BLOCK STRUCTURE

IN THE CHEBYSHEV-PADÉ TABLE

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Abstract

The problem of obtaining Chebyshev-Padé approximants for the Chebyshev series $\sum_{0}^{\infty}{}' c_k T_k(x)$ is directly related to the Padé approximation of the corresponding power series $\sum_{0}^{\infty}{}' c_k w^k$. The theory of the block structure in the Padé table is generalized for the Chebyshev-Padé table. An anomaly of "nonexistence" of Chebyshev-Padé approximants is clarified by relating it to the location of the poles of the corresponding Padé approximant.

TABLE OF CONTENTS

1. Introduction

A classical method for obtaining rational approximations for a function $g(z)$ with power series expansion

$$(1.1)\quad g(z) = \sum_{k=0}^{\infty} a_k z^k$$

is to form Padé approximants. The (m,n) Padé approximant

$$(1.2)\quad R_{m,n}(z) = p_m(z)/q_n(z)$$

for $g(z)$ is defined as that rational function with numerator of degree at most m and denominator of degree at most n such that the power series expansion of $R_{m,n}(z)$ has maximal initial agreement with the power series expansion of $g(z)$. Since $R_{m,n}(z)$ has $m+n+1$ free parameters, the (m,n) Padé approximant is normally found to satisfy

$$(1.3)\quad g(z) - R_{m,n}(z) = 0(z^{m+n+1})$$

where the notation $0(z^k)$ denotes a power series with no term of degree less than k). A method to compute $R_{m,n}(z)$ can be obtained by multiplying through in equation (1.3) by the denominator $q_n(z)$ yielding the new equation

$$(1.4)\quad q_n(z)\, g(z) - p_m(z) = 0(z^{m+n+1}),$$

where we have noted that multiplying the power series on the right of (1.3) by a polynomial (in fact, by any power series) does not reduce the order of the power series on the right. The parameters in (1.4) appear linearly; thus if

$$(1.5)\quad p_m(z) = \sum_{k=0}^{m} \alpha_k z^k \text{ and } q_n(z) = \sum_{k=0}^{n} \beta_k z^k$$

then (1.4) takes the form of $m+n+1$ linear equations in the $m+n+2$ parameters (β_0 will be fixed later). These linear equations can be specified in the following convenient form:

$$(1.6)\quad \alpha_k = \sum_{i=0}^{k} \beta_i\, a_{k-i},\ 0 \le k \le m$$

$$(1.7)\quad H_{m,n}\, \underline{\beta} = -\beta_o\, \underline{h}_{m,n}\ ,$$

where $\underline{\beta} = [\beta_n, \beta_{n-1}, \ldots, \beta_1]^T$, $\underline{h}_{m,n} = [a_{m+1}, a_{m+2}, \ldots, a_{m+n}]^T$, and where $H_{m,n}$ is the order n Hankel matrix

$$(1.8)\quad H_{m,n} = \begin{bmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_m \\ a_{m-n+2} & a_{m-n+3} & \cdots & a_{m+1} \\ \vdots & \vdots & & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n-1} \end{bmatrix} .$$

The notation here assumes the convention that $a_k = 0$ for $k < 0$ and $\beta_i = 0$ for $i > n$.

This linear-equation approach is of course not the most efficient numerical method for computing Padé approximants. There are numerous recursive schemes, like Rutishauser's Q-D algorithm or Wynn's ϵ-algorithm (Wynn [8] gives an extensive comparison of algorithms for computing Padé approximants), which require $O(n^2)$ arithmetic operations rather than the $O(n^3)$ cost of solving a linear system. However the recursive schemes are based on the assumption that the given power series is normal, by which is meant that for all m,n the Hankel matrix (1.8) is nonsingular. This normality assumption is a very strong condition; for example, if the power series coefficients are not all nonzero then the power series is not normal.

In the present investigation, we are interested in the block structure (cf. [5]) of the Padé table which arises in connection with singular Hankel matrices, and in particular with the extension of this block structure theory to the more recent Chebyshev-Padé table.

2. Rational Approximations from Chebyshev Series

It is well-known that truncated power series and Padé approximants have poor approximation properties in the space $C[-1,1]$ of functions continuous on the interval $[-1,1]$, since the approximation error increases as the distance from the point of expansion increases. In an attempt to obtain near-minimax rational approximations for a function $f \in C[-1,1]$, one starts with its expansion in terms of Chebyshev polynomials of the first kind:

$$(2.1) \quad f(x) = \sum_{k=0}^{\infty}{}' c_k T_k(x).$$

Here $T_k(x) = \cos(k \arccos x)$ and $\sum_{k=0}^{\infty}{}' u_r$ denotes the sum $\frac{1}{2}u_o+u_1+u_2+ \dots$.

Since the truncated Chebyshev series is a near-minimax polynomial approximation in $C[-1,1]$, it is reasonable to believe that an (m,n) rational function

$$(2.2) \quad r_{m,n}(x) = u_m(x)/v_n(x)$$

will be a near-minimax rational approximation if the Chebyshev series expansion of $r_{m,n}(x)$ exists and satisfies an equation of the form

$$(2.3) \quad f(x) - r_{m,n}(x) = \sum_{k=m+n+1}^{\infty} e_k T_k(x),$$

for some coefficients e_k. (The condition that the Chebyshev series expansion of $r_{m,n}(x)$ should exist is simply the condition that $r_{m,n}(x)$ should have no poles in $[-1,1]$.)

A widely used method for obtaining rational approximations from Chebyshev series is commonly known as Maehly's method. In this method, one obtains a rational function (2.2) by satisfying an equation of the form

$$(2.4) \quad v_n(x)\, f(x) - u_m(x) = \sum_{k=m+n+1}^{\infty} \hat{e}_k T_k(x)$$

(for some coefficients $\hat{e}_k$) rather than equation (2.3). This is of course a direct generalization of the Padé method and is based on the fact that (2.4) yields linear equations defining the parameters while (2.3) does not. However, unlike the Padé method where equation (1.4) is equivalent to equation (1.3) except in certain abnormal cases, equation (2.4) is not equivalent to equation (2.3). For suppose that equation (2.4) is satisfied and that $v_n^{-1}(x)$ exists as a Chebyshev series. Then multiplying through in equation (2.4) by $v_n^{-1}(x)$ will yield on the right a Chebyshev series starting with a degree-0 term, in general, which is not an equation of the form (2.3). This is due to the product rule for Chebyshev polynomials:

$$(2.5) \quad T_i(x)\, T_j(x) = (T_{i+j}(x) + T_{|i-j|}(x))/2.$$

The relative success of Maehly's method has been due to the fact that, numerically, the unwanted terms in the Chebyshev series expansion of the error function are found to have "small" coefficients (at least for "nice" functions). However, the more recent Chebyshev-Padé method of Clenshaw and Lord [2] is even simpler than Maehly's method from a computational point of view and produces more accurate approximations because it does satisfy an equation of the form (2.3).

3. The Transform Method for Chebyshev-Padé Approximation

Following the traditional Padé method, we define the (m,n) Chebyshev-Padé approximant for the function (2.1) as that rational function (2.2) with numerator of degree at most m and denominator of degree at most n such that the Chebyshev series expansion of (2.2) exists and has maximal initial agreement with the Chebyshev series expansion (2.1). It will normally be the case that an equation of the form (2.3) holds (i.e. agreement through $m+n+1$ terms) but, as in the case of Padé approximation, the maximal initial agreement will sometimes be through p terms for p less than $m+n+1$. Moreover, unlike the case of Padé approximation, there are (rare) cases where the (m,n) Chebyshev-Padé approximant is not unique.

