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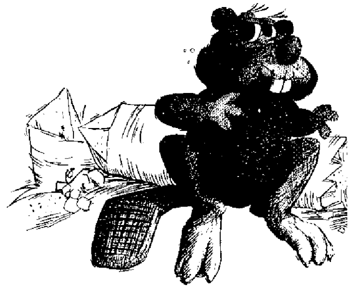

The Necessity of Double Bundle Structure in Sort Theory

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The necessity of double bundle structure in sort theory.

Introduction:

We discuss here some sort theories which are inconsistent in the sense that their collections of objects are neither sets nor proper classes in the von Neumann-Bernays-Gödel (NBG) axiom system for set theory. This results from a form of the Burali-Forti paradox, the Powerset Axiom and the Axiom of subsets of the NBG system.

The idea of a sort theory for computer science purposes goes back to Nolin 1974 [6], who wanted a semantics for programming languages with type declarations. In such a semantics data types would be treated like primitive objects (see also [9]) and not like retracts as in [11].

In [6] a treatment of this question is given, using sort collections, which are called "collections d'algorithmes" by Nolin. Our reason for calling sorts Nolin's "algorithmes" is that these objects seem much closer to some kind of generalized types, or sorts as they are known in algebra, than they are to algorithms in the usual sense, or in the complexity analysis of algorithms sense. A tutorial presentation of sort theory is given in [5].

We show in the present paper that no set satisfies the requirements for being a sort collection, and no class in the NBG axiom system is a sort collection.

In [3,4] a mathematical construction of sort domains is given, using a bundle theoretic framework, as bundle theory provides a useful tool for analyzing and unifying program semantics models. The sort domains of [3,4] verify most, but not all, sort collection requirements, and have an algebraic structure which is a double bundle structure.

In [7,8], Nolin and Le Berre present an existence theorem for informatic spaces, which they present ([7] pp 502, [8] pp 600) as a simplification of the proof in [3,4]. Their goal is to restore to the original theory some of its set-theoretical purity ("We use solely elementary set theory properties" [8] pp 600) and eliminate any "bundle theoretical presupposition" (ibidem).

This amounts, technically, to abandon the double bundle structure used in [3,4], and use only the lower bundle, which they call "ensemble convenable" (convenient set).

One shows here, by means of a counter-example, that the space they construct does not satisfy the requirements stated in their existence theorem for informatic spaces. As these requirements are roughly those defining sort collections, it is shown by a similar argument that no set or class in NBG can be an informatic space. One also shows that if a Wadsworth scheme is used as in [3,4,7,8] in order to solve the sort space construction problem, then a double bundle structure is necessary, and the minimal upper bundle structure which is necessary is produced.

I. Sort theory with sort collections:

We first recall some basic definitions from [6]. The notion of collection is left undefined inside the theory; its intuitive meaning is "aggregate" of objects.

Let T be a collection, $P(T)$ the collection of subcollections of T and $A \subseteq P(T)$ such that $\phi, T \in A$, where ϕ denotes the empty collection. One defines

(i) completed union: for any family $\{x_i\}_{i \in I} \subseteq A$, the completed union of the family is:

$$\bar{\bigcup_i} x_i = \bigcap \{y \in A : \forall i \ x_i \subseteq y\}$$

(ii) atomicity: $x \in A$ is atomic iff for any family $\{x_i\}_{i \in I}$ $x \subseteq \bigcup_i x_i \in A \Rightarrow \exists i \ x \subseteq x_i$

(iii) normal function: $f : A \to A$ is normal if $\forall x \in A \ f(x) = \bar{\bigcup}\{f(y) : y \text{ atomic} \subseteq x\}$

(iv) if $x, y \in A$, we define:

$$Fxy = \{f : A \to A \text{ normal} \mid f(x) \subseteq y\}$$

(v) A_F is defined as being the smallest collection which contains $Fxy, \forall x, y \in A$ and which is closed under infinite intersection.

(vi) evaluation: $\forall x \in A_F \ \forall y \in A \ x[y] = \bigcap \{z \in A : x \subseteq Fyz\}$

Definition: If T is a non-empty collection, and $A \subseteq P(T)$, then A is a sort collection(*) iff it verifies the following conditions:

(i) there exists a collection $\mathcal{B} \subseteq P(T)$ such that $\emptyset, T \in \mathcal{B}$ and A is the smallest subcollection of $P(T)$ which contains $\mathcal{B}$

(*) "collection d'algorithmes" in Nolin's terminology.

(as a subcollection), and is closed under infinite intersection and the operation

$$F : x,y \to Fxy$$

(ii) $FxT = T$

(iii) Let A_B be the closure of B under infinite intersection, where $\emptyset$ and T have been taken away. Then if $x \in A_B$ and $y \in A_F$, x and y are incomparable for the inclusion.

(iv) $\forall x \in A \quad x = \cup\{y \in A : y \text{ atomic} \subseteq x\}$

(v) T is atomic. □

Elements of A are called <u>sorts</u>; those of A_F are the <u>functional sorts</u> ("algorithmes propres" in [6]).

<u>Theorem 0</u> (Nolin): If $\{z_j\}_{j \in J}$ has only atomic elements, and $\bigcup_j z_j \in A$ then $(\bigcap_i Fx_iy_i)\ [\bigcup_j z_j] = \overline{\bigcup_j} \bigcap_i \{y_i : z_j \leq x_i\}$ □

<u>Theorem 1</u> (Nolin): ϕ,T, the singletons (if any) and the functional sorts are all atomic. □

The proof of this theorem uses the following lemma.

<u>Lemma 2</u>: Let f be a functional sort, then there exists a smallest family $\{Fv_kw_k : k \in K\}$ of functional sorts called <u>basis</u> of f such that:

(i) $(\forall a \in A \text{ a atomic and } a \leq v_k \text{ for some } k) \Rightarrow \exists k'.\ a = v_{k'}$

(ii) $k \neq k' \Rightarrow v_k \neq v_{k'}$

(iii) $v_k \subseteq v_{k'} \Rightarrow w_k \subseteq w_{k'}$

(iv) $\forall k \in K \quad v_k$ is atomic and $w_k \neq T$

(v) $v = \bigcap_{k \in K} Fv_kw_k$

<u>Proof</u>: Suppose $f = \bigcap_i Fx_iy_i$ - Take $\{v_k\}_{k \in K} = \{a \in A : a \text{ atomic}, f[a] \neq T\}$

and $\forall k \in K\ w_k = f[v_k]$. Then the family $\{Fv_kw_k : k \in K\}$ verifies (i) to (iv). Now

$$f = \bigcap_i Fx_iy_i = \bigcap_i F(\bigcup_k\{v_k : v_k \subseteq x_i\})(\bigcap_k\{w_k : v_k \subseteq x_i\})$$

$$= \bigcap_i \bigcap_k\{Fv_kw_k : v_k \subseteq x_i\} = \bigcap_k Fv_kw_k$$

by using theorem 0. □

Proof of Theorem 1:

Atomicity is obvious for $\emptyset$, T and the singletons. Let $v = \bigcap_{i\in I} Fx_iy_i$ be a functional sort, where $\{Fx_iy_i : i \in I\}$ is a basis of v. Assume v is not atomic, i.e.

$$v = \bigcup\{v\cap z_j : j \in J\} \quad |J| \geq 2$$

Clearly $\forall_j\ z_j \in A_F$. Let

$$z_j = \bigcap_{k\in K_j} Fv_kw_k$$

be a basis of z_j. Now since $v \nsubseteq z_j$ (otherwise $|J| = 1$), we have

$$\exists k_j \in K_j \quad v \nsubseteq Fv_{k_j}w_{k_j}$$

i.e. $\bigcap_i Fx_iy_i \nsubseteq Fv_{k_j}w_{k_j} \Leftrightarrow (\bigcap_i Fx_iy_i)(v_{k_j}) \nsubseteq w_{k_j} \Leftrightarrow$

$\bigcap_i\{y_i : v_{k_j} \subseteq x_i\} \nsubseteq w_{k_j} \Leftrightarrow$

$\bigcap_i\{y_i : v_{k_j} \subseteq x_i\} \nsubseteq w_{k_j} \Leftrightarrow$ (since Fv_kw_k and Fx_iy_i are bases,

$\bigcap_i\{y_i : v_k \subseteq x_i\} = \bigcap_i\{y_i : v_k = x_i\}$)

$\bigcap_i\{y_i : v_{k_j} = x_i\} \nsubseteq w_{k_j}$.

Now we have two cases:

1st case: $\exists i\ v_{k_j} = x_i$ and $y_i \nsubseteq w_{k_j}$

2nd case: $\nexists i\ v_{k_j} = x_i$

Define the step function:

$f(x') = v[x']$ if $x' \subseteq x_i$ for some $i \in I$, T otherwise

This function is certainly normal and

$\forall i \in I \quad f \in Fx_i y_i$, i.e. $f \in v = \bigcap_i Fx_i y_i$

In the first case we have

$f(v_{k_j}) = f(x_i) = y_i \nsubseteq w_{k_j}$

and in the second case we have $f(v_{k_j}) = T \nsubseteq w_{kj}$ ($Fv_k w_k$ basis)

Thus, since $f(v_{k_j}) \nsubseteq w_{k_j}$ for any j, we have $f \notin Fv_{k_j} w_{k_j}$. Therefore

$\forall_j \; f \notin z_j$, i.e. $f \notin \bigcup_j \{v \cap z_j : j \in J\} = v$ which is contradictory with

the fact that we have here a prepartition of v.

Therefore v is atomic. □

<u>Theorem 3</u>: The collection of normal functions $f : A \to A$ ordered extensionally by

$f \le g \Leftrightarrow \forall x \in A \quad f(x) \subseteq g(x)$

is order-isomorphic to the collection A_F of functional sorts ordered by inclusion. If we denote $[A \to A] = \{f : A \to A | f \text{ normal}\}$, the isomorphism is given by

$A_F \xrightarrow{\varphi} [A \to A] \quad \varphi(a) = \lambda x \in A \,.\, a[x]$

$[A \to A] \xrightarrow{\psi} A_F \quad \psi(f) = \bigcap_{y \in A} Fyf(y)$

<u>Proof</u>:

1. Function ψ is well defined because sort collections are closed under infinite intersection. Function φ is well defined because

$\forall a \in A_F \; \varphi(a) = \lambda x \in A \,.\, a[x]$ is a normal function, Indeed let $x = \bigcup_{i \in I} y_i \in A$. Then

$$a[x] = (\text{definition}) \cap \{u : a \subseteq F(\bigcup_i y_i)u\}$$

Now $a \subseteq F(\bigcup_i y_i)u \Leftrightarrow$

$$a \subseteq \{f : A \to A \text{ normal } | f(\bigcup_i y_i) \subseteq u\} \Leftrightarrow$$

$$a \subseteq \{f : A \to A \text{ normal } | \overline{\bigcup_i} f(y_i) \subseteq u\} \Leftrightarrow$$

$$a \subseteq \{f : A \to A \text{ normal } | \forall i \; f(y_i) \subseteq u\} \Leftrightarrow$$

$$a \subseteq \bigcap_i \{f : A \to A \text{ normal } | f(y_i) \subseteq u\} \Leftrightarrow$$

$$(\forall i \in I \;\; a \subseteq Fy_i u) \Leftrightarrow (\forall i \in I \;\; a[y_i] \subseteq u)$$

$$\Leftrightarrow \overline{\bigcup_i} a[y_i] \subseteq u$$

Thus $a[x] = a[\bigcup_i y_i] = \overline{\bigcup_i} a[y_i]$ - Therefore the function $\varphi(a) = \lambda x \in A . a[x]$ is normal. Notice that $\varphi(a)$ is also monotone.

2. φ and ψ are monotone by definition and the fact that $b \subseteq b' \Rightarrow Fyb \subseteq Fyb'$

3. $\psi \circ \varphi = id_{AF} \Leftrightarrow$

$$\forall a \in A \;\; (\psi \circ \varphi)(a) = \bigcap_{y \in A} Fy((\lambda x \in A . a[x])(y)) = \bigcap_{y \in A} Fy(a[y]) = a .$$

Now $\forall x, y \in A_F \;\; x = y \;\; \Leftrightarrow \forall z \in A \;\; x[z] = y[z]$

This holds because

$$\forall z \;\; x[z] = y[z] \Leftrightarrow (\text{definition})$$

$$\forall z \; \forall u \;\; x \subseteq Fzu \Leftrightarrow y \subseteq Fzu$$

and since both x and y are intersections of algorithms Fab this amounts to $x = y$. Now for any atomic sort u, by Theorem 0

$(\bigcap_{y\in A} Fy\ a[y])[u] = \bigcap\{a[y] : u \subseteq y\}$ = ($\varphi(a)$ is monotone) $a[u]$.

Thus $\bigcap_{y\in A} Fy(a\ [y]) = a$, which implies $\psi \circ \varphi = id_{A_F}$.

4. $\varphi \circ \psi = id_{[A \to A]} \Leftrightarrow \forall f \in [A \to A] \quad \lambda x \in A.\ (\bigcap_{y\in A} Fy\ f(y))\ [x] = f$.

