A Lower Bound for Determining the Median

J. Ian Munro

Patricio V. Poblete†

Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada
N2L 3G1

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ABSTRACT

We show $\frac{79}{43}n - O(1)$ ($\approx 1.8372n$) comparisons are necessary, in the worst case, to find the median of a set of n elements.

† On leave from the Department of Computer Science, University of Chile, Chile.

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1. Introduction

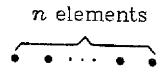
Although linear upper and lower bounds of $3n$ and $1.75n$ respectively, have been known for several years for the problem of finding the median of a set of n elements [3, 1], the gap has not been narrowed despite a number of efforts including the insights of Yap [5]. In this paper we establish a worst case lower bound of $1.8372n - O(1)$ ($1.8372... = 79/43$). The extraordinary length of the proof with respect to the improvement obtained suggests that significantly improved lower bounds, if at all possible, will require completely different proof techniques. The result does, however, confirm the expectation of Yap that a bound of $1.8333...n$ is achievable by a basic adversary strategy. We precisely state the problem as follows:

We assume we are given any algorithm that determines the median (i.e. the $\lceil n/2 \rceil^{th}$ largest element) of any totally ordered set by performing a sequence of pair-wise comparisons between elements of the set. For simplicity, we will assume that n is odd. We will construct an *adversary* [2, 4] that will provide the answers to those comparisons in such a way that the algorithm will be forced to ask at least a certain number of questions in order to determine the results. The basic restriction for the adversary is that all its answers must be consistent, i.e., there must exist at least one total order in which all the relations implied by his answers hold.

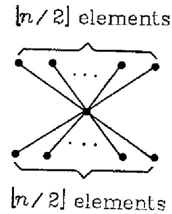
2. The Strategy of the Adversary

At any point in time, all the questions asked by the algorithm and all the answers supplied by the adversary can be summarized in a certain partial order, which we will depict by its Hasse diagram. Initially, the partial order is the unordered relation of Figure 1(a); at the end, the partial order must contain $\lfloor n/2 \rfloor$ elements above one, and $\lfloor n/2 \rfloor$ elements below it, as in Figure 1(b).

To avoid having to manage partial orders of unbounded size the adversary uses a "trimming" technique. The idea is that certain elements can be declared "too big" to be the median, and others "too small" to be the median. We say that the former elements have been "promoted", and the latter have been "demoted". Consider for instance a comparison between the two circled elements in Figure 2(a). The adversary may answer that the element at the left is greater than the element at the right, thus forming Figure 2(b); but, at the same time, it may promote two elements, as indicated in Figure 2(c), or one may be promoted and another demoted as in Figure 2(d). The promoted or demoted elements are ignored from now on (questions about them are answered in a consistent way, but are not counted). In either of our cases the remaining partial order is that of Figure 2(e).



(a) The initial configuration: No order



(b) The final configuration

Figure 1
Initial and final configurations

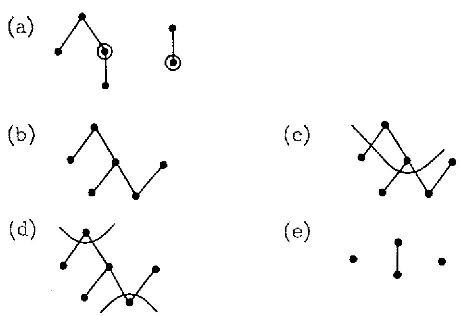


Figure 2
A comparison with subsequent promotion, demotion and simplification

When elements are promoted or demoted, we record the number of nodes and edges removed from the Hasse diagram. We use the notation "4/2" to record the fact that 4 edges and 2 elements were removed. Intuitively, the adversary tries to achieve a rate "edges/nodes" as high as possible.

The adversary must also try to delay as long as possible the deactivation of the median. To do this, the difference between the number of elements promoted and the number of elements demoted must be small. A rule like the second of the above pair is *balanced*, because one element is promoted and one is demoted. Following Schönhage [4] and later Yap [5], rules need not always be balanced if we can guarantee that their biases will ultimately cancel.

Consider for instance the comparison of Figure 3(a). One possibility is to answer as in Figure 3(b), giving 3/2 and two singletons. As we will see, the ratio 3/2 is not high enough. A better approach is to let the adversary answer as in Figure 3(c), which achieves a better ratio, 2/1, but which is biased. We write

"2/ 1(+1)" to remark that there is a difference of +1 between promoted and demoted elements. But the adversary could also have given the complementary response of Figure 3(d), and indeed *alternate* between them. We therefore have a balanced scheme with a net return of 2/ 1, as denoted by Figure 3(e).

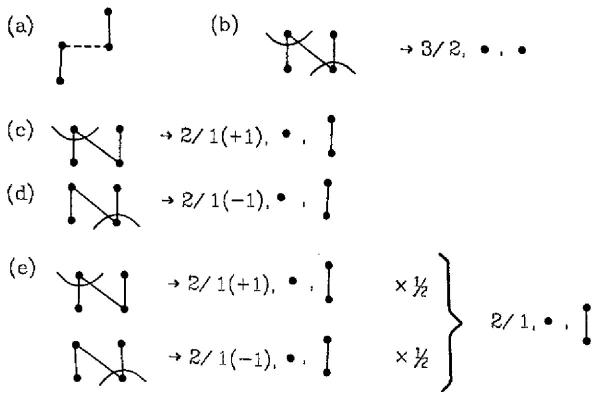


Figure 3
Biased rules leading to a balanced scheme

The fraction of time we use each rule depends on its bias. Consider, for example, Figure 4. We can apply rule (b) 1/ 3 of the time, and rule (c) 2/ 3 of the time, to achieve balance on the average.

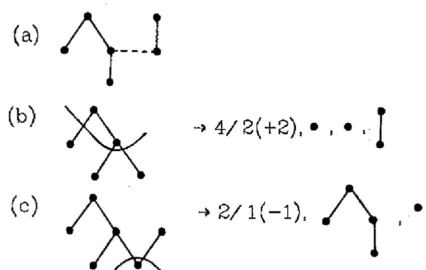


Figure 4
A 1:2 ratio in applying biased rules

The average ratio is then

$$\frac{1}{3}\times[4/\,2(+2)] + \frac{2}{3}\times[2/\,1(-1)] = \frac{8}{3}/\,\frac{4}{3}$$

Sometimes it is convenient to ignore one of the two structures. For example, when an element in any structure is compared against the top element of Figure 5(a), we can answer ">" and deactivate two elements, as in Figure 5(b). This can be viewed as a convenient shorthand, avoiding detailed examination of all possible structures compared against one.

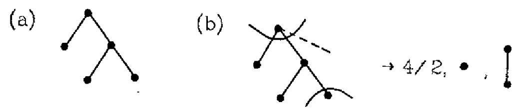


Figure 5
A stratagem in which one input is ignored

Finally, if no elements can be deactivated, an answer is given and the resulting partial order is added to the set of allowed structures. Initially, this set contains only the singleton and grows by these additions. The final set must be *closed*, in the sense that any structure generated as the result of application of a rule must be in the set.

The strategy described so far was used by Schönhage in his proof of the $1.75n$ lower bound. It has the property that the global bias is bounded by a constant at all times (the constant depends on the particular adversary).

Some structures, such as a chain of three elements, are unfavourable for the adversary, and Schönhage's adversary avoids generating them by deactivating enough elements. Yap realized that it was safe to allow the existence of these structures, provided they are generated only in contexts where a good ratio of deactivation has already been achieved. For instance, we can guarantee that every time the "three-in-a-row" structure is generated, we remove two nodes and four edges. We call this two nodes and four edges a "bonus", and we carry the structure and the bonus together (Yap calls this pair a "safe-box").

