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and
Intersection Graphs

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RESEARCH REPORT CS-81-22

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May 1981

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Abstract

An f-diagram, F, consists of the graphs $\{f_1,\ldots,f_n\}$ of n continuous functions $F_i:[0,1] \to R$. We call the intersection graph of F a function-graph (f-graph). It is shown that a graph G is an f-graph if and only if its complement G^c is a comparability graph. If a function graph G represents an f-diagram which consists of linear functions only, then G itself is a comparability graph. Some relations between f-graphs and dimensions of partial orders are given.

1. Introduction

A graph G consists of a vertex set V(G) and an edge set E(G) of unordered pairs of vertices. We consider here only graphs with no multiple edges or self loops. For two vertices $x,y \in V(G)$, we denote by $x \overset{G}{-} y$ if $(x,y) \in E(G)$ otherwise $x \overset{G}{\not-} y$. The set $\Gamma_G(v) = \{x \in V(G) \mid v \overset{G}{-} x\}$. A graph D is a directed graph if its edge set consists of ordered pairs $<x,y>$. We denote by $x \overset{D}{\to} y$ if $<x,y> \in E(D)$. For a vertex $v \in D$, $\Gamma_D^-(v) = \{x \in V(D) \mid x \overset{D}{\to} v\}$ and $\Gamma_D^+(v) = \{x \in V(D) \mid v \overset{D}{\to} x\}$. The cardinality of a set S is denoted by $|S|$.

Let V' be a finite set of curves in the plane. A graph G = (V,E) is the intersection graph of V' if for two curves $v_i, v_j \in V'$, $v_i \overset{G}{-} v_j$ if and only if the curves v_i and v_j intersect. Intersection graphs are useful in the solution of many problems such as circuit layouts, traffic control and information retrieval problems [9]. It is shown in [3] that the set of intersection graphs of curves in the plane is a proper subset of the set of all graphs. In this paper, we show that the family of all function-graphs (f-graphs) (which are the intersection graphs of families of curves obtained from sets of continuous functions F_i: [0,1] → R) is exactly the set of all complements of comparability graphs. This result is interesting since, in general, not all comparability graphs are intersection graphs of curves in the plane as will be shown in Section 3. In the case that all functions F_i are linear, their intersection graph is itself a comparability graph and it is called a permutation graph ([8]).

It is known that transitive orientations of permutation graphs represent partial orders of dimension at most two [1]. In Section 4 we study some connections between f-graphs, permutation graphs and partially ordered sets.

2. Preliminaries and Definitions

Let $P=\{P(1),\ldots,P(n)\}$ be a permutation of the numbers $1,2,\ldots,n$ and let L_1 and L_2 be two parallel lines labelled from bottom to top by $1',2',\ldots,n'$ and $P(1), P(2),\ldots,P(n)$ respectively. A permutation diagram $D(P)$ of P consists of the labelled lines L_1 and L_2 and a set of n segments of lines $\{\overline{1}, \overline{2},\ldots,\overline{n}\}$ such that segment $\overline{i}$ joins i' on L_1 with i on $L_2 (1 \le i \le n)$. A graph G represents $D(P)$ if the vertices of G can be labelled by $\{1,2,\ldots,n\}$ such that $i \overset{G}{-} j$ if and only if $\overline{i}$ and $\overline{j}$ intersect. A graph G is called a <u>permutation graph</u> (PG) if G represents at least one permutation diagram. (see figure 1(a))

In this paper the next definition of permutation diagrams will be found more useful:

Let a and b be two different points connected by two disjoint simple curves C_1 and C_2. We choose n points $1', 2',\ldots,n'$ on C_1 and $P(1), P(2),\ldots,P(n)$ on C_2, and join i' on C_1 to i on C_2 $(1 \le i \le n)$ by a curve $\overline{i}$, totally contained in the region bounded by C_1 and C_2, such that for $i \ne j$, $\overline{i}$ and $\overline{j}$ intersect at most once (See figure 1 (b)). Clearly the intersection graphs obtained from this type of permutation diagrams are the same as these obtained from permutation diagrams given by the first definition.

The class of PG was studied in Pnueli et al [8] and Even et al [4] where they were applied to model and solve problems concerning memory allocation and circuit layout. It has also been shown in [8] that PG can be characterized as those graphs where both the graph and its complement are comparability graphs. A graph G is a comparability graph, ([5],[6],[8], also called transitively orientable graph) if its edges can be oriented such that for $x,y,z \in V(G)$ $x \overset{G}{\to} y$ and $y \overset{G}{\to} z$ implies $x \overset{G}{\to} z$.

In this paper a new class of diagrams (f - diagrams), and their associated intersection graphs are introduced.

