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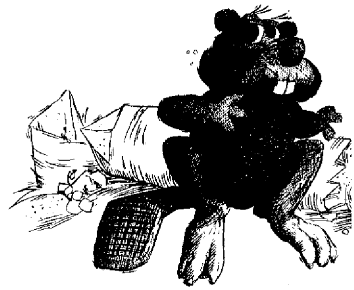

On The Definition and Computation of Rectilinear Convex Hulls

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ABSTRACT

Recently the computation of the rectilinear convex hull of a collection of rectilinear polygons has been studied by a number of authors. From these studies three distinct definitions of rectilinear convex hulls have emerged. We examine these three definitions for point sets in general, pointing out some of their consequences, and we give optimal algorithms to compute the corresponding rectilinear convex hulls of a finite set of points in the plane.

Keywords: Convex hull; rectilinear convex hull; geometry; computational geometry; maximal elements; rectilinear polygons.

1. INTRODUCTION

The computation of the convex hull of a point set has been an early and central topic in computational geometry, see [Sh] for example. Hence it is not too surprising, with the interest shown in rectilinear figures, that the

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computation of the rectilinear convex hull should eventually be studied, see [MF], [NLLW], and [OSW]. Indeed the notion of rectilinear convexity appeared quite early in the literature, see [U]. However, unlike the classical convex hull, the definition of the rectilinear convex hull is fraught with difficulties. Three different definitions have appeared in the literature. The first, which we term the classical definition was introduced in [OSW]. The second, which we term the connected definition was introduced in [MF] and [NLLW], while the third which we term the maximal definition is based on the work of [KLP].

In this note we examine these three approaches, demonstrating that each have their defects. The classical and maximal approaches allow the resulting rectilinear convex hull to be a disconnected set, while the connected approach yields, in many cases, not only a non-unique rectilinear convex hull, but also infinitely many.

In Section 2 we give the three definitions and the various examples. In Section 3 we give algorithms for the computation of the rectilinear convex hull of n points according to the three definitions. Finally in Section 4 we discuss the implications of the three alternative definitions and state our preference.

Recently, see [KS1, KS2, KS3], it has been shown that the time bound for the computation of the *mr*-convex hull can be improved to $O(n \log h)$, where h is the number of points on the hull. Moreover this has been proved to be optimal [KS2], and the same time bound has been shown to be attainable for the 3-dimensional case [KS3].

Before entering upon our discussion we need to define some basic terminology. Recall that a point set in the plane is an arbitrary subset of R^2. A point set is said to be *connected* if every two points in the set can be connected by a line (not necessarily straight) within the set. A *rectilinear (straight) line* is a straight line oriented parallel to either the x-axis or y-axis, and a *rectilinear line segment* is a line segment oriented parallel to either the x-axis or y-axis. Finally, a *rectilinear curve* is a (finite) line consisting of rectilinear line segments.

2. THE TWO DEFINITIONS AND THEIR DIFFICULTIES

In [G] and [HDK] a point set in the plane is said to be *convex* if for every pair of points in the set, the line segment they determine lies entirely in the given set. In the rectilinear case, see [MF], [NLLW], and [OSW], we have:

Definition 2.1 A point set is called *rectilinear convex* or *r-convex* if for any two of its points which determine a rectilinear line segment, the line segment lies entirely in the given set.

In Figure 2.1 we display three r-convex sets

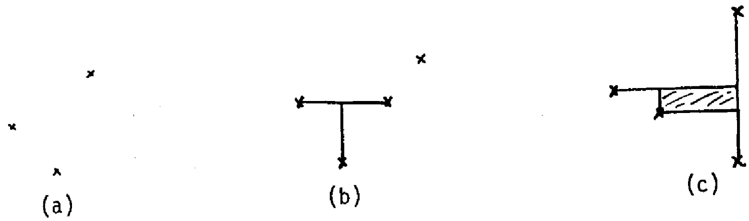


Figure 2.1

Observe that an r-convex set may be disconnected as in Figures 2.1 (a) and (b). For convex sets this is not the case. However we remark that an alternative definition of convex sets carries over to r-convex sets when suitably modified, viz:

> *A point set is r-convex if its intersection with every rectilinear (straight) line is either empty or a connected set.*

We now give three possible approaches to the definition of a rectilinear convex hull of a point set. We begin with the *classical* definition introduced by the present authors in [OSW].

