SIMULATION RELATION OF DYNAMICAL SYSTEMS*

by

Youichi Kobuchi†

and

Karel Culik II§

Research Report CS-80-06

Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada
January 1980

† On leave from Department of Biophysics
Kyoto University
Kyoto, Japan

§ Department of Computer Science
University of Waterloo
Waterloo, Ontario, Canada

* This research was supported by the Natural Sciences and Engineering Council of Canada, Grant No. A7403.

Simulation Relation of Dynamical Systems

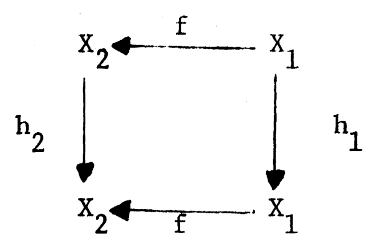
Youichi Kobuchi* and Karel Culik II

Department of Computer Science, University of Waterloo

Waterloo, Ontario, Canada

§1 Introduction

Let $<X,h>$ be a dynamical system where X is the state space and h is the mapping from X into itself. Given two such systems $<X_1,h_1>$ and $<X_2,h_2>$, we say that $<X_2,h_2>$ simulates $<X_1,h_1>$ if the dynamics of $<X_1,h_1>$ may be completely represented by a part of the dynamics of $<X_2,h_2>$. That is, we assume the existence of an injective mapping f from X_1 into X_2 such that the following diagram commutes:



In this framework we can investigate the fundamental properties of the simulation relation, for example, through its decomposition. Here the decomposition means to express a given simulation relation as a certain composition of simpler relations.

To illustrate our general idea on a concrete example, we will consider the systems where the state space is a finitely generated free monoid and the mapping is defined by a deterministic generalized sequential machine (dgsm). [1] We also assume that the injective mapping f which

---

* On leave from Department of Biophysics, Kyoto University, Kyoto, Japan.

relates the two systems under consideration is realized by a dgsm. Especially we investigate in some detail the cases where there exists a dgsm mapping $\tilde{f}$ such that $\tilde{f}$ is an inverse of f.

Analysis of simulation relation as reported here started in [2] where the mappings are restricted to be defined by monoid homomorphisms. The results of this report generalize those in [2] and add more, establishing the fundamental relationships between the decompositions of the systems and the simulation relation or its decomposition.

§2 Definitions and Preliminary Results

In what follows, capital Greek letters designate finite sets.

Let Σ be a finite set and Σ^* be the monoid generated by Σ under the operation of concatenation. Let $\Sigma^+ = \Sigma^* - \{\varepsilon\}$ where ε is the identity element of Σ^*. When Σ is partitioned into n mutually disjoint subsets $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, we write $\Sigma = \Sigma_1 \oplus \Sigma_2 \oplus \ldots \oplus \Sigma_n$.

Def. 1 Let $G(\Sigma,\Delta)$ denote the set of all dgsm mappings $h: \Sigma^* \to \Delta^*$ such that for each $a \in \Delta$ there exists $u \in \Sigma^+$ for which a appears in $h(u)$. $H(\Sigma,\Delta)$ is the subset of $G(\Sigma,\Delta)$ composed of one state dgsm mappings (i.e. monoid homomorphisms).

Def. 2 We say that $h \in G(\Sigma,\Delta)(H(\Sigma,\Delta))$ has a decomposition fg if $h(u) = f(g(u))$ for any $u \in \Sigma^*$ such that $f \in G(\Theta,\Delta)(H(\Theta,\Delta))$ and $g \in G(\Sigma,\Theta)(H(\Sigma,\Theta))$ for some alphabet Θ.

The following lemmas are easy to prove.

Lemma 1 Let Θ be a binary alphabet. Any $h \in G(\Sigma,\Delta)$ has a decomposition fg such that $f \in G(\Theta,\Delta)$ is onto Δ^* and $g \in G(\Sigma,\Theta)$.

Lemma 2 Given $h \in G(\Sigma,\Delta)$, we can have a decomposition fg such that $f \in H(\Theta,\Delta)$ and $g \in G(\Sigma,\Theta)$, where g is length preserving.

It is useful to note that a decomposition fg of $h \in H(\Sigma,\Delta)$ signifies nothing but a way of expressing $h(a)$'s as elements in $\{f(A) / A \in \Delta\}^*$ where $a \in \Sigma$. [2,3]

In this report, we restrict our attention to dynamical systems $\langle\Sigma^*,h\rangle$ where Σ is an arbitrary finite alphabet and $h \in G(\Sigma,\Sigma)$. (Note that if $h \in H(\Sigma,\Sigma)$, then the system $\langle\Sigma^*,h\rangle$ is but a DOL scheme [4]).