One way to depict the augmented structure would be that of Figure 6(a), but we prefer that of 6(b).

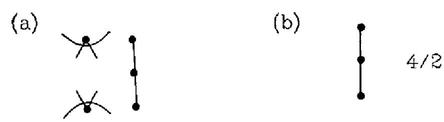


Figure 6
Denoting bonuses

This type of structure could be generated , for example, as the result of a comparison like that of Figure 7(a). We can alternate between the outcomes of 7(b), and the result of two consecutive applications of these rules is 7(c), which can be rewritten as 7(d).

A bonus needs not be unbiased, as seen in Figure 8. The rule of 8(a) is not valid (by itself) because it is biased. But we can transform it by absorbing the $3/1(+1)$ as the bonus of some structure, giving 8(b). The biased pair is trivially handled in 8(c).

The introduction of biased structures must be handled with care, since it allows for an *unbounded global bias* (note in the preceding examples the global bias was bounded at all times). While unbounded global bias is possible during the execution of the algorithm, there is a simple condition that guarantees that it will be bounded at the end. The following lemma is a slight modification of Yap's Lemma 1 [5].

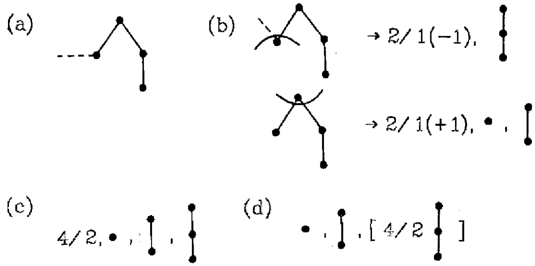


Figure 7
Generating three-in-a-row with a bonus

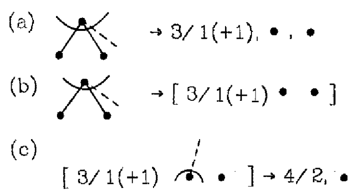


Figure 8
Biases and bonuses together

Lemma 2.1

If the number of active elements in each biased structure is strictly greater than the absolute value of its bias, then at most a constant number of biased structures can exist at the end of the execution of the algorithm.

Proof

Suppose there are t types of structures of sizes (number of active elements) $s_1, \ldots, s_t$. Suppose also their bonuses are $c_j / e_j(b_j)$ for $j=1,\ldots,t$ (we can assume structures with no bonus carry one of $0/0(0)$). Suppose the number of elements that have been promoted and are not in some bonus is p, and, analogously, d are those demoted. Recall that $|p-d|$ is bounded at all times.

If the median is still active at the end, then only one structure may exist (otherwise the median would not be uniquely determined).

If the median has been deactivated, say it has been promoted, then we can have more than one structure. Suppose we have k_j structures of type j, for $j=1,\ldots,t$. Then, the number of elements promoted is

$$UP = p + \sum_{1 \le j \le t} k_j \, \frac{e_j + b_j}{2}$$

and similarly

$$DOWN = d + \sum_{1 \le j \le t} k_j \, \frac{e_j - b_j}{2}$$

is the number of elements demoted. By assumption, since the median has been promoted, $UP \ge n/2$. Now, the total number of elements, n, is

$$n = UP + DOWN + \sum_{1 \le j \le t} k_j \, s_j$$

From this and $2 \, UP \ge n$ we get

$$UP - DOWN \le \sum_{1 \le j \le t} k_j \, s_j$$

$$p - d + \sum_{1 \le j \le t} k_j \, b_j \le \sum_{1 \le j \le t} k_j \, s_j$$

$$=> \sum_{1 \le j \le t} k_j (s_j - b_j) \le p - d \le |p - d| = O(1)$$

Now, by hypothesis, $|b_j| < s_j$ for all j, and this implies $b_j < s_j$, which in turn implies $s_j - b_j > 0$ for all j. Therefore we have that there exists a set of coefficients $\lambda_j > 0, 1 \le j \le t$, such that

$$\sum_{1 \le j \le t} \lambda_j \, k_j = O(1)$$

which implies that the k_j are bounded by a constant. ■

(Besides the notation introduced in the proof of this lemma, we will sometimes use $w_j = s_j + e_j$, the "weight" of structure j.)

Since at the end of the execution of the algorithm only a constant number of elements may still be active, we have

$$p + d = n - O(1)$$

If we can assure that each element promoted or demoted can be changed at least α comparisons, for some α, then we will have a total charge of

$$\alpha(p + d)$$

which is at least

$$\alpha n - O(1)$$

3. Accounting Techniques

Our problem is, therefore, to find a way of charging as many comparisons as possible to each element that is promoted or demoted. If we have t different types of structures, we define y_j (the "yield" of structure j) as the minimum number of comparisons that can be charged to elements in structure j. If S_j is an structure, we also denote y_j by $y[S_j]$. We want to find $y[\bullet]$.

The yields of the different structures are related by inequalities implied by the rules of the adversary.

Consider, for instance, the structures in Figure 9(a). We will assume for the purpose of this example that S_6 is formed only from S_4 and S_2 as indicated in Figure 9(b). If we decompose S_6 by the rule of 9(b), we have the rule

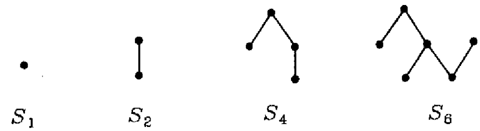
$$S_6 \to 4/2, S_1, S_1, S_2$$

and so

$$y[S_6] \le 4 + 2y[S_1] + y[S_2].$$

But since S_6 was the result of a comparison between S_4 and S_2, we will share its yield between both structures, in proportion to their weights:

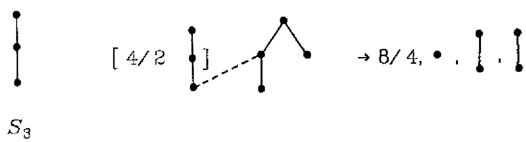
$$y[S_4] \le \frac{4}{6} y[S_6]$$



(a) Four basic structures

(b) Formation and disintegration of S_6

(c) Two alternatives for avoiding S_6



(d) S_3 and a rule for combination with S_4

Figure 9
Some structures and rules

$$y[S_2] \le \frac{2}{6} y[S_6].$$

In rules where only one structure participates, that structures gets all the yield. Thus, if the top element of S_4 is compared, we may promote it, demote one of the pair not adjacent to it, and note that

$$y[S_4] \le 4 + y[S_1] + y[S_2].$$

When there is alternation, we must average the yields proportionally. If, for instance, the rules indicated in Figure 9(c) are applied, we have

$$4/2(+2),\ S_1,\ S_1,\ S_2 \qquad \text{(1/3 of the time)}$$

and

$$2/1(-1),\ S_4,\ S_1 \qquad \text{(2/3 of the time)}$$

In the first case, the yield is at most

$$4 + 2y[S_1] + y[S_2]$$

and, in the second case, it is at most

$$2 + y[S_1] + y[S_4].$$

Therefore, the average yield is at most

$$\frac{8}{3} + \frac{4}{3}y[S_1] + \frac{1}{3}y[S_2] + \frac{2}{3}y[S_4]$$

and this must be shared between S_4 and S_2 in proportion to their weights, giving

$$y[S_4] \le \frac{16}{9} + \frac{8}{9}y[S_1] + \frac{2}{9}y[S_2] + \frac{4}{9}y[S_4]$$
$$y[S_2] \le \frac{8}{9} + \frac{4}{9}y[S_1] + \frac{1}{9}y[S_2] + \frac{2}{9}y[S_4].$$