Definition 2.2 *Classical*

Given a point set, its *r-convex hull* is the smallest r-convex set containing the given set.

In Figure 2.1 the "crosses" indicate the original points sets, whose r-convex hulls are the figures.

Observe that this definition of r-convex hull is equivalent to:

> *The r-convex hull of a point set is the intersection of all r-convex sets that contain the given set.*

This, the first definition of the r-convex hull of a set gives rise to a unique r-convex hull. Unfortunately the resulting r-convex hull may be a disconnected set, whereas the usual convex hull is always connected. Presumably both [MF] and [NLLW] had this in mind when they required the r-convex hull to be connected, giving rise to the second definition of r-convex hull, namely:

Definition 2.3 *Connected*

Given a point set, its *connected r-convex hull*, or *cr-convex hull*, is a smallest connected r-convex set containing the given set.

Observe that cr-convex hulls are not necessarily unique, a fact which [NLLW] initially overlooked. In Figure 2.2 three different cr-convex hulls of a three point set are given. There are indeed an infinite number of distinct cr-convex hulls in this case, which fact seems to have eluded [MF], who appear to believe that there are finitely many distinct cr-convex hulls of a point set.

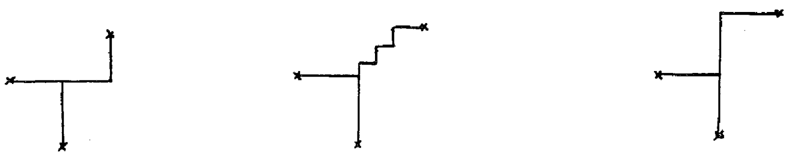


Figure 2.2

Although the non-uniqueness of cr-convex hulls is a major difficulty, there are others however. Observe that an alternative definition in terms of the intersection of all cr-convex sets which contain the given point set gives rise to neither the r-convex hull, nor a cr-convex hull, in general. For example the three point set of Figure 2.2 gives Figure 2.3. This leads to a third possible definition, namely:

Definition 2.4 *Maximal*

Given a point set, its *maximal r-convex hull*, or *mr-convex hull*, is the intersection of all closed rectilinear half-planes that contain it. In Figure 2.4 the four possible rectilinear half-planes are illustrated, while Figure 2.3 displays the mr-convex hull of the three point set in Figure 2.2.

In the following, we relate cr-convex hulls and their intersection.

Let A be a region in the plane. We define $x(left(A))$ to be the x-coordinate of the leftmost points of A. Similarly we define $x(right(A)), y(top(A))$ and $y(bottom(A))$ with the obvious meanings. The proof of the following proposition is straightforward.

Proposition 2.4 *Given a point set, the intersection of its cr-convex hulls is the mr-convex hull and is composed of* p *connected regions* $p \geq 1$ *, which can*

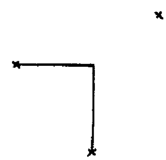


Figure 2.3

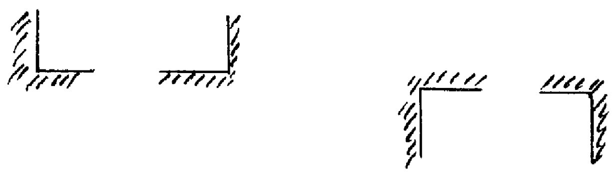


Figure 2.4

be ordered as $A_1, \ldots, A_p$ *such that for all* i , $i = 2, \ldots, p$, *either*