It is sufficient to verify for x atomic

$(\bigcap_{y\in A} Fy\ f(y))\ [x]$ = (Theorem 0) = $\bigcap\{f(y) : x \subseteq y\} = f(x)$ since every normal function is monotone. Thus the two functions f and $\lambda x\in A\ .\ (\bigcap_{y\in A} Fy\ f(y))\ [x]$ are equal. Whence the theorem. □.

<u>Proposition 4</u>: If At is the collection of atomic sorts of A, ordered by inclusion, then A_F is order-isomorphic to the collection of monotone functions over At .

<u>Proof</u>:

1. Every functional sort $a \in A_F$ defines a monotone function $\lambda x\in A.\ a_f[x]$, and thus induces a monotone function over At.

2. If f: At $\to A$ is a monotone function, then it uniquely extends to a normal function $f : A \to A$ defined by $f = \lambda y\in A\ .\ \overline{U}\ \{f(u) : u \text{ atomic} \subseteq y\}$.

The proposition follows by application of theorem 3. □

Define a poset L as being <u>non-trivial</u> iff it has at least two comparable elements i.e. $\exists x,y\in L\ x \neq y$ and either $x \leq y$ or $y \leq x$.

<u>Theorem 5</u>: There exists no non-trivial poset L such that if

$L \underset{mono}{\to} L = \{f : L \to L \mid f \text{ monotone}\}$,

then $L \underset{mono}{\to} L$ is order-isomorphic to a subposet of L (in particular $L \to L \subseteq L$ as a subposet) □ .

Proof:(*)

Since L is a non-trivial poset, $\exists x,y \in L\ x \neq y$ and $x \leq y$. The three functions

$$a = \lambda u . \text{ if } u = y \ \underline{\text{then}}\ x \ \underline{\text{else}}\ u$$

$$b = \lambda u.u$$

$$c = \lambda u. \ \underline{\text{if}}\ u = x \ \underline{\text{then}}\ y \ \underline{\text{else}}\ u$$

are all monotone from L to L, and $a \leq b \leq c$. Now assume $L \underset{\text{mono}}{\rightarrow} L \subseteq L$ as a subposet. Define

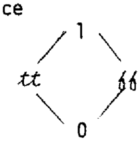
$0 = \lambda x . \ \underline{\text{if}}\ x \in \{a,b,c\} \ \underline{\text{then}}\ a \ \underline{\text{else}}\ x$

$1 = \lambda x . \ \underline{\text{if}}\ x \in \{a,b,c\} \ \underline{\text{then}}\ c \ \underline{\text{else}}\ x$

$tt = \lambda x . x$

$ff = \lambda x . \ \underline{\text{if}}\ x \in \{a,b,c\} \ \underline{\text{then}}\ b \ \underline{\text{else}}\ x$

All four functions are monotone from L to L and since $L \underset{\text{mono}}{\rightarrow} L \subseteq L$, L contains the sublattice



If $u, v \in L$ we denote $u \, I \, v$ the fact that u and v are incomparable, i.e. neither $u \leq v$ nor $v \leq u$. For example $tt \, I \, ff$.

The rest of the proof will be a transfinite induction.

(*) The author is indebted to D. Scott for the idea of this proof.

Define for every ordinal α the function

$$
f_\alpha(x) = \begin{cases} 1 & \text{if } \exists \zeta < \alpha \,.\, f_\zeta \le x \wedge \forall \zeta < \alpha \,.\, \exists y \le x \,.\, y I f_\zeta \\ tt & \text{if } \exists \zeta < \alpha \,.\, f_\zeta \le x \wedge \neg \forall \zeta < \alpha \; \exists y \le x \,.\, y I f_\zeta \\ ff & \neg \exists \zeta < \alpha \,.\, f_\zeta \le x \wedge \forall \zeta < \alpha \; \exists y \le x \,.\, y I f_\zeta \\ 0 & \text{otherwise.} \end{cases}
$$

Thus $f : \zeta \to f_\zeta$ for each ordinal ζ. We note that if for some given ordinal α we denote by $\Downarrow \alpha$ the class of all smaller ordinals, and $\downarrow x = \{y \in L : y \le x\}$, then

$$\exists \zeta < \alpha \,.\, f_\zeta \le x \Leftrightarrow f(\Downarrow\alpha) \cap \downarrow x \ne \phi$$

$\forall \zeta < \alpha \,.\, \exists y \le x \,.\, y I f_\zeta \Leftrightarrow$ every element of $f(\Downarrow\alpha)$ is incomparable with some element of $\downarrow x$.

We also note that f_α is monotone, and by induction it makes sense to write $f_\zeta \le x$ for $\zeta \le \alpha$, since $L \xrightarrow[\text{mono}]{} L$ is a subposet of L. We recall that transfinite induction is defined as follows:

Let $P(\beta)$ be a property of ordinals. Assume that for all ordinals α, if $P(\eta)$ holds for all $\eta < \alpha$, then $P(\alpha)$ holds. Then we have $P(\beta)$ for all ordinals β.

Here our property $P(\beta)$ will be:

$$P(\beta) = (\forall \zeta \;\; \zeta < \beta \Rightarrow f_\zeta I f_\beta) \wedge (f_\beta(f_\beta) = ff)$$

Let us assume that this property holds for all ordinals $\eta < \alpha$ i.e.

$$\forall \zeta, \eta < \alpha \quad (\zeta < \eta \Rightarrow f_\zeta I f_\eta)$$

$$\forall \zeta < \alpha \quad f_\zeta (f_\zeta) = ff$$

Now:

1. $\forall \zeta < \alpha \quad f_\alpha (f_\zeta) = tt$.

Indeed we first notice that if $\zeta < \alpha$ then $f(\not\downarrow\alpha) \cap \downarrow f_\zeta = \phi$ since it contains f_ζ . We also have that $f_\zeta \in f(\not\downarrow\alpha)$ is comparable with every element of $\downarrow f_\zeta$. Therefore $f_\alpha (f_\zeta) = tt$

2. $f_\alpha (f_\alpha) = ff$

We have

$f(\not\downarrow\alpha) \cap \downarrow f_\alpha \neq \phi \Leftrightarrow \exists \zeta < \alpha \quad f_\zeta \leq f_\alpha$

For such an ordinal ζ we have

$f_\zeta (f_\zeta) \leq f_\alpha (f_\zeta) = tt$

But by the induction hypothesis $f_\zeta (f_\zeta) = ff$, and $tt \ I \ ff$. Therefore $f(\not\downarrow\alpha) \cap \downarrow f_\alpha = \phi$. By using the induction hypothesis again we have $\forall \zeta < \alpha \quad \exists y \leq x \quad y \ I \ f_\zeta$ (just take $y = f_\eta$ for $\eta < \alpha \quad \eta \neq \zeta$)
Thus $f_\alpha (f_\alpha) = ff$.