Def. 3 Given two systems $\langle\Sigma^*,h\rangle$ and $\langle\Delta^*,g\rangle$, we say that $\langle\Sigma^*,h\rangle$ simulates $\langle\Delta^*,g\rangle$ (or simply, h simulates g) if there exists

$f \in G(\Delta,\Sigma)$ which is injective such that $hf = fg$. We write this relation symbolically as $h \overset{f}{\leftarrow} g$ or $g \underset{f}{\rightarrow} h$.

Note that it is decidable whether a given gsm mapping is injective by Theorem 5 in [5].

Some of the basic properties of the simulation relation are listed. The domain and the range alphabets of each mapping are assumed to be given appropriately. In the list below, lower case Greek letters denote permutations over some finite sets.

(P. 1) a) $h \overset{\mu}{\leftarrow} \mu^{-1}h\mu$

b) $h \overset{f}{\leftarrow} g \Leftrightarrow \mu^{-1}h\mu \xleftarrow{\mu^{-1}f\pi} \pi^{-1}g\pi.$

By virtue of this, we may disregard the renaming of the alphabets in considering the simulation relation.

(P. 2) a) $h \xleftarrow{f_1} g \xleftarrow{f_2} k \Rightarrow h \xleftarrow{f_1 f_2} k$

b) Let λ be an identity mapping.

$h \overset{\lambda}{\leftarrow} h \overset{f}{\leftarrow} g \Leftrightarrow h \overset{f}{\leftarrow} g \Leftrightarrow h \overset{f}{\leftarrow} g \overset{\lambda}{\leftarrow} g.$

These properties show that the set of systems under consideration makes a 'category' where the injective dgsm mappings are the morphisms.

(P. 3) Let n be an arbitrary positive integer.

a) $h \overset{f}{\leftarrow} g \Rightarrow h^n \overset{f}{\leftarrow} g^n$

b) When h is injective,

$h \overset{f}{\leftarrow} g \Leftrightarrow h \xleftarrow{h^n f} g$

c) When g is injective,

$h \overset{f}{\leftarrow} g \Rightarrow h \xleftarrow{fg^n} g$

(P. 4) Let $h \in G(\Sigma,\Sigma)$ and $\bar{h} \in G(\Delta,\Delta)$ be injective mappings.

$h = fg,\ \bar{h} = gf \Rightarrow h \underset{g}{\overset{f}{\leftrightarrows}} \bar{h}$

This is the fundamental relationship between the decomposition of injective systems and their simulation relation.

(P. 5) Given an injective $g \in G(\Sigma,\Sigma)$, we have a binary alphabet Θ, $h \in G(\Theta,\Theta)$ and $f \in H(\Sigma,\Theta)$ such that $h \overset{f}{\leftarrow} g$.

By Lemma 1, we have $g = pf$ for some $f \in G(\Sigma,\Theta)$ and $p \in G(\Theta,\Sigma)$. Then $fp \overset{f}{\leftarrow} pf$. Put $h = fp \in G(\Theta,\Theta)$.

(P. 4) and (P. 5) make the following definition meaningful.

Def. 4 Consider a simulation relation $h \overset{f}{\leftarrow} g$. If $h = fp$ for some dgsm mapping p, we call the simulation primitive.

Note that if $h \overset{f}{\leftarrow} g$ and $h = fp$, then $g = pf$.

§3 Decomposition of Simulation Relation

In this section we show two ways of decomposition of simulation relation: parallel and series. Before going into the details, we need some definitions. Let $\Sigma = \Sigma_1 \oplus \ldots \oplus \Sigma_n$. For any $x \in \Sigma^*$, $x|_{\Sigma_i}$ is the longest sparse subword of x which is entirely composed of the elements in Σ_i. That is, if $x = a_1b_1a_2b_2 \ldots a_kb_k$ where $a_j \in \Sigma_i^*$ and $b_j \in (\Sigma-\Sigma_i)^*$ for $j = 1,2,\ldots,k$, then $x|_{\Sigma_i} = a_1a_2\ldots a_k$.

<u>Def. 5</u> Let $\Sigma = \Sigma_1 \oplus \ldots \oplus \Sigma_n$, $\Delta = \Delta_1 \oplus \ldots \oplus \Delta_n$, and let π be a permutation over $\{1,2,\ldots,n\}$. Given n dgsm mappings $h_i \in G(\Sigma_i,\Delta_{\pi(i)})$ $(i=1,2,\ldots,n)$, we define a dgsm mapping $h \in G(\Sigma,\Delta)$ out of h_i's as follows:

(i) $h(a) = h_i(a)$ if $a \in \Sigma_i$, and

(ii) for any $x \in \Sigma^+$ and any $a \in \Sigma_i$

$$h(xa) = h(x)(h_i(x|_{\Sigma_i}))^{-1}h_i(xa|_{\Sigma_i}).$$

We denote $h = h_1 \otimes \ldots \otimes h_n(\pi)$. In case π is the identity mapping, we also write $h = h_1 \oplus \ldots \oplus h_n$.