With a view to understanding the "abnormal" cases which arise in Chebyshev-Padé approximation, we use an approach modelled after the "Laurent-Padé" method of Gragg and Johnson [6] rather than the approach of Clenshaw and Lord [2]. (The equations defining the Chebyshev-Padé approximant will be the same; only the derivation is different.) Thus to approximate the function (2.1) on the interval $[-1,1]$ in the x-plane, we use a transformation so that we are working with a function

$$(3.1) \qquad G(w) = \sum_{k=0}^{\infty}{}' c_k w^k$$

on the unit circle $C = \{w : |w| = 1\}$ in the w-plane, where the power series coefficients in (3.1) are precisely the Chebyshev series coefficients in (2.1). The power series (3.1) is then approximated by a technique which is essentially the ordinary Padé method and finally, transforming back to the x-plane, we get the desired rational approximation for (2.1).

The transformation being used here is the Faber mapping (cf. [7])

$$(3.2)\quad w = \phi(x) = x + \sqrt{x^2-1}$$

which maps the entire complex x-plane with the interval $[-1,1]$ deleted (i.e. the exterior of $[-1,1]$) conformally onto the exterior of the unit circle C in the w-plane. (The branch of the square root function is uniquely specified by the condition $|x+\sqrt{x^2-1}| > 1$.) We think of the unit circle C as the image under (3.2) of the interval $[-1,1]$ (traversed twice). The inverse of (3.2) is the mapping

$$(3.3)\quad x = \psi(w) = (w+w^{-1})/2.$$

Theorems 3.1 and 3.2 give the basic results for Chebyshev-Padé approximation but first Lemma 3.1 gives a useful identity.

Lemma 3.1

If x and w are related by (3.2) - (3.3) then the following algebraic identity holds:

$$T_k(x) = (w^k+w^{-k})/2.$$

Proof: Let $x = \cos\theta$ and $w = e^{i\theta}$ and the result follows.

Theorem 3.1

Given the Chebyshev series (2.1) for $f \in C[-1,1]$, let $R_{m,n}(w)$ be the (m,n) Padé approximant for the power series G(w) defined by (3.1). If none of the poles of $R_{m,n}(w)$ lies in the closed unit disc and if

$$(3.4)\quad G(w) - R_{m,n}(w) = O(z^p)$$

for some integer p then the function

$$(3.5)\quad \tilde{r}_{m,n}(x) = [R_{m,n}(x+\sqrt{x^2-1}) + R_{m,n}(x-\sqrt{x^2-1})]/2$$

is in the space $\underset{\sim}{C}[-1,1]$ and has a Chebyshev series expansion satisfying

$$(3.6)\quad f(x) - \tilde{r}_{m,n}(x) = \sum_{k=p}^{\infty} e_k\, T_k(x)$$

for some coefficients e_k.

Proof: Let $R_{m,n}(w)$ be the (m,n) Padé approximant for G(w) and let its power series expansion be

$$(3.7)\quad R_{m,n}(w) = \sum_{k=0}^{\infty}{}' \hat{c}_k w^k .$$

If none of the poles of $R_{m,n}(w)$ lies in the closed unit disc then

$$(3.8)\quad \limsup_{k\to\infty} \sqrt[k]{|\hat{c}_k|} = \rho < 1.$$

Now,

$$[R_{m,n}(w) + R_{m,n}(w^{-1})]/2 = \sum_{k=0}^{\infty}{}' \hat{c}_k (w^k + w^{-k})/2$$

and by (3.8) this Laurent series represents an analytic function for $w \in C$. Setting $w = x + \sqrt{x^2-1}$ and noting that $w^{-1} = x - \sqrt{x^2-1}$, it follows from Lemma 3.1 that the function defined by (3.5) has the Chebyshev series expansion

$$(3.9)\quad \tilde{r}_{m,n}(x) = \sum_{k=0}^{\infty}{}' \hat{c}_k T_k(x),$$

where the coefficients $\hat{c}_k$ are precisely the power series coefficients in (3.7).

Now condition (3.8) applied to the Chebyshev series in (3.9) implies that the Chebyshev series converges uniformly in [-1,1], and therefore $\tilde{r}_{m,n}(x)$ is in the space C[-1,1]. Finally, condition (3.4) states that the coefficients in (3.7) are related to the coefficients in (3.1) as follows:

$$\hat{c}_k = c_k,\ 0 \le k \le p-1 .$$

Therefore, comparing (3.9) and (2.1) shows that (3.6) holds.

Theorem 3.2

The function $\tilde{r}_{m,n}(x)$ defined in Theorem 3.1 is an (ℓ,n) rational function where

$$\ell = \max\{m,n\}.$$

<u>Proof</u>: Let the (m,n) Padé approximant for (3.1) be

$$R_{m,n}(w) = \frac{p_m(w)}{q_n(w)} = \frac{\sum_{k=0}^{m} \alpha_k w^k}{\sum_{k=0}^{n} \beta_k w^k} .$$

By forming a common denominator, we have

$$(3.10) \quad \tilde{r}_{m,n}(x) = [p_m(w)q_n(w^{-1})+p_m(w^{-1})q_n(w)] / 2\, q_n(w)q_n(w^{-1})$$

where x and w are related by (3.2) - (3.3). A little algebra shows that $q_n(w)\ q_n(w^{-1})$ takes the form

$$\sum_{k=0}^{n} \hat{\beta}_k \,(w^k+w^{-k})$$

for some coefficients $\hat{\beta}_k$, and that the numerator in (3.10) takes the form

$$\sum_{k=0}^{\ell} \hat{\alpha}_k \,(w^k+w^{-k})$$

for some coefficients $\hat{\alpha}_k$. Hence by Lemma 3.1, we have $\tilde{r}_{m,n}(x)$ expressed **as an (ℓ,n) rational function in Chebyshev form.**

The above theorems **indicate a method for obtaining the (m,n)** Chebyshev-Padé approximant if three conditions hold:

(3.11) $m \geq n$;

(3.12) the associated Padé approximant is normal - i.e. equation (3.4) is satisfied with $p = m + n + 1$;

(3.13) the associated Padé approximant has no poles lying in the closed unit disc.

For if these conditions hold, it is clear that the function $\tilde{r}_{m,n}(x)$ is the desired (m,n) rational function (2.2) which satisfies equation (2.3).

The remainder of this paper is concerned with the conditions (3.11)-(3.13). In section 4 we show how the restriction (3.11) is eliminated. Section 5 deals with the non-normal situation when condition (3.12) does not hold; this corresponds to singularity of the Hankel system and it is here that we present a theorem on block structure in the Chebyshev-Padé table. Finally, when condition (3.13) fails to hold we have the case of non-uniqueness which is discussed in section 6.

4. The Complete Chebyshev-Padé Table

The method implied by Theorems 3.1 and 3.2 requires that, given m,n, and c_k $(0 \le k \le m+n)$, we solve the Hankel system (1.7) for β_k $(1 \le k \le n)$ by setting $\beta_0 = 1$ and

$$(4.1)\quad a_0 = \tfrac{1}{2} c_0; \quad a_k = c_k \ (1 \le k \le m+n); \quad a_k = 0 \ (k < 0).$$

The rational function $\tilde{r}_{m,n}(x)$ is then determined in Chebyshev form by simple formulas for its coefficients, which are essentially the coefficients $\hat{\alpha}_k$ and $\hat{\beta}_k$ mentioned in the proof of Theorem 3.2. The drawback of this method is that if $m < n$ then $\tilde{r}_{m,n}(x)$ is an (n,n) rational function rather than the desired (m,n) rational function. Clenshaw and Lord [2], by different reasoning, show that to obtain the complete Chebyshev-Padé table the Hankel system that should be solved is as above except that (4.1) must be changed to

$$(4.2)\quad a_0 = c_0; \quad a_k = c_k \ (1 \le k \le m+n); \quad a_k = c_{|k|} \ (k < 0).$$

(Note that when $m \ge n$, zero and negative subscripts do not arise in setting up the Hankel system (1.7).) Gragg and Johnson [6] give the following interpretation which allows us to continue to work in the transform space (the w-plane) with rational functions in standard form. This greatly simplifies the understanding of the "abnormal" cases considered in section 5.

