Classes of Transducers and Their Properties*

Karel Culik II
Department of Computer Science
University of Waterloo

Christian Choffrut
Laboratoire d' Informatique
Théorique et Programmation
Université Paris VII

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Abstract

We consider the subfamilies of rational and pushdown transducers and corresponding translations (relations) which are most frequently encountered in the literature. We survey some of the known results on characterization, factorization, closure properties, decision problems and comparison of classes and give new results on these properties using either direct proofs or results from other theories such as homomorphism equivalence.

1. INTRODUCTION

One of the most natural ways of defining a relation, or translation, or transduction, of a free monoid Σ^* into another Δ^* is to use some type of transducer, i.e. some type of automaton provided with outputs. At each step of the procedure of recognition of the word $w \in \Sigma^*$, one of some available outputs in Δ^* is chosen. The concatenation of these outputs in the order they are produced during the computation defines a word $z \in \Delta^*$ which is in relation with w. Thus, finite automata yield the notion of rational relation (also known as rational transduction or finite translation) while more generally, pushdown automata define pushdown translations (relations) (cf. e.g. [1, p.228]).

Rational relations are definitely the best known family of relations due to their nice properties and to the role they play in the theory of Abstract Families of Languages where they prove to be a useful tool for studying context-free languages (cf. e.g. [14 and 3]). Unfortunately, the very success of rational relations in this theory is certainly the cause why the study of these objects for their own sake, has practically stopped, with some very few exceptions (see e.g. [18 and 19]). In our opinion, the next step towards a better knowledge of rational relations should be a systematical study of the "simplest" subfamilies of rational relations: rational partial functions, sequential and subsequential partial functions.

As far as pushdown translations are concerned, they have received less attention than they deserve. Indeed, since their general properties have been established, very little work has been done. Yet, apart from providing a model of compilation (cf. [1]), and, from a strictly mathematical point of view, posing some challenging problems, to ignore these translations would lead to exclude some natural functions such as the function which reverses each word, or characteristic functions of context-free languages, to mention only a few.

In this paper we consider the subfamilies of pushdown relations which are most frequently encountered in the literature. Since the authors assume the reader is familiar with rational relations, the emphasis will be put on other subfamilies of relations. We shall make a survey of the properties which are known so far - characterizations, factorizations, closure properties, equivalence decision problems - and give some new properties using either direct proofs or results from other theories, such as homomorphism equivalence (cf. [8]).

The definitions of the different families we shall deal with, are given in the preliminaries, in terms of transducers. Yet, when it comes to proving results, one wishes to use an alternative, more algebraic definition. Such a characterization has first been stated for rational relations by Nivat, and has easily been extended to pushdown relations (translations). It uses the notion of "bimorphism" and has given way to what in Eilenberg's terminology is called "The first factorization Theorem". In section III, we establish a similar characterization for unambiguous pushdown functions, which helps proving results in the next sections.

Section IV is concerned with properties of the closure under composition. We consider more specifically subfamilies of pushdown functions. Our results enable us to define a hierarchy of these subfamilies.

Finally equivalence decision problems are considered in section V. Some new results are shown.

II. PRELIMINARIES

II.1 Free Monoid, Relations

We denote by Σ^* the free monoid generated by the finite non-empty set - or alphabet - Σ, by ε its unit or empty word and by $\Sigma^+ = \Sigma^* \backslash \{\varepsilon\}$ the free semigroup. The length of a word $w \in \Sigma^*$ is denoted by $|w|$. Let $w = a_1 \ldots a_n$ be a word, where $n > 0$ and $a_1,\ldots,a_n \in \Sigma$. The reverse of w is the word $w^R = a_n \ldots a_1$. We define the reverse of the empty word as: $\varepsilon^R = \varepsilon$.

In order to simplify notations, all relations considered in this paper, unless otherwise stated, are from the fixed free monoid Σ^* into the fixed free monoid Δ^*. We write $f : \Sigma^* \to \Delta^*$ such a relation and view it as a function of Σ^* into the power set 2^{Δ^*}. It is length-preserving if $v \in f(u)$ implies $|u| = |v|$. If $f(u)$ possesses at most one element for each $u \in \Sigma^*$, we say that f is a partial function.

The domain of a relation $f : \Sigma^* \to \Delta^*$ is the subset $\mathrm{Dom}\, f = \{u \in \Sigma^* \mid f(u) \neq \phi\}$. Its image is the subset $\mathrm{Im}\, f = \{v \in \Delta^* \mid v \in f(u) \text{ for some } u \in \Sigma^*\}$. Its graph is the subset $\#f = \{(u,v) \in \Sigma^* \times \Delta^* \mid v \in f(u)\}$. Finally the reverse of f is the relation $f^R : \Sigma^* \to \Delta^*$ whose graph is $\#f^R = \{(u^R,v^R) \in \Sigma^* \times \Delta^* \mid (u,v) \in \#f\}$.

Given a family F of relations, F^R denotes the family of all f^R where $f \in F$.

The union of a family of relations $f_i : \Sigma^* \to \Delta^*$ $(i \in I)$ is the relation $f : \Sigma^* \to \Delta^*$ defined by: $\#f = \bigcup_{i\in I} \#f_i$.

II.2 Pushdown Relations

We refer to [1, p. 228] for all notions not explicitly defined in the sequel. We recall that a pushdown transducer, abbreviated a PDT, is an 8-tuple $T = (Q,\Sigma,\Delta,X,\delta,q_0,Z_0,F)$ where:

- Q is the set of states, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is the subset of final states
- Σ is the input alphabet
- Δ is the output alphabet
- X is the pushdown alphabet and $Z_0 \in X$ is the start symbol
- δ is a function which maps $Q \times (\Sigma \cup \{\varepsilon\}) \times X$ into finite subsets of $Q \times X^* \times \Delta^*$.

A relation $f : \Sigma^* \to \Delta^*$ is a pushdown relation (translation), abbreviated PD relation, if it is defined by some PDT.

We denote by PDR the family of all pushdown relations and by PDF the family of all pushdown partial functions.

Let $f : \Sigma^* \to \Delta^*$ be a partial function. Then it is unambiguous if it can be defined by some PDT whose underlying pushdown automaton is unambiguous (cf. e.g. [15, p. 142]). It is left deterministic (or shortly deterministic) if it can be defined by some PDT whose underlying pushdown automaton is deterministic (cf. e.g. [15, p. 139]). It is right deterministic if the partial function f^R is deterministic. Finally it is bideterministic if it is both left and right deterministic.

We denote by UPDF, DET, DET^R and BIDET respectively, the families of unambiguous, deterministic, right deterministic and bi-deterministic partial functions.

Let $L \subseteq \Sigma^*$ be a context-free language recognized by the pushdown automaton $A = (Q,\Sigma,X,\delta,q_0,Z_0,F)$. We denote by $I_L : \Sigma^* \to \Sigma^*$

the restriction of identity to L, i.e. the relation whose graph is $\{(u,u) \in \Sigma^* \times \Sigma^* \mid u \in L\}$. The restriction of identity to L is a pushdown function since it is defined by the PDT $\tau = (Q,\Sigma,\Sigma,X,\delta',q_0,Z_0,F)$ where: $(p,u,a) \in \delta'(q,a,v)$ iff $(p,u) \in \delta(q,a,v)$.

An important subfamily of PDR consists in the family RAT of all <u>rational</u> relations, i.e. of all relations $f : \Sigma^* \to \Delta^*$ whose graph is a rational subset of the product monoid $\Sigma^* \times \Delta^*$ (cf. e.g. [9, p. 236]). The family of rational partial functions is denoted by RATF. The well known facts that rational relations are particular pushdown relations and that $PDR = PDR^R$, $PDF = PDF^R$, $UPDF = UPDF^R$, $RAT = RAT^R$, $RATF = RATF^R$, can be seen for example, using Proposition III.1 of the next section.

II.3 Sequential and Subsequential Partial Functions

We now turn to the crucial notion of sequential partial functions. Since there exist in the literature all kinds of "sequential" functions, we will expose in detail the notion we will use which corresponds to Eilenberg's generalized sequential partial functions (cf. [9, Chap. XI]).

We will make use of the following convention. All partial functions f of a set X into the free monoid Σ^* shall be considered as a <u>total</u> (i.e. everywhere defined) function of X into the semiring 2^{Σ^*}. Therefore we have $f(u) = \phi$ whenever f is undefined for the value $u \in X$. Further, the product $f(u)f(v)$ equals ϕ, i.e. is undefined iff $f(u)$ or $f(v)$ is equal to ϕ.