<u>Lemma 3</u> Let $f = f_1 \otimes \ldots \otimes f_n(\pi)$ as defined in Def. 5. Then f is injective if and only if all the f_i's $(i=1,2,\ldots,n)$ are injective.

<u>Proof</u> Let f be not injective. There exist distinct $x, x' \in \Sigma^*$ such that $f(x) = f(x') = y$. When $y = \varepsilon$, there exist some f_i and $\tilde{x} \in \Sigma_i^+$ such that $f_i(\tilde{x}) = \varepsilon$, which implies f_i is not injective. When $y \neq \varepsilon$, let $y = y_1y_2\ldots y_n$ and assume $y_1 \in \Delta_i$ for some i. Then we can write $x = x_1w$ and $x' = x_1'w'$ where x_1 and x_1' are the longest prefix in Σ_i^* of x and x', respectively. As $f_i(x) = f_i(x_1')$, either $x_1 = x_1'$ or f_i is not injective. When $x_1 = x_1'$, repeat the process using w and w' instead of x and x'. It finally terminates and therefore some f_i is not injective. Conversely, if some f_i is not injective, then clearly f is not injective.

3.1. Parallel Decomposition

Theorem 1 Let $\Sigma = \Sigma_1 \oplus \dots \oplus \Sigma_n$ and $\Delta = \Delta_1 \oplus \dots \oplus \Delta_n$. Assume $h = h_1 \otimes \dots \otimes h_n (\pi)$ and $f = f_1 \oplus \dots \oplus f_n$ where $h \in G(\Sigma,\Sigma)$ and $f \in G(\Delta,\Sigma)$. Then we have for $g \in G(\Delta,\Delta)$

$$hf = fg \Leftrightarrow \begin{matrix} g = g_1 \otimes \dots \otimes g_n(\pi) \\ h_i f_i = f_{\pi(i)} g_i \quad \text{for } i=1,2,\dots,n \end{matrix}$$

Proof Let $hf = fg$, that is, $(h_1 \otimes \dots \otimes h_n)(f_1 \oplus \dots \oplus f_n) = (f_1 \oplus \dots \oplus f_n)g$. Define $g_i = g|_{\Delta_i^*}$, then g_i must be in $G(\Delta_i, \Delta_{\pi(i)})$ by the assumption. Also for $i = 1,\dots,n$, $h_i f_i = f_{\pi(i)} g_i$ must hold. Because $(h_1 \otimes \dots \otimes h_n)(f_1 \oplus \dots \oplus f_n) = h_1 f_1 \otimes \dots \otimes h_n f_n = f_{\pi(1)} g_1 \otimes \dots \otimes f_{\pi(n)} g_n = (f_1 \oplus \dots \oplus f_n)g$, g must be $g_1 \otimes \dots \otimes g_n (\pi)$. The rest of the proof is easy.

Cor. 1.1 Under the same condition as in the Theorem 1,

$$h \overset{f}{\leftarrow} g \Rightarrow h_{\pi^{v_i - 1}(i)} \cdots h_{\pi(i)} h_i \overset{f_i}{\longleftarrow} g_{\pi^{v_i - 1}(i)} \cdots g_{\pi(i)} g_i$$

$$\text{where } \pi^{v_i}(i) = i \ (i=1,2,\dots,n).$$

Cor. 1.2 Under the same condition as in the Theorem 1, and if π is identity,

$$h \overset{f}{\leftarrow} g \Leftrightarrow h_i \overset{f_i}{\longleftarrow} g_i \ (i=1,2,\dots,n).$$

In Def. 5, we defined a dgsm mapping $h = h_1 \otimes \dots \otimes h_n (\pi)$ where each h_i $(i=1,\dots,n)$ has distinct domain and range. We here define another composite mapping where the ranges of all the h_i's $(i=1,\dots,n)$ are the same.

Def. 6 Let $\Sigma = \Sigma_1 \oplus \dots \oplus \Sigma_n$ and assume that n dgsm mappings $h_i \in G(\Sigma_i,\Delta)$ $(i=1,2,\dots,n)$ are given. We define a dgsm mapping $h \in G(\Sigma,\Delta)$ as $h = \mu(\gamma_1 h_1 \oplus \dots \oplus \gamma_n h_n)$ where $\gamma_i : \Delta \to \Delta_i$, $\Delta_i = \{a^{(i)} | a \in \Delta\}$, $\gamma_i(a) = a^{(i)}$ for any $a \in \Delta$ $(i=1,\dots,n)$, and $\mu : (\bigcup_i \Delta_i) \to \Delta$ such that $\mu(a^{(i)}) = a$. We denote this composition by $h = h_1 \bar{\oplus} \dots \bar{\oplus} h_n$.

