VECTOR ADDITION SYSTEMS AND REGULAR LANGUAGES

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Abstract

Necessary and sufficient conditions are established for Vector Addition Systems to define regular languages. An algorithm is designed to decide whether these conditions are satisfied. The reachability problem for such Vector Addition Systems is shown to be decidable.

1. Introduction

Vector Addition Systems with finite reachability sets [1] clearly define regular languages. However, there also exist Vector Addition Systems with infinite reachability sets defining regular languages. This paper investigates such Vector Addition Systems.

Having established necessary and sufficient conditions for a VAS to define a regular language (Section 3), an algorithm is designed to decide whether these conditions are met (Section 4). This algorithm is based on the Karp=Miller procedure [1] for deciding whether the reachability set is finite.

A simple corollary of the main result shows that the reachability problem is decidable for a VAS defining a regular language.

2. Preliminaries

$\mathbb{Z}$ will denote the set of all integers, N the set of all nonnegative integers. A vector is usually an element of Z^n or of N^n for a fixed n. $a[i]$ denotes the i-th coordinate of the vector a. For $a,b \in Z^n$, $a+b$ is defined as usual and $a \geq b \Leftrightarrow a[i] \geq b[i]$ for $i = 1,2,\ldots,n$. 0 is the zero vector, so $a \in N^n \Leftrightarrow a \geq 0$.

For a set V of vectors, V^* denotes the set of all finite strings of elements of V, including the empty string λ. If $w = v_1 v_2 \ldots v_k$ ($v_1,v_2,\ldots,v_k \in V$), then $a+w$ will denote the vector $a+v_1+v_2+\ldots+v_k$. Also $a+\lambda = a$. If w is obtained by the concatenation of two strings x and y from V^*, i.e. $w = xy$, x is said to be a prefix of w. So the empty string λ and w itself are prefixes of w. Clearly, $a+xy = a+x+y$.

A Vector Addition System (VAS) is an ordered pair

$$A = (V,a_0)$$

where $V \subseteq Z^n$ is a finite set of integer vectors and $a_0 \in N^n$, i.e. a_0 is a vector of nonnegative integers.

A string $w \in V^*$ is said to be legal in A if for every prefix x of w, $a_0+x \in N^n$. The set of all legal strings in A is the language of A and will be denoted by $L(A)$.

Example 1: Consider the 3-dimensional VAS $A = (V,a_0)$, where

$$V = \{ v_1 = (-1,2,0),\ v_2 = (1,-3,2),\ v_3 = (0,0,-1)\},\quad a_0 = (4,0,1).$$

$w = v_1 v_3 v_1 v_2$ is a legal string in A.

Indeed: $a_0+v_1 = (3,2,1)$, $a_0+v_1v_3 = (3,2,0)$, $a_0+v_1v_3v_1 = (2,4,0)$, $a_0+v_1v_3v_1v_2 = (3,1,2)$, and all these vectors are ≥ 0.

On the other hand $v_1v_3v_2v_1$ is not legal, since $a_0+v_1v_3v_2 = (4,-1,2) \notin N^3$.

The reachability set $R(A)$ of the VAS A is the set of all vectors a_0+w in N^n, where w is a legal string in A. The Reachability Problem for Vector Addition Systems consists of finding a procedure to decide for every given VAS A and for every given vector b whether $b \in R(A)$. It is not known if this problem is solvable.

Karp and Miller [1] developed a procedure which enables us to decide if a given VAS A has a finite $R(A)$.

Given a VAS A, they construct a finite rooted tree $Tr(A)$: this is a connected directed graph with every vertex except one, the root, having exactly one incoming edge; the root has no incoming edges; the vertices are labelled by vectors in $(N \cup \{\omega\})^n$, where ω is a symbol ("infinity") such that $n < \omega$ and $n+\omega = \omega$, for every $n \in Z$.

If no label of $Tr(A)$ contains an ω, then $R(A)$ is finite. If some label contains one or more symbols ω, then there exist in $R(A)$ elements with arbitrarily large integers at the corresponding coordinates. $R(A)$ is, of course, infinite in this case. For every vector $b \in R(A)$ there exists in $Tr(A)$ a label ℓ such that $b \leq \ell$ (cf. [1]). For later use, denote by M the set of all maximal labels of $Tr(A)$ with respect to the partial ordering defined by the relation $\leq$. Since $Tr(A)$ is finite M is, of course, finite too.