A <u>sequential</u> transducer is a sextuple $S = (Q,\Sigma,\Delta,\lambda,\theta,q_0)$ where Q, Σ, Δ and q_0 are as in the definition of a pushdown transducer and where:

$\lambda : Q \times \Sigma \to Q$ is a partial function, called the <u>transition</u>

function, and

$\theta : Q \times \Sigma \to \Delta^*$ is a partial function, called the output function.

It is assumed that λ and θ have the same domain.

The partial functions λ and θ are extended to $Q \times \Sigma^*$ in the usual way (see e.g. [9, p. 297]). For all $q \in Q$ and $u \in \Sigma^*$ we write $q \cdot u$ and $q * u$ instead of $\lambda(q,u)$ and $\theta(q,u)$ respectively. This enables us, as long as no confusion may arise, to denote S by the quadruple (Q,Σ,Δ,q_0), the partial functions λ and θ being understood.

A partial function $f : \Sigma^* \to \Delta^*$ is left sequential or simply sequential if there exists some sequential transducer $S = (Q,\Sigma,\Delta,q_0)$ satisfying for all $u \in \Sigma^* : f(u) = q_0 * u$. It is right sequential if the reverse partial function f^R is left sequential. It is bisequential if it is both left and right sequential.

We shall denote by SEQ, SEQ^R and BISEQ respectively the families of sequential, right sequential and bisequential partial functions.

Sequential partial functions are particular rational partial functions as it is easily seen (cf. e.g. [9, Prop. XI, 3.1]). More precisely we have the following crucial result due to Elgot and Mezei (cf. e.g. [3, Theorem IV.5.2]):

Theorem 1

A partial function $f : \Sigma^* \to \Delta^*$ is rational, iff there exist a finite set Γ, a left (resp. right) sequential partial function $g : \Sigma^* \to \Gamma^*$ and a right (resp. left) sequential partial function $h : \Gamma^* \to \Delta^*$ such that $f(u) = h(g(u))$ holds for all $u \in \Sigma^+$.

A few words to compare the different "sequential" functions:

Ginsburg's generalized sequential machine mappings - or GSM mappings - as defined in [13, p. 93], are sequential partial functions which are total, i.e. everywhere defined. DGSM mappings, considered by several authors (cf. e.g. [16, p. 172]), are restrictions of "GSM mappings" to some arbitrary rational subset (not just the complement of a rational right ideal as for our sequential partial functions). DGSM are thus deterministic sequential transducers with a distinguished subset of "final" states.

It is clear from the definition that a left (resp. right) sequential partial function $f : \Sigma^* \to \Delta^*$ is prefix- (resp. suffix-) preserving: for each $u,v \in \Sigma^*$, $f(uv) \subseteq f(u)\Delta^*$ (resp. $f(vu) \subseteq \Delta^* f(u)$). Using this property, Ginsburg and Rose have characterized the sequential partial functions among the class of all partial functions of a free monoid into another (see e.g. [9, Theorem XI, 6.3]):

Theorem 2 (Ginsburg and Rose)

Let $f : \Sigma^* \to \Delta^*$ be a partial function such that $f(\varepsilon) \subseteq \varepsilon$. Then it is sequential iff the three following conditions hold:

1) f is prefix preserving;
2) there exists an integer $k > 0$ such that $|f(ua)| - |fu| < k$ holds for all $u \in \Sigma^*$, $a \in \Sigma$ satisfying $f(ua) \neq \phi$;
3) for each rational subset $R \subseteq \Delta^*$, $f^{-1}(R)$ is a rational subset of Σ^*.

The property of being prefix-preserving does not characterize the sequential partial functions among the class of all rational partial functions. For example, let $\Sigma = \{a,b\}$, $\Delta = \{a\}$ and consider the function defined by (cf. [13, p. 100]):

$$f(u) = \begin{cases} a^n & \text{if } u = a^n \\ a^{2n} & \text{if } u = a^n bv \text{ for some } v \in \Sigma^*. \end{cases}$$

Then f is prefix-preserving and rational but not sequential. Nonetheless, if we denote by $f_1 : \Sigma^* \to \Delta^*$ the morphism defined by $f_1(u) = a^{|u|}$ and by $f_2 : \Sigma^* \to \Delta^*$ the sequential function defined by $f_2(u) = a^{2n}$ if $u = a^n$ or $u = a^n bv$ for some $v \in \Sigma^*$, then f is the sum of two DGSM mappings: the restriction of f_1 to a^*, and the restriction of f_2 to $\Sigma^* \backslash a^*$.

We shall show a stronger result by exhibiting a rational (total) function which is both prefix and suffix preserving but which is not a <u>finite</u> union of DGSM mappings.

<u>Proposition 3</u>

Let $\Sigma = \{a,b\}$, $\Delta = \{a\}$ and consider the partial function $g : \Sigma^* \to \Delta^*$ defined by:

$$g(ba^n) = \begin{cases} a^{n+1} & \text{if } n \text{ is even} \\ a^{2n+1} & \text{otherwise ,} \end{cases}$$

and $g(u) = \phi$ if $u \notin ba^*$.

Then the subset $X = (a,a)^*(\#g)^*(b,a)(a,a)^* \cup (a,a)^* \subseteq \Sigma^* \times \Delta^*$ is the graph of a (total) function which is rational, prefix- and suffix-preserving and which is not a finite union of DGSM mappings.

<u>Proof</u>

We shall just prove that f is not a finite union of DGSM mappings, the rest resulting from standard verifications.

Assume $\#f = \bigcup_{1 \le i \le n} \#f_i$ where f_i is the restriction to a rational subset $R_i \subseteq \Sigma^*$ of a sequential function f_i'. For $1 \le i \le n$, let k_i be the integer assigned to f_i' as in the condition 2) of Ginsburg and Rose's theorem and set $K = \max\{k_i \mid 1 \le i \le n\} + 1$.

Let $M_1, M_2, \ldots, M_{n+1}$ be a sequence of integers such that $M_1 = 0$, and for all $1 \le i < n+1$, $M_{i+1} > K(M_1 + \ldots + M_i + n)$.

Consider the $n+1$ words $u_i = ba^{M_{n+1}} ba^{M_n} \ldots ba^{M_i}$, $1 \leq i \leq n+1$. Notice that if we denote by $<$ the partial order over Σ^* "prefix of", we have: $u_{n+1} < u_n < \ldots < u_2 < u_1$. Certainly there are two integers $1 \leq r < s \leq n+1$ such that u_r and u_s belong to the same subset R_i, for some $1 \leq i \leq n$. By Ginsburg and Rose's characterization we must have: $|f(u_r)| - |f(u_s)| = |f_i(u_r)| - |f_i(u_s)| < K(|u_r| - |u_s|)$. But we have: $K(|u_r| - |u_s|) = K(M_{s-1} + \ldots + M_r + (s-r)) < K(M_{s-1} + \ldots + M_1 + n) < M_s < |f(u_r)| - |f(u_s)|$ which yields a contradiction. □

We turn now to the notion of subsequential partial function.

A <u>subsequential</u> transducer is a pair (S,φ) where $S = (Q,\Sigma,\Delta,\lambda,\theta,q_0)$ is a sequential transducer and $\varphi : Q \rightarrow \Delta^*$ a partial function (see [5, p. 109]).

A partial function $f : \Sigma^* \rightarrow \Delta^*$ is <u>left subsequential</u> or simply <u>subsequential</u> if there exists a subsequential transducer (S,φ) such that $f(u) = q_0 * u . \varphi(q_0 \cdot u)$ holds for all $u \in \Sigma^+$ (notice that there exists no condition on $f(\varepsilon)$). It is <u>right sequential</u> if the reversed partial function $f^R : \Sigma^* \rightarrow \Delta^*$ is left sequential. It is <u>bisubsequential</u> if it is both left and right subsequential. Subsequential partial functions have been considered in [11] where they are called "augmented version of DGSM mapping".

Intuitively a subsequential transducer is a sequential transducer capable of guessing the end of the input word. Formally we have the following connection between sequential and subsequential partial functions:

<u>Proposition 4</u>

Let $f : \Sigma^* \rightarrow \Delta^*$ be a partial function. Then it is subsequential iff its domain $R = f^{-1}(\Delta^*)$ is a rational set and if there exist a new symbol $\$ \notin \Sigma$ and a sequential total function $g : (\Sigma \cup \{\$\})^* \to \Delta^*$ such that $f(u) = g(u\$)$ holds for all $u \in R\backslash\{\varepsilon\}$.