Lemma 4. Let $\Delta = \Delta_1 \oplus \ldots \oplus \Delta_n$. Assume $h \in G(\Sigma,\Sigma)$, $f = f_1 \bar{\oplus} \ldots \bar{\oplus} f_n \in G(\Delta,\Sigma)$, $f_i \in G(\Delta_i,\Sigma)$, $g = g_1 \oplus \ldots \oplus g_n \in G(\Delta,\Delta)$ and $g_i \in G(\Delta_i,\Delta_i)$ $(i-1,\ldots,n)$. If f is injective, then

$$h \overset{f}{\leftarrow} g \Leftrightarrow h \overset{f_i}{\longleftarrow} g_i \quad (i=1,\ldots,n)$$

3.2. Series Decomposition

Given a simulation relation $h \overset{f}{\leftarrow} g$ and a decomposition $f = f_1 f_2$ (f_1:injective), it is not necessarily possible to decompose the relation into $h \overset{f_1}{\longleftarrow} k \overset{f_2}{\longleftarrow} g$ for some system k. We have a sufficient condition for a simulation relation to be decomposed as follows.

Theorem 2 If $h \overset{f_1 f_2}{\longleftarrow} g$ and $h \overset{f_1}{\longleftarrow} k$, then we have $k \overset{f_2}{\longleftarrow} g$.

Proof. We have $f_1 f_2 g = h f_1 f_2 = f_1 k f_2$. Because of the injectivity of f_1, we get $f_2 g = k f_2$. Note that f_2 is injective when $f_1 f_2$ is injective.

Cor. 2.1 $\left.\begin{array}{l} f_1 h_1 \overset{f_1 f_2}{\longleftarrow} g \\ f_1\text{:injective} \end{array}\right\} \Rightarrow f_1 h_1 \overset{f_1}{\longleftarrow} h_1 f_1 \overset{f_2}{\longleftarrow} g$

(Note that the left hand side simulation relation is primitive.)

Cor. 2.2 $\left.\begin{array}{l} h \overset{f_1 f_2}{\longleftarrow} g \\ h \overset{f_1}{\longleftarrow} g \end{array}\right\} \Rightarrow g \overset{f_2}{\longleftarrow} g$

Cor. 2.3 $\left.\begin{array}{l} h \overset{f_1 f_2}{\longleftarrow} g \\ h \overset{f_1}{\longleftarrow} h \end{array}\right\} \Rightarrow h \overset{f_2}{\longleftarrow} g$

In view of the above Corollaries, a special type of simulation i.e., simulation by itself $(h \overset{f}{\leftarrow} h)$ seems to deserve attention. This is the case where two gsm mappings become commutative $(hf = fh)$ in the usual sense.

§4 Commutative Mappings

In this section we investigate several properties of commutative dgsm mappings. The class of commutative dgsm mappings for a given mapping is related not only to the simplest type of simulation but also to more general types as shown below.

Def. 7 Two dgsm mappings $h, g \in G(\Sigma,\Sigma)$ are said to be commutative if $hg = gh$. Let $C(h)$ denote the class of dgsm mappings which commute with fixed dgsm h, i.e., $C(h) = \{g \in G(\Sigma,\Sigma) \mid hg = gh\}$ for $h \in G(\Sigma,\Sigma)$.

Some of the properties of $C(h)$ for $h \in G(\Sigma,\Sigma)$ are listed below.

(Pc. 1) a) $f, g \in C(h) \Rightarrow fg \in C(h)$

b) $f \in C(h_1),\ f \in C(h_2) \Rightarrow f \in C(h_1h_2)$

(Pc. 2) a) $C(h) \subseteq C(h^n)$ for any integer $n \geqq 1$

b) $C(h^m) \cap C(h^n) \subseteq C(h^{m-n})$ when h is injective and m, n are integers such that $m > n \geqq 1$.

(Pc. 3) If $h = fg = gf$, then $f, g \in C(h)$.

(Pc. 4) Let $\Sigma = \Sigma_1 \oplus \ldots \oplus \Sigma_n$ and let $h = h_1 \otimes \ldots \otimes h_n\ (\pi)$ and $g = g_1 \otimes \ldots \otimes g_n\ (\mu)$.

Then $g \in C(h) \Leftrightarrow \pi\mu = \mu\pi$ and

$$h_{\mu(i)}\, g_i = g_{\pi(i)}\, h_i \quad (i=1,\ldots,n)$$

This relates the commutative mappings with some types of simulation relation. For example, we have for the case $n = 2$, π is the identity, and $\mu = (1,2)$,

Cor. Let $h_i \in G(\Sigma_i,\Sigma_i)(i=1,2)$, $f_1 \in G(\Sigma_1,\Sigma_2)$, and $f_2 \in G(\Sigma_2,\Sigma_1)$. Assume that $\Sigma_1 \cap \Sigma_2 = \phi$ and f_1, f_2 are both injective. Then

$$h_1 \oplus h_2 \in C(f_1 \otimes f_2) \Leftrightarrow h_1 \overset{f_2}{\underset{f_1}{\rightleftarrows}} h_2$$

§5 Simulations by Invertible Mappings

In this section, we restrict ourselves to the cases where the simulations are realized through an invertible dgsm mappings. We first examine properties of invertible dgsm mappings and then analyze the simulation relations utilizing the obtained properties.