An f-diagram in R^2 consists of two lines with equations $x = 0$ and $x = 1$, and a set $\{f_1, f_2, \ldots, f_n\}$ of graphs of n continuous functions $\mathcal{F}_i : [0,1] \to R$ $(1 \le i \le n)$ with the following properties:

For $i \ne j$

a) $\mathcal{F}_i(0) \ne \mathcal{F}_j(0)$, $\mathcal{F}_i(1) \ne \mathcal{F}_j(1)$

b) $\mathcal{F}_i(x) = \mathcal{F}_j(x)$ for only a finite number of points

c) If $\mathcal{F}_i(x) = \mathcal{F}_j(x)$, $x \in [0,1]$ there exists an $\varepsilon > 0$ such that for all $w \in (x-\varepsilon, x)$, and all $z \in (x, x+\varepsilon)$

$$[\mathcal{F}_i(w) - \mathcal{F}_j(w)]\,[\mathcal{F}_i(z) - \mathcal{F}_j(z)] < 0 .$$

condition (c) ensures that if the graphs of two functions intersect, they cross each other at the intersection point. (See figure 2)

Clearly if all functions F_i are linear, i.e. $F_i(x) = a_i x + b_i$ then an f-diagram is simply a permutation diagram.

A graph G represents an f-diagram if V(G) can be labelled with {1, 2, ..., n} such that $i \overset{G}{-} j$ if and only if $F_i(x) = F_j(x)$ for some $x \in [0,1]$. A graph G is called a function graph (f-graph) if G represents at least one f-diagram.

Partially Ordered Sets (POSETS)

A set X is partially ordered by a relation "$\prec$" over X if "$\prec$" satisfies

(a) $x \prec y$, $y \prec z$ implies $x \prec z$ (transitivity)

(b) $x \prec x$ (antisymmetry)

X is totally ordered if in addition "$\prec$" satisfies:

(c) For all $x,y \in X$, $x \prec y$ or $y \prec x$

X together with "$\prec$" will be called a partially ordered set, and it will be denoted by $(X,\prec)$. If in addition "$\prec$" induces a total order on X, $(X,\prec)$ will be called a linear order on X.

Given k linear orders $L_1 = (X,\prec_1)$, $L_2 = (X,\prec_2),\ldots,L_k = (X_1,\prec_k)$ on X, we define $L_1 \cap L_2 \cap\ldots\cap L_k$ as the partial order $L = (X,\prec)$ on X such that for $x,y \in X$, $x \prec y$ if and only if $x \prec_1 y$, $x \prec_2 y,\ldots,x \prec_k y$, L is called the intersection of $L_1,L_2,\ldots,L_k$.

It can be easily proved that every partial order can be obtained as the intersection of a number of linear orders. Dushnik and Miller [2] defined the dimension of a POSET $(X,\prec)$ (denoted dim $(X,\prec)$) as the smallest integer k for which there are k linear orders $L_1,L_2,\ldots,L_k$ on X whose intersection is $(X,\prec)$.

A POSET has dimension 1 if it is a linear order. Dushnik and Miller proved that the dimension of a POSET is at most 2 if and only if it is isomorphic to a set of intervals on the real line ordered by inclusion. Hiraguchi [7] showed that $\dim (X,\prec) \leq \frac{1}{2} |X|$ for $|X| \geq 4$.

Given the POSET $(X,\prec)$, a graph G can be obtained with $V(G) = X$, and $x \overset{G}{-} y$ if and only if $x \prec y$ or $y \prec x$. G is called the comparability graph of $(X,\prec)$.

Clearly the elements of a partial order $(X,\prec)$ can be labelled $\{1,2,\ldots,n\}$ such that, if $x,y \in X$, $x \prec y$, then x gets a label i which is smaller than the label j assigned to y.

From now on, we shall assume that the elements of X are labelled $\{1,2,\ldots,n\}$ such that if $i \quad j$ in $(X,\prec)$ then the integer i is smaller than the integer j (notice that the opposite is not necessarily true, since i and j may be incomparable elements in $(X,\prec)$).

If the elements of X are $\{1,2,\ldots,n\}$, a linear order $L_k = (X, \;_k)$ on X, can be obtained from a permutation $\pi_k = \{\pi_k(1),\pi_k(2),\ldots,\pi_k(n)\}$ on $\{1,2,\ldots,n\}$ such that $i \prec_k j$ in L_k if and only if $\pi_k^{-1}(i) < \pi_k^{-1}(j)$.

3. Comparability graphs and f-graphs.

The set of points $(x,y) \in R^2$ with $0 \leq x \leq 1$ will be called S. Each f_i divides S into two subsets S_i^- and S_i^+ such that $S_i^- = \{(x,y) \in S\colon \; y \leq F_i(x)\}$ and $S_i^+ = \{(x,y) \in S\colon F_i(x) < y\}$. Let us write $f_i < f_j$ $(i \neq j)$ if and only if $S_i^- \subseteq S_j^-$. This is equivalent to $f_i < f_j$ if and only if

$\mathcal{F}_i(x) \leq \mathcal{F}_j(x)$ for all $x \in [0,1]$ (Figure 3). Clearly " < " induces an order relation on the set of curves $\{f_1, f_2, \ldots, f_n\}$ of an f-diagram, hence $(\{f_1, f_2, \ldots, f_n\}, <)$ is a POSET. We now study some properties of $(\{f_1, f_2, \ldots, f_n\}, <)$ by analyzing the f-diagram obtained from $\{f_1, f_2, \ldots, f_n\}$.