3. Vector Addition Systems defining Regular Languages

Let $A = (V,a_0)$ be a VAS and $R(A)$ its reachability set. For $b,c \in R(A)$ define $b\ E\ c$ iff $(\forall w \in V^*)$ $b+w \in R(A) \Leftrightarrow c+w \in R(A)$. E is an equivalence relation. Moreover $b\ E\ c \Rightarrow [(\forall w \in V^*)\ b+w \in R(A) \Rightarrow (b+w)\ E\ (c+w)]$.

Theorem 1: $L(A)$ is regular iff $R(A)/E$ is finite.

Proof: Let $R(A)/E$ be finite. Construct a finite, non-deterministic automaton B as follows. The states of B are the elements of $R(A)/E = \{[a_1],[a_2],\ldots,[a_k]\}$. (Notice: $[a_i]$ is the equivalence class containing the vector $a_i \in R(A)$).

The input set of B is V. The transition function σ of B is defined as follows.

For $[a_i] \in R(A)/E$ and $v \in V$:

if $a_i+v \in R(A)$ then $\sigma([a_i],v) = [a_i+v]$,

if $a_i+v \notin R(A)$ then $\sigma([a_i],v)$ is not defined.

This definition of σ does not depend on the particular representative of the equivalence class. The initial state of B is $[a_0]$, and every state of B is a final state. Let $w = v_1v_2\ldots v_r \in L(A)$. Then $a_0+v_1v_2\ldots v_s \in R(A)$ for $0 \le s \le r$. Hence $\sigma([a_0+v_1\ldots v_{s-1}],v_s)$ is defined for every s $1 \le s \le r$, and $w \in L(B)$. Conversely, let $w = v_1v_2\ldots v_r \in L(B)$. Since every state of B is a final state, every prefix of w is also in $L(B)$. It follows that v_1, $v_1v_2,\ldots,w$ are all legal strings in A, i.e. $w \in L(A)$. Thus $L(A) = L(B)$, whence $L(A)$ is regular.

Let $L(A)$ now be regular. Define the relation $\approx$ on $L(A)$ as follows:

$$x \approx y \text{ iff } (\forall z \in V^*) \; xz \in L(A) \Leftrightarrow yz \in L(A).$$

It is well known [2] that $\approx$ is an equivalence relation and moreover $L(A)/\approx$ is finite. But $x \approx y \Rightarrow a_0+x \; E \; a_0+y$. Indeed, $a_0+x \in R(A)$ and $a_0+y \in R(A)$, since $x,y \in L(A)$. Now, if for some $z \in V^*$, $a_0+xz \in R(A)$, then $xz \in L(A)$; hence $yz \in L(A)$, and $a_0+yz \in R(A)$. It follows that $a_0+x \; E \; a_0+y$. Consequently, the number of equivalence classes of E is not greater than those of $\approx$, i.e. $R(A)/E$ is finite. □

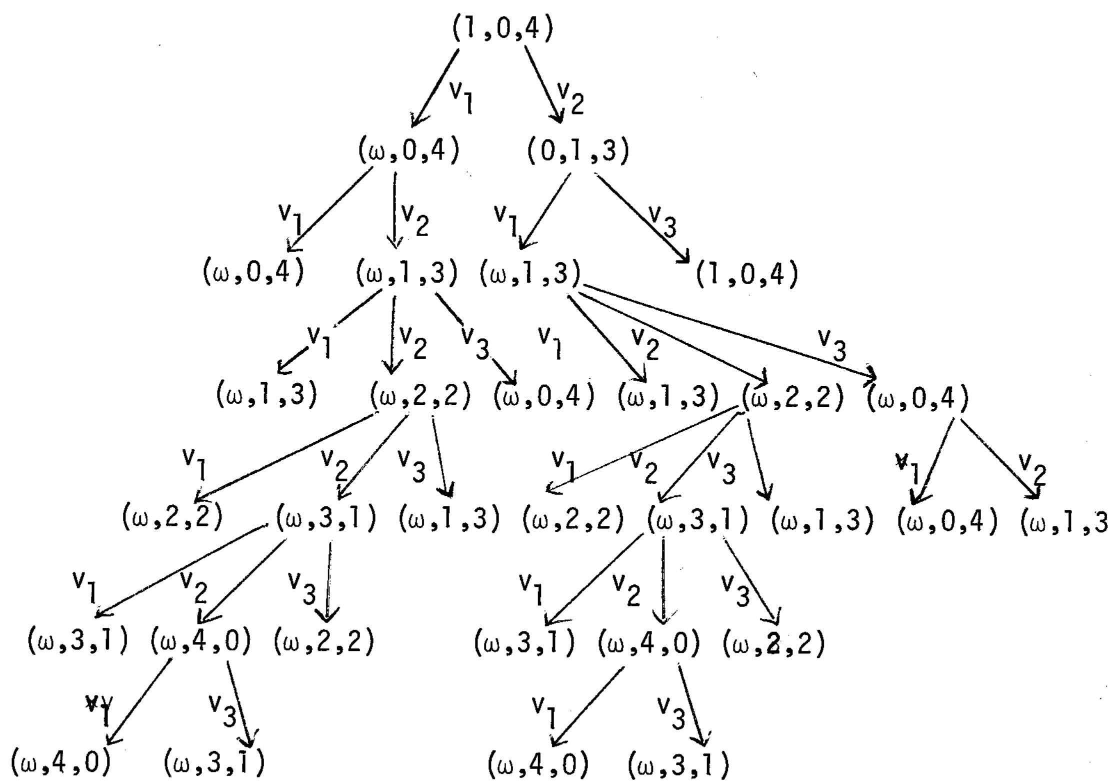
Every VAS with a finite $R(A)$ clearly satisfies the condition of the above theorem. But there evidently also exist Vector Addition Systems with infinite $R(A)$ defining regular languages. In a given VAS $A = (V,a_0)$ the i-th coordinate is said to be <u>unbounded</u> if for every $p \geq 0$ there exists a vector $b \in R(A)$ such that $b[i] > p$. This is the case iff there exists a label $m \in M$ with $m[i] = \omega$.