When a structure with a bonus participates in a comparison, its weight is its number of active elements plus the number of elements in its bonus. For instance, the yield corresponding to the rule indicated in Figure 9(d) is at most

$$8 + y[S_1] + 2y[S_2]$$

which must be shared between $[4/2\ S_3]$ and S_4 in proportion 5:4, giving

$$y[4/2\ S_3] \le \frac{40}{9} + \frac{5}{9}y[S_1] + \frac{10}{9}y[S_2]$$
$$y[S_4] \le \frac{32}{9} + \frac{4}{9}y[S_1] + \frac{8}{9}y[S_2].$$

If the inequalities for decomposing the structure S_j are $\{\, y_j \le \psi_i^{(j)} : 1 \le i \le m_j \,\}$, then we can conclude

$$y_j = \min\{\, \psi_i^{(j)} : 1 \le i \le m_j \,\}.$$

Applying this to all variables y_j, a system of equations is obtained. Each of the $\psi_k^{(j)}$ is an expression containing only constants and variables y_i, so this system can be viewed as a vector equation of the form

$$\mathbf{y} = \mathbf{F}(\mathbf{y})$$

which can be iterated to obtain a fixed point. Once the fixed point has been found by an iterative technique, a set of equalities can be extracted and solved using rational arithmetic to find the exact yields. In particular, we can find the exact value for $y[\bullet]$ under the given adversary.

4. Designing an Adversary

Although the accounting techniques from the previous section tell us how to compute the yield of a given adversary, they do not give us useful guidelines for designing one. We will look now at some heuristics that are useful during a design process.

Suppose we want to design an adversary such that $y[\bullet] > \alpha$ for some given α (e.g., 11/6). Clearly, a sufficient condition for this is $y_j > \alpha w_j$ for all j. We will find sufficient conditions for this.

Lemma 4.1 (Basic Heuristic)

If in all rules the rate edges/nodes is greater than α, then $y_j > \alpha w_j$ for all j.

Proof

We will consider only the case of one structure at the left and no alternation (the other cases are similar). Suppose the rule has the form:

$$S_j \to c / e,\ S_1^{(j)}, \ldots, S_{k_j}^{(j)}$$

where, by hypothesis, $c > \alpha e$. Note also that

$$w_j = e + w_1^{(j)} + \cdots + w_{k_j}^{(j)}$$

since every element must be accounted for somewhere. Then the corresponding inequality is

$$y_j \leq c + y_1^{(j)} + \cdots + y_{k_j}^{(j)}$$

We will prove that, if $y_j = \alpha w_j$ for all j, then this inequality is strict. In effect, we have

$$\begin{aligned} y_j = \alpha w_j &= \alpha e + \alpha w_1^{(j)} + \cdots + \alpha w_{k_j}^{(j)} \\ &= \alpha e + y_1^{(j)} + \cdots + y_{k_j}^{(j)} \\ &< c + y_1^{(j)} + \cdots + y_{k_j}^{(j)} \end{aligned}$$

Therefore, we can increase y_j for all j and still not violate this restriction. Since the same holds for all rules, we have $y_j > \alpha w_j$ for all j. ∎

There is a type of rule that does not satisfy the basic heuristic but causes no problem. These are the rules with $c / e = 0/0$ (typically, those generating a structure with a bonus, or those where two structures simply combine to form a new one). The rule has the form

$$S_j \to S_1^{(j)}, \ldots, S_{k_j}^{(j)}$$

with

$$w_j = w_1^{(j)} + \cdots + w_{k_j}^{(j)}$$

and whose associated inequality is

$$y_j \leq y_1^{(j)} + \cdots + y_{k_j}^{(j)}$$

For any $\beta > \alpha$, we can see that, if we replace y by βw, we have

$$\beta w_j \leq \beta(w_1^{(j)} + \cdots + w_{k_j}^{(j)}) = \beta w_j$$

and therefore the inequality holds. It is left to rules with $c / e \neq 0/0$ to determine how much bigger than α, β can be.

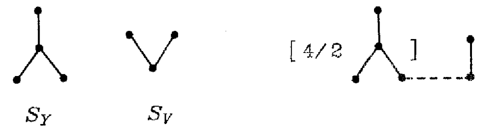
It is not always possible to use the basic heuristic directly. Consider for instance the comparison of Figure 10(a), which suggests the rules of 10(b).

After averaging, we get as right hand side

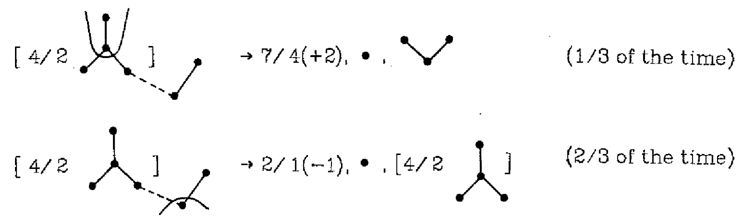
$$\frac{11}{3} / \frac{6}{3},\ S_1,\ \frac{1}{3} S_V,\ \frac{2}{3} [4/2\ S_Y]$$

which is *not* strictly above the 11/6 level according to the basic heuristic. But if at least one of the structures in this right hand side can be proved strictly above that level by using the heuristic, then we can expand it and then pass the test.

In the example, the rules of Figure 11 can be applied to S_V. After averaging, (2) becomes



a) A comparison between S_Y and S_2



b) Rules for the comparison above

Figure 10

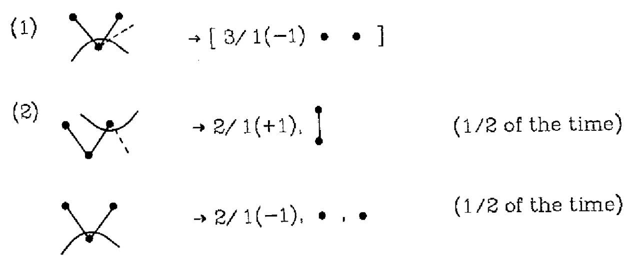


Figure 11
Rules for S_V

$$S_V \rightarrow 2/1,\ S_1,\ \frac{1}{2}\times S_2.$$

Replacing both alternatives for S_V in the right hand side, we get

$$\frac{11}{3} / \frac{6}{3},\ S_1,\ \frac{1}{3}\times[3/1(-1)\ S_1\ S_1],\ \frac{2}{3}\times[4/2\ S_Y]$$

and

$$\frac{13}{3} / \frac{7}{3},\ \frac{4}{3}\times S_1,\ \frac{1}{6}\times S_2,\ \frac{2}{3}\times[4/2\ S_Y].$$

The second alternative is at the 13/7 level, which is better than 11/6, but we still have problems with the first one. Now look at the rule for $[3/1(-1)\ S_1\ S_1]$, which demotes an S_1 on its next comparison, giving an unbiased $4/2$, S_1. Replacing this in the first alternative, we get

$$\frac{15}{3} / \frac{8}{3}, \frac{4}{3} \times S_1, \frac{2}{3} \times [4/2\ S_Y]$$

which is at the 15/8 level, better than 11/6. We are now in a position to check by using the basic heuristic that we are strictly above the 11/6 level in both cases.

5. An Adversary for $1.8372n$

Theorem 5.1

$\frac{79}{43}n - O(1)$ comparisons are necessary, in the worst case, to find the median of a set of n elements.

The proof follows immediately from the adversary (and its analysis) which is based on the preceding sections and presented in the Appendix. Clearly, the basic goal of this scheme was to prove an $(11/6)n$ bound, and the fine tuning of the previous section was helpful not only in surpassing it, but also in reaching it.