Theorem 1: A graph G is an f-graph if and only if G^c (the complement of G) is a comparability graph.

Proof: Let $\{f_1, f_2, \ldots, f_n\}$ be the curves of an f-diagram represented by G. Let H be the comparability graph of the POSET $(\{f_1, f_2, \ldots, f_n\}, <)$. Clearly $f_i \overset{H}{-} f_j$ if and only if $f_i < f_j$ or $f_j < f_i$, hence for all $x \in [0,1]$, $\mathcal{F}_i(x) \neq \mathcal{F}_j(x)$ (by condition (c) of the definition of an f-diagram). Clearly this implies that $H = G^c$, hence G^c is a comparability graph.

Conversely, let $G^c = (V,E)$ be a comparability graph and $\vec{G}^c$ a transitive orientation of it. We can assume without loss of generality that V is labelled $\{1,2,\ldots,n\}$ such that if $i \to j$ in $\vec{G}^c$, then $i < j$.

Let $(V,\to)$ be the partial order defined on V by the orientation $\vec{G}^c$ of G^c. Then for some integer k there are k linear orders $L_1 = (V,\to_1)$, $L_2 = (V,\to_2), \ldots, L_k(V,\to_k)$ on V such that $(V,\to) = L_1 \cap L_2 \cap \ldots \cap L_k$.

Each L_m $(1 \leq m \leq k)$ defines a unique permutation π_m on $\{1,2,\ldots,n\}$ such that $i \to_m j$ on L_m if and only if $\pi_m^{-1}(i) < \pi_m^{-1}(j)$

Let us build an f-diagram F for G by using

$\pi_1, \pi_2, \ldots, \pi_k$ as follows: Take $k + 1$ parallel lines $L_0, L_1, L_2, \ldots, L_k$ such that L_i has equation $x = \frac{i-1}{k-1}$ (as in Figure 4). In L_0, we choose n points labelled $1,2,\ldots,n$ from bottom to top. For $1 \le m \le k$, choose n points in L_m labelled $\pi_m(1), \pi_m(2), \ldots, \pi_m(n)$ in this order from bottom to top. We now join the point i $(1 \le i \le n)$ on L_m to i on L_{m+1} with a line segment for $0 \le m \le k-1$. Finally, we delete the lines $L_1, L_2, \ldots, L_{k-1}$. We are now left with the lines L_0, L_k and a set of n piecewise linear curves $\{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ where $\bar{i}$ joins i on L_0 to i on L_k $(1 \le i \le n)$. Clearly each $\bar{i}$ is the graph of a continuous function $F_i : [0,1] \to R$, furthermore $\{F_1, F_2, \ldots, F_n\}$ satisfy conditions (a),(b) and (c) in the definition of an f-diagram and therefore F is an f-diagram. We will now show that F is represented by G. First we notice that each curve $\bar{i}$ in F is formed by the segments of lines joining the point i on L_m to i on L_{m+1} $(0 \le m \le k-1)$. If $i \overset{G}{-} j$ and $i < j$, then since $(V, \to) = L_1 \cap L_2 \cap \ldots \cap L_k$, there exists a minimum integer $m \ge 1$ for which $\pi_m^{-1}(i) > \pi_m^{1}(j)$. Clearly the segments of lines joining the points i and j on L_{m-1} to i and j on L_m intersect.

By using similar arguments, we can show that if $i \overset{G^c}{-} j$, then the curves $\bar{i}$ and $\bar{j}$ do not intersect in F and therefore G represents F. □

As a consequence of Theorem 1, we see that the complement of any comparability graph is an intersection graph of at least one family of plane curves. This is interesting since there are comparability graphs which are not intersection graphs of curves in the plane.

Proposition 1: Not all comparability graphs are intersection graphs of curves in the plane.

Proof: Let $\tilde{G}$ be a non planar graph. Then the graph G obtained from $\tilde{G}$ by adding an extra vertex in the midpoint of each edge of $\tilde{G}$ is not an intersection graph of curves in the plane, as shown in [3 , Section 2]. We note that G does not contain any odd cycle and therefore it is a comparability graph [5]. □

4. Permutation diagrams and f-diagrams.

In Theorem 1 we constructed an f-diagram from a family of linear orders $L_1, L_2, \ldots, L_k$. This f-diagram can be viewed as a superposition of $k-1$ permutation diagrams where the i^{th} permutation diagram consists of the lines L_i, L_{i+1} and the segments which connect the points on L_i with the points on L_{i+1}.