Note here that for the first ordinal, 0, $f_0 = \lambda x.ff$ and $f_0(f_0) = ff$.

3. Thus $\zeta < \alpha \Rightarrow f_\alpha (f_\zeta) = tt$, $f_\zeta (f_\zeta) = ff$. Therefore $f_\alpha \ I \ f_\zeta$ for any ordinal $\zeta < \alpha$. We also have $f_\alpha (f_\alpha) = ff$. This is exactly the property of ordinals $P(\alpha)$ defined above.

4. From this we conclude, by transfinite induction,

$\forall \alpha , \beta \quad \alpha < \beta \Rightarrow f_\alpha \ I \ f_\beta$

Thus $\{f_\alpha : \alpha \in \text{Ord}\} \subseteq L$, where Ord is the class of all ordinals (which is a proper class and not a set), i.e. L contains a class whose elements are all incomparable, indexed by a proper class, i.e. L contains a proper class. Thus L is not a set. Contradiction. Whence the theorem. □.

Theorem 6: There exists no set which has a sort collection structure.

Proof: Let A be a sort collection. By Proposition 4, A_F contains, up to an order-isomorphism, all the monotone functions $f: At \to At$, where At is the collection of atomic sorts. By theorem 1 $A_F \subseteq At$, thus

$$A_F \overrightarrow{\text{mono}} A_F \subseteq At \overrightarrow{\text{mono}} At \subseteq A_F$$

where the inclusions denote injective order-morphisms. But $(A_F \overrightarrow{\text{mono}} A_F \subseteq A_F) \Rightarrow A_F$ is not a set by Theorem 5. Thus A is not a set. □

Thus sort collections are not sets. Are they classes in some set theory?

The NBG (von Neumann, Bernays, Godel) axiom system [2] is a set theory which is designed to handle classes. It has one single predicate letter $\in$, which is binary, and no function letters or individual constants. We use X, Y, Z, ... to represent arbitrary variables.

Definition:

1. $X \subseteq Y =_{\text{def.}} \forall Z . Z \in X \Rightarrow Z \in Y$ (inclusion)
2. $M(X) =_{\text{def.}} \exists Y . X \in Y$ (X is a set)
3. $Pr(X) =_{\text{def.}} \neg M(X)$ (X is a proper class)

☒

We shall call upon two axioms from NBG. In these axioms we use lower case letters $x, y, z, \ldots$ as special, restricted variables for sets.

<u>Powerset axiom</u> (W):

$$\forall x \ \exists y \ \forall u \ (u \in y \Leftrightarrow u \subseteq x)$$

i.e. the power class of a set is a set.

<u>Axiom of Subsets</u> (S):

$$\forall x \ \forall Y \ \exists z \ \forall u \ (u \in z \Leftrightarrow u \in x \wedge u \in Y)$$

i.e. the intersection of a set with a class is a set.

<u>Theorem 7</u>: There is no class in NBG which has a sort collection structure. □

<u>Proof</u>: Let A be a sort collection. Assume A is a class. Since $T \in A$, T is a set be definition. By axiom W, $P(T)$ is also a set and $A = A \cap P(T) \subseteq P(T)$ is a set by axiom S. Which contradicts Theorem 6. Therefore A is not a class. □

II. <u>Sort theory with informatic spaces</u>:

The basic definitions of this approach [7,8] are roughly the same as for sort collections. The new features are a closer look at the <u>convenient set</u> structure (see below) and the use of a diagram in the category theoretical sense called <u>Wadsworth scheme</u> in [4], both borrowed from [3,4].

Let T be a set and $E \subseteq P(T)$. We shall use in the sequel definitions (i) to (v) of §I, by substituting "E" to "A".

The set E is <u>convenient</u> iff:

(i) $\phi, T \in E$

(ii) E is closed for the intersection

(iii) $\forall x \in E \ \ x = \bigcup\{y \in E : y \text{ atomic} \subseteq x\}$

(iv) T is atomic. □

One defines also the following diagram (which we call a <u>Wadsworth scheme</u>, since it generalizes this author's diagram in his E_∞ construction [13]). The definition has two parts: the <u>objects</u>: E_n and the <u>arrows</u>: e_n (injections) and j_n (projections).

<u>Objects</u>: Let $E_0 \subseteq \mathcal{P}(T_0)$ be a convenient set. Assume that E_0 has an "isolated" element γ, i.e. such that $\forall\, x \in E_0 \quad x \neq \gamma \Rightarrow x \cap \gamma = \phi$.

Define $\forall\, n \in \mathbb{N}$

$G_0 = \{\gamma\}$

$F_{n+1} = \{f : E_n \to E_n \mid f \text{ is normal}\}$

$G_{n+1} = \{\downarrow f : f \in F_{n+1}\}$ with $\downarrow f = \{g \in F_{n+1} : g \leq f\}$

$T_{n+1} = T_0 \cup F_{n+1}$

$E_{n+1} = (E_0 - \{\gamma, T_0\}) \cup G_{n+1} \cup \{T_{n+1}\}$

To make the structure of these objects precise one takes the extensional order on F_n, i.e. $g \leq f \Leftrightarrow \forall x \in E_n \quad y(x) \leq f(x)$

and the subset-ordering on G_n.

<u>Arrows</u>: $\forall n \in \mathbb{N}$,

<u>injections</u>: $e_n : E_n \to E_{n+1}$

$e_n(T_n) = T_{n+1}$, $e_n(\gamma) = \overline{\bigcup}\{y : y \in G_1\}$

$e_n(x) = x$ if $x \in E_0 - \{\gamma, T_0\}$

$e_n(f) = e_{n-1} \circ f \circ r_{n-1}$ if $f \in F_n$, which extends canonically to G_n by $e_n(\downarrow f) = \downarrow e_n(f)$.

<u>projections</u>: $r_n : E_{n+1} \to E_n$

$r_n(T_{n+1}) = T_n$, $r_0(x) = \gamma$

$r_n(x) = x$ if $x \in E_0 - \{\gamma, T_0\}$

$r_n(f) = r_{n-1} \circ f \circ e_{n-1}$ if $f \in F_{n+1}$, which extends canonically to G_{n+1} by $r_n(\downarrow f) = \downarrow r_n(f)$.

The indices in e_n and r_n may be dropped when there is no ambiguity (cf Lemma 8, infra).