While this method may be extendable to achieve a bound of $(2-\varepsilon)n$, the combinatorial explosion of cases seems enormous. We feel any significant improvement beyond our bounds will have to come from radically new techniques.

6. References

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Appendix

The set of structures that the adversary allows to exist is shown in Figure 12(a) and (b), together with any associated bonuses. We have labelled the nodes with letters to simplify reference. For each structure, the "inverted" version is also allowed. If n is the number that identifies a structure, we use n' to denote its inverted companion.

Nodes labelled with the same letter (but different subscripts) are symmetric. Without loss of generality, we will assume that it is always the node with suscript one the one that participates in comparisons.

We use a tabular notation to specify the outcome of comparisons. For each possible comparison we record the *answer* given by the adversary, which elements are *promoted*, which ones are *demoted*, which structures remain *active* after these elements (and their incoming edges) are removed, the rate "c/e" associated to the rule, and a real approximation for that rate. Figure 13 shows a graphical representation of several comparisons, and Table 1 shows the corresponding tabular representation.

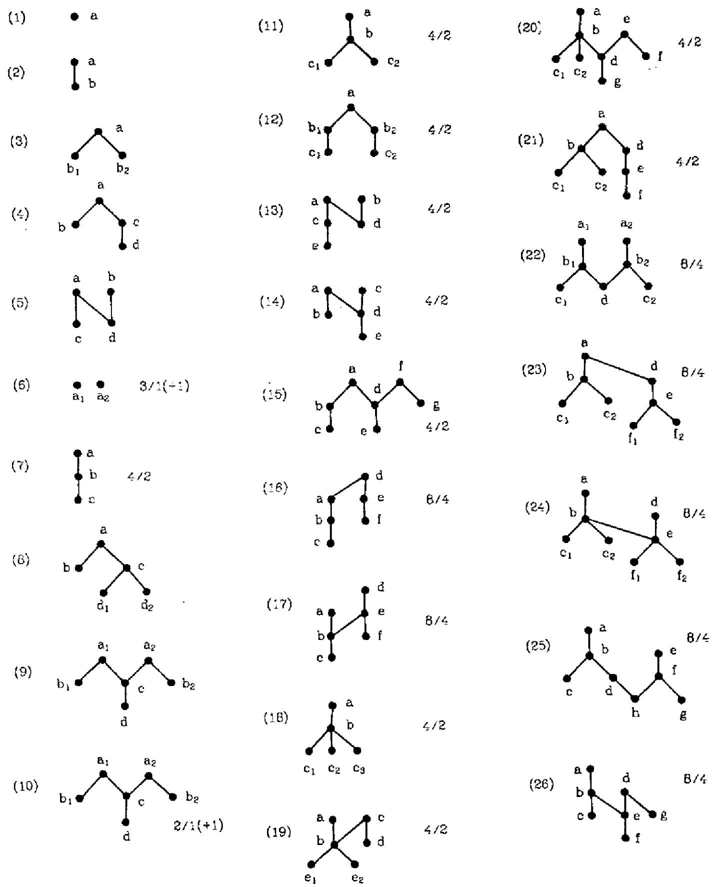

Figure 12(a)
Structures allowed by the adversary

Examples (1), (2), and (3) should be self-explanatory. Examples (4), (5), and (6) illustrate an abbreviated notation we use for rules with similar outcomes: (6) should be taken as an abbreviation for (4) and (5).

The full set of rules of the adversary are given in Table 2. If we look at that table, we observe that all rules but (127) are on or above the $\frac{11}{6}$ level. We will prove first that the actual yield per element in rule (127) is better than $\frac{11}{6}$.

Before doing that, let us consider the rules for S_3. Rule (6) gives an immediate rate of 2/1, but rule (5) gives $0/0, S_6$. However, we can use rule (15) to expand structure S_6, and get a rate of 4/2. Of these two rates, 2/1 and 4/2,

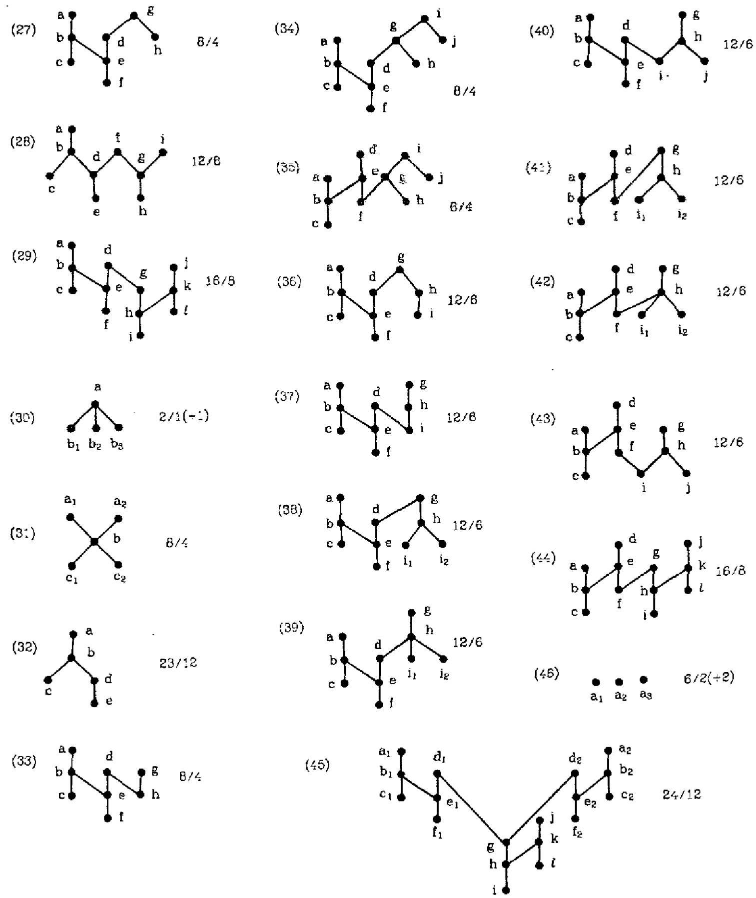

Figure 12(b)
Structures allowed by the adversary

the most unfavourable for the adversary is 2/1, so we will always use that rate when we need to expand S_3. (Note that rates are not necessarily comparable, but rates that give the same quotient are. If an structure may be expanded to give non-comparable rates, e.g. 2/1 and 19/10, we have to try all possible expansions in parallel.)

Consider now rule (127). Its right hand side has the form

$$\frac{58}{5}, \frac{32}{5}, \frac{6}{5} \times S_3, \cdots$$

Replacing S_3 by 2/1, we get a rate of

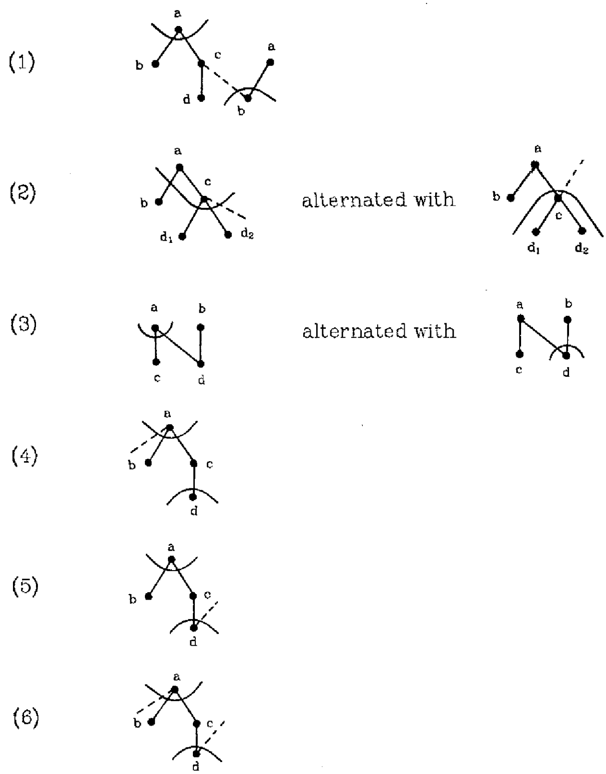

Figure 13

	comparison	answer	promoted	demoted	active	c/e	
(1)	4c ? 2b	>	4a	2b	1, 1, 2	4/2	2
(2)	8c 8c	> <	8a,c	 8c,d_1,d_2	1, 1, 1 2	$\frac{23}{5} / \frac{12}{5}$	1.9167
(3)	5 5	- -	5a	 5d	1, 2 1, 2	2/1	2
(4)	4a	>	4a	4d	1, 1	4/2	2
(5)	4d	<	4a	4d	1, 1	4/2	2
(6)	4a,d	>,<	4a	4d	1, 1	4/2	2