Def. 8 Let $f \in G(\Sigma,\Delta)$ be a dgsm mapping. We say that f is invertible if there exists a dgsm mapping $\tilde{f} \in G(\Delta,\Sigma)$ such that $\tilde{f}f$ is the identity mapping on Σ^*. $\tilde{f}$ is called an inverse of f, and we denote by f^{-1} one of the inverses of f.

Note that if $f \in G(\Sigma,\Delta)$ is invertible, it must be injective by definition. The following are elementary properties of invertible mappings with respect to their decompositions.

Lemma 5 Let $f = hg$ where $f \in G(\Sigma,\Delta)$, $h \in G(\Gamma,\Delta)$, and $g \in G(\Sigma,\Gamma)$. Then we have

(i) If h and g are invertible, then so is f.

(ii) If f is invertible, then so is g.

Proof. (i) Put $\tilde{f} = g^{-1}h^{-1}$. Then $\tilde{f}f = g^{-1}h^{-1}hg = g^{-1}g = \lambda_{\Sigma^*}$.

(ii) Put $\tilde{g} = f^{-1}h$. Then $\tilde{g}g = f^{-1}hg = f^{-1}f = \lambda_{\Sigma^*}$.

Lemma 6 Let $f = f_1 \otimes f_2 \otimes \ldots \otimes f_n \ (\pi)$ as defined in Def. 5. Then f is invertible if and only if all the f_i's $(i=1,2,\ldots,n)$ are invertible.

Proof. We show the case where $n = 2$. (It is easy to generalize the proof for arbitrary n.) Let $\Sigma = \Sigma_1 \oplus \Sigma_2$ and $\Delta = \Delta_1 \oplus \Delta_2$ be alphabets such that $f \in G(\Sigma,\Delta)$ and $f_i \in G(\Sigma_i,\Delta_{\pi(i)})(i=1,2)$. (1) Assume f_1 and f_2 are invertible and let $g_{\pi(i)} = f_i^{-1} \in G(\Delta_{\pi(i)},\Sigma_i)$ $(i=1,2)$. If we put $\tilde{f} = g_{\pi(1)} \otimes g_{\pi(2)} \ (\pi^{-1})$, then we have $\tilde{f}f = (g_{\pi(1)} \otimes g_{\pi(2)})(f_1 \otimes f_2) = g_{\pi(1)}f_1 \oplus g_{\pi(2)}f_2 = f_1^{-1}f_1 \oplus f_2^{-1}f_2 = \lambda_{\Sigma^*}$. (2) Assume f is invertible, then we have $f^{-1}(f_1 \otimes f_2) = \lambda_{\Sigma_1^*} \oplus \lambda_{\Sigma_2^*}$. It is easy to see that $f^{-1}|\Delta^*_{\pi(i)}$ is an inverse of f_i (i=1,2).

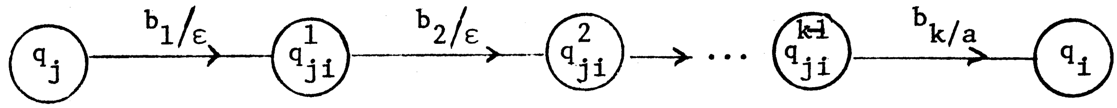
Next, we shall give a characterization of the invertible dgsm mapping f through the properties of a dgsm M_f realizing it.

Def. 9 Let $M = (\Sigma, Q, \Delta, \lambda, \mu, q_o)$ be a (completely specified reduced) dgsm where $\lambda: \Sigma \times Q \to Q$, $\mu: \Sigma \times Q \to \Delta^*$, and $q_o \in Q$. M is said to be a prefix if for every $q \in Q$, the list $\{\mu(a,q) | a \in \Sigma\}$ is a prefix. (A list of strings is a prefix if there exist no two elements in the list such that one is a (not necessarily proper) prefix of the other.)

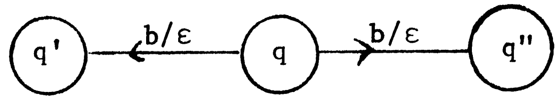
Theorem 3 Let $f \in G(\Sigma, \Delta)$ be a mapping realized by a completely specified reduced dgsm M_f. Then f is invertible if and only if M_f is a prefix.