Now define

$$F_\infty = \{(y_n)_{n\in IN} \in \prod_{i=0}^{\infty} F_i \mid y_n = r_n(y_{n+1}),\ y_0 = \gamma\}\ ;\ F = \downarrow F_\infty$$

$$E = F \cup \{\phi, F_\infty \cup (E_0 - \{\phi, T_0\})\} \cup E_0 - \{\phi, T_0\}$$

The set $E \subseteq P(T)$ (where $T = F \cup E_0 - \{\phi, T_0\}$) has ϕ and T as elements; it is closed for the intersection since

$$\forall_n\ r_n(\bigcap_i x_i) = \bigcap_i r_n(x_i)\ .$$

The atomic elements of E are those of

$F \cup \{\phi, T\} \cup \{$atomic elements of $E_0\} - \{\gamma, T_0\}$.

Thus E is a convenient set ([8] pp. 614). We shall use this set throughout the rest of this paragraph.

Note that $\forall\, z = (z_n)_{n\in IN} \in E$, $z = \cap z_n$ if $(z)_n = z_n$ is the n-th projection of z. Elements of $E_0 \cap E$ are (trivially) constant projective sequences.

The problem is now to produce an isomorphism between F and the set

$[E \to E] = \{f : E \to E \mid f$ is normal$\}$

Nolin and Le Berre's theorem ([8] pp. 619) states that:

"For any convenient set E, there exists a smallest convenient set E, which contains E as a subset, and which is closed for the operation $F : x, y \to Fxy$ (i.e. which contains every normal function from E to E)". In this statement E is meant to be the above constructed space.

Now, if we define the threshold function

$$f_{xy} = \lambda u \text{ . } \underline{\text{if}} \;\; u \sqsubseteq x \;\; \underline{\text{then}} \;\; y \;\; \underline{\text{else}} \;\; \top$$

then $Fxy = \downarrow f_{xy}$. Every threshold function is normal, thus the theorem states that

$$\forall x, y \in E \quad f_{xy} \in E \text{ .}$$

This statement is incorrect in that it does not follow from the argument which is supposed to prove it. We produce hereafter two elements x and y in E such that f_{xy} is not representable as an element of E .

In order to clarify the notations, which contribute to some extent to the error in [7,8], we use two preliminary lemmae, which link Nolin and Le Berre's notations to more classical ones ([4,10,13]).

Lemma 8: If we define for any function $f : E \to E$ the following sequence $] f [\; = (] f [_n) \; n \in \mathbb{N}$

$$] f [_0 = \top$$

$$] f [_{n+1} = \lambda y \in E_n \text{ . } (f(y))_n$$

and if $\forall \; p \in \mathbb{N}$, $x_p \in E_p$, $y \in E$ we take

$$g^p = (r^p(f_{x_p y_p}) \text{ , } \dots \text{ , } r(f_{x_p y_p}) \text{ , } f_{x_p y_p} \text{ , } f_{ex_p} \text{ , } y_{p+1} \text{ , } f_{e^2 x_p} \text{ , } y_{p+2} \text{ , } \dots)$$

then both sequences $] f [$ and g^p are projective and furthermore:

$$g^p = \;] f_{x_p y} [$$

Note: The indices in the projections r and the injections e have been dropped for the sake of clarity.

Proof:

(1) projectivity of g^p is immediate up to rank $p + 1$.

$r\,(f_{ex_p\,,\,y_{p+1}}) = r_p \circ f_{ex_p\,,\,y_{p+1}} \circ e_p =$

$\lambda z \in E_p \,.\, r_p\ (\text{if } e_p\, z \leq ex_p\ \underline{\text{then}}\ y_{p+1}\ \underline{\text{else}}\ T_{p+1}) =$

$\lambda z \in E_p \,.\ \text{if}\ e_p z \leq ex_p\ \underline{\text{then}}\ r_p(y_{p+1})\ \underline{\text{else}}\ r_p\ (T_{p+1}) =$

(since E_p is isomorphic to its image $e_p(E_p) \subseteq E_{p+1}$)

$\lambda z \in E_p \,.\ \underline{\text{if}}\ z \leq x_p\ \underline{\text{then}}\ y_p\ \underline{\text{else}}\ T_p = f_{x_p y_p}$.

Similarly

$r(f_{e^2 x_p\,,\,y_{p+2}}) = r_{p+1} \circ f_{e^2 x_p\,,\,y_{p+2}} \circ e_{p+1} =$

$\lambda z \in E_{p+1} \,.\, r_{p+1}\ (\underline{\text{if}}\ e_{p+1}\ z \leq e^2 x_p\ \underline{\text{then}}\ y_{p+2}\ \underline{\text{else}}\ T_{p+2})$

$= \lambda z \in E_{p+1} \,.\ \underline{\text{if}}\ e_{p+1}\ z \leq e^2 x_p\ \underline{\text{then}}\ r_{p+1}\ (y_{p+2})\ \underline{\text{else}}\ r_{p+1}\ (T_{p+2})$

$= \lambda z \in E_{p+1} \,.\ \underline{\text{if}}\ z \leq ex_p\ \underline{\text{then}}\ y_{p+1}\ \underline{\text{else}}\ T_{p+1}$

(ii) projectivity of] f [: we have

$r_{n+1}\ (]f[_{n+2}) = r_n \circ (]f[_{n+2}) \circ e_n =$

$r_n \circ \lambda y \in E_{n+1} \,.\ (f(y))_{n+1} \circ e_n =$

$\lambda y \in E_n \,.\ r_n\ ((f(e_n(y)))_{n+1}) =$

$\lambda y \in E_n \,.\ r_n\ ((f(y))_{n+1}) =$

$\lambda y \in E_n \,.\ (f(y))_n =]f[_{n+1}$

Hence] f [is projective.

(iii) $]\, f_{x_p y}\, [$ and g^p are equal: Since both sequences are projective, it is sufficient to show

$]\, f_{x_p y}\, [_{n+1} = g^p{}_{n+1}$ for $n = p + k$, $k > 0$

Now $g^p{}_{p+k} = f_{e^{k-1}(x_p)\,,\,y_{p+k-1}}$ by definition

$$] f_{x_p y} [_{p+k} = \lambda y \in E_{p+k-1} . (f_{x_p y}(u))_{p+k-1}$$

$$= \lambda u \in E_{p+k-1} . (\underline{if}\ \ u \leq e^{k-1}(x_p)\ \ \underline{then}\ \ y\ \ \underline{else}\ \ T\ )_{p+k-1}$$

$$= \lambda u \in E_{p+k-1} \cdot f_{e^{k-1}(x_p)}\ ,\ y_{p+k-1}\ (u)$$

$$= f_{e^{k-1}(x_p)}\ ,\ y_{p+k-1} = \text{(definition)}\ g^p{}_{p+k}$$

thus $\forall\ k > 0\ \ ]f_{x_p y} [_{p+k} = g^p{}_{p+k}$.

Whence the lemma. ☒

Operation of an Element of F over E .