In the modified approach, rather than seek the (m,n) Padé approximant for the power series (3.1) we must seek an (ℓ,n) rational function

$$(4.3)\quad R_{\ell,n}(w) = \frac{p_\ell(w)}{q_n(w)} = \frac{\sum_{k=0}^{\ell} \alpha_k w^k}{\sum_{k=0}^{n} \beta_k w^k} ,$$

where

$$\ell = \max\,\{m,n\} ,$$

such that the following two conditions hold:

$$(4.4)\quad q_n(w)\; G(w) - p_\ell(w) = 0\;(w^{m+n+1});$$

$$(4.5)\quad p_\ell(w)\; q_n(w^{-1}) + p_\ell(w^{-1})\; q_n(w) = \sum_{k=0}^{m} \hat{\alpha}_k\;(w^k+w^{-k}),$$

for some coefficients $\hat{\alpha}_k$.

The (m,n) Chebyshev-Padé approximant for (2.1) is then given by

$$(4.6)\quad r_{m,n}(x) = [R_{\ell,n}\;(x+\sqrt{x^2-1}) + R_{\ell,n}\;(x-\sqrt{x^2-1})]/2 .$$

Note that (4.4) is the analogue of (1.4) and that (4.5) simply ensures that the numerator of $r_{m,n}(x)$ will have the appropriate degree, unlike the case in (3.10). It is shown in [6] that conditions (4.4) - (4.5) are related to the following linear systems of equations:

$$(4.7)\quad \alpha_k = \sum_{j=0}^{k}{}' \; c_j\; \beta_{k-j}, \quad 0 \le k \le \ell;$$

$$(4.8)\quad H_{m,n}\; \underline{\beta} = -\beta_0\; \underline{h}_{m,n} ,$$ the Hankel system (1.7), where (4.2) defines the coefficients.

(The notation here assumes the convention that $\beta_i = 0$ for $i > n$.) We have the following results.

<u>Theorem 4.1</u>

If $\{\alpha_k\}_0^\ell$, $\{\beta_k\}_0^n$ is a nontrivial solution of (4.7) – (4.8) then it defines a rational function (4.3) satisfying conditions (4.4) - (4.5). If $\{\alpha_k\}_0^\ell$, $\{\beta_k\}_1^n$, $\beta_0 \neq 0$ defines a rational function (4.3) satisfying conditions (4.4) - (4.5) then it is a solution of (4.7) - (4.8). If $\{\alpha_k\}_0^\ell$, $\{\beta_k\}_1^n$, $\beta_0 = 0$ defines a rational function (4.3) (with $q_n(w)$ not the zero polynomial) satisfying conditions (4.4) - (4.5) then it is a solution of (4.7) - (4.8) if and only if $\det(H_{m,n}) = 0$.

<u>Proof</u>: The first two assertions are proved in [6]. The third assertion follows by similar reasoning, noting that (4.8) has nontrivial solutions with $\beta_0 = 0$ if and only if $\det(H_{m,n}) = 0$.

<u>Theorem 4.2</u>

The linear system (4.7) - (4.8) always has nontrivial solutions $\{\alpha_k\}_0^\ell$, $\{\beta_k\}_0^n$. Each such solution defines the <u>same</u> rational function (4.3) which has a unique reduced representation with constant coefficient 1 in the denominator.

<u>Proof</u>: See [6].

Recall that our definition of the (m,n) Chebyshev-Padé approximant $r_{m,n}(x)$ for $f(x)$, as presented at the beginning of section 3, involves the requirement of "maximal initial agreement" of the Chebyshev series expansions of $r_{m,n}(x)$ and $f(x)$. In the approach of Clenshaw and Lord [2] it is not clear how to obtain the (m,n) approximant if the Hankel matrix appearing in (4.8) is singular. However if the Hankel matrix is nonsingular, their approach is equivalent to obtaining the unique solution of (4.7) - (4.8) with $\beta_0 = 1$, yielding $R_{\ell,n}(w)$ in (4.3), and then defining $r_{m,n}(x)$ by (4.6).

In this case, we have satisfied conditions (4.4) - (4.5) with $q_n(0) = \beta_0 \neq 0$. Therefore $q_n^{-1}(w)$ exists as a power series and, multiplying through in equation (4.4), it follows that

$$G(w) - R_{\ell,n}(w) = 0\ (w^{m+n+1}).$$

Then, since the Chebyshev series coefficients of $f(x)$ and $r_{m,n}(x)$ are the power series coefficients of $G(w)$ and $R_{\ell,n}(w)$, respectively, we have agreement between the Chebyshev series expansions of $f(x)$ and $r_{m,n}(x)$ through $m+n+1$ terms as long as none of the poles of $R_{\ell,n}(w)$ lies in the closed unit disc (see section 6).

The results of Theorems 4.1 and 4.2 provide a precise computational definition of the (m,n) Chebyshev-Padé approximant $r_{m,n}(x)$ whether or not the Hankel matrix is nonsingular. For we simply take any nontrivial solution of (4.7) - (4.8), form the (unique) reduced representation of $R_{\ell,n}(w)$ defined by (4.3), and then define $r_{m,n}(x)$ by (4.6). In the next section, it is useful to think of this statement as the definition of the (m,n) Chebyshev-Padé approximant. Again we note that if $R_{\ell,n}(w)$ has poles which lie in the closed unit disc then a special situation exists.

5. Block Structure

We now examine the occurrence of identical Chebyshev-Padé approximants appearing in the Chebyshev-Padé table when singular Hankel matrices arise. Formally, the Chebyshev-Padé table for the function (2.1) is the doubly-infinite array with the entry in row m, column n being the (m,n) Chebyshev-Padé approximant $r_{m,n}(x)$, for $m \geq 0$ and $n \geq 0$. We define the d-table to be the corresponding doubly-infinite array of Hankel determinants, with the entry in row m, column n being $\det(H_{m,n})$ where $H_{m,n}$ is the Hankel matrix appearing in (4.8); by definition,

$$\det(H_{m,0}) = 1, \quad m \geq 0 .$$

(In [6], the table of determinants is called the c-table just as in the case of ordinary Padé approximation. However, for a given set of coefficients $\{c_k\}$, the c-table in the ordinary Padé sense is different than the d-table in the Chebyshev-Padé sense so a different terminology seems reasonable. Also, "d" is the first letter in "determinant".)

5.1. *An Example*

Consider the function

$$f(x) = \left(\frac{9703}{34000}+\frac{1333}{8160}x+\frac{2129}{51000}x^2+\frac{13}{3400}x^3+\frac{512}{6375}x^4+\frac{349}{12750}x^5-\frac{742}{6375}x^6 - \frac{464}{6375}x^7\right) / \left(1-\frac{28}{85}x-\frac{32}{85}x^2\right)$$

$$= \sum_{k=0}^{\infty}{}' c_k T_k(x)$$

with first ten Chebyshev series coefficients:

k	0	1	2	3	4	5	6	7	8	9
c_k	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{3}{80}$	$\frac{29}{2400}$	$\frac{1}{400}$	$\frac{1}{1600}$	$\frac{3}{6400}$	$\frac{1}{5120}$

The Chebyshev-Padé table for $f(x)$ is shown in Figure 1 and the d-table for $f(x)$ is shown in Figure 2.

