\in\mathbb{N}}$, if $\lim_n \mathrm{unif}(u, a_n) = tt$ then $\mathrm{unif}(u, \lim_n a_n) = tt$.

Proof: (i) It is enough to show that the inverse image

$(\lambda u.\ \mathrm{unif}(a, a))^{-1}(tt) = \{u\in\overline{H}_b:\ \exists\theta\ u = a\theta\} = U(a)$

is both open and closed if and only if a is normal. We first remark that U(a) is always closed.

Let a be non-normal; we show that U(a) is not open. Since a is not normal, then $a = p(\ldots, x, \ldots, x, \ldots)$ for some $p\in R$ and $x\in V$. Let v be an infinite tree in $\overline{H}_b$ and $\alpha_n(v)$ be the cut at height n of tree v. Then if θ is some ground closure substitution for $p(\ldots, \alpha_n(v), \ldots, \alpha_{n+1}(v), \ldots)$, $\forall n\ p(\ldots, \alpha_n(v), \ldots, \alpha_{n+1}(v), \ldots)\theta \notin U(a)$, but

$\lim_n p(\ldots, \alpha_n(v), \ldots, \alpha_{n+1}(v), \ldots)\theta = p(\ldots, v, \ldots, v, \ldots)\theta \in U(a)$.

Therefore U(a) is not open. Conversely, let a be normal. Let (u_n) be a Cauchy sequence such that $\lim_n u_n = u \in U(a)$. Let a^{-1} be the tree obtained from a by cutting off all the variable leaves of a. Then since $\forall\varepsilon>0$ $\exists N\ \forall p \geq N\ d(u_p, \lim_n u_n) < \varepsilon$, by taking $\varepsilon = 2^{-\mathrm{height}(a)}$, we have that $\forall p \geq N$, a^{-1} is a subtree of u_p. Therefore since all the variable leaves of a are distinct variables, for every $p \geq N$ we can directly and unambiguously read a substitution θ_p such that $u_p = a\theta_p$, i.e., $u_p \in U(a)$. Thus $\exists N\ \forall_p \geq N\ u_p \in U(a)$. Therefore $U(a)$ is open.

(ii) Let $a = \lim_n a_n$. This atom a has a finite number of variables $\{x_0, x_1, \ldots, x_p\}$ and each one of these variables occurs at a finite height.

<u>1st case</u>: $\{x_0, x_1, \ldots, x_p\} = \phi$

We then have the following situation: $\lim_n a_n = a \in \overline{H}_b$ and

$\exists N\ \forall p \geq N\ \ \exists\theta_p\ a_p\theta_p = u$, since $\lim_n$ unif $(u, a_n) = tt$. Now we claim $a = u$.

Indeed:

$a = \lim_n a_n \Leftrightarrow \forall\varepsilon > 0\ \exists N\ \forall p \geq N\ \ d(a_p, a) < \varepsilon$.

Since a is constant, in the sequence a_p the variables are as high as we want them to be (they are "pushed at infinity"). Therefore the distance $d(a_p, u) \to 0$. Now $d(a, u) \leq d(a, a_p) + d(a_p, u)$ and both $d(a, a_p)$ and $d(a_p, u)$ go to 0. Thus $d(a, u) = 0$ i.e., $a = u$.

<u>2nd case</u>: $\{x_0, \ldots, x_p\} \neq \phi$

Let k be the maximum occurrence height of all variables $x_0, \ldots, x_p$, and let $\varepsilon = 2^{-k}$. Then since $(a_n)_{n\in\mathbb{N}}$ is Cauchy, $\exists N_\varepsilon\ \forall p,q \geq N_\varepsilon\ d(a_p, a_q) < \varepsilon$, thus all trees a_p, for $p \geq N_\varepsilon$, coincide up to height k included. Now

$$\lim_n (u, a_n) = tt \Leftrightarrow (\exists N \; \forall q \geq N \; \exists \theta_q \quad u = a_q \theta_q)$$

By taking $q \geq \max (N_\varepsilon, N)$, we have that for each given $x \in \{x_0, \ldots, x_p\}$ and $\forall q \geq \max (N_\varepsilon, N)$ all the $\theta_q(x)$ are equal between themselves. These common values of θ_q, $q \geq \max (N_\varepsilon, N)$ over the set $\{x_0, \ldots, x_p\}$ give a substitution θ. By using the first case of this proof for the other branches of u_q, we have that substitution θ verifies $u = a\theta$. □