Element of F are used to define normal functions over E . To this end, it is sufficient to describe how they operate on atomic elements (argument similar to Proposition 4).

Define $\forall\ h \in F\ \ \ \forall\ u \in E$ atomic ([8] pp. 616)

$$h\ [u] = \bigcap_n \bigcap_i \rho_n^i\ \ (h_{i+1}(u_i))$$

where $\rho_n^i = e_{i,\ n}$ if $i < n$

id if $i = n$

$r_{i\ ,\ n}$ if $i > n$

(thus ρ_n^i injects (or projects) elements of E_i into (onto) E_n).

Lemma 9: If $h \in F$ and if we define $\forall\ u$ atomic the function $[h]$ by $[h]\ (u) = \bigcap_n h_{n+1}\ (u_n)$, then for every u which is atomic $[h]\ (u) = h\ [u]$.

Proof: This comes from the fact that

$$[h]\ (u) = \bigcap_i h_{i+1}\ (u_i) = \bigcap_n (\bigcap_i h_{i+1}\ (u_i))_n$$

$$= \bigcap_n \bigcap_i (h_{i+1}\ (u_i))_n = \bigcap_n \bigcap_i \rho_n^i\ (h_{i+1}\ (u_i))$$

This establishes the equivalence between our definition $[h]\ (u)$ ([4] pp. 213) and the one of Nolin and Le Berre for making an element h of F operate over

In [7,8] it is first shown that if we take g^p as

$$g^p =]\, f_{x_p y}\, [= (\lambda y \in E_n \,.\, f_{x_p y}\, (u))_h)n \in IN$$

Hence $[g^p] = f_{x_p y}$ (by using the definition of lemma 9). Then their argument runs as follows ([8] pp. 618-619):

"Hence, for any $x, y \in E$ and any $p \in IN$, we have $g^p = f_{x_p y}$ Thus $f_{xy} = \bigcup_p f_{x_p y}$. Indeed ... (here follows a proof of this fact).

To sum up, every normal function over E is in F".

The authors do not distinguish between the functions from E to E (such as $f_{x_p y}$) and the elements of E representing them (such as $g^p =]\, f_{x_p y}\, [$) . Thus they claim (Lemma 6, [7] pp. 501):

Since $f_{x_p y} = g^p$ and $f_{xy} = \bigcup_p f_{x_p y}$, then $f_{xy} = \bigcup_p g^p$, where $x = (x_p)_{p \in IN}$, just by using the substitutivity of equality.

Whence their theorem: E is closed for the operation $f : x,y \to f_{x,y}$. Such spaces are then called <u>informatic spaces</u> ([8] pp. 619).

The "equality" $f_{x_p y} = \bigcup_p g^p$ is written in our more precise notation:

$$f_{xy} = [\bigcup_p g^p]$$

It does not hold. Let us take $y = 1$ and $x = Y$, where 1 denotes the number "one" (it would be the singleton $\{1\}$ in this case), and $Y = (Y_n)\ n \in IN$ denotes the paradoxical combinator

$$(\lambda x \,.\, f(xx))\ \lambda f \,.\, (\lambda x \,.\, f(xx))$$

in E (E is presented as a λ-calculus model, [8] pp. 625).

Then in this case

$$g^p =]\, f_{Y_p 1}\, [\ , \text{ and}$$

[illegible]

Now we claim $\bigcup_p g^p{}_{n+1}(Y_n) = T_n$. Indeed

$$\bigcup_p g^p{}_{n+1}(Y_n) = \text{(Lemma 8)} = \bigcup_p (\lambda u \in E_n . (f_{Y_p 1}(u))_n)(Y_n)$$

$$= \bigcup_p (f_{Y_p 1}(Y_n))_n$$

$$= \bigcup_p \underline{\text{if}}\ Y_n \subseteq Y_p\ \underline{\text{then}}\ 1\ \underline{\text{else}}\ T_n = T_n$$

since there are p's larger than n and $(Y_p)_{p \in IN}$ is decreasing.

Thus

$$[\bigcup_p g^p](Y) = \bigcap_n T_n = T$$

whereas $f_{Y\,1}(Y) = 1$. Whence $f_{Y\,1} \neq [\bigcup_p g^p]$

even if $f_{Y\,1} = \bigcup_p fY_p 1$ and $f_{Y_p 1} = [g^p]$.

Since $[\bigcup_p g^p] \neq \bigcup_p [g^p]$, there is no element $a \in F$ readily available for representing the normal function f_{Y1} , i.e. such that $f_{Y1} = [a]$.

Thus the domain E is not closed for the operating $f : x,y \rightarrow f_{xy}$, and the existence theorem for informatic spaces ([7] pp. 501, [8] pp. 619) would need another proof. However the following proposition settles the matter.

<u>Proposition 10</u>: There exists no set or proper class in the NBG axiom system which has an informatic space structure. □

<u>Proof</u>: Let E be an informatic space and a set. By definition E is a complete lattice and contains every normal function from E to E . Let K be the set atomic elements of E . By definition of E , $K = F \cup \{\phi,T\} \cup (\text{Atoms}(E_0) - \{T_0\})$; and $f : E \rightarrow E$ is normal if and only

if its restriction to K is montone. Thus $K \underset{mono}{\rightarrow} K \subseteq E \underset{normal}{\rightarrow} E \subseteq F \subseteq K$

where the inclusions are poset inclusions. Therefore $K \underset{mono}{\rightarrow} K \subseteq K$ as a

poset. Which is impossible by Theorem 5. Therefore E is not a set.

Suppose E is a proper class, since $E \subseteq P(T)$ ([8] pp. 614) and $T \in E$, T is a set and $P(T)$ is also a set thus $E = E \cap P(T)$ is a set which is impossible. ⊠

3. <u>The Projective Sequence Bundle</u>:

The representation system

$] \,.\, [\; : [E \rightarrow E] \rightarrow E \;,\; f \rightarrow \,]\, f \,[\, = (\lambda u \in E_n \,.\, (f(u))_n)_{n \in \mathbb{N}}$

$[\,.\,] : F \rightarrow [E \rightarrow E] \;,\; h \rightarrow [h] = \lambda u \in E \,.\, \bigcap_n h_{n+1}(u_n)$

which was considered in Lemmae 8 and 9 has the following property.