This example exhibits a block structure of the same form as that which can appear in the ordinary Padé table (cf. [5]). Equal Chebyshev-Padé approximants are occurring in square blocks with the corresponding square block in the d-table containing zeros except in the left column and the top row. Also note that, for example, the Chebyshev series expansion of $r_{0,3}(x)$ is

$$r_{0,3}(x) = \frac{1}{2}(1)T_0(x) + \frac{1}{2}T_1(x) + \frac{1}{4}T_2(x) + \frac{1}{10}T_3(x) + \frac{3}{80}T_4(x) + \frac{29}{2400}T_5(x) + \frac{1}{300}T_6(x) + \frac{89}{144000}T_7(x) + \cdots$$

which agrees with the Chebyshev series expansion of $f(x)$ through the term $c_5T_5(x)$. Thus in the corresponding 3 by 3 block in the Chebyshev-Padé table, the approximants in positions (1,5), (2,4), and (2,5) agree with $f(x)$ through less than the expected $m+n+1$ terms; the approximant in position (1,4), which also corresponds to a single Hankel matrix, does have agreement with $f(x)$ through $m+n+1$ terms. This triangular nature is typical, with the lower right triangle in a (finite) square block containing "degenerate" approximants. Also note that if the function $f(x)$ is a rational function (as in this example), then there is an infinite block extending to the right and downward from the position containing the exact function.

m \ n	0	1	2	3	4	5	6	7	...
0	$r_{0,0}$	$r_{0,1}$	$r_{0,1}$	$r_{0,3}$	$r_{0,3}$	$r_{0,3}$	$r_{0,6}$	$r_{0,7}$	...
1	$r_{1,0}$	$r_{0,1}$	$r_{0,1}$	$r_{0,3}$	$r_{0,3}$	$r_{0,3}$	$r_{1,6}$	$r_{1,7}$	...
2	$r_{2,0}$	$r_{2,1}$	$r_{2,2}$	$r_{0,3}$	$r_{0,3}$	$r_{0,3}$	$r_{2,6}$	$r_{2,7}$	...
3	$r_{3,0}$	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$	$r_{3,4}$	$r_{3,5}$	$r_{3,6}$	$r_{3,7}$	...
4	$r_{4,0}$	$r_{4,1}$	$r_{4,2}$	$r_{4,3}$	$r_{4,4}$	$r_{4,5}$	$r_{4,6}$	$r_{4,7}$	...
5	$r_{5,0}$	$r_{5,1}$	$r_{5,2}$	$r_{5,3}$	$r_{5,4}$	$r_{5,5}$	$r_{5,6}$	$r_{5,7}$	...
6	$r_{6,0}$	$r_{6,1}$	$r_{6,2}$	$r_{6,3}$	$r_{6,4}$	$r_{6,5}$	$r_{6,6}$	$r_{6,7}$	...
7	$r_{7,0}$	$r_{7,1}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	...
8	$r_{8,0}$	$r_{8,1}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	$r_{7,2}$	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

where some of the approximants are:

$$r_{0,0}(x) = \frac{1}{2}T_0(x)$$
$$r_{0,1}(x) = \frac{3}{10}T_0(x)/(T_0(x) - \frac{4}{5}T_1(x))$$
$$r_{0,3}(x) = \frac{2697}{9010}T_0(x)/(T_0(x) - \frac{3544}{4505}T_1(x) - \frac{48}{901}T_2(x) + \frac{48}{901}T_3(x))$$
$$r_{1,0}(x) = \frac{1}{2}T_0(x) + \frac{1}{2}T_1(x)$$
$$r_{2,0}(x) = \frac{1}{2}T_0(x) + \frac{1}{2}T_1(x) + \frac{1}{4}T_2(x)$$
$$r_{2,1}(x) = (\frac{19}{58}T_0(x) + \frac{2}{29}T_1(x) + \frac{5}{116}T_2(x))/(T_0(x) - \frac{20}{29}T_1(x))$$
$$r_{2,2}(x) = (\frac{301}{1002}T_0(x) + \frac{20}{501}T_2(x))/(T_0(x) - \frac{140}{167}T_1(x) + \frac{40}{501}T_2(x))$$
$$r_{7,2}(x) = (\frac{17}{46}T_0(x) + \frac{61}{345}T_1(x) + \frac{11}{1380}T_2(x) - \frac{733}{41400}T_3(x) - \frac{601}{41400}T_4(x)$$
$$- \frac{17}{2208}T_5(x) - \frac{371}{82800}T_6(x) - \frac{29}{20700}T_7(x))/(T_0(x) - \frac{28}{69}T_1(x) - \frac{16}{69}T_2(x)) \ .$$

Figure 1: The Chebyshev-Padé table for f(x).

m \ n	0	1	2	3	4	5	6	7	...
0	1	1	$-\frac{3}{4}$	$-\frac{9}{16}$	$\frac{2697}{6400}$	$\frac{808201}{2560000}$	*	*	...
1	1	$\frac{1}{2}$	0	$\frac{3}{160}$	0	0	*	*	...
2	1	$\frac{1}{4}$	$-\frac{1}{80}$	$-\frac{1}{1600}$	0	0	*	*	...
3	1	$\frac{1}{10}$	$-\frac{1}{1600}$	$\frac{1}{48000}$	$\frac{1}{1920000}$	*	*	*	...
4	1	$\frac{3}{80}$	$-\frac{19}{96000}$	$-\frac{1}{5760000}$	*	*	*	*	...
5	1	$\frac{29}{2400}$	$-\frac{301}{5760000}$	*	*	*	*	*	...
6	1	$\frac{1}{400}$	$\frac{1}{768000}$	*	*	*	*	*	...
7	1	$\frac{1}{1600}$	$\frac{1}{1280000}$	*	*	*	*	*	...
8	1	$\frac{3}{6400}$	$-\frac{1}{10240000}$	0	0	0	0	0	...
9	1	$\frac{1}{5120}$	$\frac{1}{81920000}$	0	0	0	0	0	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

where * denotes a nonzero entry which is not entered for typographical reasons.

Figure 2: The d-table for f(x).

5.2 *Preliminaries to the Main Theorem*

There are several useful identities involving Hankel determinants which relate entries in the d-table. The most useful identity follows directly from Sylvester's determinant identity:

$$(5.1) \quad \det(H_{m,n})^2 = \det(H_{m-1,n})\det(H_{m+1,n}) - \det(H_{m,n-1})\det(H_{m,n+1}) \ , \quad m \geq 1,\ n \geq 1;$$

the corresponding identity when $m = 0$ is:

$$(5.2) \quad \det(H_{0,n})^2 = \det(H_{1,n})^2 - \det(H_{0,n-1})\det(H_{0,n+1}) \ , \quad n \geq 1.$$

Note that the first two columns in the d-table are given by

$$\det(H_{m,0}) = 1 \ , \quad m \geq 0 \ ;$$
$$\det(H_{m,1}) = c_m, \quad m \geq 0 \ .$$

The d-table can be built up iteratively in successive upward sloping diagonals using identities (5.1)-(5.2) as long as when computing $\det(H_{m,n+1})$ we do not have $\det(H_{m,n-1})$ equal to zero. (At this point, we should note that the determinants here are not identical with the determinants considered in [6]; they may differ by a sign because we have used a different permutation of rows and columns in defining the matrix $H_{m,n}$. Thus the identity corresponding to (5.1) in [6] has a plus sign where we have a minus sign. The symmetric matrix $H_{m,n}$ as we have defined it has computational advantages which can be exploited as in [4].)