Table 1

$$\frac{58}{5}/\frac{32}{5}+\frac{6}{5}\times[2/1] = 14/\frac{38}{5}$$

which is at the 1.8421 level

Using a more concise notation, the previous expansion can be summarized as follows:

(127) $\frac{58}{5}/\frac{32}{5}, \frac{6}{5}\times S_3 \Longrightarrow 14/\frac{38}{5} \approx 1.8421$

So far we have proved that the adversary achieves a rate of at least $\frac{11}{6}$. We will show now that its actual rate is even better. To do that, we will examine each one of the rules that are exactly at the $\frac{11}{6}$ level:

(22) $\frac{11}{3}/2, \frac{2}{3}\times S_{14} \Longrightarrow [5/\frac{8}{3}, \frac{71}{9}/\frac{38}{9}] \approx [1.875, 1.8684]$

(40) $\frac{11}{2}/3, \frac{1}{2}\times S_3 \Longrightarrow \frac{13}{2}/\frac{7}{2} \approx 1.8571$

(45) $\frac{11}{3}/2, \frac{1}{3}\times S_3' \Longrightarrow \frac{13}{3}/\frac{7}{3} \approx 1.8571$

(51) $\frac{22}{3}/4, \frac{2}{3}\times S_3' \Longrightarrow \frac{26}{3}/\frac{14}{3} \approx 1.8571$

(129) $11/6, \frac{1}{2}\times S_3 \Longrightarrow 12/\frac{13}{2} \approx 1.8462$

(132) $\frac{11}{2}/3, \frac{1}{2}\times S_{14}' \Longrightarrow [\frac{13}{2}/\frac{7}{2}, \frac{26}{3}/\frac{14}{3}] \approx 1.8571$

(138) $\frac{11}{3}/2, \frac{1}{3}\times S_3 \Longrightarrow \frac{13}{3}/\frac{7}{3} \approx 1.8571$

(147) $\frac{22}{7}/\frac{12}{7}, \frac{4}{7}\times S_{32} \Longrightarrow [18/\frac{68}{7}, \frac{394}{21}/\frac{212}{21}, \frac{118}{7}/\frac{64}{7}, \frac{598}{35}/\frac{324}{35}]$

$\approx [1.8529, 1.8585, 1.8438, 1.8457]$

(149) $11/6, \frac{1}{2}\times S_{26} \Longrightarrow [\frac{35}{2}/\frac{19}{2}, \frac{107}{8}/\frac{29}{4}] \approx [1.8421, 1.8448]$

(150) $11/6, \frac{1}{2}\times S_{27} \overset{*}{\Longrightarrow} \frac{79}{6}/\frac{43}{6} \approx 1.8372$

We use $\overset{*}{\Longrightarrow}$ to indicate that the rate used for S_{27} was also obtained from an expansion, in this case (138). Before continuing, we will consider some rules whose expansions will also be needed for other expansions:

(29) $0/0, \frac{1}{2}\times S_3 \Longrightarrow 1/\frac{1}{2}$

(142) $0/0, \frac{2}{3}\times S_{32} \Longrightarrow [\frac{52}{3}/\frac{28}{3}, \frac{164}{9}/\frac{88}{9}, 16/\frac{26}{3}, \frac{244}{15}/\frac{44}{15}]$

$\approx [1.8448, 1.8571, 1.8636, 1.8485]$

(176) $\frac{11}{3}/2, \frac{2}{3}\times S_{27}, \frac{1}{3}\times S_8 \overset{*}{\Longrightarrow} [\frac{62}{9}/\frac{67}{18}, \frac{364}{45}/\frac{198}{45}]$

$\approx [1.8507, 1.8571]$

(185) $\frac{11}{3}/2, \frac{1}{3}\times S_{14}' \Longrightarrow [\frac{13}{3}/\frac{7}{3}, \frac{52}{9}/\frac{28}{9}] \approx 1.8571$

We continue now with the normal sequence:

(151) $11/6,\ \frac{1}{2}\times S_{33} \Longrightarrow \frac{107}{8} / \frac{29}{4} \approx 1.8448$

(152) $11/6,\ \frac{1}{2}\times S_{34} \overset{\bullet}{\Longrightarrow} [\frac{130}{9} / \frac{283}{36},\ \frac{677}{45} / \frac{368}{45}] \approx [1.8375,\ 1.8397]$

(153) $11/6,\ \frac{1}{2}\times S'_{35} \Longrightarrow \frac{74}{5} / 8 \approx 1.85$

(154) $11/6,\ \frac{1}{2}\times S_{36} \Longrightarrow \frac{89}{6} / 8 \approx 1.8542$

(155) $11/6,\ \frac{1}{2}\times S_{28} \overset{\bullet}{\Longrightarrow}$

$$[\frac{159}{10} / \frac{43}{5},\ \frac{63}{4} / \frac{17}{2},\ \frac{35}{2} / \frac{19}{2},\ \frac{59}{3} / \frac{32}{3},$$
$$\frac{181}{9} / \frac{98}{9},\ 19 / \frac{31}{3},\ \frac{287}{15} / \frac{52}{5}]$$
$$\approx [1.8488,\ 1.8529,\ 1.8421,\ 1.8436,\ 1.8387,\ 1.8397]$$

(156) $11/6,\ \frac{1}{2}\times S_{37} \Longrightarrow \frac{119}{8} / 8 \approx 1.8594$

(157) $11/6,\ \frac{1}{2}\times S_{38} \Longrightarrow \frac{37}{2} / 10 \approx 1.85$

(158) $11/6,\ \frac{1}{2}\times S_{39} \Longrightarrow \frac{37}{2} / 10 \approx 1.85$

(159) $11/6,\ \frac{1}{2}\times S_{40} \Longrightarrow \frac{68}{5} / \frac{37}{5} \approx 1.8378$

(160) $11/6,\ \frac{1}{2}\times S'_{41} \Longrightarrow \frac{37}{2} / 10 \approx 1.85$

(161) $11/6;\ \frac{1}{2}\times S'_{42} \Longrightarrow \frac{37}{2} / 10 \approx 1.85$

(162) $11/6,\ \frac{1}{2}\times S'_{43} \overset{\bullet}{\Longrightarrow} \frac{79}{6} / \frac{43}{6} \approx 1.8372$

(163) $11/6,\ \frac{1}{2}\times S_{44} \Longrightarrow \frac{35}{2} / \frac{19}{2} \approx 1.8421$