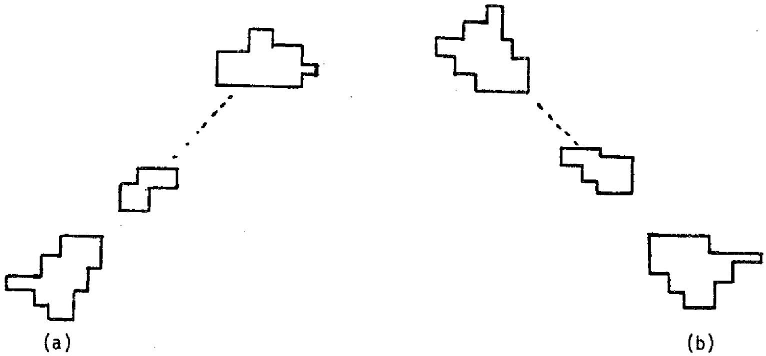
(1) $x(right(A_{i-1})) \leq x(left(A_i))$ and $y(top(A_{i-1})) \leq y(bottom(A_i))$

or

(2) $x(right(A_{i-1})) \leq x(left(A_i))$ and $y(bottom(A_{i-1})) \geq y(top(A_i))$.

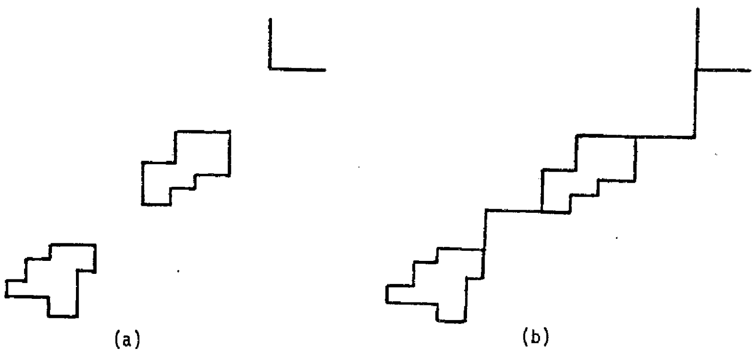
(see Figure 2.5). Moreover if (1) holds for $A_1, \ldots, A_p$ *, then the rightmost of the topmost points of each* A_i *,* $i = 1, \ldots, p-1$ *, is also a rightmost point of* A_i *, and the leftmost of the bottommost points of each* A_i *,* $i = 2, \ldots, p$ *, is also a leftmost point of* A_i *(Figure 2.5(a)). Correspondingly, if for* $A_1, \ldots, A_p$ *(2) holds, then the rightmost of the bottommost points of each* A_i *,* $i = 1, \ldots, p-1$ *is also a rightmost point of* A_i *, and the leftmost of the topmost points of each* A_i *,* $i = 2, \ldots, p$ *, is also a leftmost point of* A_i *(Figure 2.5(b)).*

Moreover, each cr-convex hull of the given set is obtained by joining $A_1, \ldots, A_p$ *with rectilinear curves (Figure 2.6).*



Two forms for the intersection of all *cr*-convex hulls.

Figure 2.5



The intersection of all *cr*-convex hulls (a) and its extension to a hull by adding rectilinear line segments (b).

Figure 2.6

A well known theorem on convex hulls, see [HDK] for example, is:

A point is in the convex hull of a set if, and only if, it is already in the convex hull of three or fewer points.

The theorem continues to hold for *r*-convex hulls and *mr*-convex hulls, but not for *cr*-convex hulls, for example Figure 2.7 gives a counter-example.