Proof

"only if"-part: Let (S,ϕ) - where $S = (Q,\Sigma,\Delta,q_0)$ - be a subsequential transducer defining the subsequential partial function $f : \Sigma^* \to \Delta^*$. Set $F = \mathrm{Dom}\,\varphi \subseteq Q$, $Q' = Q \cup \{q_t\}$ where $q_t \notin Q$ is a new symbol and $\Sigma' = \Sigma \cup \{\$\}$. The domain $R = f^{-1}(\Delta^*)$ is rational since it consists in all words $u \in \Sigma^*$ such that $q_0.u \in F$. Define the sequential transducer $S' = (Q',\Sigma',\Delta,\lambda,\theta,q_0)$ where λ and θ satisfy:

$$\lambda(q,a) = \begin{cases} q.a & \text{if } a \in \Sigma,\ q \in Q \text{ and } q.a \neq \phi, \\ q_t & \text{otherwise;} \end{cases}$$

$$\theta(q,a) = \begin{cases} q*a & \text{if } a \in \Sigma \text{ and } q \in Q \\ \varphi(q) & \text{if } a = \$ \text{ and } q \in F \\ \varepsilon & \text{otherwise.} \end{cases}$$

Let $g : \Sigma'^* \to \Delta^*$ be the sequential total function defined by S'. Then for all $u \in R$ we have:

$$f(u) = (q_0*u)\ \varphi(q_0.u) = \theta(q_0,u)\ \varphi(q_0.u) = \theta(q_0,u)\ \theta(q_0.u,\$)$$
$$= \theta(q_0,u\$) = g(u\$).$$

"if"-part: Let $g : (\Sigma \cup \{\$\})^* \to \Delta^*$ be a sequential total function defined by the sequential transducer $S' = (Q',\Sigma',\Delta,q_0)$ where $\Sigma' = \Sigma \cup \{\$\}$. Without loss of generality we may assume that the automaton underlying S recognizes the rational subset R, i.e. that there exists a subset $F \subseteq Q$ such that $R = \{u \in \Sigma^* \mid q_0.u \in F\}$. Denote by $\varphi : Q \to \Delta^*$ be partial function defined by:

$$\varphi(q) = \begin{cases} q*\$ & \text{if } q \in F, \\ \phi & \text{otherwise.} \end{cases}$$

Let $S = (Q,\Sigma,\Delta,q_0)$ be the sequential transducer obtained by restriction to Σ, of the transition and output functions of S'. For all $u \in \Sigma^*$ we have:

$$f(u) = g(u\$) = (q_0*u)(q_0.u*\$) = (q_0*u)\ \varphi(q_0.u)$$

which proves that f is subsequential. □

<u>Corollary 5</u>. Let $\$ \notin \Sigma^*$ and $f : \Sigma^*\$ \to \Delta^*$. Then f is a DGSM mapping iff f is a subsequential function.

<u>Corollary 6</u>. The family of DGSM mappings is properly between the families of total sequential functions and subsequential functions.

We refer to [5] for a systematic study of subsequential partial functions. We shall denote by SUBSEQ, SUBSEQ^R and BISUBSEQ respectively, the families of subsequential, right subsequential and bisubsequential partial functions.

The diagram in Fig. 1 shows the various strict inclusions among the families considered in this section.

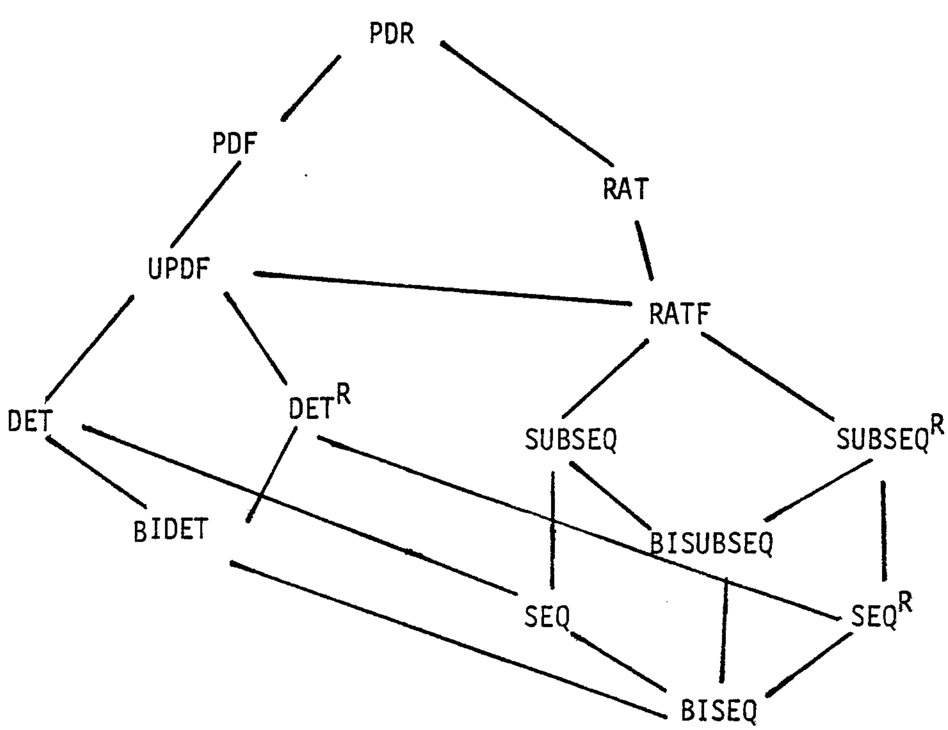


Fig. 1

III. FACTORIZATION PROPERTIES

III.1 A Useful Factorization of Unambiguous Functions

All proofs on relations involving constructions of transducers are intricate and therefore unreliable. It is desirable to use, as much as possible, an alternative, more manageable definition. This is luckily the case for rational relations which, according to what in Eilenberg's terminology is the "First Factorization Theorem", can be obtained as a composition of simpler relations: morphisms, inverse morphisms and intersections with a rational language. A similar result is true for pushdown relations, where "context-free language" has to be substituted to "rational language". We summarize in a single statement these two well-known results (for the "rational" part of the Proposition see for instance [17] or [9, Thm. IX, 2.2] and for the "pushdown" part see [12, p. 122] or [1, Thm. 3.4]). We recall that a morphism $f : \Sigma^* \to \Delta^*$ is <u>alphabetic</u> if the image of every letter of Σ by f, is either a letter of Δ or the empty word: $f(\Sigma) \subseteq \Delta \cup \{\varepsilon\}$. It is <u>strictly alphabetic</u> if $f(\Sigma) \subseteq \Delta$.

<u>Proposition 1</u>

Let $f : \Sigma^* \to \Delta^*$ be a relation. Then it is rational (resp. a pushdown relation) iff there exist a finite set Γ, a rational (resp. context-free) language $R \subseteq \Gamma^*$ and two alphabetic morphisms $h : \Gamma^* \to \Sigma^*$ and $g : \Gamma^* \to \Delta^*$ such that for all $u \in \Sigma^*$, $f(u) = g(h^{-1}(u) \cup R)$ holds.

When $f : \Sigma^* \to \Delta^*$ is a partial function - not any relation - the preceding Proposition can be made more precise (see e.g. [9, Thm. IX, 8.1]).

Theorem 2

Let $f : \Sigma^* \to \Delta^*$ be a partial function. Then it is rational iff there exist a finite set Γ, a rational subset $R \subseteq \Gamma^*$, an alphabetic morphism $h : \Gamma^* \to \Sigma^*$ which maps bijectively R onto $h(R)$ and an alphabetic morphism $g : \Gamma^* \to \Delta^*$ such that $f(u) = g(h^{-1}(u) \cap R)$ holds for all $u \in \Sigma^*$.

As a consequence, all rational partial functions are unambiguous. This is not the case of all pushdown partial functions, since the restriction of identity to an inherently ambiguous context-free language is clearly not unambiguous. However, unambiguous pushdown partial functions can be characterized in a much similar way:

Proposition 3

Let $f : \Sigma^* \to \Delta^*$ be a partial function. Then it is unambiguous iff there exist a finite set Ξ, a bideterministic context-free language $L \subseteq \Xi^*$, an alphabetic morphism $g : \Xi^* \to \Delta^*$, and an alphabetic morphism $h : \Xi^* \to \Sigma^*$ which maps bijectively L onto $h(L)$ such that for all $u \in \Xi^*$, $f(u) = g(h^{-1}(u) \cap L)$ holds.

Proof

"only if"-part: Assume we have proved that f admits the factorization

$$(1) \quad f = g \circ I_L \circ h^{-1}$$

where g and h are as in the Proposition and where L is a deterministic (not necessarily bideterministic) context-free language. By Chomsky-Schützenberger's Theorem (see e.g. [3, Th. 3.10]), since L is a deterministic, and therefore an unambiguous language, there exist a bideterministic language D (actually the intersection of a rational language with a Dyck language over n letters, for some integer $n > 0$) and an alphabetic morphism k which maps bijectively D over $k(D) = L$. The equality $I_L = k \circ I_D \circ k^{-1}$ holds. Substituting in (1) we have:

$$f = g\circ k\circ I_D\circ k^{-1}\circ h^{-1} = (g\circ k)\circ I_D\circ (g\circ k)^{-1}.$$

The conditions of the Propositions are thus satisfied.