(165) $11/6,\ \frac{1}{2}\times S_{45} \Longrightarrow [\frac{67}{4} / 9,\ 19 / \frac{31}{3},\ \frac{208}{11} / \frac{113}{11},\ \frac{251}{14} / \frac{68}{7}]$

$$\approx [1.8611,\ 1.8387,\ 1.8407,\ 1.8456]$$

We can observe that the lowest rate is $\frac{79}{6} / \frac{43}{6}$, attained in rules (150) and (162), and this concludes our proof for the $\frac{79}{43}n$ lower bound.

	comparison	answer	promoted	demoted	active	c/e	
(1)	1a?1a	>			2	0/0	
(2)	2a?1a	>			3	0/0	
	2b?1a	= (2a?1a)'					
(3)	2a?2a	>			4	0/0	
(4)	2a?2b	>			5	0/0	
	2b?2b	= (2a?2a)'					
(5)	3a	>	3a		6	0/0	
(6)	$3b_1$ 3	< -	 3a	$3b_1$	2 1,1	2/1	2
(7)	4a,d	>,<	4a	4d	1,1	4/2	2
(8)	4b 4	< -	 4a	4b	7 1,2	0/0	
(9)	4c?1a	>			8	0/0	
(10)	4c?2a	<			9	0/0	
(11)	4c?4b	>	4a	2b	1,1,2	4/2	2
(12)	4c?4c	<	4^Ra		1,10	0/0	
(13)	4'c?4c	-	4a	4'a	1,1,2,2	4/2	2
(14)	5 5	- -	5a	 5d	1,2 1,2	2/1	2
(15)	$6a_1$	<		$6a_1$	1	4/2	2
(16)	7a?1a	>			4	4/2	2
(17)	7b?1a	>			11	0/0	
	7c?1a	= (7a?1a)'					
(18)	7a?2a	>			12	0/0	
(19)	7a?2b	>			13	0/0	
(20)	7b?2a	<			14	0/0	
	7b?2b	= (7b?2a)'					
	7c?2a	= (7a?2b)'					
	7c?2b	= (7a?2a)'					
(21)	7a?4c	>			15	0/0	
(22)	7b?4c 7b?4c	< <	4a	 7b,c	1,14 1,4	$\frac{11}{3}/2$	1.8333
(23)	7c?4c	<	4a	7c	1,2,2	8/4	2
	7a?4'c	= (7c?4c)'					
	7b?4'c	= (7b?4c)'					
	7c?4'c	= (7a?4c)'					
(24)	7a?7a	<			16	0/0	
(25)	7a?7b 7a?7b	> >	7^La	 7^Rb,c	2,7 1,7	$\frac{19}{3}/\frac{10}{3}$	1.9

Table 2

	comparison	answer	promoted	demoted	active	c/e	
(26)	7a?7c 7a?7c	> >	7^{L}a	 7^{R}c	2,7 2,7	6/3	2
(27)	7b?7b	>			17	0/0	
	7b?7c	= (7a?7b)'					
	7c?7c	= (7a?7a)'					
(28)	8a,d_1	>,<	8a	8d_1	1,2	4/2	2
(29)	8b 8	< -	 8a	8b	11 1,3	0/0	
(30)	8c 8c	> <	8a,c	 8c,d_1,d_2	1,1,1 2	$\frac{23}{5}/\frac{12}{5}$	1.9167
(31)	9a_1,d	>,<	9a_1	9d	1,3	4/2	2
(32)	9b_1	<	9b_1	9a_2	1,7	0/0	
(33)	9c 9	< -	 9a_1,a_2	9c,d	2,2 1,1,2	4/2	2
(34)	10a_1 10	> -	10a_1	 10c,d	1,4 2,2	$5/\frac{8}{3}$	1.875
(35)	10b_1	<		10b_1	14	0/0	
(36)	10c,d 10	<,< -	 10a_1,a_2	10c,d	2,2 1,1,2	6/3	2
(37)	11a?1a	>			8	4/2	2
(38)	11b?1a	>			18	0/0	
(39)	11c?1a	<			14'	0/0	
(40)	11a?2a 11a?2a	> >	11a	 2b	2,3 8	$\frac{11}{2}/3$	1.8333
(41)	11a?2b 11a?2b	> >	11a	 2b	2,3 1,11	4/2	2
(42)	11b?2a	<			19	0/0	
(43)	11b?2b 11b?2b	> >	11a,b	 2b	1,1,2 1,11	4/2	2
(44)	11c_1?2a 11c_1?2a	< <	11a,b,2a	 11c_1	1,1,1 2,7	$\frac{15}{4}/2$	1.875
(45)	11c_1?2b 11c_1?2b	> >	11a,b	 2b	1,3' 1,11	$\frac{11}{3}/2$	1.8333
(46)	11a?4c 11a?4c	> >	11a	 4c,d	3,4 2,11	$5/\frac{8}{3}$	1.875
(47)	11b?4c	>			20	0/0	
(48)	11c_1?4c	<	4a	11c_1	1,2,7	4/2	2
(49)	11a?4'c	>	11a	4'a	1,2,3	8/4	2

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(50)	11b?4'c	>	11a,b		1,1,4'	6/3	2
	11b?4'c	>		4'a,c	1,1,11		
(51)	11c$_1$?4'c	>		4'a,c	1,1,11	$\frac{22}{3}/4$	1.8333
	11c$_1$?4'c	>	11a,b	4'a	1,1,3'		
(52)	11a?7a	>			21	0/0	
(53)	11a?7b	>	11a		3,7	$\frac{19}{3}/\frac{10}{3}$	1.9
	11a?7b	>		7b,c	1,11		
(54)	11a?7c	>	11a		3,7	6/3	2
	11a?7c	>		7c	2,11		
(55)	11b?7a	<	7a		2,11	$\frac{13}{2}/\frac{7}{2}$	1.8571
	11b?7a	<		11b,c$_1$,c$_2$	1,7		
(56)	11b?7b	>	11a,b		1,1,7	$\frac{15}{2}/4$	1.875
	11b?7b	>		7b,c	1,11		
(57)	11b?7c	>	11a,b		1,1,7	$\frac{20}{3}/\frac{10}{3}$	2
	11b?7c	>		7c	2,11		
(58)	11c$_1$?7a	<	7a		2,11	4/2	2
	11c$_1$?7a	<		11c$_1$	7,7		
(59)	11c$_1$?7b	<			22	0/0	
(60)	11c$_1$?7c	>	11a,b		1,4'	$\frac{23}{3}/4$	1.9167
	11c$_1$?7c	>		7c	2,11		
(61)	11a?11a	>			23	0/0	
(62)	11a?11b	>	11^La		3,11	$\frac{13}{2}/\frac{7}{2}$	1.8571
	11a?11b	>		11^Rb,c$_1$,c$_2$	1,11		
(63)	11a?11c$_1$	>	11^La		3,11	4/2	2
	11a?11c$_1$	>		11^Rc$_1$	7,11		
(64)	11b?11b	>			24	0/0	
(65)	11b?11c$_1$	>	11^La,b		1,1,11	4/2	2
	11c?11c$_1$	>		11^Rc$_1$	7,11		
(66)	11c$_1$?11c$_1$	>			25	0/0	
(67)	11a?11'a	>	11a		3,11'	6/3	2
	11a?11'a	>		11'a	3',11		
(68)	11a?11'b	>	11a		3,11'	$\frac{20}{3}/\frac{10}{3}$	2
	11a?11'b	>		11'a,b	1,1,11		
(69)	11a?11'c$_1$	>		11'a,b	1,8	$\frac{23}{3}/4$	1.9167
	11a?11'c$_1$	>	11a		3,11'		
(70)	11b?11'b	>	11a,b		1,1,11'	8/4	2
	11b?11'b	>		11'a,b	1,1,11		
(71)	11b?11'c$_1$	>	11a,b		1,1,11'	$\frac{15}{2}/4$	1.875
	11b?11'c$_1$	-		11'a,b	1,1,11		