This suggests the converse problem, i.e., that of finding the minimum number of permutation diagrams, $p(F)$, (permutation number of F) into which a given f-diagram F can be decomposed. We refer here to 'permutation diagrams' according to the second definition. Clearly, $p(F) + 1$ is an upper bound on the dimension of the partially ordered set $(\{f_1, \ldots, f_n\}, <)$.

For a given f-diagram, F, we denote by $S_\infty^+ = \bigcap_{i=1}^{n} S_i^+$ and by $S_\infty^- = \bigcap_{i=1}^{n} S_i^-$. For $u \in S_\infty^+$ and $v \in S_\infty^-$ a curve C totally contained in S which joins u and v is called an <u>f-curve</u> if:

(a) C intersects every function $f_i \in F$ exactly once.

(b) C does not intersect two functions at the same point.

A lense of F is a connected region bounded exactly by a pair of functions $f_i, f_j \in F$. Clearly if f_i and f_j intersect in $k \geq 1$ points, then they define $k-1$ lenses.

It follows from this definition that an f-diagram which does not contain any lenses is a permutation diagram. A set of f-curves is called a lense cover of F if every lense in F is intersected by at least one f-curve. Let $\ell(F)$ (lense number of F) be the cardinality of a minimum lense cover of F (see figure 5).

Theorem 2: $\ell(F)$ is equal to $p(F) - 1$

Proof: Assume that $\ell(F) = k$, then we can draw k f-curves which intersect all lenses in F. If these f-curves do not intersect each other then we have a permutation diagram between each pair of successive curves and two additional permutation diagrams one between the leftmost f-curve and the line $x = 0$ and the other between the rightmost f-curve and the line $x = 1$. Hence $p(F) \leq k+1 = \ell(F) + 1$. Also if any pair of f-curves C_i and C_j intersect each other, we can always replace them by an equivalent pair C_i' and C_j' such that C_i' and C_j' are disjoint and intersect exactly the same lenses of F as C_i and C_j (See Figure 6).

Conversely, if $p(F) = m$ then since each permutation diagram does not contain a lense it is possible to draw m-1 f-curves in F such that all lenses of F are intersected and hence $\ell(F) \leq m-1 = p(F)-1$ which proves the theorem.

□

As a result of Theorem 2, we can find p(F) by constructing a minimum lense cover for F. In [10] a good algorithm for finding the lense number of an f-diagram is given. We note that each f-curve C_i defines a linear order L_i on $\{f_1, f_2, \ldots, f_n\}$ according to the order in which the f_i's are intersected when moving from v to u along C_i. It is an open problem to find the minimum number, r , of f-curves $C_1,\ldots,C_r$ such that $L_1 \cap L_2 \ldots \cap L_r$ is equal to $(\{f_1 \ldots f_n\}, <)$. We call this minimal number, r, the dimension of the f-diagram F denoted by dim (F). The following inequalities can be easily verified:

$$\ell(F) + 1 = p(F) \geq \dim(F) \geq \dim(\{f_1 \ldots f_n\}, <).$$

5. Open Problems.

We now present an interesting open problem which is raised by this work. Let F be an f-diagram, we define the intersection number of F, I(F), to be the maximum number of intersections between any pair of functions in F. For example, I(F) = 1 implies that F is a permutation diagram and hence it represents a partial order of dimension 2. Conversely, every partial order of dimension 2 can be represented as an f-diagram F with I(F) = 1. We observe that if every pair of curves in an f-diagram F are either disjoint or have an odd number of intersection points, then the representing graph of F is a permutation graph. This follows since we can replace each curve f_i by a straight line ℓ_i between its endpoints and then ℓ_i intersects ℓ_j if and only if f_i intersects f_j. Hence for a POSET of dimension 2 we can construct an f-diagram F with an unbounded I(F). Conversely there are f-diagrams F with I(F) = 2 which represent POSETS of arbitrary dimension as shown in Figure 7. The POSET which corresponds to the f-diagram of Figure 7 has dimension $\lceil \frac{n}{2} \rceil$ (see [2]). This shows that except for permutation diagrams, the intersection number of an f-diagram is not directly related to the dimension of its corresponding POSET. It is natural to ask whether for a given POSET

$(X, \prec)$ and an integer m, is it possible to construct an f-diagram F which represents $(X, \prec)$ such that $I(F) = m$. We conjecture that for every integer m, there exists a partial order which cannot be represented by an f-diagram whose intersection is smaller than m.