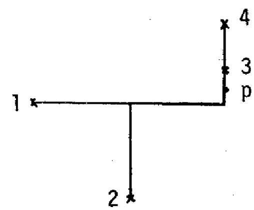


Figure 2.7

Observe, on the one hand, that p need not be in the *cr*-convex hull of $\{1,2,3\}$, cf. Figure 2.2 (c). On the other hand, if a point is in the *cr*-convex hull of $\{1,2,3\}$ it need not be in the *cr*-convex hull of $\{1,2,3,4\}$. It is difficult to see how this "theorem" statement can be modified to enable it to hold.

Up until now we have only considered the definition of (rectilinear) convex hulls, but whatever definition is chosen, we are then faced with its computation. In this note we only consider the computation of convex hulls for *finite* point sets, other cases are discussed in [MF], [NLLW], and [OSW]. Even so there are two distinct computational problems, the static and dynamic problems.

The static problem can be stated as:

Given n points in the plane, $n \geq 1$, compute their rectilinear convex hull.

The dynamic problem is:

Given an initial finite set of points, consider sequences of insertions of points, deletions of points, and rectilinear-convex-hull queries (what is the rectilinear convex hull of the current set of points).

We provide a solution to the static problem in Section 3, while in the remainder of this section we discuss briefly the dynamic problem.

It turns out, that in a dynamic environment, the *cr*-convex hull and the *mr*-convex hull fare better than the *r*-convex hull, since they are more stable. Indeed, as we show in the next section the *mr*-convex hull can be decomposed into 4 "staircases" or rectilinear curves. To maintain the *mr*-convex hull it suffices to maintain the four staircases separately. This decomposition can then be maintained as in [Ov] and [OvL] for the usual convex hull.

Now consider the *r*-convex hull of a point set in which no two points are co-rectilinear, see Figure 2.8 for example. Then its *r*-convex hull is the point set itself, that is it is completely disconnected, while its *cr*-convex hull is, of course, connected and perhaps also unique. A *cr*-convex hull corresponding to Figure 2.8 is shown in Figure 2.9, while in Figure 2.10 is displayed its *mr*-convex hull.

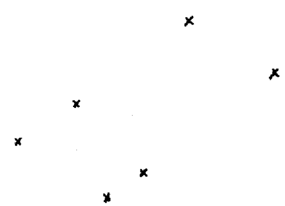


Figure 2.8

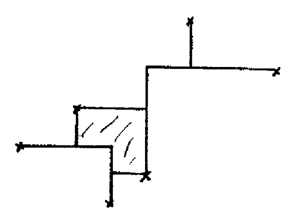


Figure 2.9

Now add one point p , to Figure 2.8, such that it is co-rectilinear with some

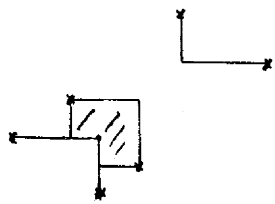


Figure 2.10

other point q , in the set. For example see Figure 2.11. In Figure 2.12 the r-convex hull of this new point set is displayed; it is *now* a connected set! Hence it is also the cr-convex hull and the mr-convex hull, and we see little change from Figures 2.9 and 2.10.

These examples show that the r-convex hull can vary tremendously after the insertion or deletion of a point. It doesn't grow and shrink smoothly as one usually expects of a convex hull, see [OvL]. Because of this instability it is to be expected that the maintenance of r-convex hulls is more time consuming than the maintenance of cr-convex hulls and mr-convex hulls. This topic is, however, beyond the scope of the present note, and is left for future investigation.

We now turn to the static problem, the computation of the (rectilinear) convex hull of n given points.

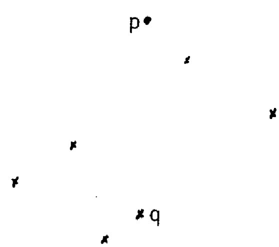


Figure 2.11

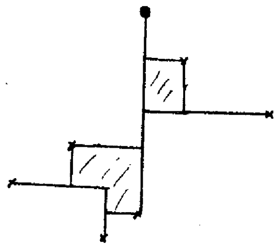


Figure 2.12

3. COMPUTING THE RECTILINEAR CONVEX HULL

In this section we provide algorithms for computing the *r*-convex hull, the *mr*-convex hull, and a *cr*-convex hull of a given set of n points, $n \geq 1$. We begin with the *cr*-convex hull, which is the more difficult of the two, because of its non-uniqueness.