Assume now we have proved that f admits the factorization:

(2) $f = g\circ I_L\circ h^{-1}$ where L is deterministic, h is as in the Proposition but where there is no assumption on the morphism g. We will show that f admits a factorization of the type (1).

First, the morphism $g : \Xi^* \to \Delta^*$ can be factorized as $g = g_2\circ g_1$, where g_1 is an injective morphism of Ξ^* into a new free monoid Γ^* and $g_2 : \Gamma^* \to \Delta^*$ is an alphabetic morphism. Indeed if we set:

$$\Gamma = \{(a,i) \in \Xi \times N \mid 1 \le i \le |g(a)|, \text{ or } g(a) = \varepsilon \text{ and } i = 0\}$$

it suffices to define g_1 and g_2 by the following conditions: for all $a \in \Xi$, if $g(a) = \varepsilon$ then $g_1(a) = (a,0)$ and $g_2 = (a,0) = \varepsilon$, otherwise, if $g(a) = b_1,\ldots,b_n$ (where $n > 0$ and $b_1,\ldots,b_n \in \Delta$) then $g_1(a) = (a,1) \ldots (a,n)$ and $g_2(a,i) = b_i$ $(1\le i\le n)$.

Substituting $g = g_2\circ g_1$ in (2) and observing that $g_1\circ I_L = I_{g_1(L)}\circ g_1$, we obtain: $f = g\circ I_L\circ h^{-1} = g_2\circ g_1\circ I_L\circ h^{-1} = g_2\circ I_{g_1(L)}\circ g_1\circ h^{-1}$. Since g_1^{-1} is a DGSM mapping, then according to [16, Theorem 12.3], $g_1(L) = (g_1^{-1})^{-1}(L)$ is a deterministic language. It suffices to verify that the partial function $h\circ g_1^{-1}$ is the restriction to its domain $D = (g_1\Xi)^* \subseteq \Gamma^*$ of an alphabetic morphism $g' : \Gamma^* \to \Sigma^*$ which is injective on $g_1(L) \subseteq D$.

Let g' be the morphism defined, for all $(a,i) \in \Gamma$ by:

$$g'(a,i) = \begin{cases} \varepsilon & \text{if } i \ne 1, \\ h(a) & \text{otherwise.} \end{cases}$$

Clearly, g' is alphabetic and for all $a \in \Xi$, we have: $g'\circ g_1(a) = h(a)$. Therefore, for all $u = g_1(a_1\ldots a_n) \in D$, we obtain: $h\circ g_1^{-1}(u) = h(a_1\ldots a_n) = g'(u)$, which shows that $h\circ g_1^{-1}$ is the restriction of g' to D. Furthermore if $u,v \in g_1(L)$ verify $h \circ g_1^{-1}(u) = h \circ g_1^{-1}(v)$ then $g_1^{-1}(u) = g_1^{-1}(v)$, i.e. $u = v$ which completes the verification.

Therefore, given any unambiguous partial function f, it suffices to prove that f admits a factorization of the type (2), which we now turn to do.

Let $\tau = (Q,\Sigma,\Delta,X,\delta,q_0,Z_0,F)$ be an unambiguous pushdown transducer defining f. For each $(q,a,x,q',x',u) \in Q \times (\Sigma \cup \{\varepsilon\}) \times X \times Q \times X^* \times \Delta^*$ such that $(q',x',u) \in \delta(q,a,x)$ we define a new symbol $[q,a,x,q',x',u]$ and we denote by W the finite set thus obtained. Let $A = (Q,W,X,\delta',q_0,Z_0,F)$ be the pushdown automaton defined for all $w = [q,a,x,q',x'u] \in W$, $p \in Q$ and $y \in X$ by: $(q',x') \in \delta'(p,[q,a,x,q',x',u],y)$ iff $p = q$ and $y = x$. Then the language L recognized by A is deterministic, and the morphism $h : W^* \to \Sigma^*$ which to each $[q,a,x,q',x',u]$ assigns its second component $a \in \Sigma \cup \{\varepsilon\}$ is alphabetic and injective on L since two distinct $w_1,w_2 \in L$ such that $h(w_1) = h(w_2)$ would correspond to two distinct computations of $w \in \mathrm{Dom}\, f$. Then it suffices to define $g : W^* \to \Delta^*$ as the morphism which to $[q,a,x,q',x',u] \in W$ assigns its last component u.

"if"-part: Since the relations $g \circ I_L \circ h^{-1}$ and $I_L \circ h^{-1}$ can be defined by two PDT having the same underlying automaton, it suffices to show that $I_L \circ h^{-1}$ is an unambiguous relation.

Let $A = (Q,\Xi,X,\delta,q_0,Z_0,F)$ be a bideterministic pushdown automaton accepting L. Define a PDT $\tau = (Q,\Sigma,\Xi,X,\delta',q_0,Z_0,F)$ as follows. For all $(q,a,x,q',x') \in Q \times (\Xi \cup \{\varepsilon\}) \times X \times Q \times X^*$ we have: $(q',x',a) \in \delta'(q,h(a),x)$ iff $(q',x') \in \delta(q,a,x)$.

Clearly, $I_L \circ h^{-1} : \Sigma^* \to \Xi^*$ is the relation defined by τ. Finally, the pushdown automaton underlying τ is unambiguous since to each $w \in \Sigma^*$ corresponds a unique factorization $h(a_1 \ldots a_n) = w$ ($n > 0$ and $a_i \in \Xi$ for $1 \le i \le n$) and for such a word $a_1 \ldots a_n$ there is a unique computation in A. This completes the proof. □

III.2 A Factorization Using Length-Preserving Partial Functions

The following is equivalent, for a certain family of deterministic partial functions, to Theorem IX, 8.4, of Eilenberg's book, which states that every rational partial function is the composition of a length-preserving rational partial function and a morphism. We think it is quite unlikely that there exists a result of this type for many reasonable families of pushdown partial functions. We recall that a subset $X \subseteq \Sigma^*$ is prefix-free if for all $u,v \in \Sigma^*$ we have: $u,uv \in X$ implies $v = \varepsilon$.

Proposition 4

Let $f : \Sigma^* \to \Delta^*$ be a deterministic partial function whose domain is prefix-free. Then there exist a finite set Ω, a length preserving deterministic partial function $g : \Sigma^* \to \Omega^*$ and a morphism $h : \Omega^* \to \Delta^*$ such that: $f = h \circ g$.

Notice that the assumptions of our Proposition cannot be weakened. Indeed let $\Sigma = \Delta = \{a,b\}$ and consider the deterministic partial function $f : \Sigma^* \to \Delta^*$ defined by:

$$\text{for all } r > 0,\ n \ge m > 0,\ f(b^r a^n b^m) = \begin{cases} b^r a^n b^m & \text{if } m < n, \\ b^r a^n b^m a^r & \text{if } m = n,, \end{cases}$$

otherwise $f(u) = \phi$.

Assume $f = h \circ g$ where $g : \Sigma^* \to \Omega^*$ is a length-preserving deterministic partial function and $h : \Omega^* \to \Delta^*$ is a morphism. Set $M = \max\{|h(x)| \mid x \in \Omega\}, g(b^M a^2 b) = u$ and $g(b^M a^2 b^2) = uv$ where by hypothesis $|u| = M+3$ and $|v| = 1$. We obtain:

$$f(b^Ma^2b^2) = h(uv) = h(u)h(v) = f(b^Ma^2b)\ h(v)$$

Since $f(b^Ma^2b^2) = b^Ma^2b^2a^M$ and $f(b^Ma^2b) = b^Ma^2b$ we have $|h(v)| = M + 1$, which yields a contradiction.

Proof of the Proposition 4

Let us note that it suffices to prove that $f = h \circ g$ where h is a morphism and g is a deterministic partial function such that $|g(u)| = 2|u|$ whenever $u \in \text{Dom } f$. Indeed, denote by Γ a copy of Ω^2 and by $[w_1w_2]$ the copy of $w_1w_2 \in \Omega^2$. Let $\gamma : \Gamma^* \to \Omega^*$ be the monomorphism defined by $\gamma([w_1w_2]) = w_1w_2$. Then γ^{-1} is a DGSM mapping and $\gamma \circ \gamma^{-1}$ is the identity over $(\Omega^2)^* \supseteq \text{Img}$. Therefore we have the factorization $f = h\gamma \circ (\gamma^{-1}g)$ where $h\gamma$ is a morphism and $\gamma^{-1}g$ is, by Proposition IV, 3. of the next section, a length-preserving deterministic partial function.