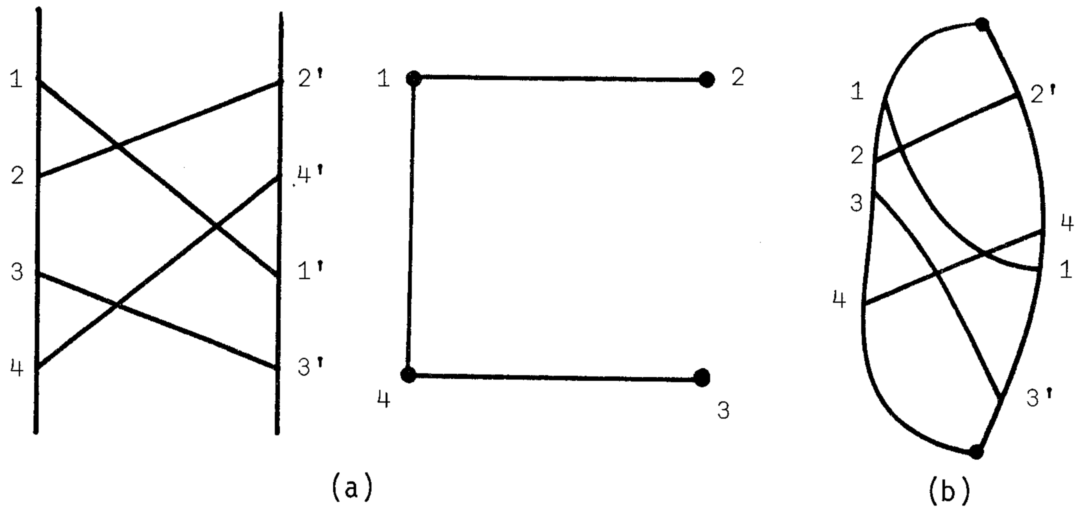

Figure 1

(a) Permutation diagram and its intersection graph

(b) A permutation diagram equivalent to (a).