To ease the non-uniqueness problem somewhat, we add the following condition to the definition of the *cr*-convex hull:

(3.1) *The cr-convex hull contains only minimal arms.*

An *arm* in a *cr*-convex hull is a rectilinear curve joining two points in the hull but which is not included in the *mr*-convex hull, see Figure 2.2 (b), and a *minimal arm* is an arm consisting of either one or two line segments, see Figures 2.2 (a) and (c).

Condition (3.1) does not guarantee uniqueness, it serves only to ensure that there are a *finite* number of alternative *cr*-convex hulls of a given finite point set. This follows from the observation that between any two points there are at most two distinct minimal arms.

But we may remove this finite ambiguity, by insisting that of the two possible minimal arms, one is always preferred.

We say └ and ┘ are *lower* minimal arms, and ┌ and ┐ are *upper* minimal arms. We now have:

(3.2) *Whenever a choice of minimal arms is possible between two points in the cr-convex hull, the lower minimal arms are chosen.*

These two conditions lead to the following proposition which follows immediately from Proposition 2.4.

Proposition 3.1 *Let S be a set of n points in the plane, $n \geq 1$. Then there is one and only one cr-convex hull of S satisfying both conditions (3.1) and (3.2).*

Although Proposition 3.1 provides a basis for computing a unique *cr*-convex hull, it is still of interest to know when the *cr*-convex hull isn't unique under the original definition. To this end we provide a characterization of these situations, by way of the following definitions.

Definition 3.2 Let S be a set of n points in the plane, $n \geq 1$, and R be the minimal rectilinear rectangle containing S. Then S can be classified according to one of the following four types:

Type 1: R is degenerate, that is a rectilinear line segment. In the remaining three types R is non-degenerate.

Type 2: There are exactly two points of S on the contour of R. These two points must be opposite corner points of R.

Type 3: There are at least three points of S on the contour of R, one of which is a corner point, and S is not Type 4.

Type 4 There are at least four points of S on the contour of R, one on each side of R.

In Figure 3.1 we display examples of the four types of S. This classification of S provides us with a means to characterize the uniqueness of the *cr*-convex hull of S. Clearly if S is of Type 1 then its *cr*-convex hull is unique. Moreover if S is of Type 2 its *cr*-convex hull is, just as clearly, non-unique, since one of the corner points needs an arm to connect it to the *cr*-convex hull of the remaining points, see Figure 2.2 once more.

When S is of Type 3 there are two subtypes to consider. Type 3(a) is two points of S are adjacent corner points of R, see Figure 3.2(a), and Type 3(b) captures the remaining situations, see Figure 3.1(c). The *cr*-convex hull of S is unique when it is of Type 3(a), by Lemma 3.3 below, and is non-unique otherwise, since it is similar to Type 2.

The only remaining type is Type 4, and again this splits into subtypes. Type 4(a) is when three or four corner points of R are in S, see Figure 3.2(b). By Lemma 3.3 the *cr*-convex hull of S is unique. Type 4(b) and 4(c) capture the remaining possibilities. Let the four points p_l, p_r, p_b, and p_t be on the left, right, bottom, and top sides of R, respectively. If ($y_l \geq y_r$ and $x_b \leq x_t$) or ($y_l \leq y_r$ and $x_b \geq x_l$), or there is a point p of S different from p_l, p_r, p_b, p_t in the set