Proof. (1) Assume $M_f = (\Sigma, Q, \Delta, \lambda, \mu, q_o)$ is not a prefix. Assume also that $\tilde{f}$ is a mapping such that $\tilde{f}f = \lambda_{\Sigma^*}$. Then for some state $q \in Q$, the list $\{\mu(a,q) | a \in \Sigma\}$ is not a prefix. That is, there exist distinct a, b $\in \Sigma$ such that $\mu(a,q) = \mu(b,q)y$ where $y \in \Delta^*$. Let $u \in \Sigma^*$ be such that $\lambda(u,q_o) = q$ and $\mu(u,q_o) = x$. Then $\tilde{f}(x\mu(b,q)) = ub$ and $\tilde{f}(x\mu(b,q)y) = \tilde{f}(x\mu(a,q)) = ua$, which shows that $\tilde{f}$ does not preserve initial subwords. Thus $\tilde{f}$ can not be a dgsm mapping. (2) Assume that M_f is a prefix. Then it is easy to see that f is injective. We construct a dgsm that realizes a mapping $\tilde{f}$ such that $\tilde{f}f = \lambda_{\Sigma^*}$. a) First, define a sequential transducer $M_f^{-1} = (\Delta, Q, \Sigma, H, q_o)$ where Δ is the input alphabet, Σ is the output alphabet, Q is the state set, $q_o \in Q$, and H is a finite subset of $Q \times \Delta^* \times \Sigma^* \times Q$ such that $(q_j, x, a, q_i) \in H$ if and only if $\lambda(a, q_i) = q_j$ and $\mu(a, q_i) = x$ where $x \in \Delta^+$ and $a \in \Sigma$. b) Next, from M_f^{-1} defined above, we are able to obtain an (incompletely specified) dgsm $\tilde{M}_f^{-1}$ as follows: i) Corresponding to each element $(q_j, b_1 b_2 \ldots b_k, a, q_i) \in H$ $(b_1, b_2, \ldots, b_k \in \Delta)$

define transitions



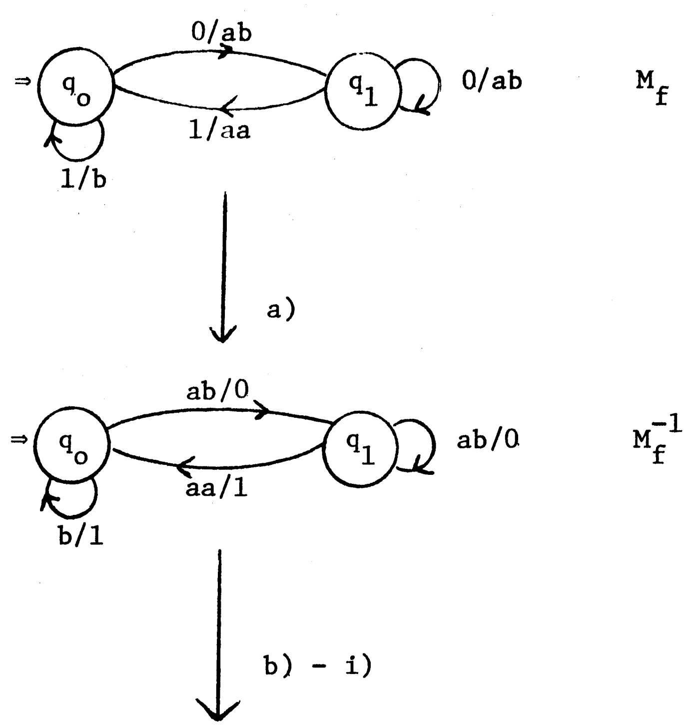
where q_{ji}^r's $(r=1,2,\ldots,k-1)$ are new states added to Q.

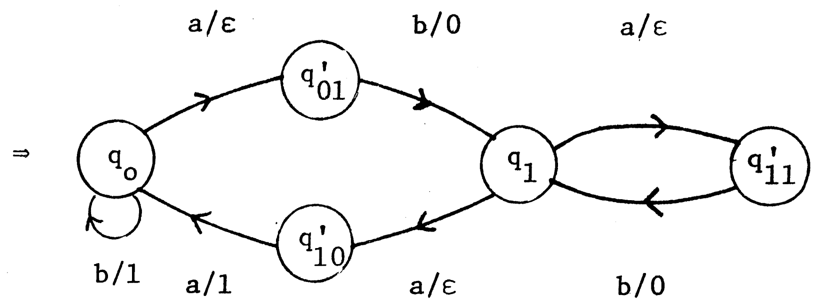
ii) After the process i), merge a pair of states q' and q" if there exist another states q such that q goes to both q' and q" by the same input with ε output. (In the diagram, 

for some $b \in \Delta$.) Repeat this process until no such merging could be possible. c) Finally, convert $\tilde{M}_f^{-1}$ to obtain a completely specified dgsm M_f-1 by adding appropriate transitions. (One of the simplest ways would be to add a self-loop with ε output label whenever necessary.)