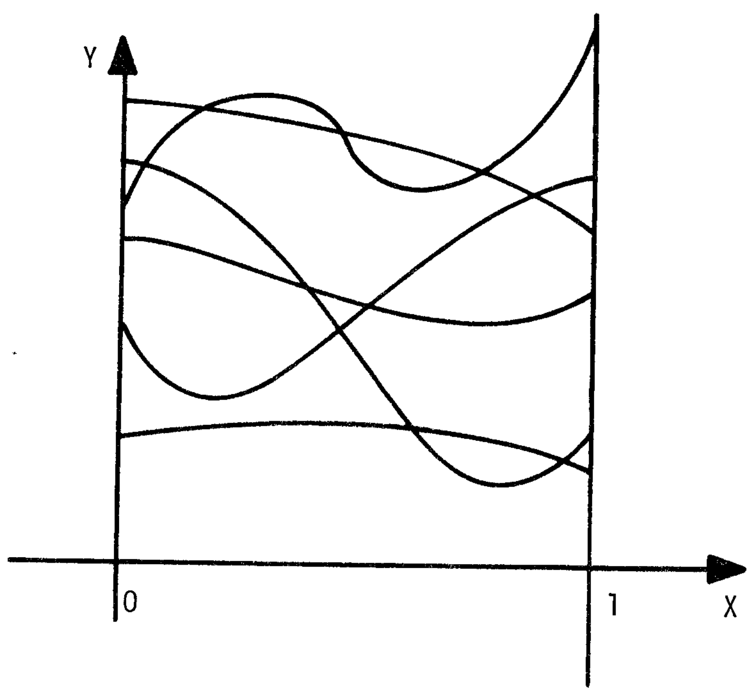

Figure 2

An f-diagram

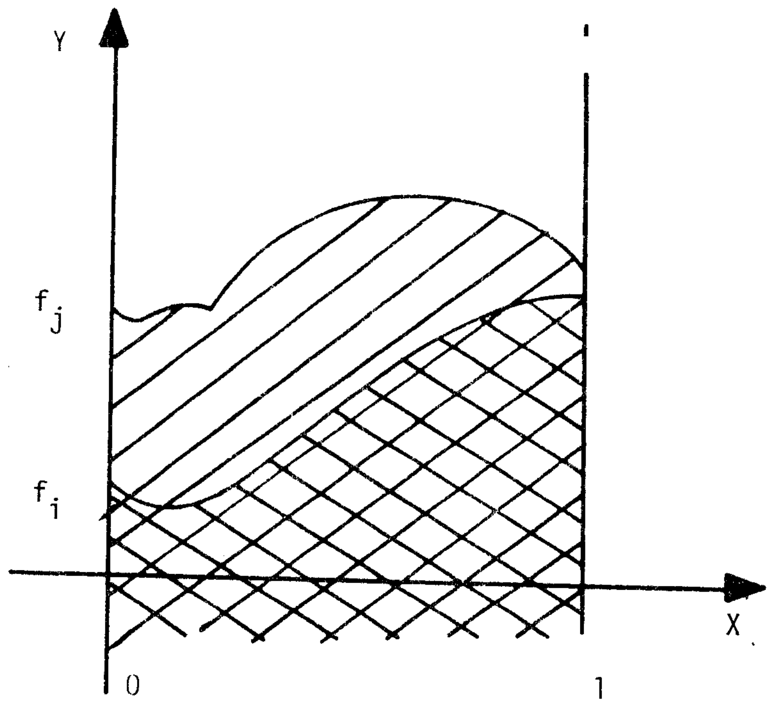

Figure 3 - $S_i^- \subset S_j^-$

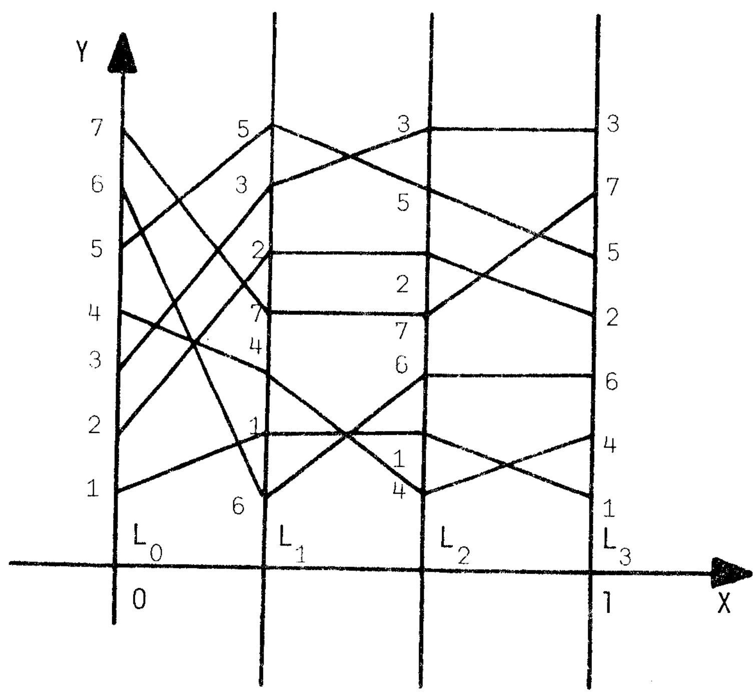

Figure 4

A construction for Theorem 1

$\pi_1 = \langle 6,1,4,7,2,3,5 \rangle$; $\pi_2 = \langle 4,1,6,7,2,5,3 \rangle$; $\pi_3 = \langle 1,4,6,2,5,7,3 \rangle$