Let $\tau = (Q,\Sigma,\Delta,X,\delta,q_0,z_0)$ be a deterministic PDT defining f. Without loss of generality, we may assume that the underlying push-down automaton recognizes Dom f by empty stack (cf. [15, Theorem 11.5.2.]). Furthermore, we may suppose that τ satisfies the following conditions:

1) all ε-moves are erasing moves (cf. [15, Exercise 5.6.6])
2) for all $q, p \in Q$, $a \in \Sigma$, $x \in X$, $y \in X^*$ and $u \in \Delta^*$ we have: $\delta(q,a,x) = (p,y,u)$ implies $|y| \leq 2$ (cf. [15, Theorem 5.4.2])
3) for all $a \in \Sigma^+$, $q \in Q$, $u \in X^*$ and $x \in X$ we have: $(q_0,a,Z_0) \overset{*}{\vdash} (q,\varepsilon,ux)$ implies $q \neq q_0$ and $x \neq Z_0$ (this can be done, if necessary, by creating a new initial state q_0', and a new start symbol Z_0'). Further more, for all $a \in \Sigma$, $\delta(q_0,a,Z_0)$ is of the form (q,x,u) where x is a letter.

Let Ω be the set consisting of a copy of Dom δ and of a new element σ. We shall denote by $[q,a,x]$ the copy of the element $(q,a,x) \in$ Dom δ. The morphism $h : \Omega^* \to \Delta^*$ of the factorization we seek, is defined by:

$h(\sigma) = \varepsilon$,

and for all $q, p \in Q$, $a \in \Sigma \cup \{\varepsilon\}$, $x \in X$, $y \in X^*$ and $u \in \Delta^*$:

$h([q,a,x]) = u$ iff $\delta(q,a,x) = (p,y,u)$.

We now give an informal explanation of how the deterministic partial function $g\colon \Sigma^* \to \Omega^*$ assigns to each word $u \in \mathrm{Dom}\, f \subseteq \Sigma^*$ a word of length $2|u|$ containing all informations about $f(u)$. A deterministic transducer τ' defining g will be obtained by modifying τ.

Assume that, at a given moment of the computation of a word $u \in \mathrm{Dom}\, f$ by the PDT τ, the current letter is $a \in \Sigma$. If this occurrence preserves the height of the stack, then the output assigned to the move in the PDT τ' will be a word of length 2. If it increases the height of the stack (by 1 according to assumption 2) then the output in τ' is a word of length 1, i.e. a letter. In this last case, the new stack symbol will eventually be removed either by an ε-move to which an output of length 1 will be assigned or by a non ε-move i.e. a move involving a letter $b \in \Sigma$, to which an output of length 3 will be assigned. In all cases the average length of an output assigned to one letter is 2 ($(1 + 3)/2$ or $1 + 1$).

Formally $\tau' = (Q,\Sigma,\Delta,X,\delta',q_0,Z_0)$ is a deterministic PDT recognizing $\mathrm{Dom}\, g = \mathrm{Dom}\, f$ by empty stack, and defined by:

i) for all $q,p \in Q$, $x \in X$ we have:

$\delta'(q,\varepsilon,x) = (p,\varepsilon,[q,\varepsilon,x])$ iff $\delta(q,\varepsilon,x) = (p,\varepsilon,u)$ for some $u \in \Delta^*$

ii) for all $p, q \neq q_0 \in Q$, $a \in \Sigma$, $x \in X$ and $y \in X^*$ we have:

$\delta'(q,a,x) = (p,y,[q,a,x])\, \sigma^{2-|y|}$ iff $\delta(q,a,x) = (p,y,u)$ for some $u \in \Delta^*$

iii) for all $q \in Q$, $a \in \Sigma$ and $y \in X^*$ we have:

$\delta'(q_0,a,Z_0) = (p,y,[q_0,a,Z_0])$ iff $\delta(q_0,a,z_0) = (p,y,u)$ for some $u \in \Delta^*$

It is left to the reader to verify that τ' works as claimed. □

IV. CLOSURE UNDER COMPOSITION

IV.1 General results

It is well known that rational relations are closed under composition (see e.g. [9, Theorem IX, 4.1]), while pushdown relations are not (if L, M are two context-free languages of Σ^* then I_L and I_M are pushdown relations, but $I_L \circ I_M = I_{L \cap M}$ might not be). Yet we have (see e.g. [12, p 115]):

<u>Proposition 1</u>

Let $f: \Sigma_1^* \to \Sigma_2^*$ be a push down relation and $g: \Delta_1^* \to \Delta_2^*$ be a rational relation. If $\Sigma_1 = \Delta_2$ (resp. $\Sigma_2 = \Delta_1$) then $f \circ g$ (resp. $g \circ f$) is a push down relation.

Obviously, the preceding Proposition holds when f and g are partial functions. Using the factorization properties established in the preceding section, we will see that it still holds in the following case:

Proposition 2

Let $f: \Sigma_1^* \to \Sigma_2^*$ be an unambiguous partial function and $g: \Delta_1^* \to \Delta_2^*$ a rational partial function. If $\Sigma_1 = \Delta_2$ (resp. $\Sigma_2 = \Delta_1$) then $f \circ g$ (resp. $g \circ f$) is an unambiguous partial function.

Proof

By Proposition III.3. and Theorem III.2. we have the factorizations

$$f = f_2 \circ I_L \circ f_1^{-1} \quad \text{and} \quad g = g_2 \circ I_R \circ g_1^{-1}$$

where $L \subseteq \Sigma^*$ is a bideterministic language, $R \subseteq \Delta^*$ is a rational language, $f_i: \Sigma^* \to \Sigma_i^*$ $i = 1,2$ and $g_i: \Delta^* \to \Delta_i^*$ $i = 1,2$ are alphabetic morphisms such that f_1 and g_1 are injective on L and R respectively.

Case 1: $\Sigma_1 = \Delta_2$

Consider the relation $h = f_1^{-1} \circ g_2: \Delta^* \to \Sigma^*$. By Lemma IX, 4.2. of [9], there exist a finite set Γ, a rational subset $K \subseteq \Gamma^*$ and two alphabetic morphisms $h_1: \Gamma^* \to \Delta^*$ and $h_2: \Gamma^* \to Z^*$ such that $h = h_2 \circ I_K \circ h_1^{-1}$. Furthermore for all $(u,v) \in \#h$, there exists exactly one element $w \in K$ such that $h_1(w) = u$ and $h_2(w) = v$.

We have:

$$\begin{aligned} f \circ g &= (f_2 \circ I_L \circ f_1^{-1}) \circ (g_2 \circ I_R \circ g_1^{-1}) = f_2 \circ I_L \circ (h_2 \circ I_K \circ h_1^{-1}) \\ &\quad \circ I_R \circ g_1^{-1} \\ &= f_2 \circ h_2 \circ I_{h_2^{-1}(L)} \circ I_K \circ I_{h_1(R)} \circ h_1^{-1} \circ g_1^{-1} \\ &= f_2 \circ h_2 \circ I_M \circ (g_1 \circ h_1)^{-1} \end{aligned}$$

where $M = h_1(R) \cap K \cap h_2^{-1}(L)$ is, according to [16 , Theorems 12.2 and 12.3], a bideterministic language.

It now suffices to show that h_1 is injective on M. Consider $z,t \in M$ such that $h_1(z) = h_1(t)$. Since $h_2 \circ I_M \circ h_1^{-1} = I_L \circ h_2 \circ I_K \circ h_1^{-1} \circ I_R$ is a partial function, we have $h_2(z) = h_2(t)$ and therefore

$z = t$, which proves that h_1 is injective on M as claimed.

Case 2: $\Sigma_2 = \Delta_1$

We have $g \circ f = g_2 \circ I_R \circ g_1 \circ f_2 \circ I_L \circ f_1^{-1}$. Since $I_R \circ g_1 \circ f_2 \colon \Sigma^* \to \Delta^*$ is a rational partial function, there exist a finite set Γ, a rational language $K \subseteq \Gamma^*$, an alphabetic morphism $h_1\colon \Gamma^* \to \Sigma^*$ which is injective on K and an alphabetic morphism $h_2\colon \Gamma^* \to \Delta^*$ such that: $I_R \circ g_1 \circ f_2 = h_2 \circ I_K \circ h_1^{-1}$. Thus we obtain:

$$\begin{aligned} g \circ f &= g_2 \circ (h_2 \circ I_K \circ h_1^{-1}) \circ I_L \circ f_1^{-1} \\ &= (g_2 \circ h_2) \circ I_K \circ I_{h_1^{-1}(L)} \circ h_1^{-1} \circ f_1^{-1} \\ &= (g_2 \circ h_2) \circ I_{K \cap h_1^{-1}(L)} \circ (f_1 \circ h_1)^{-1} \end{aligned}$$

Since $K \cap h_1^{-1}(L)$ is bideterministic, the proof is complete.