<u>Lemma 1</u>: Assume that in a VAS $A = (V,a_0)$ $k \leq n$ coordinates (say the first k coordinates) are unbounded. Assume also that there exist k nonnegative integers $h_1,h_2,\ldots,h_k$ such that for every $b \in R(A)$, every $w \in V^*$, and every $i = 1,2,\ldots,k$, $(b+w) \in R(A) \Rightarrow (b+w)[i] \geq b[i] - h_i$. Then $R(A)/E$ is finite.

<u>Proof</u>: Consider an arbitrary subset J of the set $K = \{1,\ldots,k\}$. Let $C_J \subseteq R(A)$ be the set of all vectors c in $R(A)$ such that $c[j] \geq h_j$ for every $j \in J$, and $c[i] < h_i$ for every $i \in K-J$. Let $c_1,c_2 \in C_J$ and assume that for every $i \in \{1,2,\ldots,n\} - J$ $c_1[i] = c_2[i]$. Then $c_1 \; E \; c_2$. Indeed, if $w \in V^*$, then $c_1+w \geq 0 \Leftrightarrow c_2+w \geq 0$ (the coordinates in J cannot become negative in both vectors, since these coordinates in c_1 and

c_2 are greater than the corresponding h_j; the other coordinates of c_1 and c_2 are equal). But there exists only a finite number of possible $(n - |J|)$-tuples for the coordinates $\{1,2,\ldots,n\} - J$ of the vectors in C_J, since all of them are bounded. Consequently C_J/E is finite. Now there exists only a finite number of subsets J of K, hence the number of C_J's is finite. Every $b \in R(A)$ belongs to one of these C_J's, hence $R(A)/E$ is also finite.

Example 2: Consider the VAS $A = (V,a_0)$ with $V = \{v_1 = (1,0,0),\ v_2 = (-1,1,-1),\ v_3 = (1,-1,1)\}$, $a_0 = (1,0,4)$. The tree $Tr(A)$ is shown below:



The set of maximal labels $M = \{m_1 = (\omega,0,4),\ m_2 = (\omega,1,3),\ m_3 = (\omega,2,2),\ m_4 = (\omega,3,1),\ m_5 = (\omega,4,0)\}$.

The first coordinate of this VAS is unbounded, all others are bounded. It will be shown later, that $h_1 = 4$ satisfies the conditions of Lemma 1. Thus $L(A)$ is regular in this case.

Conversely to Lemma 1, one has:

Lemma 2: Let $A = (V,a_0)$ be a VAS and assume that there exists an unbounded coordinate j such that for every $g \geq 0$ there exists a vector $b \in R(A)$ and a string $w \in V^*$ such that $(b+w) \in R(A)$ and $b[j] - (b+w)[j] > g$. Then the set $R(A)/E$ is infinite.