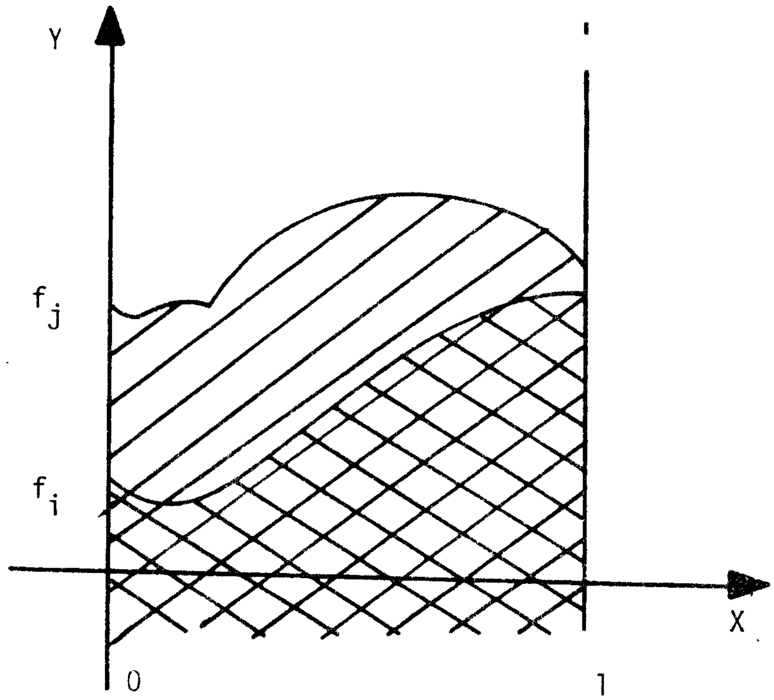

Figure 3 - $S_i^- \subset S_j^-$

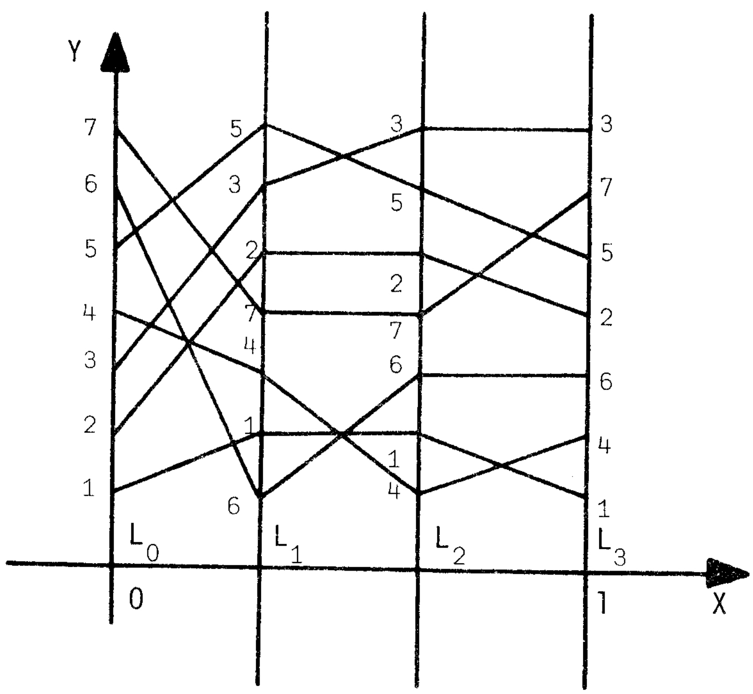

Figure 4

A construction for Theorem 1

$\pi_1 = \langle 6,1,4,7,2,3,5 \rangle$; $\pi_2 = \langle 4,1,6,7,2,5,3 \rangle$; $\pi_3 = \langle 1,4,6,2,5,7,3 \rangle$

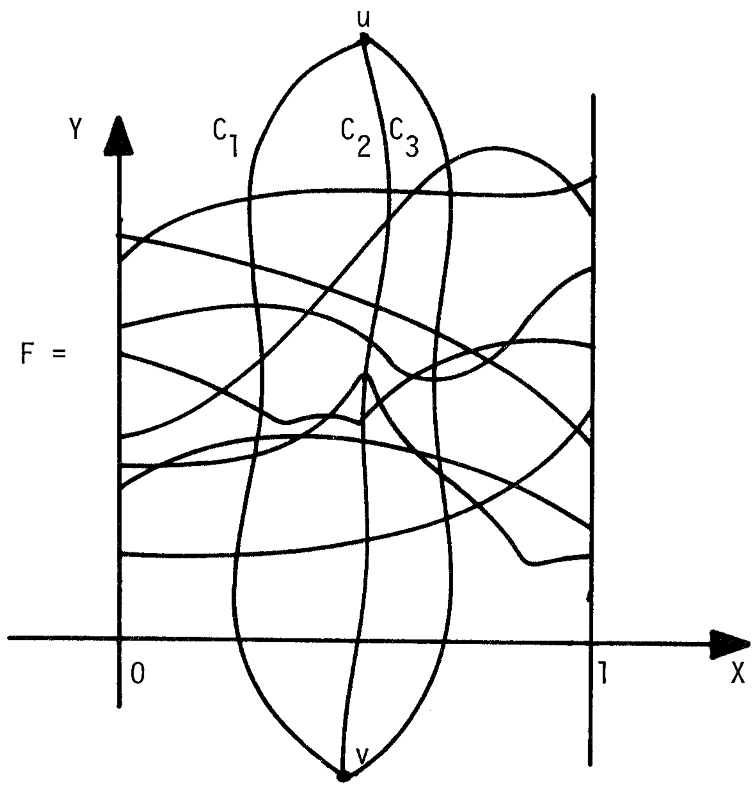

Figure 5 - C_1, C_2 and C_3 are a minimum lense cover of F.

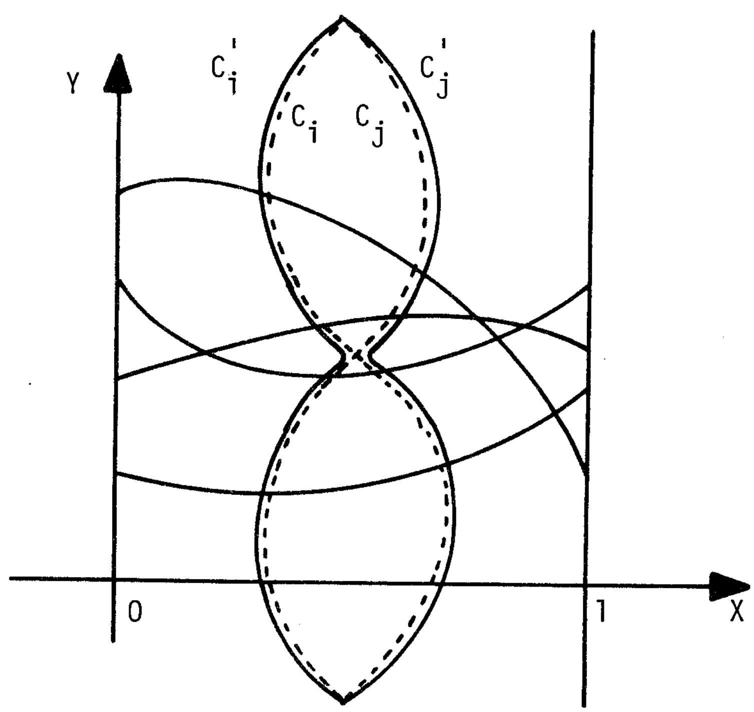

Figure 6 - Replacing the dashed curves C_i and C_j by the non-intersecting curves C_i' and C_j'.