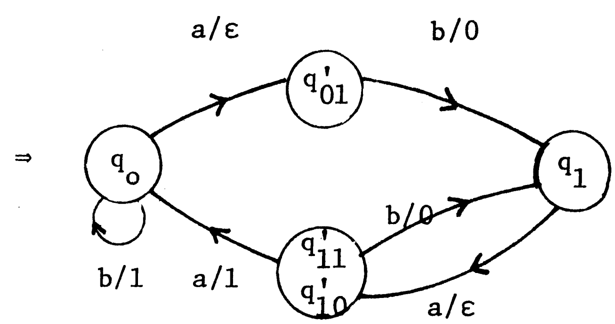
We shall give an example to illustrate the processes described in the proof (2) of Theorem 3.

Example



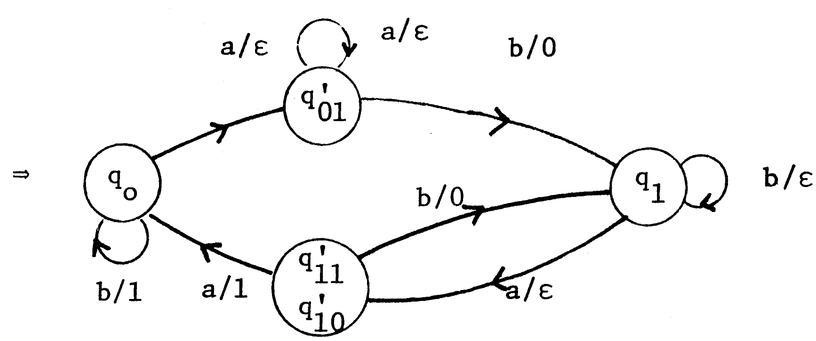


b) - ii) (q'_{11} and q'_{10} are merged)



$\tilde{M}_f^{-1}$

c)



M_{f-1}

By this theorem, we can easily determine whether a given dgsm mapping is invertible or not, although the decidability result itself is already implicit under more general context in [6].

As a corollary to the theorem, we also have a machine independent characterization of invertible dgsm mappings.

Cor. Let $f\colon \Sigma^* \to \Delta^*$ be a dgsm mapping. Then f is invertible if and only if for any $u \in \Sigma^*$ and for any distinct pair $a, b \in \Sigma$, $f(ua)$ is not a prefix of $f(ub)$.

Now we introduce a simple relation which may hold between two (dgsm) mappings having the same range as follows.

Def. 10 For $f \in G(\Gamma,\Delta)$ and $h \in G(\Sigma,\Delta)$, we say that h covers f if $h(\Sigma^*) \supset f(\Gamma^*)$.

This notion of covering plays a fundamental role when we consider the decompositions of dgsm mappings, as illustrated by the following lemmas.

Lemma 7 Let $f \in G(\Gamma,\Delta)$ and $h \in G(\Sigma,\Delta)$. Assume that h is invertible. Then the following three conditions are equivalent.

(1) h covers f;

(2) $f = hh^{-1}f$;

(3) $f = hg$ for some dgsm mapping g.

Proof (1) → (2): $h(\Sigma^*) \supset f(\Gamma^*)$ and $hh^{-1}|_{h(\Sigma^*)} = \lambda_{h(\Sigma^*)}$ imply $hh^{-1}f = f$. (2) → (3): trivial. (3) → (1): Because $f = hg$, we have $h(\Sigma^*) \supseteq f(\Gamma^*)$.

Lemma 8 Let $f \in G(\Gamma,\Delta)$ and $h \in G(\Sigma,\Gamma)$. Assume that fh is invertible. Then the following three conditions are equivalent.

(1) fh covers f;

(2) $f = fhh^{-1}$ for some h^{-1};

(3) $f = gh^{-1}$ for some (invertible) mapping g.

Proof. (1) → (2): Put $\tilde{h} = (fh)^{-1}f$. Then $\tilde{h}h = (fh)^{-1}fh = \lambda_{\Sigma*}$. That is, $\tilde{h}$ is an inverse of h. Let $h^{-1} = \tilde{h}$. Then we have $fhh^{-1} = fh(fh)^{-1}f = f$ because fh covers f. (2) → (3): trivial. (3) → (1): We have $fhh^{-1} = gh^{-1}hh^{-1} = gh^{-1} = f$, which implies that fh covers f.

Now we turn to the simulation relation where the simulation gsm mapping is invertible. When $h \in G(\Delta,\Delta)$ simulates $g \in G(\Sigma,\Sigma)$ through an invertible $f \in G(\Sigma,\Delta)$, we write $h \overset{f}{\leftarrow} g$ instead of writing $h \overset{f}{\leftarrow} g$ to emphasize the invertibility of f. First, we note the following fundamental properties.