III. Continuous transformations

3.1. Clauses of the form $a \leftarrow b$ (a, b atoms): We have the following lemma.

Lemma 5: Let $\kappa : a \leftarrow b$ be a Horn clause, and assume $\overline{H}_b$ contains an infinite tree. Then the associated function t_κ: $\overline{H}_b \rightarrow 2^{\overline{H}_b} \cup \{\phi\}$

$$t_\kappa: \quad u \rightarrow \{a\theta \in \overline{H}_b: \quad u = b\theta, \quad \theta \text{ ground substitution}\}$$

is continuous iff the atom b is normal.

Proof: (i) Assume b is not normal, i.e., $b = p(.., x, .., x, ..)$ for some $p \in R$ and $x \in V$. Then for some infinite tree v and ground substitution θ, the sequence $u_n = p(\ldots, \alpha_n(v), \ldots, \alpha_{n+1}(v), \ldots)\theta$ is Cauchy and

$\lim_n u_n = p(\ldots, v, \ldots, v, \ldots)\theta$. But $\forall n \in \mathbb{N} \;\; t_\kappa(u_n) = \phi$ and

$t_\kappa(\lim_n u_n) \ni a\theta_{x \leftarrow v}$, where $\theta_{x \leftarrow v}$ is the ground substitution obtained from θ

by modifying it at variable x by v. Hence t_κ is discontinuous.

(ii) Assume b is normal, i.e., U(b) is both open and closed. Then we have the equivalence:

$$\exists\theta\ (\lim_n u_n) = b\theta \Leftrightarrow (\exists N\ \forall p \geq N\ \exists\theta_p\ \ u_p = b\theta_p \text{ and } \lim_p \theta_p = \theta\ )$$

for every Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ of $\overline{H}_b$, where the implication $\Rightarrow$ is obtained because U(b) is open and the implication $\Leftarrow$ because U(b) is closed. This equivalence implies:

$$a\theta \in t_\kappa(\lim_n u_n) \Leftrightarrow$$

$$a\theta = a \lim_n \theta_n = \lim_n a\theta_n \in \lim_n t_\kappa\ (u_n).$$

Whence the equality $t_\kappa(\lim_n u_n) = \lim_n t_\kappa(u_n)$. □

Lemma 6: If b is a normal atom, then the transformation:

$$T_\kappa:\ 2^{\overline{H}_b} \to 2^{\overline{H}_b} \cup \{\phi\},\ S \to \{a\theta\in\overline{H}_b:\ \ b\theta\in S\}$$

associated with the clause κ: a $\leftarrow$ b is uniformly continuous for the Hausdorff distance.

Proof: (i) Assume that for $S,S'\in 2^{\overline{H}_b}$ $T_\kappa(S)$, $T_\kappa(S') \neq \phi$. Then we must show:

$$\forall\varepsilon > 0\ \exists\eta > 0\ d(S,S') < \eta \Rightarrow d(T_\kappa(S),\ T_\kappa(S')) < \varepsilon\ .$$

We have

$$d(T_\kappa(S),\ T_\kappa(S')) < \varepsilon \Leftrightarrow$$

$$\forall u \in T_\kappa(S)\ \exists v \in T_\kappa(S')\quad d(u,v) < \varepsilon \quad \text{, and}$$

$$\forall v \in T_\kappa(S')\ \exists u \in T_\kappa(S)\quad d(u,v) < \varepsilon\ .$$

Now $d(u,v) < \varepsilon \Leftrightarrow$ u and v coincide up to height $-\lg_2(\varepsilon)$; let $u = a\theta$, $v = a\sigma$. Then $b\theta \in S$, $b\sigma \in S'$ and $b\theta$ and $b\sigma$ coincide up to height:

$$-\ \lg_2(\varepsilon) - \text{height } (a) + m(b)$$

where $m(b)$ is the minimal height of occurrence of a variable in atom b. Then for $\varepsilon > 0$ given, it is enough to take $\eta = \varepsilon.\ \ 2^{\text{height}(a) - m(b)}$ in order to have $\forall\varepsilon > 0\ \ \exists\eta > 0\ \ d(S,S') < \eta \Rightarrow d(T_\kappa(S),\ T_\kappa(S')) < \varepsilon$.