Before stating and proving the Block Structure Theorem, we introduce some useful notational definitions and prove a lemma.

Definition 5.1

The <u>algebraic order</u> $\mathrm{ord}[S(w)]$ of a power series $S(w) = a_0 + a_1 w + \cdots$ is the smallest integer λ such that $a_\lambda \neq 0$. The <u>algebraic degree</u> $\deg[p(w)]$ of a polynomial $p(w) = a_0 + \cdots + a_n w^n$ is the largest integer λ such that $a_\lambda \neq 0$. The <u>Chebyshev order</u> $\mathrm{Cord}[C(x)]$ of a Chebyshev series $C(x) = \frac{1}{2}c_0 T_0(x) + c_1 T_1(x) + \cdots$ is the smallest integer λ such that $c_\lambda \neq 0$. The <u>Laurent degree</u> $\mathrm{Ldeg}[L(w)]$ of a finite symmetric Laurent series $L(w) = \alpha_0 + \alpha_1(w+w^{-1}) + \cdots + \alpha_n(w^n + w^{-n})$ is the largest integer λ such that $\alpha_\lambda \neq 0$.

Let us recall the computational method outlined in section 4 to compute the (m,n) Chebyshev-Padé approximant $r_{m,n}(x)$ for a function $f(x)$ with Chebyshev series coefficients $\{c_k\}$. The method is to define a power series $G(w)$ with coefficients $\{c_k\}$ and then to compute a rational function

$$R_{\ell,n}(w) = \frac{p_\ell(w)}{q_n(w)} = \frac{\alpha_0 + \alpha_1 w + \cdots + \alpha_\ell w^\ell}{\beta_0 + \beta_1 w + \cdots + \beta_n w^n}$$

(where $\ell = \max\{m,n\}$) which satisfies conditions (4.4)-(4.5), which can be stated in our new notation as the following two conditions:

$$(5.3) \qquad \mathrm{ord}[q_n(w)G(w) - p_\ell(w)] \geq m+n+1 \ ;$$

$$(5.4) \qquad \mathrm{Ldeg}[p_\ell(w)q_n(w^{-1}) + p_\ell(w^{-1})q_n(w)] \leq m \ .$$

Finally (assuming $R_{\ell,n}(w)$ has no poles in the closed unit disc),

$$(5.5) \qquad r_{m,n}(x) = [p_\ell(w)q_n(w^{-1}) + p_\ell(w^{-1})q_n(w)]/2q_n(w)q_n(w^{-1})$$

where we set $w = x + \sqrt{x^2-1}$. Condition (5.3) serves to guarantee that

$$\mathrm{Cord}[f(x) - r_{m,n}(x)]$$

is as large as possible and condition (5.4) serves to guarantee that $r_{m,n}(x)$ has a numerator of degree not exceeding m.

In order to prove the block structure properties of the Chebyshev-Padé table, it is necessary to make more precise statements about the cases when $q_n(0) = \beta_0 = 0$ (i.e. the cases when the Hankel matrix $H_{m,n}$ appearing in (4.8) is singular). Specifically, if the first λ coefficients of $q_n(w)$ are zero then Lemma 5.1 proves that the first λ coefficients of $p_\ell(w)$ are also zero and furthermore the right hand side of inequality (5.4) can be changed to $m-\lambda$ (which implies that the resulting Chebyshev-Padé approximant (5.5) will have a numerator of degree not greater than $m-\lambda$).

Lemma 5.1

For given integers m, n, let $\ell = \max\{m,n\}$, let $\{\alpha_k\}_0^\ell$, $\{\beta_k\}_0^n$ be a nontrivial solution of (4.7)-(4.8) and let $p_\ell(w)/q_n(w)$ be the rational function (4.3) which it defines. If $\mathrm{ord}[q_n(w)] = \lambda \geq 0$ then the following two conditions hold:

$$\text{(5.6)} \qquad \mathrm{ord}[p_\ell(w)] \geq \lambda \, ;$$

$$\text{(5.7)} \qquad \mathrm{Ldeg}[p_\ell(w)q_n(w^{-1}) + p_\ell(w^{-1})q_n(w)] \leq m-\lambda \, .$$

Proof: Equation (4.7) defines α_k in terms of $\beta_k, \beta_{k-1}, \ldots, \beta_0$ so clearly

$$\beta_k = 0 \text{ for all } k < \lambda \;\Rightarrow\; \alpha_k = 0 \text{ for all } k < \lambda \, .$$

This proves (5.6).

Turning to condition (5.7), a little algebra shows that the expression of interest takes the form

$$(5.8)\quad p_\ell(w)q_n(w^{-1}) + p_\ell(w^{-1})q_n(w)$$
$$= 2\sum_{j=0}^{\ell} \alpha_j\beta_j + \sum_{k=1}^{n}\left(\sum_{j=0}^{n-k}\alpha_j\beta_{k+j} + \sum_{j=0}^{\ell-k}\alpha_{k+j}\beta_j\right)(w^k+w^{-k})$$
$$+ \sum_{k=n+1}^{\ell}\left(\sum_{j=0}^{\ell-k}\alpha_{k+j}\beta_j\right)(w^k+w^{-k})$$

(with the usual convention that $\beta_i = 0$ for $i > n$). Now if $\lambda = 0$ then (5.7) is simply (5.4) and there is nothing to prove. So assume $0 < \lambda \le n$. Then in particular $\beta_0 = 0$, which means that the system of equations (4.8) can be written as

$$\sum_{j=0}^{n} c_{|k-j|}\beta_j = 0 \ , \qquad m+1 \le k \le m+n \ .$$

Breaking up this sum to eliminate the absolute-value sign yields

$$\sum_{j=0}^{k} c_{k-j}\beta_j + \sum_{j=k+1}^{n} c_{-k+j}\beta_j = 0$$

or, upon further transformation,

$$\sum_{j=0}^{k}{}' c_j\beta_{k-j} = -\sum_{j=0}^{n-k}{}' c_j\beta_{k+j} \ .$$

But from (4.7) we see that the left side here is precisely the expression defining α_k so we have the new relationship:

$$(5.9)\quad \alpha_k = -\sum_{j=0}^{n-k}{}' c_j\beta_{k+j} \ , \qquad m+1 \le k \le \ell \ ,$$

valid when $\lambda > 0$.

Now let us consider two separate cases.

Case $\ell = m \geq n$:

In this case we must show that in expression (5.8) the coefficient of (w^k+w^{-k}) is zero for $m-\lambda < k \leq m = \ell$. Firstly if $m-\lambda \geq n$ then in the last term of (5.8) we see that for $k > m-\lambda$,

$$\sum_{j=0}^{\ell-k} \alpha_{k+j}\beta_j = 0$$

because $\beta_j = 0$ for $0 \leq j \leq \ell-k = m-k < \lambda$ and we are done. Secondly if $m-\lambda < n$ then the last term of (5.8) vanishes as above. In the second term of (5.8) we see that for $k > m-\lambda$,

$$\sum_{j=0}^{\ell-k} \alpha_{k+j}\beta_j = 0$$

exactly as above and similarly, making use of condition (5.6), we see that for $k > m-\lambda$,

$$\sum_{j=0}^{n-k} \alpha_j\beta_{k+j} = 0$$

because $\alpha_j = 0$ for $0 \leq j \leq n-k \leq m-k < \lambda$. Therefore (5.7) is proved in case $\ell = m \geq n$.