Proof: Let $\xi = \min_{v \in V} \{v[j]\}$ (ξ is of course negative in this case). Let $g \geq 0$, $b \in R(A)$, and $w = v_1 v_2 \ldots v_r$ be as above. In the chain of vectors of $R(A)$: $b, b+v_1, b+v_1v_2, \ldots, b+w$ there can be not less than $\frac{g}{|\xi|}$ distinct vectors with decreasing j-th coordinates. No two such vectors can be E-equivalent. Indeed, let c_1 and c_2 be two such vectors, where $c_2 = c_1+x$, $x \in V^*$, and $c_2[j] < c_1[j]$. We have:

$$c_1 \ E \ c_2 \quad \text{i.e.} \quad c_1 \ E \ (c_1+x) \Rightarrow (c_1+x) \ E \ (c_1+2x) \Rightarrow \ldots$$

But for some i, $c_1 + (i+1)x$ does not belong to $R(A)$, whereas $(c_1+ix) \in R(A)$. Hence c_1 and c_2 are not E-equivalent. Since g may be arbitrarily large $R(A)/E$ is infinite.

4. <u>An Algorithm to decide whether a VAS defines a regular language.</u>

In this section an algorithm will be developed to decide whether the conditions of Lemma 1 are satisfied by a given VAS A. First notice that if $b \le c$ $(b,c \in R(A))$, then for every $w \in V^*$, $b+w \in R(A) \Rightarrow c+w \in R(A)$. Hence, if b satisfies the condition of Lemma 2 for a given g, so does c.

Let now $m \in M$. A vector $c \in R(A)$ is said to be <u>represented</u> by m iff for every i such that $m[i] \ne \omega$, $c[i] = m[i]$.

If there exist integers $h_1,\ldots,h_k$ such that the assumptions of Lemma 1 hold with respect to these h_i's for all vectors $c \in R(A)$ represented by the vectors in M, then these assumptions will hold for every vector $b \in R(A)$ with respect to the same h_i's.

The main tool which enables us to establish the required algorithm is

<u>Lemma 3</u>: Let $A = (V,a_0)$ be a VAS, and assume that for some integer $g > 0$, there exists a vector $b \in R(A)$ and a string $z \in V^*$ such that $b+z \in R(A)$ and for some i,

$$b[i] - (b+z)[i] \ge g.$$

Then there exists a maximal label $m \in M$ and a string $w \in V^*$ such that for some vector $c \in R(A)$, represented by m, $c+w \in R(A)$, and for one of its coordinates, say the j-th

$$c[j] - (c+w)[j] \ge g,$$

and, in addition, for every prefix x of w

$$(c+x)[j] \le c[j].$$

Proof: Let $w \in V^*$ be the shortest (one of the shortest) string among all strings "decrementing" by not less than g a certain coordinate of a certain vector in $R(A)$ represented by one of the elements of M. By assumption of the lemma such a string w must exist. Assume that this string w decrements by at least g the j-th coordinate of the vector $c \in R(A)$, i.e., $c+w \in R(A)$, and $c[j] - (c+w)[j] \geq g$.

Let now $w = xy$ ($x \in V^* - \{\lambda\}$) and assume that $(c+x)[j] > c[j]$.

The vector $c+x \in R(A)$ and there exists a vector d in $R(A)$ represented by some $m_1 \in M$, such that $d \geq c+x$. Now $(c+x) + y \in R(A)$, hence $d+y \in R(A)$. Moreover $d[j] - (d+y)[j] = (c+x)[j] - (c+x+y)[j] > c[j] - (c+w)[j] \geq g$.

Thus the string $y \in V^*$ which is shorter than w decrements by at least g a certain coordinate of a certain vector in $R(A)$ represented by a label of M. This contradicts the choice of w, and the lemma is proved. □

Algorithm

Step 1. Construct the tree $Tr(A)$ as in [1].

Step 2. Determine the set M.

Step 3. For every $m \in M$ and every i such that $m[i] = \omega$, construct a VAS $A(m,i)$ as follows.

Let $m \in M$ be such that $\{m[j_1], m[j_2], \ldots, m[j_k]\} \subseteq N$ and the remaining n-k coordinates of m are ω. For every i such that $m[i] = \omega$ define a $(k+1)$-dimensional VAS $A(m,i) = (V(m,i), a_0(m,i))$. For every $v \in V$ define $v(m,i) = (-v[i], v[j_1], v[j_2], \ldots, v[j_k])$. Let $V(m,i) = \{v(m,i) | v \in V\}$ and $a_0(m,i) = (0, m[j_1], m[j_2], \ldots, m[j_k])$.

Step 4. Determine (using the Karp-Miller procedure, or in any other way) whether all $R(A(m,i))$ are finite.

Theorem 2: A VAS A satisfies the assumptions of Lemma 1 iff all reachability sets of the Vector Addition Systems $A(m,i)$ defined in the above algorithm are finite.