(ii) Now let $S \in 2^{\overline{H}_b}$ and assume $T_\kappa(S) = \phi \notin 2^{\overline{H}_b}$. Then $S \cap U(b) = \phi$ where $U(b) = (\lambda u.\ \text{unif } (u,b))^{-1}\ (tt)$. Let $(S_n)_{n\in\mathbb{N}}$ be any Cauchy sequence such that $\lim_n S_n = S$. Since $U(b)$ is closed,

$$\min_{\substack{x \in S \\ y \in U(b)}}\ d(x,y) = \alpha \neq 0$$

If we take $\varepsilon = \frac{\alpha}{2}$, then since $S_n \to S$, $\exists N\ \forall p \geq N\ d(S_p,\ S) < \varepsilon$. In particular, $\forall p \geq N\ \ S_p \cap U(b) = \phi$ i.e., $T_\kappa(S_p) = \phi$. Thus we have shown:

$$T_\kappa(S) = \phi \quad \text{and } S_n \to S \quad \text{implies}$$

$$\exists N\ \forall p \geq N \quad T_\kappa(S_p) = \phi.$$

(iii) Let $(S_n)_{n\in\mathbb{N}}$ be any Cauchy sequence that $S_n \to S$ and assume $\forall_n\ T_\kappa(S_n) = \phi$, i.e., $\forall n\ S_n \cap U(b) = \phi$. Then since b is normal, $U(b)$ is open and $S\cap U(b) = \phi$. Therefore $T_\kappa(S) = \phi$.

Which completes the proof of the lemma.

3.2. General case:

The results of the preceding paragraph can be immediately extended to clauses of the general form κ: $a \leftarrow b_1\wedge\ldots\wedge b_m$ only at the cost of having the right-hand side normal in a certain sense i.e., no variable occurs more than once in $b_1\wedge\ldots\wedge b_m$. However, a weaker form of continuity is true for these clauses.

Lemma 7: Let κ: $a \leftarrow b_1\wedge\ldots\wedge b_m$, $m \geq 1$ be a Horn clause, and let $(S_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of 2^{H_b}. Then $\lim_n T_\kappa(S_n) \subseteq T_\kappa \lim_n (S_n)$.

Proof: $\lim_n T_\kappa(S_n)$ exists by construction (lemma 1). Let $x \in \lim_n T_\kappa(S_n)$. By lemma 2 this implies $\exists$ Cauchy sequence (x_p), $\lim_p x_p = x$ and $\forall p x_p \in T_\kappa(S_{k_p})$. Now since $\forall p x_p \in T_\kappa(S_{k_p})$, we have $x_p = a\theta_p$ and $b_1\theta_p,\ldots,b_m\theta_p \in S_{k_p}$ for some substitution θ_p. Now (x_p) is a Cauchy sequence implies $(b_i\theta_p)_{p\in\mathbb{N}},\ldots,$ $(b_m\theta_p)_{p\in\mathbb{N}}$ are all Cauchy sequences and $\forall p\ b_1\theta_p \in S_{k_p}$. Therefore

$$\forall i \lim_p (b_i\theta_p) = b_i \lim_p \theta_p \in \lim_p S_{k_p} = \lim_n S_n \quad .$$

Since $\forall i\ b_i(\lim_p \theta_p) \in \lim_n S_n$, we have $a(\lim_p \theta_p) = \lim_p x_p = x \in T_\kappa(\lim_n S_n)$.

Hence $\forall x \in \lim_n T_\kappa(S_n)$, $x \in T_\kappa(\lim_n S_n)$. Whence the lemma.

Corollary: If κ is a definite Horn clause, and if (S_n) is a decreasing sequence in $2^{\overline{H}_b}$, than $\bigcap_n T_\kappa(S_n) = T_\kappa(\bigcap_n S_n)$.