Case $\ell = n > m$:

In this case the last term of (5.8) is null so it remains to show that in the second term of (5.8) the coefficient of (w^k+w^{-k}) is zero for $m-\lambda < k \leq n$. The expression consists of two terms which in this case take the form

$$(5.10) \qquad \sum_{j=\lambda}^{n-k} \alpha_j\beta_{k+j} + \sum_{j=\lambda}^{n-k} \alpha_{k+j}\beta_j$$

where we have used the fact (from (5.6)) that

$$\alpha_j = \beta_j = 0 \quad \text{for} \quad 0 \le j < \lambda .$$

This time the two terms do not individually vanish but we can show that one term in (5.10) is precisely the negative of the other term in (5.10) when $k > m-\lambda$. To prove this we use the fact that the α_i's are functionally dependent on the β_i's as specified by equations (4.7) and also, for a restricted range of subscripts, (5.9). Noting that in the second term of (5.10) if $k > m-\lambda$ then the subscript range for α_{k+j} satisfies

$$m+1 \le k+j \le n = \ell ,$$

we see that it is valid to use equation (5.9) in that term. Using also equation (4.7) in the first term, expression (5.10) becomes

$$\sum_{j=\lambda}^{n-k}\left(\sum_{i=0}^{j}{}' c_i\beta_{j-i}\right)\beta_{k+j} - \sum_{j=\lambda}^{n-k}\left(\sum_{i=0}^{n-k-j}{}' c_i\beta_{k+j+i}\right)\beta_j$$

for k lying in the range $m-\lambda < k \le n$. Finally by expanding the expressions on the left and on the right of the minus sign, we find that the two expressions are identical and therefore the entire expression is zero, yielding the desired result.

5.3 *The Main Theorem*

We are now ready to state and prove the Block Structure Theorem for the Chebyshev-Padé table. Noting that Theorem 5.1 applies only for Chebyshev series with nonzero constant term c_0, we make the following practical observation. If $c_0 = 0$ in the Chebyshev series expansion of a function $f(x)$ then $f(x)$ must have at least one zero in the interval $[-1,1]$ (see [1], p. 110, Theorem 5). As a numerical approximation, the (truncated) Chebyshev series will have a large relative error in a neighbourhood of any point where $f(x)$ vanishes and therefore all zeros of $f(x)$ should be divided out before applying this method of approximation (see [3], pp. 116-118).

<u>Theorem 5.1</u> (Block Structure Theorem)

Let $f(x) = \sum_{k=0}^{\infty}{}' c_k T_k(x)$ be a Chebyshev series with $c_0 \neq 0$. Let $u_s(x)/v_t(x)$ be a Chebyshev-Padé approximant for $f(x)$ defined as in (4.6) from the unique reduced form of $R_{\ell,n}(w)$ in (4.3) determined from a nontrivial solution of (4.7)-(4.8) for some particular choice of m, n. (We are assuming that for these particular values m, n the rational function $R_{\ell,n}(x)$ has no poles lying in the closed unit disc.) Suppose that

(5.11) $\deg[u_s(x)] = s \; ; \quad \deg[v_t(x)] = t \; ;$

(5.12) $\mathrm{Cord}[f(x) - u_s(x)/v_t(x)] = s + t + k + 1$

(where $k = \infty$ is allowed). Then the following statements are true:

(i) $k \geq 0$;

(ii)
- $\det(H_{s,n}) \neq 0$ for $t \leq n \leq t+k$;
- $\det(H_{m,t}) \neq 0$ for $s \leq m \leq s+k$;
- $\det(H_{m,n}) = 0$ for $(s < m \leq s+k$ and $t < n \leq t+k)$;

(iii)
- $r_{m,n}(x) = u_s(x)/v_t(x)$ for $(s \leq m \leq s+k$ and $t \leq n \leq t+k)$;
- $r_{m,n}(x) \neq u_s(x)/v_t(x)$ for $(m < s$ or $n < t)$;
- $r_{m,n}(x) \neq u_s(x)/v_t(x)$ for $(m > s+k$ or $n > t+k)$ except possibly in the anomolous cases (see section 6) where the rational function $R_{\ell,n}(w)$ in (4.3) determined from a nontrivial solution of (4.7)-(4.8) has one or more poles lying in the closed unit disc (in which case $r_{m,n}(x)$ is not defined from $R_{\ell,n}(w)$ as in (4.6)).

Proof: To determine the (m,n) Chebyshev-Padé approximant $r_{m,n}(x)$ for $f(x)$, we consider the power series

$$G(w) = \sum_{k=0}^{\infty}{}' c_k w^k .$$

Let $\ell = \max\{m,n\}$ and let $p_\ell(w)/q_n(w)$ be an (ℓ,n) rational function (4.3) determined from a nontrivial solution of (4.7)-(4.8). Let $\lambda = \mathrm{ord}[q_n(w)]$. Suppose that the reduced form of $R_{\ell,n}(w) = p_\ell(w)/q_n(w)$ is $a_\sigma(w)/b_t(w)$ with

$$(5.13) \quad \deg[a_\sigma(w)] = \sigma; \quad \deg[b_t(w)] = t;$$

and with $b_t(0) = 1$. Now $q_n(w) \neq 0$. Also, $p_\ell(w) \neq 0$ since we must have $R_{\ell,n}(0) = \frac{1}{2}c_0 \neq 0$. Hence $\gcd(p_\ell(w), q_n(w))$ exists, and factoring out this greatest common divisor yields

$$(5.14) \quad p_\ell(w) = w^\lambda(d_0 + d_1 w + \cdots + d_\eta w^\eta) a_\sigma(w)$$

$$(5.15) \quad q_n(w) = w^\lambda(d_0 + d_1 w + \cdots + d_\eta w^\eta) b_t(w)$$

with $d_0 d_\eta \neq 0$. Denote the middle factor by $d(w)$. Conditions (5.3) and (5.7) now take the form

$$(5.16) \quad \mathrm{ord}[w^\lambda d(w) b_t(w) G(w) - w^\lambda d(w) a_\sigma(w)] \geq m + n + 1$$

$$(5.17) \quad \mathrm{Ldeg}[d(w)d(w^{-1})\{a_\sigma(w) b_t(w^{-1}) + a_\sigma(w^{-1}) b_t(w)\}] \leq m - \lambda .$$

Now $d^{-1}(w)$ exists as a power series (since $d_0 \neq 0$) and multiplying through in the expression in (5.16) yields

$$(5.18) \quad \mathrm{ord}[w^\lambda\{b_t(w)G(w) - a_\sigma(w)\}] \geq m + n + 1 .$$

Also, since $\mathrm{Ldeg}[d(w)d(w^{-1})] = \eta$, (5.17) becomes

$$(5.19)\quad \mathrm{Ldeg}[a_\sigma(w)b_t(w^{-1}) + a_\sigma(w^{-1})b_t(w)] \le m - \lambda - \eta \ .$$