<u>Proposition 11</u>: If $\forall f : E \rightarrow E$ normal, we define the projective sequence $]\,f\,[$ as in Lemma 8, and if $\forall h \in F$, function $[h]$ is defined as in Lemma 9, then

$\forall f \in [E \rightarrow E]$ $[\,]\,f\,[\,] = f \Leftrightarrow$ for every atomic projective sequence $(z_n)_{n \in \mathbb{N}} \in E$ we have $f(\bigcap_n z_n) = \bigcap_n f(z_n)$. ⊠

<u>Proof</u>: For $z = (z_n)_{n \in \mathbb{N}} = \bigcap_n z_n$ atomic,

we have:

$$[\,]\,f\,[\,]\,(z) = \bigcap_n \,]\,f\,[_{n+1}\,(z_n) =$$

$$= \bigcap_n (\lambda y \in E_n \,.\, (f(y))_n\,(z_n) = \bigcap_n (f(z_n))_n$$

$$= \bigcap_{n,k} (f(z_n))_k = \bigcap_n \bigcap_k (f(z_n))_k = \bigcap_n f(z_n)$$

Hence $[\,]\,f\,[\,]\,(z) = f(z) \Leftrightarrow f(\bigcap_n z_n) = \bigcap_n f(z_n)$ ⊠

This statement establishes the equivalence between the representability of normal functions from E to E and their regularity for the following bundle structure defined by the projective sequences:

1. spectrum function:

   $\forall\ z = (z_n)_{n \in \mathbb{N}} \in E$ spectrum$(z) = \{z_n\}_{n \in \mathbb{N}}$

2. limit function:

   The limit function is the greatest lower bound. This a monic ordered bundle, with $\bigcup_{n \in \mathbb{N}} E_n$ as its set of rational elements. It is distinct from the convenient domain structure of E, and is intrinsically embedded, together with the regularity property "a la Scott" attached to it:

$$f(\bigcap_n z_n) = \bigcap_n f(z_n)$$

in the construction of E and the representation system:

$$] \,.\, [\; : [E \to E] \to E$$

$$[\,.\,] : F \to [E \to E]$$

The solution adopted in the first sort domain construction [3,4] was to strengthen this regularity property, which only concerns projective subsets of E, in order to have a full Scott - continuity property, which concerns every directed subset of E. This leads to 2-bundles with an algebraic upper bundle, thus linking sort theory to denotational semantics [12]. Another reason for choosing this bundle was to introduce some more structure in the ground data type objects of E_0, for computability reasons, and the upper Scott topology seemed a good start to this end.

Function $f_{x_p y}$ verifies the condition $f_{x_p y}(\bigcap_n z_n) = \bigcap_n f_{x_p y}(z_n)$ since $x_p \in E_p$ is finite for the projective sequences:

$\forall\ (z_n) \in E \quad x_p \geq \bigcap_n z_n \Rightarrow \exists n\ x_p \geq z_n$

This holds also for every normal function f whose domain $\mathrm{Dom}(f) = \{x \in E : f(x) \neq T\}$ contains only inductive sequences. The function

$$f_{Y\ 1} = \lambda u\ .\ \underline{\text{if}}\quad u \leq Y\ \ \underline{\text{then}}\ \ 1\ \ \underline{\text{else}}\ \ T$$

does not verify the condition of Proposition 11 since

$$Y \geq \bigcap_n Y_n \Leftrightarrow \exists n\ Y \geq Y_n$$

is not true.

All this can be made more precise in the following way.

Let X be a set, partially ordered by $\leq$, and $\sqcup : S \to \sqcup S$ (resp. $\sqcap : S \to \sqcap S$) the partial function which takes least upper bounds (resp. greatest lower bounds) of subsets S of X, whenever they exist.

Definition: (i) A monic ordered bundle (m.o.b.) over X is a couple $<\mathrm{lim}\ ,\ s>$ where $\mathrm{lim} \in \{\sqcup\ ,\ \sqcap\}$ and $s : X \to P(x)$ are such that $\forall\ x \in X \quad x = \mathrm{lim}\ (s(x))$

(ii) An elementary m.o.b. over X is a m.o.b. $<\mathrm{lim}\ ,\ s>$ such that $\forall\ x\ \ \forall\ u \in s(x)\ \ u \in s(u)$ □

This definition is slightly different from the one given in [4].

Examples:

(i) the trivial m.o.b. is defined by $\mathrm{lim} = \sqcup$ and $s(x) = \downarrow x = \{y : y \leq x\}$

(ii) any algebraic c.p.o. defines an elementary m.o.b. by $\mathrm{lim} = \sqcup$ and $s(x) = \{y \leq x :\ y \text{ compact}\}$

(iii) any convenient set as defined in section II of this paper is an elementary m.o.b. by $\mathrm{lim} = \cup$ and $s(x) = \{y \subseteq x :\ y \text{ atomic}\}$.

Definition: Let $<\lim_x , s_x>$ be a m.o.b. on X and $<\lim_y , s_y>$ be a m.o.b on Y . Then a function $f : X \to Y$ is regular iff

$$\forall x \in X \qquad f(x) = \lim_y f(s_x(x)) \qquad \square$$

We denote by $[X \to Y]$ the set of regular functions from X to Y (ordered extensionally).

Examples:

(i) The regular functions for trivial m.o.b. are the monotone functions.

(ii) The regular functions for algebraic c.p.o. are the Scott-continous function.

(iii) The regular functions for convenient sets are the normal functions. $\square$

It is easily shown that m.o.b.'s are closed under product and coproduct [4].

Now define the following Wadsworth scheme. Let D be any elementary m.o.b. which has a top element. Define

Objects:
$$A_0 = D$$
$$A_{n+1} = D + [A_n \to A_n] \quad n \geq 0$$

where $[A_n \to A_n]$ is supplied with the trivial bundle and A_{n+1} with the coproduct bundle.

Arrows:
$$i_o : A_o \to A_1 \ , \ i_o\ (x) = x$$
$$i_{n+1} : A_n \to A_{n+1} \ , \ i_{n+1}\ (x) = x \quad \text{if} \quad x \in D$$
$$= i_n \circ x \circ j_n \quad \text{otherwise}$$

$$j_o : A_n \to A_o \ , \ j_o\ (x) = x \quad \text{if} \quad x \in D$$
$$= T \quad \text{otherwise}$$

$$j_{n+1} : A_{n+1} \to A_n \ , j_{n+1} \ (x) = x \quad \text{if} \quad x \in D$$

$$= j_n \circ x \circ i_n \quad \text{otherwise.}$$

We take the set-theoretical limit of this scheme:

$$A_\infty = \{(y_n)_{n \in IN} : y_n = j_n \ (y_{n+1})\}$$

This can be decomposed in fact into

$$A_\infty = \{(y_n)_{n \in IN} : y_n = y_{n+1} \in D\} \ +$$

$$\{(y_n)_{n \in IN} : yo = T \ , y_n = j_n(y_{n+1}) \ , \ y_{n+1} \in [A_n \to A_n]\}$$

$$= D + \Delta_\infty$$

where we define Δ_∞ as being the second part (in fact the functional part) of the decomposition.