□

The families of (left) deterministic partial functions and of right sequential partial functions are incomparable. Therefore, if we compose a deterministic partial function with a right sequential partial function (or more generally with a rational partial function), the result is not necessarily a deterministic partial function. However we have:

<u>Proposition 3</u>

Let $f\colon \Sigma_1^* \to \Sigma_2^*$ be a deterministic partial function and $g\colon \Delta_1^* \to \Delta_2^*$ a DGSM mapping. If $\Sigma_1 = \Delta_2$ (resp. $\Sigma_2 = \Delta_1$), then $f \circ g$ (resp. $g \circ f$) is a deterministic partial function.

<u>Proof</u>

Let $T = (Q,\Sigma_1,\Sigma_2,X,\delta,q_0,Z_0,F)$ be a deterministic pushdown transducer defining f an $S = (P,\Delta_1,\Delta_2,\lambda,\theta,p_0,H)$ a DGSM (sequential transducer with final states) defining g. We remind our convention from section II.3. that for all $q \in P$ and $u \in \Delta_1^*$ we write $q.u$ and $q * u$ instead of $\lambda(q,u)$ and $\theta(q,u)$, respectively.

Case 1: $\Sigma_2 = \Delta_1$

It suffices to notice that the partial function $g \circ f: \Sigma_1^* \to \Delta_2^*$ is defined by the deterministic pushdown $\tau' = (QXP, \Sigma_1, \Delta_2, X, \delta', (q_0, p_0), Z_0, F\times H)$ where δ' is defined, for all $q, q' \in Q$, $p \in P$, $a \in \Sigma_1 \cup \{\varepsilon\}$, $u \in \Delta_2^*$, $x \in X$ and $x' \in X^*$ by:

$((q', p.u), x', p*u) \in \delta'((q,p),a,x)$

iff $(q', x', u) \in \delta(q,a,x)$

Case 2: $\Sigma_1 = \Delta_2$

We shall first consider three particular subcases.

subcase i) g is length preserving (which is equivalent to saying that for all $p \in P$ and $a \in \Delta_1$, we have $p * a \subseteq \Delta_2$).

Then $f \circ g$ is defined by the deterministic pushdown transducer $\tau_1 = (Q\times P, \Delta_1, \Sigma_2, X, \delta', (q_0, p_0), Z_0, F\times H)$ where for all $q, q' \in Q$, $p \in P$, $a \in \Delta_1 \cup \{\varepsilon\}$, $x \in X$, $x' \in X^*$ and $u \in \Sigma_2^*$ we have:

$((q', p.a), x', u) \in \delta'((q,p),a,x)$ iff $(q', x', u) \in \delta(q, p*a, x)$

subcase ii) For all $a \in \Delta_1$, we have $g(a) = (a,1)\ldots(a,n)$ for some integer $n > 0$ (in particular $|g(a)| = n$).

Consider the set $R = \{(q,a,i) \in Q\times\Delta_1\times\mathbb{N} \mid 1 \le i \le |g(a)|\}$ and define the pushdown transducer

$\tau' = (R, \Delta_1, \Sigma_2, X, \delta', (q_0, a_0, |g(a_0)|), Z_0, F)$ - where a_0 is a fixed arbitrary letter - in the following way:

- for all $q, q' \in Q, a, b \in \Delta_1, x \in X, x' \in X^*$ and $u \in \Sigma_2^*$ we have:

$((q', a, 1), x', u) \in \delta'((q, b, |g(b)|), a, x)$ iff $(q', x', u) \in \delta(q, (a,1), x)$

- for all $q, q' \in Q$, $a, b \in \Delta_1, x \in X, x' \in X^*, u \in \Sigma_2^*$ and $1 < i < |g(a)|$ we have:

$((q', a, i+1), x', u) \in \delta'((q,a,i), \varepsilon, x)$ iff $(q', x', u) \in \delta(q, (a,i), x)$

- for all $q, q' \in Q, a \in \Delta_1, x \in X, x' \in X^*, u \in \Sigma_2^*$ and $1 \le i \le |g(a)|$ we have:

$((q',a,i),x',u) \in \delta'((q,a,i),\varepsilon,x)$ iff $(q',x',u) \in \delta(q,\varepsilon,x)$.

It suffices to note that τ' is deterministic and defines $f \circ g$.

subcase iii) g is an alphabetic morphism. Let $Q' \subseteq Q$ be the subset of all states defining ε-moves. Define the pushdown transducer $\tau' = (Q,\Delta_1,\Sigma_2,X,\delta',q_0,Z_0,F)$ in the following way:

- for all $q \in Q', q' \in Q, x \in X, x' \in X^*$ and $u \in \Sigma_2^*$ we have: $(q',x',u) \in \delta'(q,\varepsilon,x)$ iff $(q',x',u) \in \delta(q,\varepsilon,x)$
- for all $q \in Q', q' \in Q, a \in \Delta_1, x \in X, x' \in X^*$ and $u \in \Delta_2^*$ we have:

  if $g(a) \neq \varepsilon$, then $(q',x',u) \in \delta'(q,a,x)$ iff $(q',x',u) \in \delta(q,g(a),x)$

  if $g(a) = \varepsilon$ then $(q,x,\varepsilon) \in \delta'(q,a,x)$

It is easy to check that τ' is deterministic and that it defines $f \circ g$.

To complete the proof, it suffices to note that every sequential partial function g can be factorized as: $g = g_2 \circ g_1$ where g_1 is a length-preserving sequential partial function and g_2 is a morphism. But g_2 can be factorized as $g_2 = g_2'' \circ g_2'$ where g_2' is as in subcase ii) and g_2'' is alphabetic. □

Let $R \subseteq \Sigma^*$ be a rational subset and $f: \Sigma^* \to \Delta^*$ an unambiguous partial function. Since $I_R: \Sigma_1^* \to \Sigma^*$ is a rational partial function, by Proposition 2, the restriction $f \circ I_R$ of f to the subset R is an unambiguous partial function. We can deduce from the next Proposition, a similar result for deterministic partial functions.

We recall that a subset $R \subseteq \Sigma^* \times \Delta^*$ is <u>recognizable</u> (cf.[9], p.68]) if there exist a morphism φ of $\Sigma^* \times \Delta^*$ into a finite monoid M and a subset $N \subseteq M$ such that $R = \varphi^{-1}(N)$.

<u>Proposition 4</u>

For each unambiguous (resp. deterministic) partial function $f: \Sigma^* \to \Delta^*$ and each recognizable subset $R \subseteq \Sigma^* \times \Delta^*$, the partial function

$g: \Sigma^* \to \Delta^*$ whose graph is $\#g = \#f \cap R$, is an unambiguous (resp. deterministic) partial function.

Proof

Let $T = (Q,\Sigma,\Delta,X,\delta,q_0,Z_0,F)$ be an unambiguous (resp. deterministic) PDT defining f and let $R = \varphi^{-1}(N)$ be as in the above definition. Then it suffices to observe that g is defined by the unambiguous (resp. deterministic) PDT $T' = (Q\times M,\Sigma,\Delta,X,\delta',q_0\times\{1\},Z_0,F\times N)$ where we have denoted by 1 the unit of the monoid M and where δ' is defined by:

for all $q,q' \in Q, m \in M, a \in \Sigma \cup \{\varepsilon\}, x \in X, y \in X^*$ and $u \in \Delta^*$

we have:

$((q',m.\varphi(a,u)),y,u) \in \delta'((q,m),a,x)$ iff $(q',y,u) \in \delta(q,a,x)$ □

As a consequence we have:

Corollary 5

For each unambiguous (resp. deterministic) partial function $f: \Sigma^* \to \Delta^*$ and each rational subset $R \subseteq \Sigma^*$, the restriction of f to R is an unambiguous (resp. deterministic) partial function.

Proof

Indeed we have $\#(f \circ I_R) = \#f \cap (R \times \Delta^*)$ □

Corollary 6

For each unambiguous (resp. deterministic) partial function $f: \Sigma^* \to \Delta^*$ and each rational subset $R \subseteq \Delta^*$, $f^{-1}(R)$ is an unambiguous (resp. deterministic) context-free language.

Proof

Indeed, $f^{-1}(R)$ is the domain of the partial function $g: \Sigma^* \to \Delta^*$ whose graph is: $\#g = \#f \cap (\Sigma^* \times R)$ □

Finally, we recall that sequential and subsequential partial functions are closed under composition (see e.g. [3], Proposition IV,2.5]).