Now if $r_{m,n}(x) = u_s(x)/v_t(x)$ then (except for possible anomalous cases as noted in the statement of the theorem) it must be derived as in (4.6) from $R_{\ell,n}(w) = p_\ell(w)/q_n(w)$ with reduced form $a_\sigma(w)/b_t(w)$ satisfying the above properties. Note also that $\sigma \le \max\{s,t\}$. (For to obtain $r_{s,t}(x)$ we would set $\ell = \max\{s,t\}$ and form the rational function $R_{\ell,t}(w)$ which must have reduced form $a_\sigma(w)/b_t(w)$, whence $\sigma \le \ell$.) From (5.12) it follows that

$$(5.20)\quad \mathrm{ord}[G(w) - a_\sigma(w)/b_t(w)] = s + t + k + 1$$

since the power series coefficients of $G(w)$ and $a_\sigma(w)/b_t(w)$ are precisely the Chebyshev series coefficients of $f(x)$ and $u_s(x)/v_t(x)$, respectively. Hence, since $b_t(0) \ne 0$, (5.20) yields

$$(5.21)\quad \mathrm{ord}[b_t(w)G(w) - a_\sigma(w)] = s + t + k + 1 \ .$$

(5.18) then yields

$$(5.22)\quad \lambda + s + t + k + 1 \ge m + n + 1 \ .$$

Also, since the expression in (5.19) becomes $u_s(x)$ when (4.6) is applied, (5.19) yields

$$(5.23)\quad s \le m - \lambda - \eta \ .$$

We have established the following inequalities:

$\eta \ge 0$; $\lambda \ge 0$; by definition in (5.14)-(5.15)

$\lambda + \eta + \sigma \le \ell$; $\lambda + \eta + t \le n$; from (5.14)-(5.15)

$m + n \le \lambda + s + t + k$; from (5.22)

$\lambda + \eta \le m - s$; from (5.23)

or equivalently,

$$\eta \geq 0;$$
$$\lambda \geq \max\{0, (m-s) + (n-t) - k\};$$
$$\eta + \lambda \leq \min\{m-s, \ell-\sigma, n-t\}.$$

Finally, recalling that $\ell = \max\{m,n\}$ and using the fact that $\sigma \leq \max\{s,t\}$ (as noted above) we see that $\ell - \sigma \geq \min\{m-s,n-t\}$ so that the final inequality above simplifies. We have:

$$(5.24) \quad \begin{cases} \eta \geq 0; \\ \lambda \geq \max\{0, (m-s) + (n-t) - k\}; \\ \eta + \lambda \leq \min\{m-s, n-t\}. \end{cases}$$

Also, by Theorem 4.1, if $\lambda > 0$ in (5.24) then we must have $\det(H_{m,n}) = 0$.

Conversely, given $r_{s,t}(x)$ if m, n are integers such that (5.24) has a solution (η,λ) and if $\det(H_{m,n}) = 0$ in case $\lambda > 0$ then $r_{m,n}(x) = r_{s,t}(x)$. For by defining $p_\ell(w)$, $q_n(w)$ by (5.14)-(5.15), for arbitrary $d(z) = d_0 + \cdots + d_\eta w^\eta$ with $d_0 d_\eta \neq 0$, we have an (ℓ,n) rational function satisfying (4.4)-(4.5) and by Theorem 4.1, (4.7)-(4.8) also holds.

Thus the question of whether $r_{m,n}(x) = r_{s,t}(x)$ is essentially reduced to the question of whether (5.24) has a solution (η,λ) with $\det(H_{m,n}) = 0$ in case $\lambda > 0$.

Proof of Property (i)

Since $r_{s,t}(x)$ is a Chebyshev-Padé approximant, there exist integers satisfying (5.24). Hence,

$$k \geq (m-s) + (n-t) - \lambda \geq (\eta+\lambda) + (\eta+\lambda) - \lambda = 2\eta + \lambda \geq 0 .$$

Proof of Property (ii)

First note that (5.12) immediately implies that $r_{s,n}(x) = r_{s,t}(x)$ for $t \leq n \leq t+k$ and $r_{m,t}(x) = r_{s,t}(x)$ for $s \leq m \leq s+k$ (i.e. $a_\sigma(w)/b_t(w)$ provides a solution of (4.7)-(4.8) as long as $s+t \leq m+n \leq s+t+k$). Now if $\det(H_{m,n}) = 0$ for these ranges of m, n then (4.8) has a nontrivial solution with $\beta_0 = 0$, whence (4.4)-(4.5) has a solution with $\lambda > 0$ in (5.14)-(5.15), whence (5.24) implies that $\min\{m-s, n-t\} > 0$, whence $m > s$ and $n > t$. Thus we conclude that

$$\det(H_{s,n}) \neq 0 \quad \text{for} \quad t \leq n \leq t+k;$$

$$\det(H_{m,t}) \neq 0 \quad \text{for} \quad s \leq m \leq s+k.$$

Let $\underline{\beta}$ be the (unique) solution of (4.8) with $\beta_0 = 1$ when $m = s$, $n = t$:

$$(5.25) \quad H_{s,t}\underline{\beta} = -\underline{h}_{s,t} \; .$$

Let $H_{s,t}^{(k)}$ denote the $(t+k)$ by t matrix formed by adding k rows to $H_{s,t}$ in the "natural" way:

$$H_{s,t}^{(k)} = \begin{bmatrix} c_{s-t+1} & \cdots & c_s \\ \vdots & & \vdots \\ c_s & \cdots & c_{s+t-1} \\ c_{s+1} & \cdots & c_{s+t} \\ \vdots & & \vdots \\ c_{s+k} & \cdots & c_{s+t+k-1} \end{bmatrix} .$$

Now (5.12), or equivalently (5.20), implies that the solution vector $\underline{\beta}$ in (5.25) satisfies k equations in addition to the t equations in (5.25); i.e.

$$H_{s,t}^{(k)}\underline{\beta} = -\underline{h}_{s,t+k} \; .$$

Thus the vector $\underline{h}_{s,t+i}$ is a linear combination of the column vectors of $H^{(i)}_{s,t}$, for $0 \leq i \leq k$. But if $i > 0$, the $(t+1)^{st}$ column vector in the matrix $H_{s+i,t+i}$ is precisely the vector $\underline{h}_{s,t+i}$, while the first t column vectors in $H_{s+i,t+i}$ form the matrix $H^{(i)}_{s,t}$. Thus there is a linear dependency in the column vectors of $H_{s+i,t+i}$ for $0 < i \leq k$; i.e.

$$\det(H_{s+i,t+i}) = 0 \quad \text{for} \quad 0 < i \leq k \ .$$

We now have the following structure in the d-table:

$$\begin{array}{cc|cccc} & t & \multicolumn{4}{c}{\overbrace{\hphantom{* \ * \ \cdots \ *}}^{k}} \\ s & * & * & * & \cdots & * \\ \hline \multirow{4}{*}{$k\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right.$} & * & 0 & & & \\ & * & & 0 & & \\ & \vdots & & & \ddots & \\ & * & & & & 0 \end{array} \qquad (* \neq 0)$$

A straightforward application of identity (5.1) to successive sub-diagonals and super-diagonals in this block now shows that

$$\det(H_{m,n}) = 0 \quad \text{for} \quad s < m \leq s+k \text{ and } t < n \leq t+k \ .$$

Proof of Property (iii)

If we consider the standard cases where for given m, n, a nontrivial solution of (4.7)-(4.8) defines a rational function $R_{\ell,n}(w)$ as in (4.3) which has no poles lying in the closed unit disc, then for these cases the (m,n) Chebyshev-Padé approximant $r_{m,n}(x)$ is defined from $R_{\ell,n}(w)$ as specified by (4.6). In such cases it follows that $r_{m,n}(x) = r_{s,t}(x)$ if and only if (5.24) has a solution (η,λ) with $\det(H_{m,n}) = 0$ in case $\lambda > 0$. Now (5.24) has a solution (η,λ) if and only if

$$\max\{0,(m-s)+(n-t)-k\} \leq \min\{m-s,n-t\} \ ;$$

that is, if and only if $s \leq m \leq s+k$ and $t \leq n \leq t+k$. Also, from (5.24), we can have $\lambda > 0$ only if $\min\{m-s, n-t\} > 0$; i.e. only if $m > s$ and $n > t$, in which case $\det(H_{m,n}) = 0$ by property (ii).