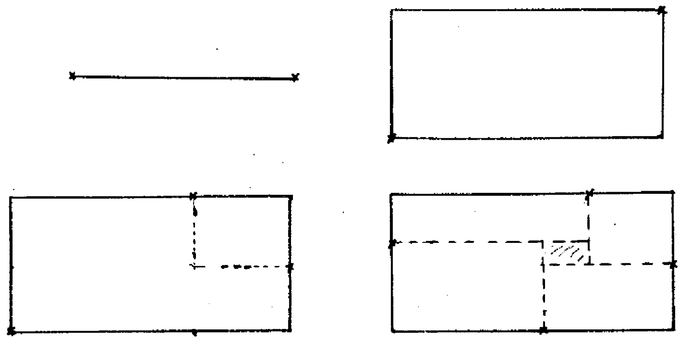


Figure 3.1

$$\Big\{ (x, y) : (x \ge x_b > x_t \text{ and } y \ge y_l > y_r) \text{ or } (x \le x_l < x_b \text{ and } y \le y_r < y_l)$$

$$\text{or } (x \le x_b < x_t \text{ and } y \ge y_r > y_l) \text{ or } (x \ge x_t > x_b \text{ and } y \le y_l < y_r) \Big\}$$

then S is of Type 4(b) and the *cr*-convex hull of S is unique, see Figure 3.1(d) for one example. If S is neither Type 4(a) nor Type 4(b) then it is Type 4(c). Again Type 4(c) defines a non-unique convex hull, see Figure 3.2(c). The rectilinear dashed curves given in Figure 3.2(c) are always in the *cr*-convex hull but the turning points of these curves can be connected in an arbitrary manner.

Types 3(a), 4(a) and 4(c) provide for unique *cr*-convex hulls by means of the following:

Lemma 3.3: *Let S be a set of points, and R be its minimal enclosing rectilinear rectangle. If three corner points of R are in S, then the cr-convex hull of S is unique, and equals the mr-convex hull of S.*

Proof: Since three corner points of R are in S, then two adjacent edges of R must be in the *cr*-convex hull of S. Consider an arbitrary point p in S which is not on these two edges (if there are no such points the two

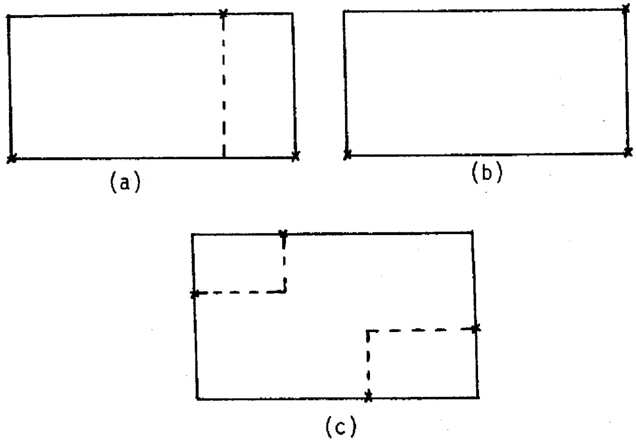


Figure 3.2

edges form the unique *cr*-convex hull of S). The two rectilinear line segments joining p to the two edges must be in the *cr*-convex hull of S. Furthermore all the points within the rectangle formed by these two edges and two line segments must also be in the *cr*-convex hull of S. Now consider the union of the two edges and the rectangles formed by all points p in S which are not on the two edges. This is clearly *r*-convex, connected, unique, and the minimal such set containing S, and, hence, is the unique *cr*-convex hull of S. The equality with the *mr*-convex hull follows from Proposition 2.4. □

If S is of Type 3(a), then the *cr*-convex hull of S is the union of two *cr*-convex hulls of the kind specified in the lemma, see Figure 3.2(a). For Type 4(b) a similar, but more complex, decomposition is possible, see Figure 3.1(d).