Having in mind Proposition 11, we supply A_∞ with a <u>double bundle</u>, or <u>2-bundle</u>, <u>structure</u> defined as follows:

(i) "lower bundle" of A_∞ : it is induced by the m.o.b. structure of $\prod_{i \in IN} A_i$, of which A_∞ is a subset. Thus:

$$\lim = \sqcup \ , \ s((y_n)_{n \in IN}) = (s_n \ (y_n))_{n \in IN}$$

where $< \sqcup, s_n >$ is the m.o.b. structure of A_n .

(ii) "upper bundle" of A_∞ : is defined by $\lim = \sqcap$ and

$$s^*((y_n)_{n \in IN}) = \{y_{n+1} : n \in IN\}$$

if we identify each A_n with its injective image in A_∞ .

<u>Definition</u>: A function $f : A_\infty \to A_\infty$ is regular iff it is regular for the upper and lower bundles i.e.

$$\forall \ x \in A_\infty \quad f(x) = \sqcup \ f(s(x))$$

$$f(x) = \sqcap \ f(s^*(x))$$

□

Since $A_\infty = D + \Delta_\infty$, as far as regular functions are concerned, the upper bundle brings nothing to the D-part of A_∞ , since in this case $s^*((y_n)_{n\in IN})$ is a singleton, and the lower bundle brings nothing to the Δ_∞ - part of A_∞ since in this case

$$s((y_n)_{n\in IN}) = (\downarrow y_n)_{n\in IN} = \downarrow (y_n)_{n\in IN}$$

and regularity amounts to monotonicity.

Let $[A_\infty \to A_\infty] = \{f: A_\infty \to A_\infty | f \text{ regular}\}$. Then $[A_\infty \to A_\infty]$ itself is equipped with a 2-bundle structure.

More precisely $\forall f \in [A_\infty \to A_\infty]$ define $]f[= (]f[_n)_{n\in IN}$

$$]f[_0 = T$$

$$]f[_{n+1} = \lambda y \in A_n . (f(y))_n$$

This is similar to the definition of lemma 8, and one shows similarly that $]f[\in \Delta_\infty$.

Define also $\forall h \in \Delta_\infty$

$[h] : A_\infty \to A_\infty$, $[h](u) = \sqcap_n h_{n+1}(u_n)$ if $u \in \Delta_\infty$ or $u \in D$ and $u \in s(u)$

$= \sqcap [h](s(u))$ otherwise.

<u>Theorem and Definition</u>: The couple $[\,.\,]$, $]\,.\,[$ defines a 2-bundle isomorphism between Δ_∞ and $[A_\infty \to A_\infty]$ □.

<u>Proof:</u>

1. $\forall h \in \Delta_\infty$ $[h] \in [A_\infty \to A_\infty]$, this is easily verified, by definition.

2. $\forall f \in [A_\infty \to A_\infty]$ $]f[\in \Delta_\infty$, the proof is similar to the one of lemma 8.

3. $[\,.\,]$ and $]\,.\,[$ are both monotone. Equivalently, if $[A_\infty \to A_\infty]$ is supplied with the trivial m.o.b. as its <u>lower bundle</u>, then

$[\ .\ ]$ and $]\ .\ [$ are regular for the lower bundle structures of Δ_∞ and $[A_\infty \to A_\infty]$

4. The couple $<\lim, s>$ defined by $\lim = \sqcap$ and

$\forall f \ \ s(f) = \{[\ ]\, f\ [_{n+1}] : n \in \mathbb{N}\}$

defines an upper bundle structure over $[A_\infty \to A_\infty]$. Indeed, let

$K(A_\infty) = \Delta_\infty \cup \{u \in D : u \in s(u)\}$

Then $\forall\, n \in \mathbb{N}$, $[\ ]\ f\ [_{n+1}]$ is canonically defined by

$$[\ ]\ f\ [_{n+1}] = \lambda\, u \in K(\Delta_\infty)\ .\ \sqcap_p\ (]\ f\ [_{n+1})_{p+1}\ (u_p)$$

On the other hand

$$\sqcap_n\ [\ ]\ f\ [_{n+1}] = \sqcap_n\ \lambda\, u \in K(\Delta_\infty)\ .\ \sqcap_p\ (]\ f\ [_{n+1})_{p+1}\ (u_p)$$

$$= \lambda\, u \in K(\Delta_\infty)\ .\ \sqcap_n \sqcap_p\ (]\ f\ [_{n+1})_{p+1}\ (u_p)$$

$=$ (since $(]\ f\ [_i)\ (u_j)$ is decreasing in both indices) $=$

$$= \lambda\, u \in K(\Delta_\infty)\ .\ \sqcap_k\ (]\ f\ [_{k+1})\ (u_k)$$

$$= \lambda\, u \in K(\Delta_\infty)\ .\ f\ (u) = f$$

since f is regular for the upper bundle (same argument as for Proposition 11). Therefore $f = \sqcap\, s(f)$.

5. We just showed that $\forall\ f \in [A_\infty \to A_\infty]\ \ f = [\ ]\ f\ [\ ]$.

6. $\forall\ h \in \Delta_\infty\ \ ]\ [h]\ [\ = h$ since

$]\ [h]\ [_{n+1} = \lambda\, y \in A_n\ .\ (\ [h]\ (u))_n =$ (if $K(A_n) = A_n \cap K(\Delta_\infty)$)

$$= \lambda\, u \in K(A_n)\ .\ (\sqcap_p\ h_{p+1}\ (u_p))_n$$

Now $\forall\ v \in\ K(A_n)$

$$]\ [h]\ [_{n+1}\ (v) = (\sqcap_p\ h_{p+1}\ (v_p))_n = h_{n+1}\ (v)$$

Thus $]\,[h]\,[_{n+1} = h_{n+1}$ as functions, therefore $]\,[h]\,[= h$.

7. $]\,.\,[$ is regular for the upper bundles since

$]\,s(f)\,[=]\,\{\,[\,]\,f\,[_{n+1}] : n \in \mathbb{N}\}\,[$

$= \{\,]\,[\,]\,f\,[_{n+1}]\,[: n \in \mathbb{N}\}$ = (Point 6.)

$\{\,]\,f\,[_{n+1} : n \in \mathbb{N}\} = s\,(]\,f\,[\,)$

similarly $[\,.\,]$ is shown to be regular for the upper bundles by computing $s\,([h])$ for $h \in \Delta_\infty$:

$s\,([h]) = \{[\,]\,[h]\,[_{n+1}] : n \in \mathbb{N}\} = \{h_{n+1} : n \in \mathbb{N}\} = \{[h_{n+1}] : n \in \mathbb{N}\}$

$= [\,s\,(h)\,]$. Which comples the proof. □

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