Proof: Because T_κ is monotonic, we have $T_\kappa(\bigcap_n S_n) \subseteq \bigcap_n T_\kappa(S_n)$. Now by the preceding lemma $\lim_n T_\kappa(S_n) \subseteq T_\kappa(\lim_n S_n)$ and by Lemma 3, $\bigcap_n T_\kappa(S_n) = T_\kappa(\bigcap_n S_n)$. □

Since $\overline{H}_b$ is a compact metric space, the notion of a limit of closed subsets in the Hausdorff distance sense coincides with the notion of a limit in the Kuratowski-Painlevé sense, i.e., for any sequence $(S_n)_{n\in\mathbb{N}}$ of $2^{\overline{H}_b}$, $\lim_n S_n$ exists in the Hausdorff distance sense if and only if $\lim_n S_n = \underset{n}{LS}(S_n) = \underset{n}{LI}(S_n) = \underset{n}{L}(S_n)$ where LS (resp. LI, resp. L) denotes the limit sup. (resp. the limit inf, resp. the limit in the Kuratowski-Painlevé sense) of sequence (S_n). This result is given in Hausdorff [4] pp. 170-172.

We recall the definition of these limits.

limit inf:

$$x \in \underset{n}{LI}\,(S_n) \Leftrightarrow \forall \text{ open neighbourhood } V(x) \text{ of } x$$

$$\exists N > 0 \quad \forall p \geq N \quad V(x) \cap S_p \neq \phi$$

lim sup:

$$x \in \underset{n}{LS}\,(S_n) \Leftrightarrow \forall \text{ open neighbourhood } V(x) \text{ of } x$$

$$\forall N > 0 \quad \exists p \geq N \quad V(x) \cap S_p \neq \phi$$

Kuratowski-Painlevé limit:

If $\underset{n}{\mathrm{LI}}\ (S_n) = \underset{n}{\mathrm{LS}}\ (S_n)$ then by definition $\underset{n}{\mathrm{L}}(S_n) = \underset{n}{\mathrm{LI}}\ (S_n) = \underset{n}{\mathrm{LS}}\ (S_n)$.

Lemma 8: Let P be a logic program.

(i) For any sequence $(S_n)_{n\in\mathbb{N}}$ of $2^{\overline{H}b}$, $\underset{n}{\mathrm{LS}}\ T_P(S_n) \subseteq T_P\ (\underset{n}{\mathrm{LS}}\ (S_n))$

(ii) For any Cauchy sequence $(S_n)_{n\in\mathbb{N}}$ of $2^{\overline{H}b}$, $\underset{n}{\mathrm{LI}}\ T_P(S_n) \subseteq T_P(\underset{n}{\mathrm{LI}}\ (S_n))$

Proof: (i) Let $x \in \underset{n}{\mathrm{LS}}\ T_P(S_n)$. Then $x \in \underset{n}{\mathrm{LS}}\ T_P(S_n) \Leftrightarrow$

$\forall\varepsilon > 0\ \ \forall N > 0\ \ \exists p \geq N\ \ \exists y \in T_P(S_p)\ \ d(x,y) < \varepsilon$. Take $\varepsilon < 2^{-h(P)}$, where $h(P)$ is the maximum height of the left-hand sides of P. Then $d(x,y) < \varepsilon \Rightarrow$ $\exists$ clause κ: $(a \leftarrow b_1 \wedge \ldots \wedge b_m) \in P$ such that $x = a\sigma$, $y = a\theta$ where σ and θ are substitutions and furthermore, since $y \in T_P(S_p)$, $b_1\theta,\ldots,b_m\theta \in S_p$.

Let us take a decreasing sequence of ε's such that $\varepsilon < 2^{-h(P)}$. Each ε will give some y such that $d(x,y) < \varepsilon$. Since P is finite, infinitely many such y's will be such that $y = a\theta$ and $b_1\theta,\ldots,b_m\theta \in S_p$. Now

$$\lim_{\varepsilon\to 0} y = x = \lim_{\varepsilon\to 0} a\ \theta = a \lim_{\varepsilon\to 0} \theta = a\sigma \quad .$$

Thus the sequence θ converges in Θ towards σ, and $a\sigma \leftarrow b_1\sigma \wedge \ldots \wedge b_m\sigma$ is an instantiation of κ.

Now we have $b_1\sigma,\ldots,b_m\sigma \in \underset{n}{LS}\ (S_n)$ by construction of these limits $b_i\sigma = b_i\ (\lim_{\varepsilon\to 0} \theta)$. Thus $x = a\ \sigma \in T_\kappa(\underset{n}{LS}\ (S_n)) \subseteq T_P\ (\underset{n}{LS}\ (S_n))$. Therefore $\underset{n}{LS}\ (T_P(S_n)) \subseteq T_P\ (\underset{n}{LS}\ (S_n))$.