Given a set of points the determination of its type can be obtained straightforwardly, resulting in:

Proposition 3.4 *Given a set of* n *points in the plane,* $n \geq 1$ *, it can be classified in* $O(n)$ *time and space, that is in optimal time and space. Hence it is decidable in* $O(n)$ *time and space, whether or not its cr-convex hull is unique.*

Propositions 3.1 and 3.4 lead to two different methods of dealing with cr-convex hulls. The first is to compute the unique restricted cr-convex hull, and the second is to compute the cr-convex hull only if it is unique. However rather than giving an algorithm specific to the unique case, we consider computing the cr-convex hull in general, that is under conditions (3.1) and (3.2).

The key observation is that we can always decompose the computation of the cr-convex hull into at most four similar computations, cf. the computation of maximal elements in [B], [KLP], [Ov] and [OvL]. The decomposition gives at most four Type 2 situations, for each of which we need to compute either the lower or upper cr-convex hull. Since the lower and upper cr-convex hull algorithms are almost identical we only give the one for the upper hull in detail. After presenting the algorithm we discuss how the various decompositions are obtained.

Given a Type 2 set of points it follows that any upper (and lower) cr-convex hull is a non-decreasing rectilinear curve (or "staircase"). The following algorithm makes essential use of this fact, see [MF] for a similar algorithm.

Algorithm *UPPER CR-CONVEX HULL*

Input: n points in the plane, $n \geq 1$, such that the corner points (x_l, y_b) and (x_r, y_t) of R are also points of S.

Output: The vertices of an upper cr-convex hull of S.

begin

Step 1: Sort the n points in non-descending order by their x-coordinates, and for two or more points with equal x-coordinates they are sorted in decreasing y-coordinate order. This requires $O(n \log n)$ time and $O(n)$ space.

Step 2: Scan the sorted points keeping only those points which form a maximal monotonic increasing sequence with respect to their y-coordinates. This can be carried out in a single scan of the sorted sequence in $O(n)$ time and space.

Step 3: Extend this sequence of points to include the additional points which are necessary to specify the endpoints of the rectilinear line segments. This is done so that each added point has the same y-coordinate as the previous point in the sequence. This sequence is the upper cr-convex hull, and this step requires $O(n)$ time and space also.

end *UPPER CR-CONVEX HULL.*

To compute a cr-convex hull of an arbitrary set of n points in plane, we consider each type in turn:

Type 1: The lower *cr*-convex hull is simply the reverse of the upper hull.

Type 2: The lower *cr*-convex hull is computed in a similar manner to the upper hull, except points with equal x-coordinates are sorted in increasing order of y-coordinate. Also the new points introduced in Step 3 are added to the set of input points.

Type 3: In this case there can be two upper portions, however the second can be computed by scanning from right to left rather than left to right. The lower hull can be computed as in Type 2, see Figure 3.1(c).

Type 4: In this case there can be two upper portions as in Type 3, and two lower portions, for which the Type 2 technique can be adapted, see Figure 3.1(d).

Since the computation of the *mr*-convex hull is analogous we have:

Theorem 3.5 *A cr-convex hull and the mr-convex hull of an n point set in the plane, $n \geq 1$, can be computed in $O(n \log n)$ time and $O(n)$ space. Moreover for reasonable models of computation this is optimal.*

Optimality follows by observing that the usual lower bound proof for convex hull computation can be adapted to the rectilinear case, see [OvL] for example.

Having dealt with the *cr*-convex hull, we now turn to the computation of the *r*-convex hull. If no two points, in the given set S, are co-rectilinear, then the *r*-convex hull is S itself, hence we consider what happens when two points in S are co-rectilinear. Immediately the line segment joining them must be in the *r*-convex hull. For example if the two points p and q, say, have the same x-coordinate, then $[p,q]$ is a vertical line segment. Consider the swath cut by this linear segment when sweeping it horizontally through S. If this swath contains no new points from S, and there are no other co-rectilinear points in S, then the *r*-convex hull has been found, see Figure 3.3. However, if there is at least one new point r say, in the swath, then it together with $[p,q]$ defines a horizontal line segment which must be in the *r*-convex hull, and this cuts a vertical swath, and so on.