Thus the first statement of property (iii) is proved without exception. The second and third statements of property (iii) are also proved for the standard cases. It remains only to note that the exception need not attach to the second statement since by (5.11) $u_s(x)/v_t(x)$ is not an (m,n) rational function if $m < s$ or $n < t$.

6. The Case of Non-Uniqueness

We now turn finally to a brief discussion of an anomaly which can occur in the Chebyshev-Padé table. It has been noted several times in the preceding sections that in defining the Chebyshev-Padé approximant $r_{m,n}(x)$ by (4.6), it is necessary to assume that none of the poles of the rational function $R_{\ell,n}(w)$ lies in the closed unit disc. The need for this assumption is evident in the proof of Theorem 3.1. Now since the given Chebyshev series (2.1) represents a function $f(x)$ in the space $C[-1,1]$, it follows that

$$\limsup_{k\to\infty} \sqrt[k]{|c_k|} = \rho < 1$$

and therefore the power series (3.1) represents a function $G(w)$ which has no poles lying in the closed unit disc. The rational function $R_{\ell,n}(w)$ in (4.3) will therefore have no poles lying in the closed unit disc for $m+n$ large enough, since the poles of $R_{\ell,n}(w)$ will converge to the poles of $G(w)$ by virtue of the "Padé property" (4.4).

The anomaly which can occur is that for a particular Chebyshev series and for particular values of m and n, it may happen that the rational function $R_{\ell,n}(w)$ in (4.3) has one or more poles lying in the closed unit disc. In such a case the definition (4.6) of $r_{m,n}(x)$ breaks down, in the sense that $r_{m,n}(x)$ will not have a valid Chebyshev series expansion agreeing with the function $f(x)$ as desired. In such a case, there exists no (m,n) rational function having a valid Chebyshev series expansion agreeing with the Chebyshev series expansion of $f(x)$ through $m+n+1$ terms.

The nature of this anomaly can be best appreciated by appealing to an example. Clenshaw and Lord [2] give some numerical results for the function $\Gamma(1+x)$, $x \in [0,1]$, or equivalently, the function

$$f(x) = \Gamma((3+x)/2), \quad x \in [-1,1]$$
$$= \sum_{k=0}^{\infty}{}' c_k T_k(x) .$$

The first six Chebyshev series coefficients for $f(x)$ are (to five decimal places):

k	0	1	2	3	4	5
c_k	1.88357	.00442	.05685	-.00422	.00133	-.00019

The d-table for $f(x)$ has no zero entries and each entry in the Chebyshev-Padé table is normal (i.e. $r_{m,n}(x)$ agrees with $f(x)$ through exactly $m+n+1$ terms of the Chebyshev series) except for the $(1,1)$ entry. Clenshaw and Lord refer to the $(1,1)$ entry as "nonexistent" but we see below that in the context of the preceding sections it should be referred to as "nonunique".

The $(0,0)$, $(1,0)$, and $(0,1)$ entries can be found to be:

$$r_{0,0}(x) = .94179\ T_o(x);$$

$$r_{1,0}(x) = .94179\ T_o(x) + .00442\ T_1(x);$$

$$r_{0,1}(x) = .94177\ T_o(x)/(T_o(x) - .00470\ T_1(x)).$$

To compute the $(1,1)$ entry we solve the linear systems (4.7)-(4.8), yielding the rational function (4.3) as

$$R_{1,1}(w) = (.94179 - 12.109w)/(1 - 12.862w) .$$

$R_{1,1}(w)$ has a pole at $w = .07775$ which lies in the unit disc and therefore (4.6) will not yield a rational function with the desired property. Note that $R_{1,1}(w)$ is the correct $(1,1)$ Padé approximant for the power series with coefficients $c_0/2, c_1, c_2, \ldots,$ since for Padé approximation the location of the poles is irrelevant. If we "blindly" form $r_{1,1}(x)$ by (4.6) in the usual manner, we obtain the rational function

$$r_{1,1}(x) = (.94146T_0(x) - .14554T_1(x))/(T_0(x) - .15456T_1(x)) .$$

Note that $r_{1,1}(x)$ itself does not have a pole in the interval $[-1,1]$. However, the Chebyshev series expansion of $r_{1,1}(x)$ is of the form

$$r_{1,1}(x) = \frac{1}{2}(1.88292)T_0(x) - .00003T_1(x) - .000002T_2(x) - .0000002T_3(x) - \cdots$$

which certainly does not agree with the Chebyshev series expansion of $f(x)$ through the desired three terms. Clenshaw and Lord suggest that the "nonexistence" of the $(1,1)$ approximant in this situation can be detected by the presence of a pole of $r_{1,1}(x)$ in the interval $[-1,1]$ and hence by the divergence of the Chebyshev series expansion of $r_{1,1}(x)$. But Clenshaw and Lord's method produces the same rational function $r_{1,1}(x)$ as above and it clearly has no such pole. The confusion arises because the practical method given by Clenshaw and Lord for estimating the error in a Chebyshev-Padé approximant $r_{m,n}(x)$ requires the computation of the Chebyshev series coefficients of $r_{m,n}(x)$ but their method actually computes the power series coefficients of $R_{\ell,n}(w)$. If $R_{\ell,n}(w)$ has none of its poles lying in the closed unit disc then the two sets of coefficients are identical; otherwise, the computed coefficients (the power series coefficients of $R_{\ell,n}(w)$) do indeed show divergence (i.e. numerical growth in the size of the computed coefficients).

For the function $f(x)$ above, there exists no $(1,1)$ rational function having a Chebyshev series expansion agreeing with $f(x)$ through the term $c_2T_2(x)$. If the (m,n) Chebyshev-Padé approximant is defined to be the (m,n) rational function whose Chebyshev series agrees with $f(x)$ through $m+n+1$ terms then we would, with Clenshaw and Lord, refer to the $(1,1)$ approximant for the above function as "nonexistent". However we have seen in previous sections that it is more appropriate to define the (m,n) Chebyshev-Padé approximant to be the (m,n) rational function whose Chebyshev series has maximal initial agreement with $f(x)$. Thus for the function considered at the beginning of section 5, the (2,5) Chebyshev-Padé approximant is "nonexistent" in the sense of Clenshaw and Lord, but by our definition the (2,5) approximant is the rational function $r_{0,3}(x)$ of that section. The "maximal initial agreement" in that case is through the term $c_5T_5(x)$ (i.e. through six terms). For the gamma function of this section, the maximal initial agreement that can be realized by a (1,1) rational function is through two terms and this agreement is provided by either of the approximants $r_{1,0}(x)$ or $r_{0,1}(x)$ listed earlier in this section. Thus the (1,1) Chebyshev-Padé approximant for this function is not unique. In view of Theorem 4.2, we are guaranteed that the <u>only</u> situation where the (m,n) Chebyshev-Padé approximant is not unique is the case when $R_{\ell,n}(w)$ has one or more poles lying in the closed unit disc.

7. Conclusion

The problem of obtaining Chebyshev-Padé approximants for a given Chebyshev series expansion has been viewed in a context that directly relates the problem to Padé approximation of a corresponding power series expansion. It was shown that the occurrence of singular Hankel matrices in the formulas for computing Chebyshev-Padé approximants gives rise to a block structure in the Chebyshev-Padé table, such that the approximant corresponding to a singular Hankel matrix is identical to an approximant corresponding to a nonsingular Hankel matrix. Also, an anomaly of "nonexistence" mentioned in the work of Clenshaw and Lord was clarified by relating it to the location of the poles of the intermediate "Padé-type" approximant which arises.

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