Lemma 9 Let $f \in G(\Sigma,\Delta)$ be invertible.

(i) For any $h \in G(\Delta,\Delta)$, there exists a unique $g \in G(\Sigma,\Sigma)$ such that $h \overset{f}{\leftarrow} g$, if and only if f covers hf. (In fact, $g = f^{-1}hf$.)

(ii) For any $g \in G(\Sigma,\Sigma)$, there exists $h \in G(\Delta,\Delta)$ of the form fgf^{-1} such that $h \overset{f}{\leftarrow} g$. (Note that this is a primitive simulation.)

Proof. (i) Let $hf = fg$. Then $g = f^{-1}hf$ and we have $hf = ff^{-1}hf$, which means f covers hf by Lemma 7. If f covers hf, then there exists g such that $hf = fg$ by Lemma 7. The uniqueness of g is also guaranteed by the fact that f covers hf. (ii) Let $h = fgf^{-1}$ for an inverse f^{-1}. Then $hf = fgf^{-1}f = fg$.

As to the decomposition of simulation relation when the coding gsm is invertible, we have the following two basic results.

Theorem 4 Let $h \overset{f_1 f_2}{\longleftarrow} g$ for $h \in G(\Sigma,\Sigma)$, $g \in G(\Delta,\Delta)$, $f_1 \in G(\Theta,\Sigma)$, and $f_2 \in G(\Delta,\Theta)$. Assume that f_1 is invertible. Then the simulation relation can be decomposed as $h \overset{f_1}{\longleftarrow} k \overset{f_2}{\longleftarrow} g$ for some $k \in G(\Theta,\Theta)$ if and only if f_1 covers hf_1.

Proof. By Theorem 2 and Lemma 7.

Theorem 5 Let $h_1h_2 \overset{f}{\leftarrow} g$ for $h_1 \in G(\Gamma,\Sigma)$, $h_2 \in G(\Sigma,\Gamma)$, $f \in G(\Delta,\Gamma)$, and $g \in G(\Delta,\Delta)$. Assume that h_1 is invertible and h_2 is injective. Then the simulation can be decomposed as $h_1h_2 \overset{h_1}{\longleftarrow} h_2h_1 \overset{h_1^{-1}f}{\longleftarrow} g$ if and only if h_1 covers f.

Proof. When the decomposition is occurring, we have $h_2h_1h_1^{-1}f = h_1^{-1}fg = h_1^{-1}h_1h_2f = h_2f$. Thus $h_1h_1^{-1}f = f$ which is equivalent to ' h_1 covers f ' by Lemma 7. If h_1 covers f, then $h_1h_1^{-1}f = f$ by the same lemma. Then we have $h_1h_2h_1h_1^{-1}f = h_1h_2f = fg = h_1h_1^{-1}fg$, which implies $h_2h_1h_1^{-1}f = h_1^{-1}fg$.

§6 Concluding Remarks

In general systems theory, the principle of simulation plays an essential role [7]. We presented some basic properties of simulation relations of dynamical systems in the case where the dynamics and the simulation mapping are given by dgsm's. We paid a special attention to the cases where the simulation mapping is invertible.

Although the treatments reported here are by no means complete, we hope the concepts and the methods introduced are useful in revealing the fundamental dynamical structure of systems.

Some of the relevant problems we have not touched upon in this report include:

1) For given two systems decide if there holds the simulation relation through some simulation mapping.

2) Establish suitable complexity measure of gsm mappings and relate it with their decomposition structures.

3) Investigate the properties (e.g., invertibility, decomposition structure, etc.) of gsm mappings with endmarker [6] and generalize the results of this report for that case.

References

[1] Hopcroft, J.E. and Ullman, J.D. Introduction to Automata Theory, Languages, and Computation. Addison-Wesley Publ. Comp. (1979).

[2] Kobuchi, Y. Decomposition of Homomorphism and Embedding of DOL Scheme. 10th Summer Language and Automata Symposium 26-29 July 1979 Fukuoka, Japan.

[3] Ehrenfeucht, A. and Rozenberg, G. Simplifications of Homomorphisms. Inform. and Contl. 38, 298-309 (1978).

[4] Herman, G.T. and Rozenberg, G. Developmental Systems and Languages. North-Holland Publ. Comp. (1975).

[5] Culik II, K. Some Decidability Results About Regular and Pushdown Translations. Information Processing Letters 8, No. 1 5-8 (1979).

[6] Choffrut, C. Une Caracterisation des Fonctions Sequentielles et des Fonctions Sous-sequentielles en tant que Relations Rationnelles. Theoretical Computer Science 5, 325-337 (1977).

[7] Klir, G.J. (ed.) Trends in General Systems Theory. Wiley-Interscience (1972).