Proof: If $R(A(m,i))$ is infinite, then for every $g > 0$ there exists a $w(m,i) \in V^*(m,i)$ such that $(a_0(m,i) + w(m,i))[1] \geq g$. Let c be a vector in $R(A)$ represented by the above $m \in M$, and such that the coordinates of c corresponding to the ω-coordinates of m are sufficiently large (cf. Theorem 4.2 of [1]). Let $w \in V^*$ be the string corresponding in the obvious way to $w(m,i)$. Then $c+w \in R(A)$ and $c[i] - (c+w)[i] = (a_0(m,i) + w(m,i))[1] \geq g$. Since this will occur for every g the VAS A cannot satisfy the assumptions of Lemma 1.

Conversely, assume that all $R(A(m,i))$ are finite. Let h be greater than every coordinate in all vectors of all $R(A(m,i))$.

Assume that in the VAS A some coordinate of a certain vector may be decremented by more than h. Then by Lemma 3, there exists a $m \in M$ and a string $w \in V^*$, such that for some vector $c \in R(A)$ represented by m, $c+w \in R(A)$ and say $c[i] - (c+w)[i] > h$ and for every prefix x of w $(c+x)[i] \leq c[i]$. Consider the VAS $A(m,i)$, and the string $w(m,i) \in V^*(m,i)$ corresponding to the above w. Now $a_0(m,i) + w(m,i) \in R(A(m,i))$. Indeed, for every prefix $x(m,i)$ of $w(m,i)$ one has

$$(a_0(m,i) + x(m,i))[1] = - ((c+x)[i] - c[i]) \geq 0,$$

and for all other coordinates $\ell = 2,3,\ldots,k+1$ $(\bar{a}_0(m,i) + x(m,i))[\ell] = (c+x)[j_{\ell-1}]$.

Now $(a_0(m,i) + w(m,i))[1] = -((c+w)[i] - c[i]) > h$ but this is impossible, because h was larger than any coordinate in any $R(A(m,i))$. This contradiction shows that the VAS A satisfies the assumptions of Lemma 1, with $h_1 = h_2 = \ldots = h_k = h$.

Example 2 cont'd.: Apply the algorithm to the VAS $A(V,a_0)$ defined in Example 2.

Step 1 and Step 2 have already been performed.

Step 3. Take one of the labels of M with an ω-coordinate, say $m_2 = (\omega,1,3)$. The VAS $A(m_2,1)$ is as follows:

$$V(m_2,1) = \{v_1(m_2,1) = (-1,0,0),\ v_2(m_2,1) = (1,1,-1),\ v_3(m_2,1) = (-1,-1,1)\}$$
$$a_0(m_2,1) = (0,1,3).$$

Similarly one constructs the other $A(m,i)$ for all the other $m \in M$ and their ω-coordinates.

Step 4. Consider the above $A(m_2,1)$. Let $w(m_2,1) \in V^*(m_2,1)$ and consider the vector $d = a_0(m_2,1) + w(m_2,1)$. In order to obtain say $d[1] \geq 10$, the number of occurrences of $v_2(m_2,1)$ in $w(m_2,1)$ must exceed the number of occurrences of $v_3(m_2,1)$ at least by 10. But in order to have $d[3] > 0$, the number of occurrences of $v_2(m_2,1)$ in $w(m_2,1)$ can exceed the number of occurrences of $v_3(m_2,1)$ only by 3, so one shall never get $d[1] \geq 10$. Similarly $d[2]$ and $d[3]$ are clearly bounded say by 10. Consequently $R(A(m_2,1))$ is finite.

One may check similarly that all other $R(A(m,i))$ are also finite for this A; hence $L(A)$ is regular.

Corollary: The reachability problem for a VAS A with L(A) regular is decidable.

Proof: Let $A(V,a_0)$ be a VAS with L(A) regular. By Theorem 1, R(A)/E is finite, hence by Lemma 2, the decrements of all coordinates of the vectors in R(A) are bounded. Let h be the integer determined in the proof of Theorem 2; then for every unbounded coordinate i the decrements of the i-th coordinate of every vector in R(A) are bounded by h. Notice that h can be determined by a finite procedure.

Let now $x = (x_1, x_2, \ldots, x_n) \in N^n$. In order to determine if $x \in R(A)$, construct a labelled rooted tree similar to that in the Karp-Miller procedure, but rather than introducing ω, continue the process. A node in the tree is an end (i.e. it does not have successors) in the following cases:

(1) The label of the node is d and for every $v \in V$, at least one coordinate of d+v is negative.

(2) Its label equals that of one of its ancestors.

(3) One of the coordinates of its label exceeds the corresponding coordinate of the vector x by h.

The tree is clearly finite, and $x \in R(A)$ iff x is a label of one of the nodes of the constructed tree. □

References

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