Apart from simplifications of detail, the process described above determines a subset of S which gives rise to a *cr*-convex subset of the *r*-convex hull of S. Thus our algorithm first finds the *r*-convex hull partitioning of S, and then secondly finds the *cr*-convex hull of each element of the partition.

Algorithm *R-CONVEX HULL*;

Input: A set of n points in the plane, $n \geq 1$.

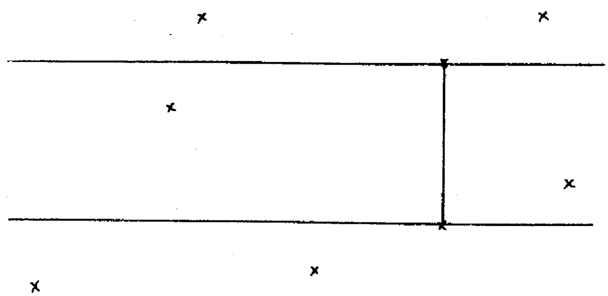


Figure 3.3

Output: The r-convex hull of the set.

begin

Step 1: Sort the points by x-coordinate;
Sort the points by y-coordinate.
If both sorted lists have unique values, then the input set is the r-convex hull, hence output its n cr-convex hulls immediately.

Step 2: Consider all repeated x-values in the x-sorted list. These form a number of disjoint intervals or swaths in their projection on the y-axis. Choose one of them, initializing the connected set C to consist solely of their corresponding points. Remove their entries from the two lists.

Step 3: Find all points in the y-list which fall within the y-swath of the chosen y-interval. If there are none then C has been completed, goto Step 5, otherwise add them to C, removing their entries from the lists. The leftmost and rightmost points in C define an x-swath.

Step 4: Find all points in the x-list which fall within the x-swath of Step 3. Again if no such points exist, then C has been completed, goto Step 5, otherwise add the points to C, removing their entries from the lists. The bottommost and topmost points in C define a y-swath, goto Step 3.

Step 5: On completion C is a subset of S which has a cr-convex hull in the r-convex hull of S. Compute and output the cr-convex hull of C. If the two lists are not empty, then return to Step 2, unless no repeated values remain, when the remaining points are their own cr-convex hulls in the r-convex hull of S, and they should be output individually.

end *R-CONVEX HULL.*

That the above algorithm correctly computes the r-convex hull of a finite point set is straightforward to ascertain from the basic definitions. That it computes the r-convex hull in $O(n\log n)$ time can be seen by observing that finding points in a swath is simply a range search query. Using any balanced search tree scheme each such query can be implemented in $O(\log n+k)$ time, where k is the number of reported points, moreover deletions can be carried out in $O(\log n)$ time. Since each point is reported only once, there can be overall at most n reports, hence we obtain:

Theorem 3.6 *The r-convex hull of an n point set in the plane, $n \geq 1$, can be computed in $O(n\log n)$ time and $O(n)$ space, and these bounds are optimal.*

4. CONCLUSIONS

We have considered three possible definitions of the rectilinear convex hull of a point set each of which has its advantages and disadvantages. The following table summarizes their properties:

	r-convex hull	cr-convex hull	mr-convex hull
connected	not necessarily	yes	not necessarily
unique	yes	not necessarily	yes
consistent	yes	no	yes
stable	no	yes	yes

where "consistent" means consistent with alternative definitions.

We have shown that, by adding two restrictions, the cr-convex hull of a finite point set can be uniquely defined. We have also given simple (in the complexity sense) necessary and sufficient conditions for the cr-convex hull to be unique. Finally we have given time- and space-optimal algorithms to compute the r-convex hull, a cr-convex hull, and the mr-convex hull of a finite point set.

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