(ii) By definition of these limits, we have $\underset{n}{LI}\ (S_n) \subseteq \underset{n}{LS}\ (S_n)$

Now if $(S_n)_{n\in\mathbb{N}}$ is Cauchy, then by Hausdorff's theorem we have $\underset{n}{LI}\ (S_n) = \underset{n}{LS}\ (S_n) = \lim_{n}\ (S_n)$. Now we have by (i):

$$\underset{n}{LI}\ T_P(S_n) \subseteq \underset{n}{LS}\ T_P(S_n) \subseteq T_P(\underset{n}{LS}\ (S_n)) = T_P\ (\underset{n}{LI}\ (S_n))$$

which completes the proof. □

Remark 1: If the sequence (S_n) is not Cauchy, then $\underset{n}{LI}\ T_P(S_n) \subseteq T_P\ (\underset{n}{LI}\ (S_n))$ does not hold, as shown by the following example.

Take P: $\{p(b) \leftarrow y\ (f(x),\ f^2(x))$

$p(b) \leftarrow r\ (f(x),\ f^3(x))\ \}$

and the sequence of sets $(S_n)_{n\in\mathbb{N}}$ defined by:

$$S_{2n} = \{q(f^n(b),\ f^{n+1}(b))\}$$

$$S_{2n+1} = \{r(f^n(b),\ f^{n+2}(b))\}$$

Then $\underset{n}{LI}\ (S_n) = \phi$ thus $T_P\ (\underset{n}{LI}\ (S_n)) = \phi$, and $\underset{n}{LS}\ (S_n) = \{q(f^\omega, f^\omega),\ r(f^\omega, f^\omega)\}$ thus $T_P(\underset{n}{LS}\ (S_n)) = \{p(b)\}$.

On the other hand:

$$T_P\ (S_{2n}) = \{p(b)\} = T_P\ (S_{2n+1})\ .$$

Thus:

$$\underset{n}{LI}\ (T_P(S_n)) = \underset{n}{LS}\ (T_P(S_n)) = \{p(b)\}\ .$$

Therefore

$$\{p(b)\} = \underset{n}{LI}\ (T_P\ (S_n)) \nsubseteq T_P\ (\underset{n}{LI}\ (S_n)) = \phi\ .$$

Remark 2: If $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, then $(T_P\ (S_n))_{n\in\mathbb{N}}$ is not necessarily a Cauchy sequence. Consider the following example:

$$P: \{p(a) \leftarrow q(u,v)$$

$$p(b) \leftarrow q(f(x),\ f^2(x))\ \}$$

$$S_{2n} = \{q(f^n(a),\ f^{n+1}(a))\}$$

$$S_{2n+1} = \{q(f^n(a),\ f^{n+2}(a))\}$$

Then $\mathrm{LI}_n (S_n) = \mathrm{LS}_n (S_n) = \{q(f^\omega, f^\omega)\} = \lim_n S_n$

$$T_P (\mathrm{LI}_n (S_n)) = T_P (\mathrm{LS}_n (S_n)) = \{p(a), p(b)\}$$

But we also have

$$T_P (S_{2n}) = \{p(a), p(b)\}$$

$$T_P (S_{2n+1}) = \{p(a)\}$$

Therefore

$$\mathrm{LI}_n (T_P (S_n)) = \{p(a)\}$$

$$\mathrm{LS}_n (T_P (S_n)) = \{p(a), p(b)\}$$

and the Hausdorff limit of $(T_P (S_n))$ does not exist.

Theorem 1: Let $P = \{\kappa_i: i \in I\}$ be a logic program. Then:

(i) For every Cauchy sequence $(S_n)_{n \in \mathbb{N}}$ of $2^{\overline{H}_b}$, if $\lim_n T_P (S_n)$ exists, then

$$\lim_n T_P(S_n) \subseteq T_P (\lim_n S_n).$$

(ii) T_P is $\cap$-continuous i.e., for every decreasing sequence $(S_n)_{n \in \mathbb{N}}$ of $2^{\overline{H}_b}$,

$$T_P (\bigcap_n S_n) = \bigcap_n T_P (S_n) \qquad \square$$

Proof: (i) Follows immediately from the above theorem and Hausdorff's theorem. A more direct proof is as follows:

$$\text{Let } x \in \lim_n T_P(S_n) = \lim_n (\bigcup_i T_{\kappa_i}(S_n)).$$

Then by lemma 2(ii), there exists a Cauchy sequence $(x_p)_{p\in\mathbb{N}}$ such that $(\lim_p x_p) = x$, and $\forall p\ x_p \in \bigcup_i T_{\kappa_i}(S_{k_p})$. Let us take $\varepsilon > 0$ such that:

$$-\lg_2(\varepsilon) = (\text{maximal height of left-hand sides of } P) + 1.$$

Then we get

$\exists N\ \forall p \geq N\ \ x_p = a\theta_p$ for some fixed left-hand side a of P. Let κ: $a \leftarrow b_1 \wedge \ldots \wedge b_m$ be the clause containing this left-hand side. Then:

$$\forall p \geq N\ \ x_p = a\theta_p \in T_\kappa(S_{k_p}) \text{ and } x = a \lim_p \theta_p = a\theta$$

Hence $\forall p \geq N\ \ \exists\ b_1\theta_p, \ldots, b_m\theta_p \in S_{k_p}$ and the sequences $(b_1\theta_p)_{p\in\mathbb{N}}, \ldots,$ $(b_m\ _p)_{p\in\mathbb{N}}$ are all Cauchy sequences. Their limits $b_1\theta, \ldots, b_m\theta$ are all in $\lim_{k_p} S_{k_p} = \lim_n S_n$ according to Lemma 2(i). Hence:

$$a\theta \leftarrow b_1\theta \wedge \ldots \wedge b_m\theta$$

is an instantiation of κ, therefore:

$$x = a\theta \in T_\kappa(\lim_n S_n) \subseteq T_P(\lim_n S_n)$$

i.e., $x \in T_P(\lim_n S_n)$. Whence $\lim_n (T_P(S_n)) \subseteq T_P(\lim_n S_n)$ whenever the first limit exists.

(ii) It is enough to show that for every decreasing sequence $(S_n)_{n\in\mathbb{N}}$ of $2^{\overline{H}_b}$, $T_P(\bigcap_n S_n) \subseteq \bigcap_n T_P(S_n)$, but this is given by the monotonicity of T_P. From (i) and Lemma 3 we get $\bigcap_n T_P(S_n) \subseteq T_P(\bigcap_n S_n)$. Whence the theorem. □

IV. <u>Greatest fixpoint theorem</u>:

<u>Theorem 2</u>: Let P be a logic program; then $\bigcap_n T_P(\overline{H}_b)$ is the greatest fixpoint of T_P.

<u>Proof</u>: The sequence $(T_P{}^n(\overline{H}_b))_{n\in\mathbb{N}}$ is decreasing and T_P is $\cap$-continuous; whence:

$$T_P(\bigcap_n T_P{}^n(\overline{H}_b)) = \bigcap_n T_P{}^{n+1}(\overline{H}_b) = \bigcap_n T_P{}^n(\overline{H}_b)$$

The fact that this fixpoint is the greatest one is obvious. □

V. <u>An example</u>

We consider "Hamming's problem" as discussed in Dijkstra [3]: construct the sorted list of all natural numbers ≠ 0 containing no prime factors other than 2, 3 or 5. This can be formalized by using a logic program. We first give our notation:

u.x	is the list resulting from inserting atom u at the beginning of list x.
< , >	are relation symbols with their usual meaning (infix notation is used here)
Prod(a,b,c)	means $c = a * b$
F(x,n,y)	= list y is obtained by multiplying all the elements of list x by n (scalar multiplication)
M(x,y,z)	= sorted list z is obtained by merging without repetition sorted lists x and y.