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(72)	$11c_1?11'c_1$ $11c_1?11'c_1$	< <	$11'c_1$	 $11c_1$	7,11 7,11'	2/1	2
(73)	$12a,c_1$	>,<	12a	$12c_1$	1,2	8/4	2
(74)	$12b_1$ 12	< -	 12a	$12b_1,c_1$	7 2,2	$5/\frac{8}{3}$	1.875
(75)	13 13	- -	13a	 13d	2,2 1,7	4/2	2
(76)	14a,e	>,<	14a	14e	1,2	8/4	2
(77)	14b 14	< -	 14a	14b	11' 1,7	2/1	2
(78)	14c 14	> -	14c	 14d,e	4 1,2	$\frac{19}{3}/\frac{10}{3}$	1.9
(79)	14d 14	< -	 14a	14d,e	1,2 1,7	4/2	2
(80)	15a,e	>,<	15a	15e	2,3	8/4	2
(81)	15b 15	< -	 15a	15b,c	14 2,4	$5/\frac{8}{3}$	1.875
(82)	15c	<	15a	15c	1,4	8/4	2
(83)	15d 15	< -	 15a,f	15d,e	2,7 1,2,2	6/3	2
(84)	15f	>	15f	15e	1,4	8/4	2
(85)	15g	<	15a	15g	2,7	4/2	2
(86)	16a 16a	> <	16a,d	 16a,b,c	2,2 7	$\frac{52}{5}/\frac{28}{5}$	1.8571
(87)	16b 16	< -	 16d	16b,c	4 2,7	$\frac{23}{3}/4$	1.9167
(88)	16c,d	<,>	16d	16c	2,2	12/6	2
(89)	16e 16	< -	 16d	16c,e,f	7 2,7	$\frac{13}{2}/\frac{7}{2}$	1.8571
(90)	16f	<	16d	16f	1,7	8/4	2
(91)	17b,c 17	<,< -	 17d,e	17b,c	1,7 1,7	$\frac{15}{2}/4$	1.875
	17d,e	= (17b,c)'					
(92)	17a?1a	>			26	0/0	
	17f?1a	= (17a?1a)'					
(93)	17a?2a	<			27	0/0	

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(94)	17a?2b	>			33	0/0	
	17f?2a	= (17a?2b)'					
	17f?2b	= (17a?2a)'					
(95)	17a?4c	<			34	0/0	
(96)	17f?4c	<			35	0/0	
	17a?4'c	= (17f?4c)'					
	17f?4'c	= (17a?4c)'					
(97)	17a?7a	<			36	0/0	
(98)	17a?7b	>			28	0/0	
(99)	17a?7c	>			37	0/0	
	17f?7a	= (17a?7c)'					
	17f?7b	= (17a?7b)'					
	17f?7c	= (17a?7a)'					
(100)	17a?11a	<			38	0/0	
(101)	17a?11b	<			39	0/0	
(102)	17a?11c_1	>			40	0/0	
(103)	17f?11a	<			41	0/0	
(104)	17f?11b	<			42	0/0	
(105)	17f?11c_1	>			43	0/0	
	17a?11'a	= (17f?11a)'					
	17a?11'c_1	= (17f?11c_1)'					
	17f?11'a	= (17a?11a)'					
	17f?11'b	= (17a?11b)'					
	17f?11'c_1	= (17a?11c_1)'					
(106)	17a?17a	>			29	0/0	
(107)	17a?17f	>			44	0/0	
	17f?17f	= (17a?17a)'					
(108)	18a	>	18a		30	4/2	2
(109)	18b 18b	> <	18a,b	 18b,c_1,c_2,c_3	1,1,1 1	$9/\frac{14}{3}$	1.9286
(110)	18c_1 18	< -	 18a,b	18c_1	11 1,1,1	4/2	2
(111)	19a 19	> -	19a	 19b,e_1,e_2	8 1,2	$\frac{13}{2}/\frac{7}{2}$	1.8571
(112)	19b 19	< -	 19c	19b,e_1,e_2	1,2 1,11	$\frac{15}{4}/2$	1.875
(113)	19c,e_1	>,<	19c	19e_1	1,7	4/2	2
(114)	19d 19	< -	 19c	19d	31 1,11	0/0	

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(115)	20a,b,e 20	>,>,> -	20a,b,e	 20d,g	1,1,1,2 2,11	$\frac{31}{5}/\frac{16}{5}$	1.9375
(116)	$20c_1$,d,g 20	<,<,< -	 20a,b	$20c_1$,d,g	2,7 1,1,4	$\frac{34}{5}/\frac{18}{5}$	1.8889
(117)	20f 20	< -	 20a,b	20d,f,g	1,11 1,1,4	$\frac{34}{5}/\frac{18}{5}$	1.8889
(118)	21a,b,e,f	>,>,<,<	21a,b	21e,f	1,1,1	15/8	1.875
(119)	$21c_1$	<	21a	$21c_1$	2,7	8/4	2
(120)	21d 21d	> <	21a,d	 21d,e,f	2,3 11	$\frac{52}{5}/\frac{28}{5}$	1.8571
(121)	$22a_1$,d	>,<	$22a_1$	22d	2,7	8/4	2
(122)	$22b_1$ 22	> -	$22a_1,b_1$	 22d	1,11 7,7	4/2	2
(123)	$22c_1$ 22	< -	 $22a_1,b_1$	$22c_1$,d	2,7 1,11	$\frac{15}{2}/4$	1.875
(124)	$23a,f_1$	>,<	23a	$23f_1$	3,7	8/4	2
(125)	23b 23	> -	23a,b	 $23e,f_1,f_2$	1,1,11 8	$\frac{49}{5}/\frac{26}{5}$	1.8846
(126)	$23c_1$	<	23a	$23c_1$	2,11	8/4	2
(127)	23d 23	> -	23a,d	 $23e,f_1,f_2$	3,3 8	$\frac{58}{5}/\frac{32}{5}$	1.8125
(128)	23e 23e	> <	23a,d,e	 $23e,f_1,f_2$	1,1,3 8	13/7	1.8571
(129)	24a,b,d 24	>,>,> -	24a,b,d	 $24e,f_1,f_2$	1,1,3 1,11	11/6	1.8333
(130)	$24c_1,e,f_1$ 24	<,<,< -	 24a,b	$24c_1,e,f_1,f_2$	1,7 1,1,11	$\frac{26}{3}/\frac{16}{3}$	1.8571
(131)	25a,b,d 25	>,>,> -	25a,b,d	 25h	1,11 7,11	$\frac{15}{4}/2$	1.875
(132)	25c,h 25	<,< -	 25a,b	25c,h	7,7 1,14'	$\frac{11}{2}/3$	1.8333
(133)	25e,f 25	>,> -	25a,e,f	 25h	1,4 7,11	$\frac{19}{4}/\frac{5}{2}$	1.9
(134)	25g 25	< -	 25a,b	25g,h 25h	2,11 1,1,7	$\frac{26}{3}/\frac{14}{3}$	1.8571