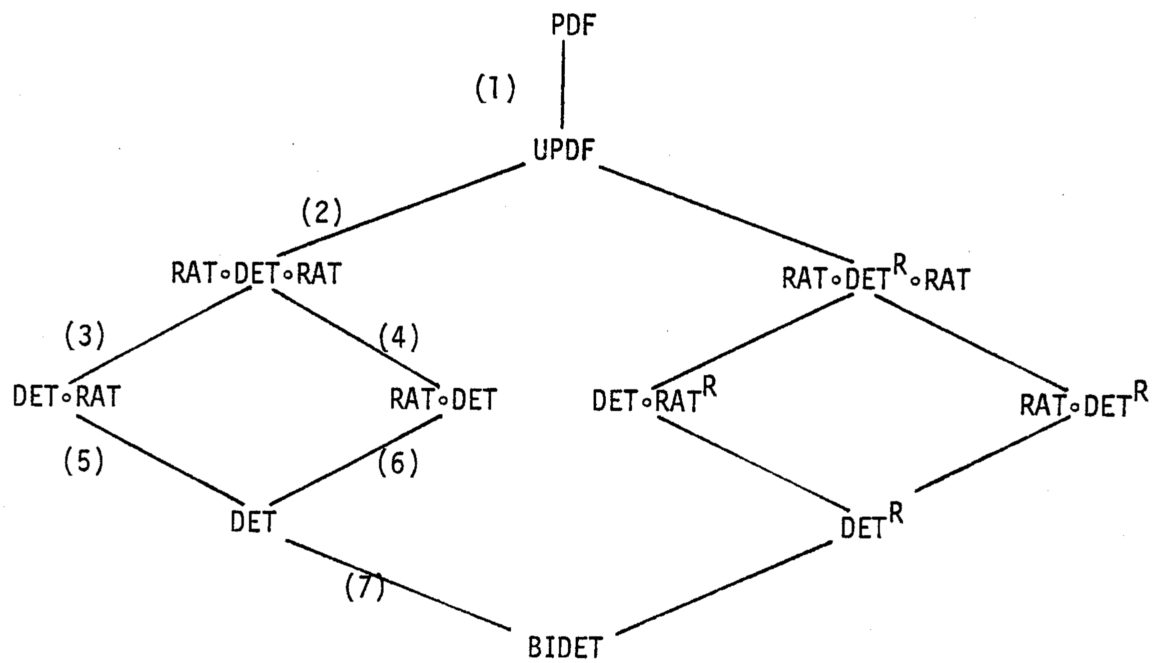
IV.2 A hierarchy of pushdown partial functions

Given two families of partial functions F_1 and F_2, we shall use the customary notation $F_2 \circ F_1$ to indicate the family of all partial functions of the form $f_2 \circ f_1$ with $f_1 \in F_1$ and $f_2 \in F_2$.

Proposition 2 of the preceding paragraph shows that: UPDF ∘ RATF = RATF ∘ UPDF = UPDF. This implies in particular, that DET, DET ∘ RATF, RATF ∘ DET and RAT ∘ DET ∘ RAT are subfamilies of UPDF. We will see that all inclusions among these families are strict.

<u>Proposition 7</u>

The following strict inclusions hold:



Proof

For obvious reasons of symmetry it suffices to consider the left part of the diagram.

Strict inclusions (1) and (7) can be established arguing on the domain of the different partial functions. The same holds for strict inclusion (2). Indeed, by Theorem II.1. and Proposition 3 of the preceding paragraph we have $RATF \circ DET \circ RATF = SEQ^R \circ DET \circ SEQ^R$ which shows that the domain of a partial function $f \in RATF \circ DET \circ RATF$ if the inverse image of a (left) deterministic language by a right sequential partial function. But we do not get in this way all unambiguous languages (cf. e.g. [21, p.40]).

Strict inclusions (5) and (6) are consequences of the fact that RATF and DET are incomparable.

Strict inclusions (3) and (4) will follow from the fact that $RATF \circ DET$ and $DET \circ RATF$ are incomparable which we will turn to prove now.

In order to show that $DET \circ RATF \not\subseteq RATF \circ DET$, let $\Sigma = \{a,b,c,d\}$ and set $L = \{a^n b^n c \mid n > 0\} \cup \{a^n b^{2n} d \mid n > 0\}$. Then $I_L : \Sigma^* \to \Sigma^*$ belongs to $DET \circ RATF$ as is easily seen, but does not belong to $RATF \circ DET$, since its domain is not a deterministic context free language.

We are now left with proving $RATF \circ DET \not\subseteq DET \circ RATF$. This will be done in two steps.

Lemma 8

Let $\Sigma = \{a,b\}$ and $\Delta = \{c,d\}$. The partial function $f: \Sigma^* \to \Delta^*$ defined by:

$$\text{for all } m,n \geq 0,\ f(a^n b a^m b) = \begin{cases} c^n & \text{if } n \geq m, \\ d & \text{otherwise,} \end{cases}$$

and $f(u) = \phi$ if $u \notin a^*ba^*b$,

is not deterministic.

Proof

Assume f is defined by a deterministic pushdown transducer $T = (Q,\Sigma,\Delta,\delta,q_0,Z_0,F)$. We shall assign to T a deterministic pushdown automaton A recognizing a language $L \subseteq (\Sigma \cup \Delta)^*$ consisting of words belonging to the shuffle of a word $u \in \Sigma^*$ with its image $f(u) \in \Delta^*$.

Formally, we consider the pushdown automaton $A = (Q \cup Q',\Sigma \cup \Delta, X,\delta',q_0,Z_0,F)$ where the set Q' of new states disjoint from Q and δ' are defined by replacing each relation $(q',y,u) \in \delta(q,a,x)$ by a set of relations according to the following rules:

i) if $u = \varepsilon$, then the collection is reduced to:

$$(q',y) \in \delta'(q,a,x)$$

ii) if $u = u_1 \ldots u_p$ $(u_i \in \Delta \quad 1 \leq i \leq p)$ then the collection is:

$$(q'_1,x) \in \delta'(q,a,x)$$

$$(q'_{i+1},x) \in \delta'(q'_i,u_i,x) \qquad 1 \leq i < p$$

$$(q',y) \in \delta'(q'_1,u_p,x)$$

where $q'_1,\ldots,q'_p$ are new states.

If we denote by L the (deterministic) context-free language recognized by A, then $L \cap \{a,b,c\}^* \subseteq \{a,b\}^*c^*$ since the occurrence c cannot be output before A knows for sure that $n \geq m$, i.e. before it reads the second occurrence of b. More precisely $L \cap \{a,b,c\}^* = \{a^nba^mbc^n \mid n \geq m\}$ which contradicts the fact that L is context-free. □

Lemma 9

Let $\Sigma = \{a,b\}$, $\Delta = \{c,d\}$ and consider the partial function $f: \Sigma^* \to \Delta^*$ defined by:

$$f(u) = \phi \text{ if } u \notin a^*b^*$$

and $$f(a^n b^m) = \begin{cases} c^n & \text{if } n \geq m \geq 0 \\ d & \text{otherwise} \end{cases}$$

Then $f \in (RATF \circ DET) \setminus (DET \circ RATF)$.

Proof

i) $f \in RATF \circ DET$

Let $g\colon \Sigma^* \to \Delta^*$ be the deterministic partial function defined by:

$g(u) = \phi$ if $u \notin a^* b^*$

and $$g(a^n b^m) = \begin{cases} c^n & \text{if } n \geq m > 0, \\ c^n d^{m-n} & \text{otherwise.} \end{cases}$$

Consider now the right sequential partial function $h\colon \Delta^* \to \Delta^*$ satisfying, for all $u \in \Delta^*$: $h(uc) = uc$ and $h(ud) = d$. Then it suffices to note that: $f = h \circ g$.

ii) $f \notin DET \circ RATF$

By Theorem II.1 and Proposition 3 of this section, it suffices to verify that f can not be factorized as $f = h \circ g$ where $g\colon \Sigma^* \to \Gamma^*$ is a right sequential partial function and $h\colon \Gamma^* \to \Delta^*$ a deterministic partial function.

Assume this is the case and let φ be the canonical morphism of Σ^* onto the transition monoid of a sequential transducer defining g (cf. e.g. [10, p 157]).