From this follows the following logic program P with query Eq(x,y):

[0. F(nil, n, nil) ←]

1. $F(a.x,\ n,\ b.y) \leftarrow Prod(a,\ n,\ b) \wedge F(x,\ n,\ y)$
2. $M(u.x,\ u.y,\ u.z) \leftarrow M(x,\ y,\ z)$
3. $M(x_1.x,\ y_1.y,\ x_1.z) \leftarrow (x_1 < y_1) \wedge M(x,\ y_1.y,\ z)$
4. $M(x_1.x,\ y_1.y,\ y_1.z) \leftarrow (y_1 < x_1) \wedge M(x_1.x,\ y,\ z)$
5. $Eq(x,y) \leftarrow F(y,\ 3,\ u) \wedge F(y,\ 5,\ v) \wedge M(u,\ v,\ w) \wedge F(y,\ 2,\ z) \wedge M(w,\ z,\ x)$
6. $\leftarrow Eq(x,\ 1.x)$

The desired infinite list, which is computed by fair derivations

$\ell = 2.3.5.6.8.9.10.12.15.16.18.20.24.\ \ldots$

verifies $Eq(\ell,\ 1.\ell) \in \bigcap_{n \in N} T_P{}^n\ (\overline{H}_b)$ i.e., $\forall\ n \in N\ Eq(\ell,\ 1.\ell) \in T_P{}^n(\overline{H}_b)$,

where $\overline{H}_b$ is the completed Herbrand base associated with the above program.

The proof is as follows: For n = 0, it is obvious. Let $n \neq 0$. To make things simpler let us assume that <, > and Prod produce their results "instantaneously". We may remark that:

$$\forall q>0, q \in N\ \{F(\vec{\alpha}.\vec{u},\ m,\ \vec{\beta}.\vec{v}) : \vec{\alpha}, \vec{u}, \vec{\beta}, \vec{v} \text{ lists},\ |\vec{\alpha}| = |\vec{\beta}| = q,\ \vec{\alpha} = m \star \vec{\beta}\} \subseteq T_P{}^q(\overline{H}_b)$$

From this we deduce, if $(\ell/n+1)$ denote the initial segment of length n+1 of list ℓ, that

$$\{Eq((\ell/n+1).\ \vec{u},\ 1.(\ell/n+1).\ \vec{v}):\ \vec{u}, \vec{v} \text{ lists}\} \subseteq T_P^{n+1}\ (H_b)$$

by applying the fifth clause once.

Whence $Eq(\ell,\ 1.\ell) \in T_P^{n+1}\ (\overline{H}_b)$.

In fact we even have:

$$[Eq(x,\ 1.x)] \cap (\bigcap_n T_P{}^n\ (\overline{H}_b)) = \{Eq(\ell,\ 1.\ell)\} \qquad (\star)$$

where $[Eq(x,\ 1.x)]$ denotes the subset of $\overline{H}_b$ obtained by replacing every occurrence of a free variable in Eq(x, 1.x) by every possible element of the Herbrand universe.

The property $(\star)$ above is just an occurrence of a more general result shown in [8], which can be stated as follows. Let P be a finite logic program, and t a query; assume Δ is a generic name for finite derivations which are fair up to $q > 0$, and where $\theta_1, \ldots, \theta_n$ is the sequence of most general unifiers associated with Δ.

Then:

$$\left(\bigcup_{\substack{\Delta \text{ finite derivation} \\ \text{fair up to } q}} [t\theta_1 \ldots \theta_n] \right) = [t] \cap T_p{}^q(\overline{H}_b)$$

When q goes to $+\infty$, this becomes

$$\bigcup_{\Delta \text{ fair derivation}} [t\theta_1 \ldots \theta_n \ldots] = [t] \cap (\bigcap_q T_p{}^q (\overline{H}_b))$$

which is a generalization of $(\star)$.

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Bibliography:

[1]. Apt, K.R. and van Emden, M.H., Contributions to the theory of logic programming, JACM 29, 3 (1982), pp. 841-862.

[2]. Arnold, A. and Nivat, M., The metric space of infinite trees. Algebraic and topological properties, Fundamenta Informatica 3, 4 (1980), pp. 445-476.

[3]. Dijkstra, E.W., A discipline of programming, Prentice Hall, (1976).

[4]. Hausdorff, F., Set theory, Chelsea Publishing Co., New York (1962).

[5]. Kowalski, R., Logic for problem solving, Elsevier North Holland, New York, (1979).

[6]. Kowalski, R. and van Emden, M.H., The semantics of predicate logic as a programming language, JACM 23, 4 (1976), pp. 733-742.

[7]. Nait Abdallah, M.A., Sort theory, University of Waterloo, TR CS-82.

[8]. Nait Abdallah, M.A. and van Emden, M.H., Algorithm theory and logic programming (draft manuscript).