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(135)	26a,b,d 26	>,>,> -	26a,b,d	 26c,e,f	1,1,2 2,2	13/7	1.8571
(136)	26c,e,f 26	<,<,< -	 26a,b,d	26c,e,f	2,2 1,1,2	13/7	1.8571
(137)	26g 26	< -	 26a,b,d	26g	17 1,1,2	$\frac{19}{4}/\frac{5}{2}$	1.9
(138)	27 27	- -	27g	 27e,f	1,17 3,7	$\frac{11}{3}/2$	1.8333
(139)	28a,b,f 28	>,>,> -	28a,b,f	 28g,h	1,2,7 1,17	$\frac{49}{5}/\frac{26}{5}$	1.8846
(140)	28g,h 28	<,< -	 28a,f	28g,h	1,17 4,7	$\frac{19}{2}/5$	1.9
(141)	28c,d,e 28	<,<,< -	 28a,b,f	28c,d,e	2,11' 1,2,7	13/7	1.8571
(142)	28i 28	> -	28f,i	 28d,e,g,h	2,32 1,1,7	0/0	
(143)	29a,d 29	>,> -	29a,d	 29e,f,h,i	4,17 2,7,7	$\frac{38}{3}/\frac{20}{3}$	1.9
(144)	29b 29b	> <	29a,b,d	 29b,c,e,f,h,i	1,2,17 1,2,7	$\frac{49}{3}/\frac{26}{3}$	1.8846
(145)	29e,f,h,i 29	<,<,<,< -	 29a,b,d	29e,f,h,i	2,7,7 1,2,17	$\frac{97}{7}/\frac{52}{7}$	1.8654
(146)	29c 29	< -	 29a,b,d	29c,e,f,h,i	2,2,7 1,2,17	$\frac{125}{8}/\frac{17}{2}$	1.8382
(147)	29j,k 29	>,> -	29d,j,k	 29e,f,h,i	1,7,32 2,7,7	$\frac{22}{7}/\frac{12}{7}$	1.8333
(148)	29ℓ	<	29a,b,d	29h,i,ℓ	1,1,2,2	26/14	1.8571
(149)	29g?1a 29g?1a	< >	 29a,b,d	29g,h,i	1,7,17 1,2,26	11/6	1.8333
(150)	29g?2a 29g?2a	< <	 29a,b,d	29g,h,i	2,7,17 1,2,27	11/6	1.8333
(151)	29g?2b 29g?2b	< >	 29a,b,d	29g,h,i	2,7,17 1,2,33	11/6	1.8333
(152)	29g?4c 29g?4c	< <	 29a,b,d	29g,h,i	4,7,17 1,2,34	11/6	1.8333
(153)	29g?4'c 29g?4'c	< >	 29a,b,d	29g,h,i	4',7,17 1,2,35'	11/6	1.8333

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(154)	29g?7a 29g?7a	< <	29a,b,d	29g,h,i	7,7,17 1,2,36	11/6	1.8333
(155)	29g?7b 29g?7b	< >	29a,b,d	29g,h,i	7,7,17 1,2,28	11/6	1.8333
(156)	29g?7c 29g?7c	< >	29a,b,d	29g,h,i	7,7,17 1,2,37	11/6	1.8333
(157)	29g?11a 29g?11a	< <	29a,b,d	29g,h,i	7,11,17 1,2,38	11/6	1.8333
(158)	29g?11b 29g?11b	< <	29a,b,d	29g,h,i	7,11,17 1,2,39	11/6	1.8333
(159)	29g?11c 29g?11c	< >	29a,b,d	29g,h,i	7,11,17 1,2,40	11/6	1.8333
(160)	29g?11'a 29g?11'a	< >	29a,b,d	29g,h,i	7,11',17 1,2,41'	11/6	1.8333
(161)	29g?11'b 29g?11'b	< >	29a,b,d	29g,h,i	7,11',17 1,2,42'	11/6	1.8333
(162)	29g?11'c 29g?11'c	< <	29a,b,d	29g,h,i	7,11',17 1,2,43'	11/6	1.8333
(163)	29g?17f 29g?17f	< >	29a,b,d	29g,h,i	7,17,17 1,2,44	11/6	1.8333
(164)	29g?17a	<			45	0/0	
(165)	29g?29g 29g?29g	< >	29^La,b,d	29^Lg,h,i	7,17,29 1,2,45	11/6	1.8333
(166)	29'g?29g 29'g?29g	< <	29a,b,d,g,j,k	29'a,b,d,g,j,k	1,1,2,2,29' 1,1,2,2,29	26/14	1.8571
(167)	30a	>	30a		46	0/0	
(168)	$30b_1$	<		$30b_1$	3	4/2	2
(169)	$31a_1$ 31	> -	$31a_1$	$31b,c_1,c_2$	11 1,1	$\frac{15}{2}/4$	1.875
(170)	31b 31b	> <	$31a_1,a_2,b$	$31b,c_1,c_2$	1,1 1,1	13/7	1.8571
	31c	= (31a)'					
(171)	32a,e	>,<	32a	32e	3	26/14	1.8571
(172)	32b 32b	> <	32a,b	32b,c,d,e	1,2 1	$\frac{82}{3}/\frac{44}{3}$	1.8636
(173)	32c 32	< -	32a,b	32c,e	7 1,2	24/13	1.8462
(174)	32d 32d	> <	32a,b,d	32d,e	1,1 7	$\frac{122}{5}/\frac{66}{5}$	1.8486

Table 2 (continued)

	comparison	answer	promoted	demoted	active	c/e	
(175)	33	-	33a,b,d		1,2,2	$\frac{19}{4}/\frac{5}{2}$	1.9
	33	-		33h	1,17		
(176)	34	-	34i		1,27	$\frac{11}{3}/2$	1.8333
	34	-		34e,f	7,8		
(177)	35	-	35g,i		1,1,17	$\frac{38}{5}/4$	1.9
	35	-		35b,c,f	1,2,4		
(178)	36	-	36g		2,17	$\frac{23}{3}/4$	1.9167
	36	-		36e,f	4,7		
(179)	37	-	37a,b,d		1,2,7	$\frac{31}{4}/4$	1.9375
	37	-		37i	2,17		
(180)	38	-	38g,h	38e,f	1,1,1,7	15/8	1.875
(181)	39	-	39g,h	39e,f	1,1,1,7	15/8	1.875
(182)	40	-	40a,b,g,h		1,1,4	$\frac{26}{5}/\frac{14}{5}$	1.8571
	40	-		40i	7,17		
(183)	41	-	41g,h	41b,c	1,1,1,7	15/8	1.875
(184)	42	-	42g,h	42b,c	1,1,1,7	15/8	1.875
(185)	43	-	43d,e		7,14'	$\frac{11}{3}/2$	1.8333
	43	-		43i	7,17		
(186)	44	-	44g,j,k		1,2,17	13/7	1.8571
	44	-		44b,c,f	1,2,17		
(187)	$45a_1,b_1,d_1$	>,>,>	$45a_1,b_1,d_1$		1,2,29	$\frac{23}{2}/6$	1.9167
	45	-		45g,h,i	7,17,17		
(188)	45g,h,i	<,<,<		45g,h,i	7,17,17	$\frac{23}{2}/6$	1.9167
	45	-	$45a_1,b_1,d_1$		1,2,29		
(189)	$45c_1,e_1,f_1$		$45c_1,e_1,f_1,g,h,i$	1,2,7,17		$16/\frac{26}{3}$	1.8462
	45	-	$45a_1,b_1,d_1$		1,2,29		
(190)	45j,k	>,>	$45a_1,a_2,b_1,b_2,d_1,d_2,j,k$		1,1,1,2,2,7	$\frac{174}{11}/\frac{94}{11}$	1.8511
	45	-		45g,h,i	7,17,17		
(191)	$45l$	<		$45g,h,i,l$	2,17,17	$\frac{97}{7}/\frac{52}{7}$	1.8654
	45	-	$45a_1,b_1,d_1$		1,2,29		
(192)	$46a_1$	<		46a,	6	4/2	2

Table 2 (continued)