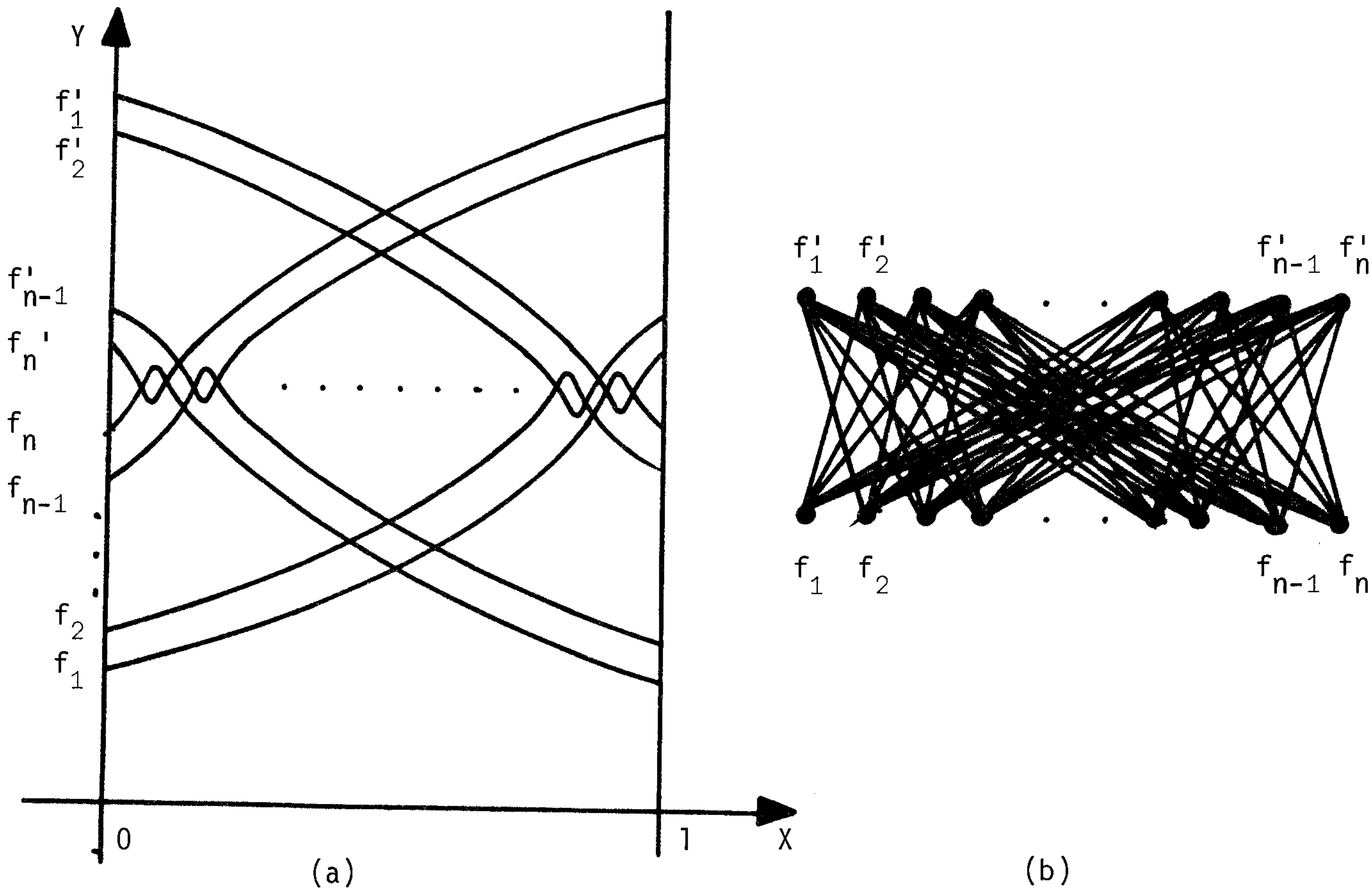

Figure 7(a) - An f-diagram with 2n functions for a POSET of dimension n with $I(F) = 2$. For every pair i,j $(i \neq j)$ $1 \leq i, j \leq n$ f_i intersects f_j and f_i' intersects f_j'. In addition f_i intersects f_i' but does not intersect f_j', $j \neq i$.

Figure 7(b) - The comparability graph of the POSET $(\{f_1,f_2,\ldots,f_n,f_1',f_2',\ldots,f_n'\},<)$

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