Choose an integer $n > 0$ satisfying the conditions:

(1) $\varphi(a^n) = \varphi(a^{2n})$ and $\varphi(b^n) = \varphi(b^{2n})$. Then there exist four words $u,v,w,z \in \Gamma^*$ such that for all $r,s \geq 0$ we have: $g(a^{n(r+1)}\, b^{n(s+1)}) = u^r v w^s z$. This implies the following:

$$h(u^r vw^s z) = \begin{cases} c^{n(r+1)} & \text{if } r \geq s, \\ d & \text{otherwise.} \end{cases}$$

Consider the sequential partial functions $h_1: \Sigma^* \to \Gamma^*$ and $h_2: \Delta^* \to \Delta^*$ defined by:

$$h_1(t) = \begin{cases} u^r vw^s z & \text{if } t = a^r ba^s b, r,s \geq 0 \\ \phi & \text{otherwise;} \end{cases}$$

and

$$h_2(t) = \begin{cases} c^r & \text{if } t = c^{n(r+1)}, \\ t & \text{otherwise.} \end{cases}$$

Then by Proposition 3 of the section, the partial function: $h_2 \circ h \circ h_1: \Sigma^* \to \Delta^*$ is deterministic. But this contradicts the preceding lemma since this partial function is precisely the one defined in this lemma. □

V. DECISION PROBLEMS

Let F_1 and F_2 be two subfamilies of pushdown relations and consider $f_1 \in F_1$ and $f_2 \in F_2$. We are concerned in this section with the two following decision problems:

problem 1 : $f_1 \in F_2$?

problem 2 : $f_1 = f_2$?

Problem 1 is known to be undecidable in case $F_1 = F_2 = \mathrm{RAT}$ (see e.g. [3, Theorem III, 8.4.]. Several authors have independently proved that Problem 1 is decidable when $F_1 = \mathrm{RAT}$ and $F_2 = \mathrm{RATF}$ (cf. [4] and [20]).

<u>Proposition 1</u>

Given an arbitrary rational relation $f : \Sigma^* \to \Delta^*$ it is decidable whether it is a partial function i.e. whether $f(u)$ contains at most one element for all $u \in \Sigma^*$.

In particular this proves that Problem 2, under the same assumption, is decidable. Indeed, $f_1 = f_2$ iff the two rational languages Dom f_1 and Dom f_2 are equal and if the union of f_1 and f_2 is again a partial function. Therefore we have (cf. the same references):

<u>Proposition 2</u>

Given two arbitrary rational partial functions $f_i : \Sigma^* \to \Delta^*$, $i = 1,2$, it is decidable whether $f_1 = f_2$, i.e. whether $f_1(u) = f_2(u)$ holds for all $u \in \Sigma^*$.

Assume $F_1 = \mathrm{RAT}$ and F_2 is any of the families SEQ, SEQ^R, BISEQ, SUBSEQ, SUBSEQ^R, BISUBSEQ. For each of these cases Problem 1 can be decided (cf. [6]):

<u>Proposition 3</u>

Given an arbitrary rational relation $f : \Sigma^* \to \Delta^*$ each of the following problems is decidable:

1) $f \in SEQ$?

2) $f \in SEQ^R$?

3) $f \in BISEQ$?

4) $f \in SUBSEQ$?

5) $f \in SUBSEQ^R$?

6) $f \in BISUBSEQ$?

A stronger result than Proposition 2 has been shown in [7, Theorem 7], namely that Problem 2 is decidable for $F_1 = RATF$ and $F_2 = UPDT$. We will further strengthen this result in two ways.

<u>Proposition 4</u>

Given an arbitrary rational relation $f_1 : \Sigma^* \to \Delta^*$ and an unambiguous pushdown function $f_2 : \Sigma^* \to \Delta^*$, it is decidable whether $f_1 = f_2$.

<u>Proof</u>

By Proposition 1 and Theorem 7 in [7].

<u>Proposition 5</u>

Let $f_1 : \Sigma^* \to \Delta^*$ be a pushdown relation and $f_2 : \Sigma^* \to \Delta^*$ a rational partial function such that $\mathrm{Dom}\, f_2 \subseteq \mathrm{Dom}\, f_1$. Then it is decidable whether f_1 and f_2 are equal.

<u>Proof</u>

Observe that under the assumptions of the Proposition, $\mathrm{Dom}\, f_2 = \mathrm{Dom}\, f_1$ iff the context-free language $\mathrm{Dom}\, f_1 \setminus \mathrm{Dom}\, f_2$ is empty.

This is known to be decidable. We shall from now on assume that $\mathrm{Dom}\ f_1 = \mathrm{Dom}\ f_2$.

Let Δ_1 and Δ_2 be two disjoint copies of Δ such that $\Delta_1 \cap \Sigma = \Delta_2 \cap \Sigma = \phi$. The idea of the proof is to define a pushdown relation $f : \Sigma^* \to (\Delta_1 \cup \Delta_2)^*$ with the following property. For each $u \in \Sigma^*$, every word in $f(u)$ belongs to the shuffle of the copy (in Δ_1^*) of some word of $f_1(u)$ with the copy (in Δ_2^*) of some word of $f_2(u)$. In particular, if p_1 and p_2 are the canonical projections of $(\Delta_1 \cup \Delta_2)^*$ over Δ^* , we will have for all $u \in \Sigma^* : f_1(u) = p_1 \circ f(u)$ and $f_2(u) = p_2\ f(u)$. Therefore the initial decision problem will be reduced to testing whether the two homomorphisms p_1 and p_2 agree over the context-free language $\mathrm{Im}\ f = f(\Sigma^*)$.

Denote by $j_1 : \Delta^* \to \Delta_1^*$ and $j_2 : \Delta^* \to \Delta_2^*$ the canonical isomorphisms. By Proposition III.1. the pushdown relation $j_1 \circ f_1 : \Sigma^* \to \Delta_1^*$ and the rational partial function $j_2 \circ f_2 : \Sigma^* \to \Delta_2^*$ admit the factorizations:

$$j_1 \circ f_1 = g_1 \circ I_L \circ h_1^{-1} \quad \text{and} \quad j_2 \circ f_2 = g_2 \circ I_R \circ h_2^{-1}$$

where $L \subseteq \Xi_1^*$ is a context-free and $R \subseteq \Xi_2^*$ a rational language, and where $h_1 : \Xi_1^* \to \Sigma^*$, $g_1 : \Xi_1^* \to \Delta_1^*$, $h_2 : \Xi_2^* \to \Sigma^*$ and $g_2 : \Xi_2^* \to \Delta_2^*$ are alphabetic morphisms. We will assume, without loss of generality that Ξ_1, Ξ_2, Σ, Δ_1 and Δ_2 are pairwise disjoint.

Let $g_1' : \Xi_1^* \to (\Sigma \cup \Delta_1)^*$ be the morphism defined for all $x \in \Xi_1$ by : $g_1'(x) = h_1(x) g_1(x)$, and set : $f_1' = g_1' \circ I_L \circ h_1^{-1}$.

Consider next the rational subset $R' \subseteq (\Xi_2 \cup \Delta_1)^*$ which is the shuffle of $R \subseteq \Xi_2^*$ with Δ_1^* (cf. e.g. [9, Proposition II, 3.4.]). Define the alphabetic morphisms $h_2' : (\Xi_2 \cup \Delta_1)^* \to (\Sigma \cup \Delta_1)^*$ and $g_2' : (\Xi_2 \cup \Delta_1)^* \to (\Delta_2 \cup \Delta_1)^*$ by setting:

$$h_2'(x) = \begin{cases} x & \text{if } x \in \Delta_1 \\ h_2(x) & \text{if } x \in \Xi_2 \end{cases} \qquad g_2'(x) = \begin{cases} x & \text{if } x \in \Delta_1 \\ g_2(x) & \text{if } x \in \Xi_2 \end{cases} .$$

Then $f_2' = g_2' \circ I_{R'} \circ h_2'^{-1} : (\Sigma \cup \Delta_1)^* \to (\Delta_2 \cup \Delta_1)^*$ is a rational relation and we leave it to the reader to verify that the pushdown relation $f = f_2' \circ f_1' : \Sigma^* \to (\Delta_1 \cup \Delta_2)^*$ satisfies the condition: $f_1 = p_1 \circ f$ and $f_2 = p_2 \circ f$ as claimed.

Assume $f_1 = f_2$ and consider $x \in f(\Sigma^*)$. Then we have $x \in f(u)$ for some $u \in \Sigma^*$. This yields:

$$p_1(x) \in p_1 f(u) = f_1(u) \quad \text{and} \quad p_2(x) \in p_2 f(u) = f_2(u) .$$

Since $f_1(u)$ and $f_2(u)$ are two equal singletons, we obtain $p_1(x) = p_2(x)$.

Conversely, assume that $p_1(x) = p_2(x)$ holds for all $x \in f(\Sigma^*)$. Then for all $u \in \Sigma^*$ we obtain:

$$f_1(u) = p_1 f(u) = p_2 f(u) = f_2(u) .$$

In other words, $f_1 = f_2$ iff the two morphisms are equivalent over the subset $f(\Sigma^*)$. Since this subset is context-free (cf. e.g. [12, p. 116]), this last problem is decidable by [8] or [2]. □

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